In this chapter I recap and discuss the results and conclusions of the reported research and development work, that is, I go into the main aspects of scientific inquiry and authenticity addressed in the case studies and into the tool design of an integrated computer environment for inquiry-oriented mathematics and science education. Finally, I reflect on my work as a whole and go into possible implications for future research and development.

4.1 Analysis Framework

My reflection on the outcomes of the exploratory case studies is to a large extent oriented toward answering the following two questions, which were driving the research and development work:

- How can the use of ICT and in particular of an integrated computer learning environment contribute to the realization of challenging, cross-disciplinary practical work of good quality, in which pre-university students can work with real data, apply mathematical methods and techniques in concrete problem situations, improve their mathematical and scientific knowledge and skills, and increase their mathematical and scientific literacy?

- What integrated tools should the computer learning environment provide for inquiry-oriented mathematics and science education? What are the requirements for the computer learning environment from a mathematical point of view and do they link up with requirements coming from science fields?

In other words, my reflection is on the following goals of my research and development work: (1) better understanding of how, why, and to what extent ICT tools can support students in their learning and practice of scientific inquiry; and (2) more insight in what it takes to develop an integrated computer environment for learning mathematics and science in an inquiry-oriented approach, the usability of which is explored within educational practice.

As was explained in Section 1.4, the exploratory case studies served many goals in this respect:
• They were meant to gain insight in the needs of secondary school students for doing authentic inquiry work.

• They helped me specify requirements for an integrated computer learning environment from a mathematical point of view.

• They served to test the usability and scope of (prototypical) implementations of particular tools for collecting, processing, and analyzing data.

• They gave an impression of the potential of ICT regarding the realization of challenging, cross-disciplinary practical work in which secondary school students engaged in activities such as experimenting, data collection, and data analysis in much the same way as scientists and practitioners (without requiring that they behave exactly as professionals).

The answering of the driving questions—How can ICT contribute to the realization of authentic inquiry activities in which students develop mathematical and scientific literacy? What kind of tool-based environment could support inquiry-oriented mathematics and science education?—depends much on the kind of scientific inquiry by students that one has in mind and one’s view on how students can best learn to do inquiry and develop research abilities. In my research study, I focussed on ICT-supported quantitative mathematical modeling activities, that is, on activities in which students explore mathematical models with the support of ICT tools in order to come to grips with natural phenomena and to interpret real data, which they often collect themselves.

In most classroom case studies, I opted for a structured and guided inquiry approach, interpreted here as instruction in which students are first directed in their work—mostly in order to brush up required knowledge and skills, to make them familiar with ICT tools needed in the investigation, or to direct the pathway of inquiry—and in which students at the later stage of the investigation can choose an optional activity or carry out a more open task in which they have to make own decisions on how to proceed. From a research point of view, because of the explorative nature of my study, I explored how the designed instructional materials and the ICT tools functioned in school practice, whether learning goals were achieved, what obstacles students encountered in their work, what recommendations for improvement of the instructional materials and the ICT tools could be made, and so forth.

In the field experiments and the usability studies, the type of inquiry more often resembled an open investigation in which students could autonomously explore a phenomenon instead of using instructional materials prepared in advance to set out the mathematical modeling route. In these case studies, I explored the potential of ICT for realizing challenging, authentic practical work. They can also be considered as sample activities showing teachers and students what is possible when appropriate tools are available and inspiring them to let their students do similar inquiry activities. By the way, three of these cases studies were strongly linked with student profile projects:

1. Gait analysis via electromyography (Section 3.4.1).
2. Understanding the motion of a falling chain (Section 3.5.4).
3. Exploring the giant circle on the high bar (Section 3.5.7).
4.2. Aspects of Scientific Inquiry and Authenticity

My reflection on the outcomes of the field experiments and usability studies concerns to a large extent the design of an appropriate tool-based environment to support inquiry-oriented mathematics and science education.

With regard to scientific inquiry, and in particular quantitative mathematical modeling, I structure my reflection around the following two aspects of practical work:

- Abilities that are considered central in scientific inquiry and quantitative mathematical modeling, and that the students practiced in the classroom activities and field experiments. One may think of abilities like planning, carry out and evaluating an experiment, handling ICT tools, handling data, graph sense, symbol sense, making inferences and drawing conclusions, mathematizing, setting up a computer model, representational fluency, and so forth.

- Characteristics of practical investigations such as the level of authenticity, the role of ICT, the developers' objectives, the instructional design and activity specification, the quality of students' work, students' obstacles and blockages in their investigations, and the level of engagement of the students.

My reflection on an appropriate tool-based environment for inquiry-oriented mathematics and science education is directed toward the design and implementation of such an environment, based on the experiences obtained from the development of version 5 and 6 of Coach. I discuss how new ICT tools and improved versions of existing tools functioned in practice and what potential they have for students' practical investigations. I concentrate on video analysis, computer modeling, the combination of these two types of activities, and on data analysis.

4.2 Aspects of Scientific Inquiry and Authenticity

In this section I reflect on the outcomes of the exploratory case studies presented in the previous chapters concerning aspects of scientific inquiry and authenticity in the practical investigations, field experiments, and usability studies. The ordering in which the findings are presented is irrelevant. I start with quantitative mathematical modeling abilities addressed and playing an important role in the practical investigations, and then I continue with discussing some characteristics of the practical work, including the authentic nature of the student activities.

4.2.1 Quantitative Mathematical Modeling Competency

I go into empirical inquiry and mathematical modeling competency via a framework consisting of two strongly connected cycles and I go into other competencies relevant for quantitative mathematical modeling.

Empirical Inquiry and Mathematical Modeling Competency

One could say that a person possesses quantitative mathematical modeling competency if (s)he is across several contexts able to autonomously and insightfully carry through all aspects of a quantitative mathematical modeling process in order to come to grips with natural phenomena and to interpret real data in a given situation (cf.,
Blomhøj and Jensen, 2003, 2007; Jensen, 2007). It develops over a long series of quantitative mathematical modeling activities inside and outside school, starting in inquiry-oriented mathematics and science education with small scaffolded tasks and ending with large autonomous investigations. The two major components of quantitative mathematical modeling, which each have their own cycle of sub-processes involved, are (1) mathematical modeling, and (2) the design and conduct of experiments to generate useful data. In the instructional design of some mathematical modeling activities described in Chapter 2 I used the mathematical modeling cycle of Blum and Leiß (2005, p. 1626), shown in Figure 2.5 (p. 43), for the description of the cognitive processes involved in the mathematical modeling part. But too little attention is actually paid in this framework to the process of creating and handling real data for the purpose of comparing model results with empirical data. An inquiry cycle for experimental investigative work should be added as a second dimension in the model of modeling. Figure 4.1 combines an empirical inquiry cycle (with transitions 1, 2, iii, iv, v, 6, 7) and the mathematical modeling cycle of Blum and Leiß (with transitions 1, 2, 3, ..., 7) into a single framework. It resembles the MEI framework of theoretical and empirical modeling (MEI/OCR, 2003) and the modeling-experimental bi-cycle of Fuchs (2007, 2008) as a visual model for a form of the scientific method (Extended by Fuchs to a quadruple-cycle by inclusion of processes related to hypothesizing and generating good questions). It can be used for the following purposes:

- Structuring quantitative mathematical modeling activities.
- Analyzing students’ work and their modeling route. This includes identification, documentation, and analysis of students’ blockages, obstacles, or missed opportunities (cf., Galbraith & Stillman, 2006; Schaap, Vos, & Goedhart, 2011a; Stillman, Brown, & Galbraith, 2010).
- Elaborating one’s view on quantitative mathematical modeling competency.

In Table 4.1 I exemplify the various stages and transitions in this framework in the context of the student activity of mathematical exploration of shapes of objects, in this case of a suspension bridge and a hanging chain (Section 2.3.2). These stages have been discussed to a certain extent in Section 2.8 in the context of bouncing balls. For another explanation of the theoretical modeling in a different context, the interested
4.2. Aspects of Scientific Inquiry and Authenticity

reader is referred to a similar illumination of the steps in the modeling process by Leiß, Schukajlov, Blum, et al. (2010).

<table>
<thead>
<tr>
<th>General description of the transition</th>
<th>Sample result of the transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. First, the real situation has to be understood, for example by starting with a photograph. An individual mental model of the real situation must be constructed, which can often be expressed in a schematic drawing.</td>
<td><img src="image1" alt="Sample result of the transition" /></td>
</tr>
<tr>
<td>2. By structuring, making assumptions, and simplification one has to turn the situation into a real model with a concrete presentation of a question: How does a nonelastic chain with objects of equal weight, symmetrically attached at equal horizontal distance, hang under gravity (ignoring the mass of the chain)?</td>
<td><img src="image2" alt="Sample result of the transition" /></td>
</tr>
<tr>
<td>3. Using basic concepts of mathematics and physics one mathematizes and transforms the real model into a mathematical model (e.g., a differential equation). In case of a small number of weights, a more appropriate approach is a free-body diagram involving gravitational and tension forces, and one can reason about slopes of chain segments.</td>
<td>ODE is $y'' = \frac{1}{c}$ for certain constant $c$. The positive slopes of consecutive chain segments between two weights have a fixed ratio, namely 1:3:5:7:...</td>
</tr>
<tr>
<td>4. Then one applies mathematical knowledge, skills, and tools to obtain mathematical results (e.g., a solution of a differential equation). The mathematical model may also be transformed into a computer model that is used to obtain numerical results.</td>
<td><img src="image3" alt="Sample result of the transition" /></td>
</tr>
<tr>
<td>5. By activating basic ideas again, one interprets the mathematical results in the real world as real results for the given problem.</td>
<td>The kinks in the chain are on a parabola. By using a large number of weights in the computer model, the shape of a suspension cable of a bridge can be approximated.</td>
</tr>
<tr>
<td>6. The next step in the modeling process is the validation of the real results according to the situational model, centered around questions such as: Is the solution of the problem satisfactory, i.e., do the real results reasonably and accurately describe or predict reality? Are the assumptions or simplifications in the model adequate or must they be reviewed?</td>
<td>Since the ratio of positive slopes is independent of the positioning of the weights (as long as symmetry is maintained), the approach is applicable for describing the shape of a flexible chain hanging under gravity in mathematical terms.</td>
</tr>
<tr>
<td>7. The modeling ends with an exposition of a final answer to the original problem.</td>
<td>The shape of the main span of a suspension bridge is well described by a parabola. The shape of a flexible chain hanging under gravity is certainly not well described by a parabola.</td>
</tr>
</tbody>
</table>

Table 4.1: Steps in the model of quantitative mathematical modeling, illustrated for the student activity of exploring the mathematical shape of a suspension bridge (continued on the next page).
### General description of the transition

iii. For exploration of the real model one makes an **experimental design**. In this case, it can be an image measurement in an experimental setting of a suspended bathtub chain with five equal weights attached in such manner that the horizontal distances between the weights and the horizontal distances between the outer weights and suspension points are all equal.

![Sample result of the transition](image1.png)

### Sample result of the transition

iv. Next, data are collected, processed, and analyzed to obtain **experimental results**. Here, mathematical analysis of a digital image of the experimental setting does the job and the data processing is linked with the theoretical model.

![Sample result of the transition](image2.png)

v. The experimental results are interpreted as **real results**. Here, regression analysis can support the idea that the suspension points and the kinks in the chain are on a parabola.

![Sample result of the transition](image3.png)

viii. One should not forget to compare experimental and mathematical results by means of visual inspection and a thorough analysis of the differences between these results in order to judge if a satisfactory solution of the problem is already in sight. Parameters in mathematical models are adjusted to optimize the matching of results.

![Sample result of the transition](image4.png)

<table>
<thead>
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<td>iii. For exploration of the real model one makes an <strong>experimental design</strong>. In this case, it can be an image measurement in an experimental setting of a suspended bathtub chain with five equal weights attached in such manner that the horizontal distances between the weights and the horizontal distances between the outer weights and suspension points are all equal.</td>
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<td>viii. One should not forget to compare experimental and mathematical results by means of visual inspection and a thorough analysis of the differences between these results in order to judge if a satisfactory solution of the problem is already in sight. Parameters in mathematical models are adjusted to optimize the matching of results.</td>
</tr>
</tbody>
</table>

The seven steps in the mathematical modeling cycle presented in Figure 4.1 can be related to the following modeling sub-competencies (Maaß, 2006): understanding, simplifying, mathematizing, working mathematically, interpreting, validating, and presenting. The first and last two steps in this modeling cycle are also present in the empirical inquiry cycle and they are extended with the three steps that can be related to the following inquiry sub-competencies: planning and designing an experiment, conducting an experiment, representing, processing, and analyzing data, and interpreting experimental results. Two stages, one in each cycle, are directly linked when experimental data are compared with model results.

It must be kept in mind that in reality the two cycles in this framework of quantitative mathematical modeling are not as disjoint as suggested: On the one hand, the chosen mathematical model determines to a large extent how one sets up an experiment and handles the collected data; see, for example, the examination of the various models of air resistance for a badminton shuttlecock (Section 3.5.2). On the other hand, data handling may suggest a particular mathematical model; see, for example, the sinusoidal modeling of joint angle-time profiles of human locomotion (Section 2.4), the bi-exponential modeling in quantitative pharmacology (Section 2.7), and the bi-exponential modeling of the decay of beer foam in a glass (Section 3.5.3). In other words, one can go in any order through both cycles. One can even switch half-way...
4.2. Aspects of Scientific Inquiry and Authenticity

the theoretical modeling cycle, when a mathematical model has been derived, to an experimental cycle to verify whether the model is promising. This can also be linked to the parsimony principle that guides modeling: Simple models are in general preferred and they are to be developed and validated first to arrive at feasible models of low complexity. This is why the modeling of alcohol metabolism in humans, presented in Section 2.7, started with 1-compartment models based on simple mathematical formulas for absorption and elimination processes, and was continued with the exploration of multi-compartment models when a more accurate and detailed description of the metabolic process was needed. Another example is the modeling of the decay of the head of a beer in Section 3.5.3, which started with a simple model of exponential decay, but continued with more complex models when comparison of the model results with experimental results revealed a deviation in the first stage of the decay. In practice, quantitative mathematical modeling is often not a systematic, linear process from problem to solution, nor a simple matter of going repeatedly through in itself clear inquiry and mathematical modeling cycles. In order to overcome blockages in steps of the empirical and theoretical modeling cycles, one may need to stray from the straight and narrow, and first explore experimental techniques or study theory.

The results obtained by quantitative mathematical models depend very much on the decisions made during the inquiry and modeling process. For example, it depends on the personal understanding of the problem situation, on the assumptions made, on the way one structures and simplifies the problem(s), on the design choices made for the experiment, on the choice of mathematical tools and representations, and so forth. A good illustration is the mathematical modeling of running presented in Section 3.5.5: Two types of models for the same phenomenon were discussed, namely, kinematic models based on Newton’s second law of motion and kinetic models based on an energy balance for a runner. Whereas a kinematic approach focuses on the physics of motion of the human body and body parts in running events, the kinetic approach aims at strengthening the physiological basis of the mathematical models of running and at developing methods to quantitatively evaluate the influence of technical, physiological, and environmental variables on the performance in sports events.

In my opinion, students must and can be made aware of the dependency of outcomes on the choices made during the inquiry and modeling process; it is part of their learning about scientific inquiry and modeling. This is one of the literacy aspects that can be easily overlooked in secondary education, namely learning about the many ways in which professional modelers work and think. After all, the majority of teachers and educators have only limited personal experience with mathematical modeling, or more generally, with scientific inquiry, and they depend on external resources to bring this aspect to the fore in their instructional activities and in their guidance of students in practical work.

Other Competencies Relevant for Quantitative Mathematical Modeling

It is also clear that other competencies besides empirical inquiry and mathematical modeling abilities play an important role in quantitative mathematical modeling such as self-regulation, collaboration, reasoning, appraisal of scientific literature, communication, and a critical attitude. Self-regulation, or self-regulated learning, is “an active, constructive process whereby learners set goals for their learning and then attempt to monitor, regulate, and control their cognition, motivation, and behavior, guided and
constrained by their goals and the contextual features in the environment” (Pintrich, 2000, p. 453). Basic components are metacognition (i.e., the awareness, knowledge, and control of cognition), motivation, and volition. It also plays a major role in the decision making of when a satisfactory model is found and modeling can stop. For example, in the video analysis of moving coins presented in Section 3.2.3, one could easily have stopped work after carrying out the experiment and giving the first and most obvious explanation of the motion. Only after a detailed inspection of empirical results it turns out that the motion of the coins was more complex than initially thought. Intrigued by this unexpected problems, one may be motivated to continue with a more detailed analysis of the problem situation. In the image analysis of a hanging Slinky (Section 3.2.1), the quality of the mathematical model was also iteratively improved, for example, by taking the thickness of the Slinky into account. These two examples illustrate that the perception of the quality of a mathematical model depends to a large extent on the accuracy that the modeler requires for his or her description, prediction, explanation, and understanding of the phenomenon, and on the modeler’s willingness to invest time and effort.

Regarding the experimental part of the framework, a critical attitude is essential for good planning, design, conduct, and evaluation of an experiment. I consider the following two types of understanding necessary for development of empirical inquiry competencies, including a critical attitude toward experimental work:

- Understanding the aim and nature of an investigation, the relevant disciplinary concepts and ideas, the involved practical techniques, and the criteria used for evaluation and testing the quality of empirical data. These are the components of the PACKS (Procedural and Conceptual Knowledge in Science) model developed by Millar, Lubben, Gott, and Duggan (1994).

- Understanding of concepts of evidence about design, measurement, data handling, and evaluation of the complete experimental task. This framework was proposed by Gott and Duggan (1995) and expanded to a framework for practical work in science and scientific literacy through concepts of evidence and argumentation (Gott & Duggan, 2007; Gott, Duggan, & Roberts, 2003). It takes the point of view that one of the most important factors of success and good quality in empirical inquiry work is the experimentalist’s ability to underpin his or her work on the basis of a set of ideas about claims and evidence, that is, the experimentalist’s ability to express the thinking behind the doing.

I concur with Robert and Duggan (2007) that students’ understanding of ideas about evidence (in the context of experimental work) can be developed by explicitly and progressively introducing them the concepts of evidence required to conduct an investigation, by doing this separately from teaching subject matter knowledge, and by gradually increasing the openness of the tasks until they can do a holistic investigation such as the student profile project. The classroom practical investigations based on video analysis (summarized in Section 2.4 and 2.5) are to a large extent examples of this approach, in which students learned to effectively use a video analysis tool through problems that were not overwhelming difficult regarding the substantive ideas required to solve them. Only the authentic nature of the student activities and the occasional introduction of a mathematical method and technique (such as numerical differentiation and regression) added a substantive component to the tasks. In the
open elements of the students’ tasks, such as the selection of the investigated motion, the setting up of an experiment, the choice and use of options in a video measurement, the description of collected data in connection with the captured motion, and the comparison of motion data sets, students were more intensively confronted with concepts of evidence about design, measurement, data handling, and evaluation than in the closed tasks.

Examples of self-regulation, collaboration, reasoning, appraisal of work of other researchers, criticism, and communication abilities are: finding similar examples or phenomena that one has explored before or knows about; looking for and appraising known results that have been presented in scientific papers or books; taking one’s time to mess around with the given problem until it becomes clearer and one gets ideas; diagnosing obstacles encountered and thinking about steps to overcome them; selecting and using appropriate mathematical representations and procedures; selecting and using appropriate ICT tools for parts in the investigation; arguing and justifying decisions made in an investigation; planning and managing time and effort put into (steps of) an investigation; monitoring the investigation and keeping the focus on the chosen target; reflecting about and controlling one’s own thinking and the inquiry process; understanding the role of empirical data and mathematical models in the process of exploring a given problem situation; discerning between relevant and irrelevant data and handling lack, superfluity, or inconsistence of data; dealing with uncertainty and ambiguity; understanding that more than one model may apply (depending on the target of the investigation); comparing own results with those of other researchers; working systematically; realizing when to seek help or support; working in a team; communicating effectively; being aware of own strengths and weaknesses; being curious; looking for alternative models or explanations; and so forth. The list of quantitative mathematical modeling sub-competencies and abilities seems endless, and the students must have a willingness to put these into action (cf., Maaß, 2006, p. 116–117).

Henceforth, I only discuss a few aspects of quantitative mathematical modeling competency because in the classroom case studies presented in Chapter 2 students hardly went spontaneously and autonomously through the full inquiry and modeling cycles, and because I did not assess their performance in investigative work for a variety of problem situations. I limit the discussion to sub-competencies and abilities in which the use of ICT plays a role and makes a difference. The purpose of the field experiments and usability studies presented in Chapter 3 was more to explore the design, implementation, and educational potential of an integrated computer environment for quantitative mathematical modeling than to explore students’ competencies. I only use them occasionally to support my elaboration of certain aspects of scientific inquiry and authenticity.

4.2.2 Design and Conduct of Experiments, and Basic Data Handling

I present empirical findings about the students’ abilities to design and conduct an experiment, and to work with experimental data. I focus on general findings, acknowledging that individual or team differences in the quality of experimental work exist. For example, in the practical investigation of analyzing acids in various Cola
drinks via automated titration (Section 3.3.2; pp. 140–142) it was observed in the classroom and noticed in the students’ written reports that the quality of both experimentation and reporting varied. This was in this particular case not considered a big issue, at least when students paid attention to the quality of their experimental results, when they could figure out what went wrong during the experiments, and could formulate or make improvements. The main focus was still to learn to do experiments of good quality and to report about the results in a clear way. But when students are more familiar with the experimental techniques and possess a larger amount of subject matter knowledge, one may expect more from the students’ quality of practical work.

Henceforth I discuss in this section only video analysis activities. The main conclusions regarding these activities (summarized in Section 2.4 and 2.5) are the following:

- If one considers a student who is able to carry out sub-processes of empirical modeling on request as competent, then most participants in the classroom case studies showed a sufficient level of competency. However, from an understanding of evidence perspective (Gott & Duggan, 2007; Gott et al., 2003) there was still room for improvement of the quality of the students’ inquiry work.

- A sufficient level of empirical modeling competency seems not reserved for upper-level pre-university students only. Students in pre-vocational secondary education were also able to get an impression of what it takes to do a scientific inquiry and develop inquiry abilities by carrying out a small investigation task at their own educational level using digital video technology.

- The radius of action of video analysis, that is, the range of situations in which a person is able to activate his or her video analysis competency, seems large. Students who learned and practiced video analysis in one situation (e.g., gait analysis) seemed to have no difficulty in applying video technology in other situations (at least not in situations that were close enough to contexts in which the technology was introduced, such as in strongly related student profile projects).

The surprisingly quick uptake of video analysis technology and motion analysis by secondary school students does not mean that there are no comments on the quality of the students’ work and on the support level of the video analysis tool. Below I underpin the above findings by going more deeply into some classroom observations and some findings obtained by inspection of students’ written accounts of the investigation.

**Students’ Video Analysis Sub-Competencies**

I go into some observed shortcomings of students in their experimental work. They concern the experimental set-up, the data handling, and interpretation of measurements. This leads to a simple recommendation for an instructional sequence of video analysis.

*Students’ Shortcomings in Setting Up a Video Analysis Experiment*

All students seemed to be aware of the fact that the quality of the video analysis depends much on the accuracy of measurements in the video clip or digital image: They put great effort in correctly clicking on points of interest in the digital media. One would expect that they also paid much attention and put great effort in realizing the best setting for capturing a video or digital image. But I seldom noticed the latter
behavior amongst the participants of the practical classroom studies. For example, not many students did a provisional video recording, as a try-out to see how capturing would go and what could be improved in the experimental setting (including the camera settings). No students repeated an experiment for reasons of reliability; all of them seemed to be happy and satisfied with the first successful recording of a motion. Also, a common mistake was to forget to place a calibration object in the scene. In case of investigations related to body motion, this was afterwards repaired by calibration on the basis of measurements of human body segments or on the basis of default settings in a scene (e.g., the default size of equipment used in the experiment). Students seemed to understand that it is advantageous to position the camera fronto-parallel to the plane of motion in order to minimize perspective distortion, but apparently did not take the distance of the camera to the plane of motion or object of interest much into account. I consider the students’ shortcomings in designing and conducting an experiment which involves video recording and analysis of a motion more as lack of experience in experimental work than as obstacles that are difficult to overcome.

Students’ Shortcomings in Basic Video Data Handling
I noticed some shortcomings in the various ways students collected, handled, and interpreted video data. For example, although all students were able to identify problems in video clips for further analysis and could easily work with video editing tools to improve video clips by applying geometrical transformations or by making changes in brightness and saturation, they seldom took initiatives to improve the video analysis in this way. In addition, students often ignored the problem of dropped frames in their video capture and just accepted the data as collected. In the early experiments with the video analysis tool of COACH, students were actually hindered by the fact that video capture and editing were not yet supported in the tool; these components of a video activity had to be done outside the environment with a program like VirtualDub (Lee, 2011). Students also had technical problems with file management because of insufficient support in COACH at that moment for a broad range of video compression formats; files often were difficult to handle because of their size. These technical problems and lack of support in the integrated tool-based environment certainly had an impact on the students’ data collection in video activities.

Students’ Difficulties in Interpreting Results
Most students did not spontaneously and autonomously choose an appropriate coordinate system for their video and image analysis, but relied more on the default setting of the video analysis tool. They were apparently not aware of the possibility of reducing the complexity of the data analysis by a smart choice of coordinate system. The same holds for the graphical representations: Many students just accepted the default display of video and image data generated by the video tool and they put little effort in improving it by other means than automatic zoom-to-fit. Most students achieved a basic level of interpretation of experimental results and linking data with the recorded motion, but many of them did not spontaneously and autonomously use the replay option or the video scrubbing technique to find more details about the motion and the connection of events in the video clip with features in the graphical displays. However, the real problem is here not the lack of students’ benefit of features of a video analysis tool, but their limited graph sense and underdeveloped representational fluency (i.e., limitations in comprehension of multiple representations). This also came to the fore when students compared data sets: They tended to focus on differences in
basis features of motion graphs and to forget about what these differences meant in terms of the recorded and analyzed motions and how the differences (and similarities) could support claims made about the motions. Anyway, lack of graph sense and representational fluency seemed to hinder students in extracting all information that was intrinsically available in several linked representations and in evaluating the quality of their experimental work (e.g., reflecting on how to represent data in order to arrive at or underpin claims and conclusions).

A Recommendation for Instruction of Video Analysis

The observed students' weaknesses in data collection and data handling in video analysis activities bring me to the following recommendation: In an introduction of students to video analysis one better pays much attention to various aspects of experimental set-up and data handling, and to the (further) development of graph sense and data sense. Based on my experience in the e-class setting, I hypothesize that screen casts of worked-out examples of video analysis will help students develop the required flexibility in applying standard methods and techniques and a sense of quality of experimental work and data handling.

Video Analysis at Pre-Vocational Level

It goes without saying that upper-level pre-university students outperformed pre-vocational secondary school students in understanding concepts of evidence in experimental work. But the video measurement project of vmbo tl-3 students, summarized in Section 2.5, is an example of a rather open inquiry that is nevertheless within reach of the target group of students at their own educational level. After a guided introduction of video technology, the students were able to

- select a motion of their own interest,
- pose a question about this motion that could be investigated and answered by means of video analysis,
- set up an experiment and record a video clip of the motion,
- use the video clip to measure and analyze the motion by means of a graph,
- draw a conclusion in the form of an answer to the posed question and underpin it with the video analysis results, and
- write a short report in the form of a screen shot of their video activity containing the research question, the answer to the posed question, annotated graphs, and comments that underpin the answer.

This classroom case study clearly showed that quantitative mathematical modeling is not reserved for some privileged persons with high abilities, but can be done by students with variable ability level, although the scientific nature and sophistication of their work may differ enormously. In secondary education, students with different levels of knowledge and skills can carry out inquiry activities, provided that the activities and expectations are fine-tuned to the target group. For example, one of the lessons learned from the video measurement project of vmbo tl-3 students was that for this type of students it is even more important that technicalities in video capturing
are reduced to the minimum and that video capturing is a built-in facility of the integrated tool-based computer environment for inquiry-oriented mathematics and science education. Another lesson learned was that these pre-vocational students apparently need explicit instruction and time to practice before they are ready for open tasks. On the other hand, a short investigation like the one tried out in the classroom case study may make students aware of quality aspects of experimental work and motivate them to learn to improve their performance.

**The Radius of Action of Video Analysis**

The classroom case studies revealed that students had not much difficulty with transferring their knowledge and skills about video analysis from one context to another. For example, they could easily plan to use an already learned and practiced video technique in a new activity and then carry out the required work. This is important because, as the many examples of Chapter 3 illustrate, there is a broad range of application areas of video analysis. The only problem was their tendency to forget to apply their sub-competencies in experimental work and the need of a teacher to remind them. This supports the common sense that the critical attitude required for the design, performance and evaluation of experimental work is only developed in small steps and that this learning process depends much on the number and quality of opportunities that one gets during a school career to learn and practice inquiry abilities.

Another point, which popped up in the classroom study about bouncing balls (Section 2.8) when students had to study data from a sound experiment, is that one may not expect that students can use secondary empirical data without much difficulty. Students may lack a full understanding of how the data were collected because they did not do the experiment themselves and they may not have enough experience with the particular kind of experimental work to know what went on. This finding might even hold in cases where the data collection technology is the same. Therefore, it seems advisable to let students often carry out all aspects of video analysis (including the capturing of the video clip), before requiring that they can deeply analyze given video clips. This is in line with the finding of Hug and McNeill (2008) that students respond differently when evaluating data collected by themselves compared with secondary data from others’ work and that they are more engaged with primary data.

Actually, my experience with vwo students who did a profile project in which video analysis played a large role—examples related to the classroom gait analysis activity were projects concerning comparison of gait patterns (e.g., running, walking, hopping) and gaits under various experimental conditions (e.g., change of type of shoes, inclination angle of the ground surface), a study of the motion of the arms during human gait, an analysis of horse gait patterns, and a motion analysis of a speed skater—was that they apparently had developed through practical work at school enough video analysis competencies or enough inquiry competencies to carry out their experimental work at an appropriate level of quality and without a long learning curve during their project. But this sample of students, who did their profile projects within the framework of outreach activities of the Faculty of Science of the University of Science, was probably not representative for the majority of vwo students.
Chapter 4. Findings and Conclusions

4.2.3 Graph Sense

I summarize observations and findings about the graph sense of the students who participated in the classroom case studies and I position these results in the wide range of research literature on graphs and graphing. I briefly discuss the envisioned role of technology in the development of graph sense and how the design of a graphing tool influences the students’ performance.

The Notion of Graph Sense and its Importance

I defined ICT-supported quantitative mathematical modeling as an exploration of mathematical models with the support of ICT tools in order to come to grips with natural phenomena and to interpret real data. The key aspect of this type of human activity is making links between two worlds: (1) the world of real objects and observable or measurable things, in which one can do experiments, observe phenomena, and collect data; and (2) the world of ideas, in which there is room for theories, models, concepts, methods, and so forth, (cf., Millar, Tiberghien, & Le Maréchal, 2002). In his modeling theory of cognition, Hestenes (2008, 2010) made the distinction between mental models and conceptual models, that is, between private subjective constructions and manipulations in the mind of an individual and more objective scientific knowledge generated and communicated in the world of scientists. This separation between a mental world and a conceptual world can be useful to distinguish between private mental conceptions or common sense and scientifically grounded cognition.

Like Bisdikkian and Psillos (2002, p. 193), I consider graphical representations as bridges facilitating the world of real objects and observable or measurable things and the world of ideas during any data handling process connected to students’ practical work. The linking between graphical representations and each of these two worlds involves comprehension and construction skills employed to bridge concepts, methods, and ideas with real world phenomena by means of graphing. Conventional graphical representations are one of the symbolic forms used to encode model structures and therefore are part of Hestenes’ conceptual world. Invented graphical representations and private sketches belong to the mental world. In my discussion of graph sense I ignore the possible separation of the world of ideas into the mental and conceptual worlds because I focus on conventional graphs used in science and learned at school. What is more relevant is that I consider graphical representations as extremely valuable tools for both making sense of information and for generating and communicating information. This is expressed well by the maxim that “a picture tells more than a thousand words.”

In this perspective, graph comprehension and graphs construction abilities are essential for becoming a mathematically and scientifically literate person. Delmas, Garfield, and Ooms (2005) defined graph sense as “the ability to recognize components of graphs, speak the language of graphs, understanding relationships between tables and graphs, respond to questions about graphs, recognize better graphs, and contextual awareness of graphs.” In this definition, graphs sense consists of a wide range of abilities and it not only involves reading and making sense of graphs, but also constructing or inventing graphs that best convey information and having a critical attitude toward the use of graphs. This critical attitude is needed because different types of graphs are appropriate for different purposes and one consequently has to
make many decisions about how to visualize data. Friel, Curcio, and Bright (2001), who introduced the construct of graph sense in statistical educational research from a cognitive perspective, considered graph sense more as representing certain ways of thinking with and about graphs than as a body of knowledge and skills that can be transmitted to others. They expressed their view that “graph sense develops gradually as a result of one’s creating graphs and using already defined graphs in a variety of problem contexts that require making sense of data” (p. 145). They also reviewed frameworks for characterizing the kinds of questions that graphs can be used to answer and the instructional implications of the identified levels of graph comprehension. I use one of these frameworks to discuss the graph comprehension skills of the students who participated in the classroom case studies.

Levels of Graph Comprehension

The framework of Curcio (1981, 1987), which is much cited in mathematics education research, consists of the following three kinds of levels of tasks, behaviors, and actions:

1. **Reading data**: locating presented information and translating it from graphical into another form of communication such as text or a tabular representation. The action is local and specific: For example, a student reads literally direct factual information such as basic graphing resources (title, caption, legend, labels, axes, scales, units, etc.) and single data values. Students must at least understand the basic structure of a graph.

2. **Reading between data**: interpreting and integrating presented information. This includes comparing data values, applying simple operations to data values, looking for relationships between quantities as shown on a graph, telling a story about the graph, and so forth. This means that a student not only understands the basic structure of a graph (described above), but also understands the relationship contained therein.

3. **Reading beyond data**: generating information. Students extend, predict, or infer from data and/or tap existing schemata for information that is only implicitly present on the graph. Typical examples are interpolation and extrapolation of given data, noting trends and patterns, predicting, data smoothing, and regression analysis. This requires that a student understands the context in which the data are presented, in addition to being able to read between data.

One can also look upon this categorization as a hierarchy of (1) an elementary level (locating, translating); (2) an intermediate level (interpreting, comparing, integrating); and (3) an advanced level (extending, predicting, inferring). Shaughnessy, Garfield, and Greer (1996; see also, Shaughnessy, 2007) advocated the addition of a fourth level to the above framework of graph comprehension: **Reading behind data**. This refers to looking critically at graph use and connecting the graphical information with the context by deep analysis and causal reasoning that is based on subject matter knowledge and experience. Typical examples of actions at this level are examination of the data quality and the data collection methodology, suggestion of a possible explanation, and elaboration on alternative models and graphical representations. This level of graph comprehension is more challenging than all other identified levels and it is most strongly promoted and addressed in inquiry-oriented activities in which students...
produce and interpret their own data and report their findings. The field experiments
and usability studies in Chapter 3 illustrate this type of behavior many times in activ-
ities that range from connecting concentration graphs with the progress of a chemical
reaction on the basis of theory of chemical kinetics (Section 3.3.1) to using own expe-
riences to interpret motion graphs in the context of making a backward giant circle
on the high bar (Section 3.5.7)

Graph Comprehension Skills Observed in the Classroom

Many research studies (See, for example, Curcio, 1987, 2010; Åberg-Bengtsson &
Ottosson, 1995, 2006) have indicated that, although a majority of school children, even
at young age, seem to be able to understand the most prominent features of simple
graphical representations and can cope with the reading-off of values, students of all
ages often have notable problems in going beyond that elementary level and in drawing
conclusions from the data represented in a graph. The students who participated in
the classroom case studies were no exception: They had difficulties at all levels of
graph comprehension, except the first level of reading data.

Reading Between Data

For example, at the level of reading between data, I observed during the gait analysis
activities of vwo-5 students and noticed in the students’ writings (Section 2.4.1) that
they had difficulties in interpreting graphs in the context of leg motion; this was more
challenging than constructing the graphs. For many a student it was quite difficult to
learn from the instructional materials alone how a cyclogram represents a relationship
between hip and knee joint angles, and how it essentially summarizes the shape of
the leg during a gait cycle. Further explanation of the meaning of a cyclogram by the
teacher improved the students’ abilities in this respect. But even hereafter, comparison
of hip-knee cyclograms of self-recorded gait patterns often just resulted in writing
down differences between diagrams and focussing on local information presented in
the graphs, instead of describing how cyclograms summarized different leg motions
and taking a global view of the presented motion data.

It turned out in this classroom study that reading between the data was more dif-
ficult for many a student than I had expected, especially concerning the interpretation
of a hip-knee cyclogram. The fact that a cyclogram is a parametric curve with time
only implicitly involved certainly makes the graph interpretation tasks of analyzing a
cyclogram and comparing two cyclograms more challenging for students than doing
such tasks for angle-time profiles. But I also noticed that the students sometimes
seemed to have forgotten that they could benefit from the video scrubbing technique
to connect graphical features with events in the video clip. Especially when comparing
two cyclograms, many students did not first analyze each gait pattern by use of the
synchronized link between the recorded motion in the video clip and the cyclogram
before continuing with a discussion of the differences and similarities between the cy-
clograms and between the corresponding gaits. Instead, they immediately stepped into
the comparison of two graphical representations, the cognitive load of which is higher
without the synchronized video clips at hand. Maybe the students were hindered
here by the fact that their video analysis software did not support synchronization of
representations with more than one video clip at the same time.
4.2. Aspects of Scientific Inquiry and Authenticity

The student performance in reading between data improved in the second exploratory case study about gait analysis by students (Section 2.4.2) after I had improved the instructional materials in the sense of a more gentle introduction to the hip-knee cyclogram and after I had realized a practical setting in which students could record and explore more than one gait cycle in a video analysis activity. I think that the longer duration of the recorded motion helped the students better connect the graphs with the motion and describe the graphs in a more accurate way.

**Reading Beyond Data**

At the level of reading beyond data, I observed that vwo students apparently had few difficulties with linking the real world context with graphs and reasoning about these links in case the connections were direct, as in graphs of measured quantities with respect to time. When the connections were more indirect, as in graphs of derived quantities and in less common graphs such as cyclograms, the students had more difficulty in understanding the link between the visual representation and the real world object of study and in utilizing the connection for explanatory purposes.

For example, in the practical investigation about human growth (Section 2.2) I observed that vwo-4 students had no problems with graph interpretation tasks up to intermediate level of difficulty, but that they did not consider increase diagrams as an obvious and natural way to investigate a process of change; they only used these diagrams when explicitly asked. This may well illustrate that the students were not used to think of change of a quantity as an interesting quantity itself. In other words, it may reveal a weakness in the students’ ability to link the general concept of a changing variable with a graphical representation of change of a quantity. This point needs attention in instruction because increase diagrams and numerical differentiation of data are important, generally applicable mathematical methods and techniques to investigate processes of change.

Another example of students’ difficulties with reading and interpreting graphs comes from the classroom experiments with vmbo tl-3 students doing video analysis activities (Section 2.5). Many students had problems with understanding horizontal displacement on a vertical axis, that is, with the traditional choice of axes in time profiles of quantities. Once time was plotted against horizontal position in the instructional materials, these students were more able to successfully interpret the measured graph in relation with the recorded motion. This indicates that these students still preferred an iconographic representation, that is, a graph in which the direction of motion is the same as the direction of the axis on which the measured quantity is projected, with the effect that the graph looks like a map of the recorded motion.

**Reading Behind Data**

Having a look behind the data was promoted in various ways in the classroom studies. For example, the activities in the computer-based investigations of mathematical shapes (Section 2.3) went beyond the level of empirical modeling by means of regression analysis and causal reasoning: Students had to use physics knowledge to underpin the models. In the instructional materials about quantitative pharmacology (Section 2.7), I made use of the broad range of models for intake and clearance of alcohol in the human body to ensure that students had the opportunity to practice evaluation and revision of models, and in this way would get acquainted with the concept of model progression. Similarly, a lot of attention went in the instructional activities about
bouncing balls (Section 2.8) to the validation of (intermediate) mathematical models, with the hope and expectation that looking at various models of one and the same phenomenon would promote a critical attitude of students. High speed video technology was applied to get a closer look at the kinematics of a bouncing ball and to raise the quality of the model. I hoped and expected that, by explicitly paying attention in the tasks to looking behind the data, the students could develop the critical attitude that is necessary for successful modeling of biological, chemical, or physical phenomena.

The classroom case studies in which students had the toughest challenge to look behind the data were the spreadsheet-based data handling activities in which vwo-5 students worked with censored clinical data (Section 2.6.1) and in which vwo-3 students explored precipitation times series data (Section 2.6.2). In both activities, students had to critically analyze available data, create their own hypotheses and data models, and compare statistical methods. But graph comprehension played a smaller role in these activities than the understanding, interpretation, evaluation, and manipulation of data. Therefore I come back to this issue when I discuss the students’ notion of data sense.

Critical Factors Affecting Graph Comprehension

Friel et al. (2001) considered four critical factors that affect graph comprehension:

1. The purpose of using graphs (analysis, communication).
2. Task characteristics (visual decoding, perceptual judgment, relating graph features with a context).
3. Viewer characteristics (prior mathematical knowledge, familiarity with the context, proficiency with graphing tools, the repertory of strategies with visualization).
4. Discipline characteristics (data type and size of a data set, the way a representation provides structure for data, graph complexity).

They are not much different from the factors that according to Shah, Freedman, and Vekiri (2005) influence graph comprehension: task (user’s goal, extraction of information, scientific visualization, . . . ), display characteristics (format, use of symbols and colors, . . . ), data (complexity, . . . ), and person characteristics (prior knowledge about content, graphical knowledge, scientific literacy, . . . ).

The Purpose of Using Graphs

I took the identified factors that affect graph comprehension seriously into account when I designed classroom practical investigations and when I improved or designed new tools of the integrated computer learning environment for inquiry-oriented mathematics and science education. I deliberately put emphasis in most investigation tasks on making sense of data by graphs and graphing. I always emphasized that, in quantitative mathematical modeling, graphs have meaning in relation to the real world context and are not just pretty pictures. I exposed students as much as possible to problem situations in which graphs clearly convey information that may stay under the surface in other representations. In particular, I provided them with opportunities to experience that a data graph may make a trend in a data set more evident than a table of recorded quantities (cf., Shah & Freedman, 2010; Zacks & Tversky, 1999).
4.2. Aspects of Scientific Inquiry and Authenticity

To illustrate this, the following examples from the classroom case studies may suffice. Increase diagrams of height-age growth diagrams reveal a puberty growth spurt and allow easy comparison of height growth of boys and girls (Section 2.2, p. 32). Hip-knee cyclograms summarize gait patterns in an information-rich visual way and offer an objective way to compare gaits and to reason about leg motion (Section 2.4.2, pp. 53–54). Concentration profiles in quantitative pharmacological studies illustrate how intake and elimination of drugs takes place in the human body and in a graphical format it is relatively easy to compare measured data with results from computer simulations of pharmacokinetics (Section 2.7, p. 79).

Many examples of the expressive power of graphs can also be found in the field experiments and usability studies presented in Chapter 3; I mention a few. In the context of chemical kinetics (Section 3.3.1, p. 128, p. 134) modeled concentration profiles reveal how a chemical reaction system adapts to changing chemical experimental conditions. In gait analysis via electromyography, the dynamic linking of video clips of gaits with EMG recordings enables the determination of muscle activity during various gait phases (Section 3.4.1, p. 153). The differences between squat jump and countermovement jump can be explored by examination of force-time graphs (Section 3.4.2). Inspection of velocity-time graphs of a falling badminton shuttlecock is an important tool in deciding which model of air resistance best describes the motion of the falling object (Section 3.5.2, p. 169). Details of the initial motion of a sprinting person can be found in the velocity-time graph obtained by high speed video recording: increasing and decreasing parts of the graph correspond with the propulsion during foot contact and the aerial phase when both feet are airborne, respectively (Section 3.5.5, p. 187). These examples illustrate my point of view that a graph is a means rather than an end in itself.

**Task Characteristics**

I selected the contexts of the practical investigations such that the participating students could be expected to be already familiar with them or could easily recognize the need to become more familiar with the related discipline characteristics. For example, the case studies about human growth, human locomotion, and quantitative pharmacology always served a dual purpose: (1) learning to work with data in a graphical way and (2) becoming familiar with the ways scientists and practitioners do investigations, collect data, and use visual representations in their work (cf. Miller, 1998). The context of human locomotion (Section 2.4) was deliberately chosen to let students explore a complex motion with which they were all familiar, but for which it was also obvious that there does not exist a definitive mathematical and scientific model that explains each and every detail. Yet the students were expected to learn that, with their mathematics knowledge at secondary school level, they could make sense of recorded data, use graphs in data analysis, relate graph features to events in the motion, and could interpret and construct graphs in much the same way as motion scientists do.

**Viewer Characteristics**

In retrospect I consider the chosen instructional approach in the practical investigations successful insofar as the students became familiar and more proficient with commonly used mathematical methods and techniques, learned to use generally applicable ICT-tools in practical work, and got an impression of what it takes to do an investigation. It turned out to be more complicated than expected to bridge the gap between the informal knowledge and mental models of students about real-world
phenomena on the one hand and the scientific practice in which these phenomena are explored on the other hand. In the instructional materials for the case studies about human locomotion and quantitative pharmacology, for example, I introduced the students to the scientific vocabulary used in these disciplines with the intention that they would develop some sense of the benefits of a clear and unambiguous terminology. But in the students’ reports I noticed that the orientation to the scientific practice had failed in the sense that most students kept using natural, imprecise language. They apparently were not motivated enough to use scientific vocabulary; only in student profile projects I noticed the willingness to use more frequently the language of mathematics or science.

**Discipline Characteristics**

Concerning the design of the integrated tool-based computer learning environment, much attention was paid to how graph interpretation and construction tasks could be supported. Two findings from past research directed my work. Firstly, Cleveland (1984) noted that the majority of scientific visualizations consists of graphs, in particular two-variable graphs (83% of all graphs) in which one variable is plotted against another, such as scatter plots, time series plots, and function plots. Secondly, research has indicated that scatter plots and line graphs are one of the most important, but also most complicated types of graphs for secondary school students to master. It is misleading to think that these graphs are for non-experts easy to read and to interpret (See in the context of motion graphs, for example, Beichner, 1994, 1996; Billings & Klanderman, 2000; McDermott, Rosenquist, & Van Zee, 1987; Radford, 2009; Testa, Monroy, & Sassi, 2002; Zollman & Brungardt, 1995). The multiplicity of graphing in the Cartesian plane—Ainsworth & Van Labeke (2004) distinguished in the context of graphical representation of processes of change between (1) a time-persistent representation showing a range of values overtime; (2) a time-implicit representation, which shows a range of values but not the time when these values occurred; and (3) a time-singular representation showing only a single point of time—introduces a challenge of moving flexibly among these graphical representations (See also, Yerushalmy & Shternberg, 2006).

These two findings underpin the restriction to this type of graphs in the remainder of this chapter. This does not mean that other graphical representations are second-class representations and hardly occur in school practice. As a matter of fact and to great surprise, researchers have found differences in visualization practices between textbooks and scientific publications (See, for example, Roth, Bowen, & McGinn, 1999; van Eijck, 2006). In scientific research papers, more Cartesian graphs are used and more graphing resources (labels, scales, units, legends, captions, etc.) to facilitate graph reading are provided than in textbooks. In the practical investigations I promoted the use of scatter plots and line graphs, and I paid much attention to ways that help making the dynamic nature of graphs more obvious to students. In Section 4.3 I present a few features of Coach 6 that are intended to help students in their work with graphs.

**Disciplinary Differences in Contexts and Language for Graphing**

Friel et al. (2001) considered discipline characteristics as one of the critical factors that affect graph comprehension. But they were not thinking of the fact that students
4.2. Aspects of Scientific Inquiry and Authenticity

must overcome obstacles for applying what they learned about graphs and graphing in mathematics lessons to other disciplines like science and economics. Research studies have revealed that there exists a transfer problem in using graphs in various disciplines. This was one of the reasons for Roth (2003) and coworkers (Roth & Bowen, 2001, 2003; Roth & Lee, 2004; Roth & McGinn, 1998) to take a sociocultural orientation toward graphing as social practice (in a school setting, at the workplace, etc.), instead of the cognitive-psychological perspective. I refer to (Ellermeijer & Heck, 2002; Heck & Ellermeijer, 2010) for a short discussion of differences between mathematics and physics regarding contexts and language for graphs and graphing. Table 4.2 and Table 4.3 list some identified differences in construction and purpose of graphs, and the use of graphing language, respectively, between mathematics and physics.

<table>
<thead>
<tr>
<th>Mathematics</th>
<th>Physics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph</td>
<td>Represents a relationship between two quantities. Main purpose is to explore or to present the relationship between quantities.</td>
</tr>
<tr>
<td>Axes</td>
<td>Values of quantities are expressed in some unit. Scaling is a matter of choice and may be nonlinear. Dual-scaling is used.</td>
</tr>
<tr>
<td>Origin</td>
<td>Arbitrary position.</td>
</tr>
<tr>
<td>Plot range</td>
<td>Determined by the ranges of the quantities.</td>
</tr>
<tr>
<td>Slope/Gradient</td>
<td>Represents the change of a quantity with respect to another and is on its turn a quantity with a unit.</td>
</tr>
</tbody>
</table>

Table 4.2: Different contexts of graphing in mathematics and physics.

<table>
<thead>
<tr>
<th>Mathematics</th>
<th>Physics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangent line, slope.</td>
<td>Gradient, steepness.</td>
</tr>
<tr>
<td>Origin (of coordinate system).</td>
<td>Zero (of a quantity).</td>
</tr>
<tr>
<td>Domain and range (of a function).</td>
<td>Range (of quantity values).</td>
</tr>
<tr>
<td>Graph of (y(x)), (x-y) graph.</td>
<td>(v-t) diagram, (x-t) diagram.</td>
</tr>
<tr>
<td>Set of 2-tuples ((x,y)).</td>
<td>Plot of (y) vs. (x).</td>
</tr>
</tbody>
</table>

Table 4.3: Some different languages for graphs.

**ICT-Based Approaches to Develop Graph Sense**

There are several approaches to mastery of graphing: For example, the spreadsheet-based, active graphing approach of Pratt (1995), Ainley (2000), and of Ainley, Nardi, and Pratt (2001), and the data logging approach of Nicolaou, Nicolaïdou, Zacharia, and Constantinou (2007), Philips (1997), and of Van den Berg, Schweikert, and Manneveld (2010). Research evidence suggests that when computer-based graphs replace the drawing of graphs by hand, school children can learn to work with line graphs and scatter graphs at an earlier age than with traditional progression because a lower skill level of graphing is possible through the facilitation of data collection and data analysis and the stimulation for students to talk to each other about the graphs on the screen. This seems to contradict the reported difficulties of secondary school students with interpretation and construction of graphs in paper-based activities (Leinhardt, Zaslavsky, & Stein, 1990). Apparently, the affordances of graphing software make
the difference. Phillips (1997, p. 56) suggested that “the development of graphical interpretation skills are more likely to be promoted by exposing children to a rich and varied diet of graphical experiences, than by detailed attention to graphical syntax.”

Graphing Software and Spreadsheet Programs
Ainley (2000, p. 367) attributed success of an ICT-based active graphing approach to the user’s control of the appearance of a graph: “Although, inevitably, any software will be designed to use default options which determine the initial appearance of the graph, the user’s power to control the appearance of the graph is a significant aspect of the nature of computer-based graphs.

• Computer-based graphs are dynamic, in the sense that their size and proportions can be altered by dragging the corners of the graph. Unless the user has specified, otherwise, the scales shown on the axes will change as the graph is distorted.

• The same set of data can be quickly and easily displayed in a wide range of graphical forms, and the appearance of the graph can be changed through menus, which control the scales on the axes, the orientation, the style of markings and labels, and so on.

• Computer-based graphs can be created interactively: once the graph has been produced, it will change as data on the spreadsheet is changed.

• Depending on the particular software being used, it may be possible to use drawing tools to superimpose a line upon a graph, to facilitate reading the graph or the interpretation of trends.”

Data Logging, Video Analysis, and Computer Simulation Systems
The fact that Ainley (2000) only considered graphing software and spreadsheet programs limited her thinking. In data logging, video analysis, and computer simulation systems, the dynamics of graphs goes a step further: While one collects or computes data, or while one replays the data creation, one can simultaneously see graphs and tables growing in size. In other words, real time display of data is provided. This helps students see a time series graph not simply as a static picture, but more as a dynamic representation of a process of change, and it helps student link a graph with a real-world phenomenon. Depending on the data logging system, it may be possible to use drawing tools to predict a graph before a data graph is created by a measurement. The last two features of graphing with data logging systems (dynamic representation and linking with a real-world phenomenon) are important because from research it is known that there are several areas of difficulties that students have with graphs and graphing. For example, students may have difficulties in connecting graphs with the real world, in linking graphs with concepts, and in transitioning between a graph and events described by the graph. Students may also have difficulties in constructing appropriate graphical representations to support conclusions or predictions (See, for example, Tairab & Khalaf Al-Naqbi, 2004).

Reported Effects of Data Logging on Graph Comprehension
I take the standpoint that when students at upper secondary education level carry out a practical investigation they must already have developed some level of graph sense in order to fully benefit from the graphing facilities, that is they must already sense what graphs represent and what purpose these representations serve, be able
4.2. Aspects of Scientific Inquiry and Authenticity

To read and interpret graphs in various contexts, have knowledge and skills to select, construct and manipulate graphical representations that aid in making sense of data, and be able to work sufficiently fluently and flexibly with conventional mathematical graphs. Research has shown that the use of ICT, and in particular systems for data logging, video analysis, and for computer modeling, simulation and animation, helps improve students’ graph comprehension.

For example, Svec (1999) concluded that data logging engendered conceptual change in undergraduate physics students and improved students’ graph interpretation skills. These findings were in line with results obtained in many other studies with students of all ages and educational background, in which the use of data logging regularly led to better scores on conceptual tests (See, for example, Boujaoude & Jurdak, 2010; Lapp & Cyrus, 2000; Lo Cicero & Spagnolo, 2009; Linn, Layman, & Nachmias, 1987; McRobbie & Thomas, 2000; Mokros & Tinker, 1987; Newton, 2000; Nicolaou et al., 2007; Rogers, 1996; Russell et al., 2003, 2004; Stylianou et al., 2005; Thornton & Sokoloff, 1990; Trumper, 1997). Lindwall and Ivarsson (2004, 2010) claimed that data logging worked better in this sense than computer simulation. They concluded that the two main difference between the learning environments, as observed in the analysis, were: (1) the kind of interpretive resources that were used in the coordination of actions and in the completion of tasks; and (2) the kind of discourse that each environment promoted. But maybe the most important lesson learned from such micro-analytic studies is the following: (1) details of the used technology and the designed activities strongly influence the learners’ actions and the learning outcomes; and (2) with regard to the design of educational technology and the effectiveness of the technology in practice, the proof of the pudding is in the eating.

Reported Effects of Video Analysis on Graph Sense

The positive effects of data logging on graph comprehension are also reported in studies about the effects of the use of video analysis tools on the students’ graph sense (cf., Beichner, 1996, 1999; Escalada & Zollman, 1997; Larkin-Hein & Zollman, 2000; Koleza & Pappas, 2008; McCullough, 2000; Zollmann & Brungardt, 1995). I refer to Section 3.1.3 for a brief discussion of the educational benefits of video analysis in mathematics and science education. Struck and Yerrick (2011) reported that there were even indications that high school student achievement was higher for velocity-time and acceleration-time graphs using the video analysis method compared with the data logging method. However, their results may be influenced by their choice to use two different tools, namely, the PASCO XPLORER GLX hand-held for data logging and the Vernier’s LOGGERPRO software on a desktop computer for video analysis.

A Comment on the Effects of Data Logging and Video Analysis on Graph Sense

The positive effects of data logging and video analysis technology on the development of graph sense does not mean that graphs and graphing are easy for students and does not take away misconceptions and graphical illusions or misinterpretations of graphs in connection with real-world phenomena. Rodrigues, Pearce, and Livett (2001) for example noted in their study with first year university physics students that the use of video analysis and data logging in practical work motivated their students and helped them understand physics concepts and linking the graphs with the real-world motions investigated, but also that the students used elements of the video analysis and data logging to reinforce already existing ideas rather than challenging the robustness of their existing ideas.
Chapter 4. Findings and Conclusions

Students’ Alternative Conceptions about Graphs and Graphing

Leinhardt et al. (1990) analyzed research on the interpretation and construction tasks associated with functions and some of their representations. They distinguished the following three main categories of students’ difficulties with learning to focus more broadly on the overall shape of the graph or parts of graphs (pp. 37–39):

- Interval/point confusion, when students narrow their focus on a single point in cases where they better consider a range of points.

- Slope/height confusion, which means that students confuse gradients (slopes) with maxima and minima.

- Iconic interpretation, when students interpret a graph of a situation as a literal picture of the situation.

In the classroom studies at vwo level I only encountered slope/height confusion in the activities about human growth (Section 2.2), when students were asked the question “Who grows faster, boy or girl, at what age?” This does not mean that the students did not suffer from the other two identified difficulties, but only that they might not have easily come to the surface in the exploratory case studies.

Leinhardt et al. (1990) categorized various other misconceptions and difficulties in students’ understanding and use of graphical representations of functions. Their list contains issues with notation and grammar, use of variables, and allowable types of correspondences, as well as overemphasis of linear graphs through the origin and difficulties with scaling. From my experiences in the classroom case studies, I conclude that the ability to read scales seems to be one of the most important factors affecting the interpretation of graphs created by means of visualization tools.

For good understanding of a graph created in graphing software or a graphing calculator one must realize what visual features of the graph will not change under changes of the viewport, the data window, and the scaling used in the coordinate systems (For example, intercepts of lines), and what will change when alterations are made (For example, physical slopes of lines and angles between lines). Selecting and constructing a suitable shape of a two-variable graph is an issue, particularly in interpretation of graphs in situated contexts such as science. Leinhardt et al. (1990), in their review of functions, graphs and graphing, noted the following:

- The issue of scale becomes more fundamental when using graphing technologies (p. 17).

- Scale is an issue when using graphs for scientific data analysis and in computer-based instruction, but usually is not an issue when introducing graphing in mathematics classes (p. 17). The scale is often assumed or given in mathematics instruction (Normally the scale is the same on each axis).

- Change of scale is one of the main sources of graphical illusions that may become obstacles in the process of abstracting from graphs as visual-concrete representations to graphs of symbolic representations (p. 19).

The use of graphing technology hardly played a role in the review of Leinhardt et al. (1990) because the graphing calculator had not yet entered into the classroom...
4.2. Aspects of Scientific Inquiry and Authenticity

...at the time of writing of the reviewed research papers. But the common difficulties students experience in understanding graphs, including the difficulties in choosing, constructing, and scaling axes, and the difficulties in understanding transformations of functions and changes of scale have been repeatedly found in subsequent research studies on graph sense and on the use of graphing technology. Like Mitchelmore and Cavanagh (2000) found in their research study about interpretation of linear and quadratic graphs on a graphing calculator, I also observed in classroom experiments and noticed in students’ reports that students often had a tendency to accept a graphic image constructed in the software environment uncritically, without attempting a more appropriate scaling and shaping of the diagram, and that they apparently did not fully grasp the processes used by the software to display graphs. Only when students started to author Coach activities as a form of reporting results or used Excel to create graphical representations of data to be included in a report or a PowerPoint presentation, they seemed to be more conscious of the main choices that could be made in graphical displays (including selection of colors, symbols, and other display characteristics). But they were driven then by decorative goals instead of mathematics.

4.2.4 Data Sense

I go into some aspects of data handling. The main topic is how students collected and made sense of data in the classroom case studies, for example by video analysis, computer modeling, and regression analysis.

Levels of Making Sense of Data

Making sense of data is a research ability that is central in quantitative mathematical modeling because modeling results are always compared with real data. Data sense can be defined as the ability of persons to (1) derive meaning from data created by others or by themselves, and (2) to generate data sets that help to come to grips with an phenomenon or a situated problem. It refers to an understanding of data, the way they are collected, organized, displayed, processed, created, analyzed, compared, related to observed events, and how they are used in making inferences and drawing conclusions. Because visual representations of data are the dominant scientific format of representing data, data sense is closely linked with graph sense. Shaughnessy (2007, p. 988) noted in the context of statistics learning and reasoning that “graphs are critical for data representation, data reduction, and data analysis.” This explains why these two notions are usually intertwined in research on mathematics and statistics education and why the categorization of tasks, behaviors, and actions associated with graph comprehension by Curcio (1981, 1987) and why Shaughnessy (2007; see also, Friel et al., 2001) used the word ‘data’ in the labeling of the levels of graph comprehension. Curcio (2010) recently referred to data-graph comprehension, to emphasize the direct link between graph comprehension and data comprehension.

Data Collection and Data Creation

Data collection was part of most classroom case studies, but also secondary data originating from scientific research studies were used. For example, published growth data of the 4th Dutch Growth Study of 1997 (Fredriks, 2004; Wit, 1998) were used in...
Chapter 4. Findings and Conclusions

the case study about human growth (Section 2.2) and time-series data published in
the Climate Atlas of the Netherlands (Heijboer & Nellestijn, 2002) were used in the
spreadsheet-based data handling activity outlined in Section 2.6.2. One of the reasons
to use real data was to raise the level of authenticity of the activities and make students
aware of the importance of the quality of data. Bottom-line was the thought that when
designers of learning activities are not taking great care of the validity of their data
and just invent data that fit the learning targets instead of relying on trustworthy data
resources, one cannot expect that students are serious about and feel responsible for
the quality of data in their practical work.

Data creation was also essential in the students’ computer modeling activities be-
cause in computer simulations data were generated for visual comparison with empir-
ical data. Students had to make in these investigations all kinds of decisions about
parameter settings in the mathematical model, time steps in the algorithm, duration
of the simulation, choice of numerical integration method, and so forth. My expe-
rience was that in general students were able to get the job done, but more by the
method of trial and improvement, than by a systematic approach, underpinned by
theoretical arguments, or by referencing to literature data. For example, the use of
regression analysis as a method to get initial estimates for parameters in a computer
model was hardly used; it is a technique that must explicitly taught. Students also
relied much on the default settings of the computer modeling tool and they changed
the numerical integration algorithm without giving it much thought and relying just
on past experiences.

Regression Analysis

The focus of many practical investigations was on getting a global view of data. In
exploratory data analysis, global understanding of data commonly refers to the ability
to search for, recognize, describe, and explain general patterns in a set of data (change
over time, trends) by observation and intuition, or by statistical techniques. The
classroom case studies, especially the ones about human growth (Section 2.2) and
mathematical modeling of shapes (Section 2.3), revealed that the students had informal
ideas about association and correlation of quantities, but did not have many formal
concepts and skills connected to these notions. This came as no surprise because
regression was for the participants in the case studies not a curriculum topic in their
mathematics education, or at best it was a hidden topic. I go into some consequences
that popped up in the classroom case studies and recommend inclusion of regression
into the mathematics curriculum.

Students’ Informal Understanding of Data Fitting

Students understood the idea of a best line fit of data (perhaps from science laboratory
work), but only based this on visual judgment in data graphs and not on criteria of
goodness of fit. They were readily willing to accept other regressions formulas—they
often tried them out when they used a regression tool—but when the instructional
material referred to simple formulas that fit the data, they automatically favored
linear function fits and took other function types such as quadratic functions less into
consideration. A strong emphasis in mathematics education on linearity is probably
causing this effect, even in cases where a nonlinear approach is more appropriate.
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This also explained some observed weaknesses in the students’ actions when they applied nonlinear regression: They relied mostly on default settings of the regression tool and seemed to be unaware that nonlinear data fitting had been implemented in COACH as a two-phase process in which firstly good initial estimates of parameters in the regression model must be determined (manually by the user or on the basis of data handling methods) and secondly these estimates are used in a systematic improvement of the intermediate result (For example, by applying the Levenberg-Marquardt method for nonlinear regression). This concept of data fitting in two distinguished phases of initial parameter estimation followed by systematic improvement must be explicitly taught before students can effectively apply the regression tool.

Students’ Understanding of the Method of Peeling-Off Functions

In the classroom activities about gait analysis (Section 2.4) and quantitative pharmacology (Section 2.7) students were introduced to the least squares method of peeling-off functions (Foss, 1969) for finding appropriate regression curves that are formulated as a sum of two or more mathematical functions. They had no difficulty in following the procedural steps. When explicitly asked to inspect residual data of a function fit and to consider a consecutive fit of these residual data, students could do this. When this technique was applied to distinguish between a clearly separated global trend and a superimposed function—for example, in the task to describe the position-time data of the hand of a person who walks with constant speed on a straight line by a mathematical function (Figure 2.6, p 46)—the students had no difficulty in relating the summands in the regression formula to components of the described phenomenon and in open tasks of this kind they were more inclined to take initiatives for doing this. In other cases (e.g., quantitative pharmacology), when it is less obvious that a global trend and a superimposed function can be separated, students seemed to neglect the possibility of a regression model consisting of a sum of mathematical functions. Maybe this behavior is caused by the fact that the regression tool of COACH, graphing calculators, and most graphing softwares only (or mainly) offer single mathematical functions as regression models.

I consider it important that students understand the idea of decomposing a signal into several components because it is a technique that is often applied in regression analysis. Examples discussed in Chapter 3 are the harmonic analysis of tides (Section 3.3.3, pp. 144–147), the decomposition of the motion of a yoyo into a translational and rotational component (Section 3.5.1, p. 163), the bi- and tri-exponential regression models of the decay of beer foam in a glass (Section 3.5.3, p. 172), and sinusoidal fits of the body rotation angle, shoulder joint angle, and hip joint angle during a backward giant circle on the high bar (Section 3.5.7, p. 199).

A Too Global View in Data Fitting

Students hardly made use of the feature of the regression tool of COACH that they could zoom in on part of the data set for applying regression analysis to the particular data selection. Again, they only did this when explicitly instructed to do this (for example in bi-exponential fits of pharmacological data), but they hardly made the first move to do this. This technique could also have helped students to describe data by a piecewise-defined function, but this type of regression model is even more out of sight than a model consisting of a sum of two or more functions. Anyway, I noticed a tendency in the students’ actions and reports to forget that they could split data in parts that could be modeled separately.
However, I consider it important that students understand the idea of taking a component-wise view in data fitting because it is a technique that is often applied in research. Examples discussed in Chapter 3 are the distinction of the constant velocity phase of a falling badminton shuttlecock (Section 3.5.2, p. 167) and the distinction between the contact phase and aerial phase during the motion of vertical bouncing on a jumping stick (Section 3.5.6, p. 190), resulting in a combination of a sine fit with a parabolic fit for the two phases, respectively.

A Recommendation
The findings of students’ performance in regression analysis indicate that improvement can only be realized when regression becomes a curriculum element and teachers explicitly pay attention to the methods and the underlying theory. I consider regression an important subject for secondary mathematics education: In my opinion, secondary school students should learn that mathematics can help decompose data into a signal to be recovered and irrelevant information present in the data as noise. This idea of data being a mixture of signal and noise is perhaps one of the most fruitful and fundamental ideas of data analysis. It also underpins the mediating between structure and randomness for data smoothing and it can help students find an appropriate balance between signal and noise in their data. When more attention is given to this idea, students are expected to much better understand, utilize, and appreciate the numerical differentiation methods based on data smoothing, instead of always first trying to find a good algebraic function fit and use this function for computing derivatives. Thus, it will probably contribute to the students’ data analysis competency.

Reading Behind Data
I promised earlier (p. 220) to discuss the outcomes about students’ reasoning with with censored clinical data (Section 2.6.1) and precipitation times series data (Section 2.6.2). In both classroom case studies, students had to critically analyze available data, create their own hypotheses and data models, and compare statistical methods. They used the spreadsheet program Excel as a calculating tool so that they could focus on statistical reasoning, that is, on making sense of statistical information and on interpreting results, instead of on filling out tables. I discuss the students’ competency in reading behind the data.

Survival Analysis of Censored Clinical Data
Students were challenged to handle censored observations originating from a clinical study in a hospital. This was much different from the statistics and data analysis presented in their mathematics textbook, which may give the wrong impression that data handling is not much more than applying standard recipes. Here, students had to come up with their own statistical model for handling the data, use their own model to compute survival probabilities, and draw conclusions. All students who participated in this classroom study had chosen the Culture & Society stream, had no confidence in their mathematical competencies, and had motivational problems with studying mathematics. Notwithstanding this student profile, they performed extraordinarily well in the classroom experiment. They understood the basics of survival analysis and censored data, grasped simple methods of computing survival probabilities, and comprehended the life-table or actuarial method and its dependency on the researcher’s point of view on censoring. The latter achievement was evidenced by the students’
success in evaluating three naive points of view, reflecting on two given points of view on censoring in the actuarial method, and in proposing and underpinning an alternative point of view.

I think that the instructional materials and the instructional setting contributed much to the success of the students’ reading behind the data. The instructional materials hardly contained closed questions with a single correct answer, but instead contained open-ended tasks that gave students on the one hand the impression that they could carry them out and stimulated them on the other hand to go one step further and think critically. Students could overcome their math anxiety, in particular their fear of using mathematical formulas, and their lack of confidence in their mathematical abilities through the use of a spreadsheet program and by working in small teams. The spreadsheet program provoked the students to work in an active way and to talk extensively and thoughtfully about the statistical and computational problems involved within their team, with the teacher, and with other groups of students. The computer results also revealed that the students had sometimes found good solutions to the exercises, but that they had simply not been able to express their answers in terms of formulas. Listening to the conversations amongst the students and discussing the answers with them revealed that the students’ understanding of the subject was better than their written reports alone suggested.

The most important conclusion that I draw from this classroom case study is that one can challenge pre-university students to read behind the data and get back good results, even with students from whom one would not expect it on the basis of the achievements in regular mathematics lessons.

Handling Weather Data
In this classroom case study I explored whether vwo students at lower gymnasium level could successfully work with data sets of medium size (as opposed to the small data sets in their mathematics textbook) by means of a spreadsheet program and use various types of moving averages of precipitation time series data for the answering of concrete questions about the weather. Students had to concentrate on computing with averages over various, possibly overlapping, time periods and noticing what were the effects of their choices. The notion of moving average was new to them, both in terms of calculation and application, and the instructional materials deliberately did not contain the term. The tasks were grouped into closed questions at the beginning and more open tasks further on.

The students were able to do the closed tasks and vivid discussions amongst students arose about the various choices made in the exercises. Thus, when given clear tasks, the students understood the concept of moving average, were able to do computations, and could draw conclusions based on the various choices made with regards to the type of moving average. But the question remained whether they would spontaneously and autonomously show the same behavior in open tasks. Alas, this could not be answered in the classroom case study because the final open task was not formulated clear enough for the eyes of the students. Besides of this, classroom observations revealed that the student did not have enough experience with Excel to use it effectively. Especially the use of absolute or relative cell references confused students and hindered them in their work. Despite of all difficulties, some students could formulate their reasoning in clear words. This indicates that reading behind the data may not be beyond the competency level of vwo students at lower gymnasium level, even when
the problem is a little bit more complex than usual, the data set is medium-sized, and computational tools are needed.

4.2.5 Symbol Sense

A precise definition of symbol sense does not exist. In analogy with number sense, Arcavi (1994, 2005) described a number of behaviors that characterize symbol sense in the domain of high school algebra. His list included amongst others the ability to appreciate the power of symbols, to know when the use of symbols is appropriate, and the ability to manipulate and make sense of symbols in a range of contexts. Drijvers (2003, 2006, 2011) confined the interpretation of symbol sense to the understanding of the meaning and structure of algebraic formulas and expressions, which involves (1) the strategic abilities to arrive at a problem approach and to maintain an overview of this process, (2) the capacity to view symbolic expressions globally, and (3) the capacity of algebraic reasoning. This links up with Zorn's (2002) description of symbol sense as a very general ability to extract mathematical meaning and structure from symbols, to encode meaning efficiently in symbols, and to manipulate symbols effectively to discover new mathematical meaning and structure. Symbol sense is in this view the flexible algebraic expertise that (1) directs algebraic manipulation and engineering of symbolic expressions from the background, and (2) includes the insight in underlying concepts. How students best acquire this algebraic expertise—by practice or by focusing on reasoning and strategic problem solving in lesson activities—and the role of ICT in this is much debated (See, for example, Schoenfeld, 2004).

Pierce and Stacey (2004) pointed out that many of the symbol sense behaviors described by Arcavi (1994) are unaffected by the availability of computer algebra systems because these ICT tools only perform manipulations and calculations facilitated by algorithmic routines, but do not create a mathematical model, nor decide how to work mathematically, nor interpret mathematical results in the real world as real results. However, in order to direct and monitor this work the user needs according to these authors the part of symbol sense what could be called algebraic insight. Pierce and Stacey (2004) distinguished two aspects of algebraic insight: First the thinking which allows a person to monitor working within the symbolic mode of operating, that is so-called algebraic expectation; and second the ability to link representations, in this case to link the symbolic with graphical or numeric representations. In their opinion, once a basic level of algebraic insight has been reached regarding both aspects, computer algebra systems may assist in the further development of students’ algebraic insight. I concur with this viewpoint, not only for use of computer algebra systems, but also for other ICT tools used in mathematics and science education. For example, when students in the case studies about gait analysis (Section 2.4) used wrong formulas to compute knee- and hip joint angles from measured joint positions, no teacher had to tell them because they could immediately recognize physiologically impossible angle.

In my R&D work I did not focus on school algebra and the learning of it; I only explored whether participants in the classroom case studies had developed sufficient algebraic expertise for carrying out their practical investigations, what obstacles they encountered, whether they were bothered by differences between the use of variables and functions in mathematics, science, and in computers, and whether the ICT-supported practical investigations provided opportunities for students’ growth of algebraic ex-
pertise and give them an aesthetic feel for the power of symbols and a feeling for when to abandon the use of symbols and turn to other representational forms.

The main message that I intended to send to students in the practical investigations was that mathematical formulas and algebra are not just a hobby of mathematics and physics teachers, but that they play an important role in models and modeling. Algebra may reduce the complexity of mathematical models: Parameter scaling, in which dimensional analysis is used to reduce the number of parameters, is for example a common technique applied by professional modelers to simplify a mathematical model and its analysis. I did not pursue this technique in the practical investigations, but instead I always emphasized general applicability of algebraic methods. For instance, in the computer-based investigations of mathematical shapes of real objects (Section 2.3), students could easily understand that although they were exploring particular objects, they were investigating a prototypical shape and that the set of mathematical methods and techniques which they applied were oriented toward solving and understanding a more general case. Regression was quite often applied to determine good initial estimates of parameters for use in computer models. As shown in the examples of modeling human running via a kinematic approach (Section 3.5.5, pp. 181–182) and of modeling upward hopping (Section 3.5.6, p. 190), initial estimation of parameters in a model often benefits from some algebraic pre-processing that leads to first approximations of solutions or fruitful relationships between parameters in the mathematical model.

Students’ Algebraic Skills

One might expect that upper-level secondary students have acquired basic algebraic skills and understand how to use basic laws of algebra when solving problems in which mathematical formulas play a role. However, in the classroom case studies it was often observed in class and noticed in students’ written reports that the students had rather weak algebraic skills and lacked confidence in using mathematical formulas. Illustrative were the difficulties of

- the vwo-5 students in the case study about bouncing balls (Section 2.8) with writing down a formula for the percentaged loss of energy at a bounce of the table tennis ball (p. 95), despite their Mathematics B background in the Nature & Technology profile, and

- the vwo-5 students from the Culture & Society stream in the case study about survival analysis (Section 2.6.1) with pencil-and-paper based algebraic manipulation and entering of formulas in an Excel sheet (p. 67), leading to a behavior of avoiding the use of mathematical formulas.

In both cases, I had the strong impression that the algebraic expectation of the students was underdeveloped, hindered them in their work, and led in some cases to a behavior of guessing a formula without giving it much thought.

The Versatility of Mathematical Language and Notation

Mathematics and science teachers as well as developers of a versatile computer learning environment that offers integrated tools for mathematics, science, and technology are faced with the following two difficult questions:
Chapter 4. Findings and Conclusions

1. How to deal with the versatility of mathematical language and mathematical notation, and in particular, how to deal with the variability of the concept of variable in mathematics and science?

2. How to deal with the differences in language between mathematics and science?

The interested reader is referred to discussions of these issues by Heck (2001) and by Ellermeijer and Heck (2002); here I only summarize some findings.

Regarding the notion of variable, I concluded that the meaning of variable is variable. In mathematics, three uses of variables (and parameters) can be distinguished and students must learn all three for effective algebraic work:

1. A variable used as a polyvalent name, that is, a name for an object than can take a multitude of values. For example, if $n$ is a divisor of 6, the letter stands for any of the numbers 1, 2, 3, and 6. In the task “solve $x^2 = 2$”, the variable $x$ is a priori an indeterminate and it can take a posteriori two values.

2. A variable used as a placeholder, which denotes the places in an expression where the same object is meant. For example, in the formula $(a + b)^2 = a^2 + 2ab + b^2$, the variables $a$ and $b$ stand for any number. In $f(x) = x^2$, the variable $x$ is only used to define the function $f$ and has no special meaning.

3. A variable used as a variable object, that is, a symbol for an object with varying value. For example, the object can be a physical quantity such as time, position, and temperature, or an economic quantity such as price, capital, and income.

A variable object may be related with others. One speaks of independent variables, whose values one is free to choose, and of dependent variables, whose values one can compute given the values of the independent variables. The roles of independent and dependent variables are often not fixed during a computation. For example, studying the knee joint angle in human locomotion, one may on the one hand consider it as a function of time, but on the other hand describe it as a function of the hip joint angle (in a hip-knee joint cyclogram; see, for example, Figure 2.8). In the computer modeling of acid-base titration (Section 3.3.2), when a titration curve was constructed (p. 140), it turned out to be convenient to use pH as independent variable instead of time for the computation of the so-called progress curve. In the analysis of the motion of a yoyo (Section 3.5.1) it even was advantageous to consider functions in $t^2$ instead of time $t$. One of the big ideas in calculus, and in mathematics in general, is the freedom of choosing independent and dependent variables.

In science, a variable is most often used as a name for a quantity that can vary (often with respect to time) and that in many cases can be measured. There is in science a much stronger conventional use of names of variables than in mathematics. One striking feature of the symbolic writings in science is that function, sample of function values, and single function value are mixed up easily and apparently without much harm: A physical quantity is sometimes a function of time, in other occasions a finite sample of values measured at different times, and sometimes a function value at a certain fixed time. A value of a physical quantity actually consists of three parts, namely the numerical value (a number), the precision (the number of significant decimals or the margins of error), and the unit that is used to measure the quantity.
This makes quantity arithmetic more difficult to learn and to use than reference-free number arithmetic. The following example, taken from (Van der Kooij, 1999), illustrates this. Compare the given answers to the following problem:

“Peter and John walk in a straight line in the same direction from the same starting point, with the same speed of \( 2 \text{ m/s} \). Peter starts first at time \( t = 0 \) and John 3 seconds later. Give the formula for the distance \( s \) walked by John after \( t \) seconds, for \( t > 3 \text{ sec} \).”

Answer 1: \( s = 2(t - 3) \).  
Answer 2: \( s = 2t - 6 \).

In generalized arithmetic, the two expressions are equivalent: a factored and expanded form. In quantity arithmetic, which takes dimensions into consideration, the first answer \( s = 2(t - 3) \) represents a time-approach of the problem via a distance = speed \( \times \) time formula and the second answer represents a distance-approach \( s = 2t - 6 \) via a distance = distance – distance formula. So, thinking about dimensions and units of measurement makes clear that in real life problems not only the variables but also the numbers have contextual meanings. Especially the number 1 is tricky because it is always left out of expressions: No one writes the formula \( N = N_02^t \) of exponential growth as \( N = N_02^{(t/1)} \) to show that the doubling time is 1 time unit and that dimensions are actually correct in this formula. When manipulating the formulas in a real world context, a student better keeps the dimensions in mind to verify work. In the instructional materials for the classroom case study about quantitative pharmacology (Section 2.7) much attention was payed to students’ practice with different specifications of growth rates and converting between various units. In the case study on models of sprinting (Section 3.5.5), I adopted the common approach in biomechanics and normalized quantities to body mass in order to make the algebra manipulation a little bit easier by reduction of the number of variables in the model.

Table 4.4 lists some essential differences in terminology and notational systems between science and mathematics (See, for more details, Ellermeijer & Heck, 2002).

<table>
<thead>
<tr>
<th>Mathematics</th>
<th>Science</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized arithmetic with dimensionless variables is used.</td>
<td>Quantity arithmetic with its own rules and use of units is dominant.</td>
</tr>
<tr>
<td>Names of variables are free to choose and changing names in an expression does not change its meaning.</td>
<td>A variable is often related with some scientific concept and its name is an abbreviation of this notion.</td>
</tr>
<tr>
<td>Irrational numbers like ( \sqrt{2} ), ( \pi ) and ( e ) are important; floating-point numbers are without accuracy: ( 1.0 = 1.000 ).</td>
<td>In measurements, only natural numbers and floating-point numbers with accuracy occur: ( 1.0 \neq 1.000 ).</td>
</tr>
<tr>
<td>There is a strong focus on special properties of functions (e.g., on asymptotic behavior).</td>
<td>Properties and assumptions may rule out parts of mathematical interest.</td>
</tr>
<tr>
<td>Words like “big”, “small”, “negligible” have little meaning.</td>
<td>A small change of a quantity ( Q ) is also a quantity ( \Delta Q ) with its own arithmetic.</td>
</tr>
</tbody>
</table>

Table 4.4: Some differences between the use of variables in mathematics and science.

The following illustrative example of confusion amongst students caused by the difference between mathematical language in mathematics and science occurred in the classroom case study about mathematical shapes of objects: When some students were making graphs invisible in a plot, they got to their surprise weird diagrams with no coordinate system or no labels near the axes. They were apparently thinking of a graph as a representation of a function, that is, as a representation of a single object, so...
that it suffices to work with one variable. This is common in mathematics (exemplified by the statement “plot the graph of $x^2$”). In science however, a graph represents a relation between quantities. Then one must work with at least two variables.

In general, a computer variable
- stores a numerical value or points to an object;
- may play different roles in a statement, for example in the assignment $i = i + 1$;
- may have a special, non-mathematical meaning, for example, a reference to previous results;
- obey unusual manipulation rules (e.g., by the ordering of commands or automatic simplification);
- is often a finite representation of a variable in the mathematical sense. The most complicated representation concerns the concept of variable object. In many a computer environment this is either a finite indexed list of values or an algorithm expressed in finite terms.

A user of mathematical and scientific software must be aware of these differences between computer variables and variables as they are used in mathematics and science. The differences are not obvious at all. In Heck (2001) recommendations were made to teachers interested in using computer algebra systems in their instructions.

Regarding the use of a system-dynamics based modeling tool one better not underestimates the obstacles students initially may have. For example, in a text-based modeling tool, they must understand that in the assignment $t = t + dt$, the following task is specified: Take the current value of time $t$, add the time step $dt$, and store the result in the variable $t$. Here, the symbol $t$ plays two roles: On the left-hand side it can be pictured as a lettered box for storing a value, and on the right-hand side it denotes a value retrieved from a lettered box. In mathematics however, a variable need not have a value and its role, when it appears in an expression or definition, is always fixed. The assignment would probably be represented in mathematics by a recursive definition like $t_{n+1} = t_n + dt$. In the graphical, system dynamics-based modeling tool of Coach 6, and in modeling tools like Stella and Powersim, this sequence could be represented by the graphical model shown in Figure 4.2, where the annotations underneath the icons display the hidden formulas. The left and right flow arrows represent the replacement of the current value in the lettered box by the formula shown under the left flow arrow in each step of the iteration. This formula is the sum of the current value of the lettered box and the variable $dt$. On the one hand, this graphically represents the two roles of the variable $t$ in the underlying code of the iteration step by means of two types of icons (level and flow); on the other hand, it lets the user focus on one variable ($t$) with the change of the values of this variable specified in the incoming and outgoing flow arrows.

![Figure 4.2](image)

**Figure 4.2**: A graphical model of the recurrence relation $t_{n+1} = t_n + dt$. 
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In the perspective of a variable and its change in each iteration step one can simplify the graphical model for the recurrence relation \( t_{n+1} = t_n + dt \) into the one shown in Figure 4.3.

![Figure 4.3: A simplified graphical model of \( t_{n+1} = t_n + dt \).](image)

This peculiarity of two graphical models representing one and the same recurrence relation also popped up in the classroom case study about modeling in the context of quantitative pharmacology (p. 81), when vwo students were introduced to the graphical modeling tool and explored the recurrence relation \( S_t = 0.8 \cdot S_{t-1} + 16, \ S_0 = 100 \). There I could link the various graphical models of the recurrence relation to the level of understanding of the students and their ability to manipulate the recurrence relation. However, the simple fact that there is not a single graphical model and consequently not a single computer program that computes a sequence of values specified by a recurrence relation is something that students and teachers have to get used to before they can appreciate the flexibility in computer modeling.

An important factor determining the success of computer modeling by students seems to be their ability to develop and maintain a clear view of the variables involved in the model and their relation to one another. In the classroom case studies I noticed that students tended to use explicit functional relationships between variables in the computer models instead of describing the change of variables: For example, they preferred to write down in a computer model the statement \( y = x^2 \) instead of specifying the sequence of statements \( dy = 2x \ dx; \ x = x + dx; \ y = y + dy \). Van Buuren, Uylings, and Ellermeijer (2010) also noticed this in the context of construction of physics-related models: In the researchers’ first pilot project, students tended to use in their computer models explicit formulas derived as solutions of particular problems studied earlier, instead of using fundamental equations, such as the definitions of velocity and acceleration, and Newton’s second law of motion. One could interpret this as a preference for analytical models instead of numerical models, but I think that it was more caused by lack of familiarity and experience with computer modeling. Actually, I consider it more as a hurdle to make the use of the variables \( x \) and \( y \) as variable objects more explicit by the introduction of small increases \( dx \) and \( dy \). This is something that the participants of the classroom case study about modeling shapes of real objects experienced.

**Symbol Sense and the Use of a Spreadsheet Program**

In the spreadsheet-based classroom case studies (Section 2.6), vwo students had to relate the numbers in one column of the data sheet with those in another column and to formulate an explicit dependency. With some effort most of them succeeded, mainly because the spreadsheet program allowed them to work with formulas in an active way, by creating formulas through pointing to cells. Neuwirth (1995) called this technique the gestural description of mathematical formulas, emphasizing that a formula can describe a process of computing a result instead of counting on the look at formulas as mathematical objects. It is known that this shift in focus from process
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to object character of formulas and functions is difficult for many students (Tall et al., 2001). This certainly held for the participants in the classroom case studies whose algebraic thinking was either weak and full of alternative conceptions (vwo-5 students having a Culture & Society profile) or was still in early development (vwo-3 students at gymnasium level). Although the use of the spreadsheet program helped the students understand the process character of a mathematical formula, this did not mean that working effectively with such software was always easy for them. In particular, the students’ understanding of absolute and relative cell references in Excel turned out to be weak in the classroom experiments. This was in accordance with results from educational research literature (e.g., Haspekian, 2003, 2005a,b) about the versatile use of cells in spreadsheet programs, alternative conceptions of students, and the instrumental genesis of the spreadsheet tool.

4.2.6 Representational Fluency

When students learn mathematical and scientific concepts, interacting with multiple representations such as videos, diagrams, graphs, formulas, computer models, and so forth, can on the one hand bring unique benefits, but on the other hand it may also make the learning process more difficult. Yet, it is inevitable that students must develop representational fluency. For example, Pierce and Stacey (2004) considered the ability to link the symbolic with graphical or numeric representations as one of the aspects of algebraic insight that is important in mathematical modeling. I discuss what is meant by representational fluency, reasons for emphasizing multiple representations in education, difficulties associated with the use of of multiple forms of representation, and a number of heuristics that I used to design multi-representational activities.

A Definition of Representational Fluency

Sandoval, Bell, Coleman, et al. (2000, p. 6) provided a comprehensive definition of representational fluency, which was inspired by an earlier description by Lesh (1999):

“We view representational fluency as being able to interpret and construct various disciplinary representations, and to be able to move between representations appropriately. This includes knowing what particular representations are able to illustrate or explain, and to be able to use representations as justifications for other claims. This also includes an ability to link multiple representations in meaningful ways.”

Reasons for Emphasizing Multiple Representations in Education

Links between several representations help a student especially when all aspects of a complex idea cannot be adequately represented with a single system, which is often the case in learning and doing mathematics and science, and when the meanings of actions in one representation system can be illuminated by exhibiting their consequences in another representation. This also reflects that in scientific practice heavy use of multiple representations is made for studying a problem and describing a phenomenon. Mathematicians and scientist often use multiple representations because

- different kinds of information can be conveyed with specific types of representations (for example, phenomena with simulations, animations, or video clips);
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- interaction with multiple representations supports various ideas, strategies, and processes in problem solving;

- different representations of a problem are seldom equivalent computationally, even when they contain equivalent information (Experts know this and make use of it);

- use of multiple representations promotes deeper, abstract, and general understanding.

In practice, mathematicians and scientists select those representations that they feel comfortable with, match best with their working style, and that they expect to be convenient and effective in studying or solving a particular problem. In my opinion, this use of multiple representations must also be reflected in inquiry-oriented mathematics and science education. The main idea is that multiple representations act to enrich the activities from which a student gains experience and understanding, and that they serve as a language with which the student organizes and reorganizes experiences about mathematical and scientific phenomena and concepts, hopefully giving the student the opportunity to use representations that match best with his or her learning style. Students learn that a single representation system does not suffice for problem solving and modeling in the most interesting cases, simply because it cannot cover all aspects of a scientific phenomenon or concept. In other words, multiple representations can support learning by allowing for complementary information or complementary roles. For example, tables make information explicit, allow quick and accurate read off (of single values), and facilitate pattern recognition. In diagrams, a lot of information can be grouped together such that certain aspects of a phenomenon can be quickly recognized (For example, linearity vs. nonlinearity, acceleration, deceleration, and constant velocity, and so forth). A mathematical formula is a compact, but precise way of describing a quantitative relationship between variables. An additional strength of a mathematical formula is that it is general and abstract, and therefore applicable in many cases.

The following two examples of the role of multiple representations to provide complementary information originate from the classroom case studies. In the study of the mathematical shape of the main span of a bridge (Section 2.3.1), the process of data acquisition is lost when position data collected for the span are only imported and shown in a mathematical exercise. By providing an image of the bridge and if necessary its rectified image, a student is expected to get a better understanding what the data stand for and how they are produced. If needed, (s)he can add more data. The same holds for a video analysis activity: Once the data have been collected, one could move on with the data alone. But the availability of the video clip allows the student to review the recorded process as many times as (s)he wants during the analysis of the data. Dynamic linking between a video and a recorded graph is a bigger leap forward. For example, after collecting data about human locomotion (cf., Section 2.4), the dynamic link between the video clip and a data graph (e.g., a hip-knee cyclogram) helps the user connect graphical features with events in the motion. When collecting and interpreting data about muscle activity via electromyography (Section 3.4.1), it is essential that one can simultaneously record a video clip and do a measurements, and that one can afterwards replay the experiment and use the video scrubbing technique to relate peaks in the graph of a processed EMG signal with certain gait events.
In addition to the complementary roles of multiple representations, students should learn that multiple representations can offer a source of referential accuracy by providing redundancy and that one representation can constrain interpretation of another. For example, in the animation example shown in Figure 2.32 (p. 97), the animation of the bouncing ball in the upper-right window with the in reality invisible velocity vector can constrain the understanding of the velocity-time diagram and help the student understand that positive and negative velocities mean upward and downward directions of motion, respectively. I believe that students’ understanding of instructional content can grow when combinations of representations are used, and that multiple representation can support the construction of deeper understanding when students relate those representations to identify strengths and weaknesses of particular representations and shared invariant features of all representations in use. The underpinning idea is that a persons understanding of a phenomenon, a problem, or a concept is refined the more representations (s)he can interact with. Being able to move flexibly across representations and perspectives when the task warrants it, knowing or identifying strengths and weaknesses or differences and similarities of various external representation systems, and thoughtful decision making about which representation to turn to next during a problem solving or modeling activity are personal abilities that must be learned, practiced, and maintained in inquiry-oriented mathematics and science education.

In summary, I concur with the functional taxonomy of multiple representations in teaching and learning developed by Ainsworth (1999, 2006, 2008) in which she grouped the roles that multiple representations can play in the following main categories: (1) a complementary role; (2) a constraining function; and (3) a constructing function toward deeper understanding. Practical investigations and student research projects are in my opinion essential in giving students first-hand experiences with the use of multiple representations, provided that these students have already learned the basics of the external representations, and these activities contribute to consolidation and solidification of the representational fluency of students.

Difficulties Associated with Multiple Representations

How strong the motivation for using multiple representations in mathematics and science education may be, it does not mean that one can close one’s eyes for difficulties associated with using multiple representations. The cognitive load is definitely enlarged when multiple representations come into play and it has been reported in many research studies (cf., Ainsworth, 2006, 2008; Ploetzner et al., 2008, 2009) that students find retrieving information from representations, moving between and within representations, and coming up with appropriate representations difficult. The students who participated in the classroom case studies were no exception. For example, in video activities many of the students did not spontaneously and autonomously use the replay option or the video scrubbing technique to find more details about the recorded motion and the connection of events in the video clip with features in the graphical displays. Only when they were advised to do this, they would follow the suggestion without much difficulty. Lack of graph sense and representational fluency seemed to hinder students in extracting all information that was intrinsically available in several linked representations (such as in hip-knee cyclograms of recorded human gaits) and in evaluating the quality of their experimental work (such as the quality of the experimental set-up).
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Heuristics to Guide the Design of Multi-Representational Activities

I believe that teachers can guide and support their students in learning to read and use information from representations and to work effectively with multiple representations. I also concur with Kaput (1992, pp. 533-543) that computer technology, through the dynamic linking of representations and immediate feedback, can assist students in their learning process from concrete experiences to ever more abstract objects and relationships of more advanced mathematics and science, and can support visualization and experimentation with aspects of investigated phenomena. Ainsworth (2008) summarized a number of heuristics that could be used to guide design of effective multi-representational systems [Between brackets I place labels of the connected principle(s) of multimedia learning listed by Mayer (2009)]:

- Minimize the number of representations employed and avoid too similar representations (the coherence and redundancy principle).

- Carefully assess the skills and experiences of the intended learners in order to decide on support of constraining representations to stop misinterpretation of unfamiliar representations, and to avoid unnecessary constraining representations (pre-training principle).

- Select an ordering and sequencing of representations that maximizes their benefits by allowing learners to gain knowledge and confidence with fewer representations before introducing more (segmenting principle).

- Consider extra support like help files, instructional movies, exercises, and placement of related representations close to one another on the computer screen, to help learners overcome the cognitive tasks associated with learning with multiple representations (guided activity principle, worked-out example principle, segmenting principle, modality principles, navigation principles, spatial and temporal contiguity principle).

In the instructional materials designed for the COACH-based classroom case studies I applied the above recommendations. For example, the authoring facilities of the COACH environment allowed me to adjust activities to the students level (e.g., on the one hand to incorporate guidance and on the other hand to handle the prior knowledge effect). The consistent (semi-)automatic linking between representations within the computer environment, the default suggestion in COACH of not using more than four tool windows at the same time on the computer screen, the pedagogical organization of activities in projects, and the user control to pace the presentation of the instructional materials (the pacing principle) were other signs of design choices that were in line with the above heuristics. The last guideline was explicitly applied in the e-class setting for the case study about quantitative pharmacology (Section 2.7) by means of screencasting.

4.2.7 Instructional Design

I briefly discuss the chosen instructional approach in the classroom activities, the learning outcomes, the progressive instructional approach to mathematical modeling, the experiences with four types of tool instruction, and a hypothetical learning route for the use of the graphical, system-dynamics based modeling tool.
Structured and Guided Inquiry Learning

Although I did not focus in my research on the construction of a framework for teaching and learning practical work, scientific inquiry, or modeling, nor designed and researched a learning trajectory for studying a particular subject or domain, I explored in most classroom case studies a structured and guided inquiry approach. This is interpreted here as instruction in which students are first directed in their work—mostly in order to brush up required knowledge and skills, to make them familiar with ICT tools needed in the investigation, or to direct the pathway of inquiry—and in which students at the later stage of the investigation can choose an optional activity or carry out a more open task in which they have to make own decisions on how to proceed. In the closed part, students were strongly guided in their learning process by asking questions that directed attention to certain aspects of the problem situations or specific mathematical methods and techniques that were useful for studying the problem(s) at hand. The main idea behind the closed part of an activity was that it would help students get started with inquiry work and modeling. This worked well in the spreadsheet-based classroom studies (Section 2.6) because it turned out to be possible to stimulate students to think critically. In the open part, less instructional support was provided in order to leave space for students’ own initiatives. Many students reported that they appreciated optional activities and the freedom in the more open parts of the instruction.

The Learning Outcomes

With regards to the learning outcomes of structured and guided inquiry I conclude that this worked reasonably well, but suffered from known difficulties with this approach if not enough moments of reflection on the practical work as a whole class activity are organized by the teacher. The biggest problem was the following: Students successfully carried out the closed tasks in an activity without really reflecting on the reasons of doing these tasks, their place in the inquiry route, or their links with the main research question in the activity, even though the authors had put great effort in positioning the tasks in the inquiry cycle and linking them with the research question for which an answer was sought. The side effect was limited transfer of knowledge and skills to similar activities without explicit instructions on inquiry steps.

For example in the classroom activity about human growth (Section 2.2), students learned the construction, use, and meaning of increase diagrams in the context of height growth of Dutch boys and girls. But when they chose the optional task to explore weight growth of Dutch boys and girls, they did not spontaneously look at the change of weight nor paid attention to the hint at the beginning of the text outlining the activity and giving the required data. It seemed that they insufficiently noticed the resemblance of this open activity with the closed one about human height growth done before. A possible explanation could be that the students had not been early enough involved in getting a perspective on the inquiry route. A classroom discussion at the end of the compulsory part of the activity might have helped to improve this orientation on inquiry work. However, one may not expect from students that they learn from one example that quite often the change of a quantity is more interesting than the quantity itself, and that increase diagrams and numerical differentiation of data are important instruments.
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A Progressive Instructional Approach to Mathematical Modeling

The limited transfer of inquiry and modeling abilities from one or two inquiry activities was also one of the reasons for me to design modeling activities in which students could go several times through the modeling cycle while investigating the same phenomenon, starting with simple models first and then improving them by making small changes or adding details. The idea was to let students develop some competencies related to steps of the mathematical modeling process, to let them gain experience in implementing a computer model based on the mathematical model, and to guide them first a couple of times through the whole modeling process, also with respect to reasoning and metacognitive competencies, before turning to the acquisition and practising of broader modeling competencies and before letting students do a more complicated, multi-staged modeling task that builds on application of a broad basis of mathematical and scientific knowledge and skills. Positive experiences and satisfactory results with this instructional approach were obtained in the classroom experiments in which students explored mathematical models of alcohol metabolism (Section 2.7) and explored the motion of bouncing balls (Section 2.8). This approach of exploring various models of the same phenomenon was also expected to contribute the students' development of a critical attitude and a feel for the parsimony principle that guides modeling, which means that simple models are in general preferred and are developed and validated first to arrive at feasible models of low complexity. The modeling of a bouncing gait (Section 3.5.6) is a nice illustration of a problem solving strategy based on a gradual increase of complexity: first a related problem of vertical motion with a jumping stick is explored, followed by vertical human hopping, before the real problem of forward hopping is attacked. The model progression is also clearly recognizable in the case study about the decay of beer foam in a glass (Section 3.5.3), where first a simple exponential decay model is validated before going to more complicated bi- or tri-exponential models in which a distinction has been made between quantities referred to as dry and wet foam height. Whether the instructional goals of a progressive modeling approach to mathematical modeling were met in the classroom experiments was difficult to judge, but positive indications were found in the classroom observations and the students' reports.

Tool Instruction

I also explored four types of instruction for teaching effective use of ICT tools in an inquiry-oriented activity. There were structured inquiry activities of the following kind:

1. The teacher gave a quick introduction into tool use and the students hereafter worked with instructional materials that contained detailed information about how to use the ICT tools. Tools instructions were not only presented on paper, but also inside the digital activities. This style of instruction was applied in the classroom studies about human growth, mathematical shapes of of real objects, and gait analysis (Sections 2.2–2.4).

2. The teacher gave a long classroom introduction into tool use with students making their own notes, which they could consult when they did their practical work. This instructional design was applied in the video-based practical work at pre-vocational secondary school level (Section 2.5).
3. The inquiry route was outlined in the instructional materials, but these materials only contained minimal explanations or hints about the tool use because the students were expected to have already acquired the necessary ICT skills. Students who needed more support or brushing up of their ICT skills could use an enclosed short manual. This instructional design was applied in the classroom studies that concerned spreadsheet-based data-handling (Section 2.6).

4. Instructions for learning to work with software and demonstrations of worked-out examples were given through screencasts, and in which students were furthermore supported by communication tools offered in an e-class setting. This style was to a certain extent applied in the classroom studies about pharmacology (Section 2.7) and about bouncing balls (Section 2.8).

Although all instructional designs worked out quite well in practice, it was found that:

- Detailed instructional notes had two drawbacks: (1) students reported that explanations were sometimes too vague for them; and (2) even simple mistakes in the explanation of a tool—sometimes related to a new version of the software—occasionally led to confusion and blockages because students rightly assumed that they had to do exactly what was written down. Too much detail in instruction is error-prone.

- Many vmbo tl-3 students had to learn to make notes of a teacher-led demonstration of tool use in the classroom case study presented in Section 2.5. While they were seeing how things were done (in this particular case, capturing a motion with a webcam and improving the quality of the video clip for measurement with a video editing tool), they made only a few notes because it all appeared quite simple. But when they had to carry out the experiment on the next school day, some students discovered that they had failed to write down some essential steps. Other students, more surprisingly, seemed to have forgotten about their note taking and carried out the experiment by the method of trial and error and on the basis of what they recalled from the demonstration on the previous day. Nevertheless, I still consider the strategy of students making their own notes as an interesting option; however, it may well be more suitable for pre-university students instead of pre-vocational students.

- When students were already sufficiently acquainted with an ICT tool (in this case a spreadsheet program), the use of optional software guides seemed to work well.

- Screencasting was appreciated by all students in the e-class setting and they quickly learned in this way the basics of using a graphical system-dynamics modeling tool. Screencasts seemed to work much faster and better than user manuals, written instruction, or help pages. I underpinned this success of screencasting by research-based principles of multimedia learning (cf., Mayer, 2005, 2009) and cognitive load theory (cf., Kalyuga, 2009, ch. 2; Mayer, 2005).

**A Hypothetical Learning Route for Using a Graphical Modeling Tool**

I do not present a hypothetical learning trajectory in the sense of Simon (1995) "made up of three components: the learning goal that defines the direction, the learning activ-
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ities, and the hypothetical learning processes—a prediction of how the students’ thinking and understanding will evolve in the context of the learning activities” (p. 136). Van Buuren, Uylings, and Ellermeijer (2011) have developed such a learning path on graphical, system dynamics-based modeling that is integrated into the Dutch physics curriculum from lower-secondary education (age 13-15 years). Here I only briefly discuss the approach of introducing upper-secondary pre-university students to the graphical, system dynamics-based modeling tool of COACH 6 used in the classroom case studies. This approach is rooted in basic understanding of the mathematics before computer programming is done and therefore emphasizes that the graphical model is in essence nothing more than a graphical specification of a computer program.

In the classroom experiment in which vwo students modeled shapes of bridges and hanging chains (Section 2.3.2), they used the text-based version of the modeling tool of COACH 5 to explore the parabola and catenary. The students were able to understand, adapt, and construct simple computer models. This implies that vwo students can understand the very basics of computer programming and can accept a construction of a function by means of an algorithm instead of a formula.

In the classroom experiment of modeling alcohol metabolism and pharmacokinetics (Section 2.7), students were in an e-class setting first introduced to graphical computer modeling of recurrence relations. They started first with models in which the value of one variable was iteratively updated in each step by replacing the current value by another one computed by a formula. For example, the recurrence relation \( S_n = 0.8S_{n-1} + 16, S_0 = 100 \) was graphically represented by the model shown in Figure 4.4.

![Figure 4.4: A graphical model of the recurrence relation \( S_t = 0.8 \cdot S_{t-1} + 16, S_0 = 100 \).](image)

This method also works for systems of first-order equations. For example, the Fibonacci sequence defined by the recurrence relations \( F_{n+2} = F_{n+1} + F_n, F_1 = F_2 = 1 \) can be introduced in COACH 6 by graphical modeling of the following system of first-order recurrence relations: \( F_{n+1} = G_n, G_{n+1} = G_n + F_n, F_1 = G_1 = 1 \). The model and a short table of values are shown in Figure 4.5; the hidden formulas for the flows are written underneath the icons.

![Figure 4.5: Introducing the Fibonacci sequence via the graphical modeling tool.](image)

The next step of reducing the number of arrows and making the graphical model as connected as possible leads to the model shown in Figure 4.6. Admittedly, this is not the easiest and most common ICT-based approach to number sequences and recurrence relations. In this particular case, using a text-based model would be easier, but in the
graphical model essentially happens the same; this can be verified by converting from graphical to text-based mode.

Figure 4.6: Introducing the Fibonacci sequence via the graphical modeling tool.

A study of simple autonomous differential equations could be the next step, for example an exponential growth or decay model. This was one of the early models investigated by students in the classroom experiment about alcohol metabolism (Section 2.7). The Euler algorithm for solving a differential equation should be understandable for upper-level pre-university students. This would give students a basic idea of what goes on in numerical solving of differential equations. In physics lessons, one could explore simple models of moving bodies subject to forces like the gravitational force, a spring force, and resistance forces. Starting with simple models has the advantage that the numerical solution can still be compared with analytical models. Students would have the opportunity to experience that they each time use fundamental laws of physics in the specification of the computer model. An example is the motion of a free falling object (Figure 2.26, p. 88). Once this is understood enough, more complicated systems for which no analytical solution is possible or within reach of secondary students come into sight. A model for a bouncing ball is a nice example of what students could explore (Figure 2.27, p. 88). Other topics of physics-oriented modeling activities discussed in Chapter 3 and considered within reach of some pre-university students under guidance of a physics teacher are the motion of a yoyo, a badminton shuttlecock, a chained object, an athlete in the 100 meter sprint, a person performing a bouncing gait, and an artistic gymnast on the high bar.

Compartmental modeling is another generally applicable model perspective because of its simplicity and versatility, and because the common manner of sketching compartmental models in textbooks and articles has a strong resemblance with the way in which the models would be constructed in a graphical, system dynamics-based modeling tool. Compartmental modeling was the main method explored in the classroom experiment about alcohol metabolism and quantitative pharmacology (Section 2.7) and the participants in this case study could work with the models in a satisfactory way. The 3-compartment model of Pieters et al. (1990), of which a graphical model is shown in Figure 2.21 (p. 79), is a nice example of what kind of realistic models can be investigated by students.

As one may notice, I stayed in this outlined learning route away from some metaphor, such as the commonly applied hydraulic metaphor of a bathtub-water flow system (See, for example, Booth Sweeney & Sterman, 2000; Sterman, 2000), for introducing graphical, system dynamics-based modeling to students. Research (See, for example, Booth Sweeney & Sterman, 2000, 2007; Cronin & Gonzalez, 2007; Cronin, Gonzalez, & Sterman, 2009; Pala & Vennix, 2005) shows that many people, even in the context of a bathtub-water flow system, do not understand the distinction between levels and flows and are unable to infer correctly the behavior of a level from the behavior of its inflows and outflows, or vice versa to infer the behavior of the
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4.2.8 Authenticity

Authentic inquiry implies that one does not restrict oneself to routine solution strategies and standard recipes, that more than one approach is possible, and that a single, definitive solution of the problem may not exist or cannot be found. Yet, mathematics, science, and technology provide useful tools to explore open, ill-structured problem situations in a scientific manner and come to grips with them (even in cases where a solution is beforehand clearly out of sight). In practice, often a combination of technological tools is used to observe phenomena and to collect, process, and analyze data.
In summary, authenticity can be related to the nature of the students’ activities and to the characteristics of the tools that they use. I interpreted the authentic nature of the practical investigations as the opportunity for students to work on real-world problems, with the goal to come to grips with phenomena through scientific methods. Students worked directly with high-quality data in much the same way as scientists and practitioners do. One of the aims was that the tool use reflected innovation in mathematics, science, and technology. Many times I guided the students in their use of the same theoretical framework, nomenclature, research methods, and techniques as professionals (e.g., perspective correction in crime scene photography, Section 3.2.2, or automated titration and computerized analysis of a chemical composition of a substance in a chemical laboratory, Section 3.3.2). In essence, I tried to make their mathematics and science learning resemble common practice in these disciplines, in which investigations can often be characterized as being challenging, complex, open-ended, and cross-disciplinary, and as requiring a strong commitment of participants plus a broad range of skills. This was also the reason why I (almost obsessively) chose to let students work with real data that they collected themselves (in digital image and video analysis activities) or received as secondary data that I obtained from the research literature for classroom experiments about human growth, survival analysis, weather data, alcohol metabolism, and quantitative pharmacology.

I refer to an authentic student research project as an independent investigation that is carried out by a student or a team of students and that has many of the following characteristics:

- Students work on a (preferably self-chosen) rather challenging, ill-defined or ill-structured, open-ended problem that is rooted in a real life situation instead of a more abstract or ideal situation.

- Students do not follow some standard recipes, but they examine their problem from different perspectives, using a variety of resources and high-order skills. Think, for example, of research abilities such as choosing a manageable problem, formulating a good research question, structuring work, and so forth.

- A broad range of competencies is required to make the project a success. Think of making good use of ICT for information gathering, data acquisition, data processing and analysis, problem-solving, and reporting.

- The students’ work is open-ended in the sense that there exist multiple methods or approaches to obtain many possible or even competing results. The student researchers actually decide if the investigation is finished for whatever reason.

- It offers students the opportunity to be in contact with contemporary, cross-disciplinary research and to learn about the nature of mathematics and science.

- Students disclose their own understanding through a portfolio or a polished product like a report, paper, or presentation.

The authenticity of the student profile projects concerning the motion of a bungee jumper (Section 3.5.4) and gait analysis via electromyography (Section 3.4.1) has been discussed in (Heck, 2010; Heck, Uylings, Kędzierska, & Ellermeijer, 2010). Here I only
elaborate on the authenticity of the students’ profile project on the backward giant circle on the high bar presented in Section 3.5.7.

The work of the student researchers who carried out the backward giant circle project resembles the work of sports scientists regarding affective and cognitive processes in authentic inquiry. To begin with, there were (1) the students’ intrinsic motivation to select the subject of their investigation; and (2) the generation of their own research question(s). The students teamed up to investigate the mechanics of the backward giant circle on the high bar, triggered by their own hobby and interests. To get started, they studied some biomechanics literature provided to them (e.g., Hiley & Yeadon, 2001; Tsuchiya, Murata, & Fukunaga, 2004) and searched for information on Internet (e.g., www.mannaweb.net and www.coachesinfo.com). Hereafter they formulated the following main purpose of their study: “We investigate the influence of the shoulder and hip joint angles on the angular velocity in a backward giant circle on the high bar and we investigate how a gymnast can optimize these and other factors in his performance in order to achieve the highest possible angular velocity and still perform well in the eyes of the gymnastics jury.”

For the purpose of exploring the influence of the shoulder and hip joint angles on the angular velocity in a backward giant circle with both quantitative and qualitative methods, the student researchers designed an experiment in which they could collect position, angle, velocity, and time data for a backward giant circle on a high bar through video measurement and in which they could use video tools for analyzing video clips of various types of swing motions. They planned to do the experiments in the practice room of their gymnastics club where the high bar apparatus could be used on a quiet moment during daytime. In order to be able to make several full swings after another, the students used a training tool for the apparatus that reduced the friction when the gymnast circled about the bar, made it easier to do a full swing, and simplified the control over experimental conditions. This is one aspect of authentic inquiry: Researchers are responsible for their experimental set-up and they must decide how to collect and analyze data.

All in all, this indicates a research design in which procedures to address a problem were determined, variables to investigate were selected, control of variables was thought of, and experiments and data collection were planned. Finally the student researcher wrote their profile report, which formed the basis of a published article about their work. This research project offered the students the opportunity to personally experience the challenges faced in sports science.

In all three student profile research projects discussed in this thesis, the role of ICT in investigative work was to allow the students to collect real-time data of good quality, to work in much the same way as professionals do, and to compare results from experiments, models, and theory. Furthermore, the students could develop and practice through the activities their research abilities, and be in contact with experts in the field of study. The fact that they had to apply their knowledge of mathematics and science in a meaningful way in a concrete context led at the same time probably to deepening and consolidation of this knowledge. Through this kind of practical investigation the students practiced the following important research abilities:

- Formulate good research questions that guide the work.
- Design and implement an experiment for collection of relevant data.
• Apply mathematical knowledge and techniques, and science concepts in new situations.

• Construct, test, evaluate, and improve mathematical models, and have insight in their role in science.

• Interpret and theoretically underpin results.

• Collaborate with others in an investigation task and reflect on the work.

ICT played an important role in enabling the students to carry out investigations at a high level of quality. I consider the student-driven and ICT-supported experimental design, the modeling process, the underlying thinking processes, and the discussions with peers during the research as more important in the students' work than the obtained results. All the same, it is joyful when experiment, model, and theory are in agreement, as was the case in the projects of understanding the physics of bungee jumping, analyzing muscle activity in human gait, and investigating the influence of a gymnast's body posture during a backward giant circle on the angular motion.

4.3 Aspects of Tool Design

In my development work I focussed on the design and implementation of an integrated computer environment for learning mathematics and science in an inquiry-oriented approach. In a one-sentence characterization, I specified the envisioned environment as a single, activity-based, open computer working environment that is designed for the educational setting and that offers its users a versatile set of integrated tools for the study of natural phenomena, mathematics, science, and technology. The versatility of the tools comes already to the fore in the area of working with data: data collection, through measurement with sensors or by collection of data on video clips and digital images; graphical representation of data; processing and analysis of data, including regression analysis; and construction, simulation, and validation of computer models. It forms the heart of the so-called STOLE concept, acronym for Scientific and Technical Open Learning Environment (Ellermeijer, 1988), outlined in Section 1.2 (pp. 13–14) and exemplified by its implementation in the hard- and software environment named COACH (Heck, Kędzierska, & Ellermeijer, 2009). The development of COACH and, on a smaller scale, of particular tools in this environment was characterized in Section 1.4 as a case-based design of educational hard- and software. This case-based design approach may give the wrong impression of a somewhat unstructured development process that is not guided by ideas about mathematics and science education and the role of ICT herein, but only progresses by the method of trial and improvement in a rather unpredictable direction. In this section I try to take away this alternative conception of case-based design of educational hard- and software. To this end I discuss the role of ICT in quantitative mathematical modeling and exemplify it by the case studies presented in this thesis. Hereafter I reflect on some ICT-supported mathematical representations and on transitions between various representations. Finally I summarize my contributions to development of the video analysis tool, the data processing and analysis tools, and the modeling tool, in the hope and expectation that this also gives a more honest impression of the case-based design process.
4.3. Aspects of Tool Design

4.3.1 The Role of ICT in Quantitative Mathematical Modeling

The model of quantitative mathematical modeling presented in Section 4.1 (pp. 206–208), which consists of an empirical inquiry cycle and a mathematical modeling cycle, contains no reference to ICT or other forms of technology. In essence each transition in the cycles is unaffected by the availability of technology because a tool only supports its user and does not take over the job to be done: Technology itself does not design an experiment, nor creates a mathematical model, nor decides how to work mathematically, nor interprets empirical or mathematical results in the real world as real results. This does not mean that ICT and technology play no role. For each transition in the model of quantitative mathematical modeling one can list options of meaningful use ICT and other forms of technology. In this section I want to list some computer-based activities for students that are part of quantitative mathematical modeling, without having the pretension of making an exhaustive listing.

Understanding the Task

For orientation on the real situation and the task or problem, a student must on the one hand evoke his or her knowledge or experience concerning the subject and on the other hand actively search for new information that could be useful. When confronted with a quantitative mathematical modeling task it is always wise first to check what one already knows (or thinks to know) about the situated problem and whether one can draw inspiration or a lesson from prior experiences with quantitative mathematical modeling. Discussion with peers, teachers, or experts is very useful, but in the absence of direct contact this may be realized by email, chat, or within a virtual learning environment. This use of ICT as a communication tool comes more to the fore in the process of collecting information: An Internet search provides access to a wealth of information on a subject or the context of an investigation, and it may lead to contacts with persons who share the same or similar interest in the subject or have strong subject knowledge or experience. For example, in the project of understanding the physics of bungee jumping (Section 3.5.4), the student researchers contacted one of the authors of a scientific paper on the subject. In one profile research project about human growth that arose from the classroom case study presented in Section 2.2, the student researchers consulted a doctor working in a child health center.

At this stage of coming to grips with the problem situation, ICT can be a source of information in various forms: A video or animation on a DVD or on Internet offers the opportunity to get a clearer picture of the problem or task at hand, or allows its viewer to observe a phenomenon more closely. An animation or a computer simulation may allow its user to experiment and gain experience. For example, playing with an animation of a ball bouncing on an oscillating platform as shown in Figure 2.32 (p. 97) may illustrate the phenomenon to be investigated and introduce students into the concept of chaos. High speed video technology provides the opportunity to observe motions in more detail. From profile research projects it is clear that students spend a substantial amount of time on searching and finding information on Internet. Usually, they spend less time on explorative, self-devised experiments that could provide an orientation basis. Experiments of this kind are predominantly of qualitative nature and are mostly oriented toward answering questions such as “What essentially happens if one of the conditions in an experiment changes?” In the context of a bouncing ball, for example, one may explore the change in motion of the bouncing ball when the
Chapter 4. Findings and Conclusions

ground surface or the material, size, weight, or temperature of the ball is different. In
general, ICT plays in these explorative experiments only a minor role.

Simplifying and Structuring
In order to turn a situation model into a real model one must structure the existing,
possibly incomplete knowledge and experiences, do simple experiments, make assump-
tions, and apply simplifications. In an investigative study about mathematical shapes
of objects (cf., Section 2.3), for example, the simple experiment of trying to hold up
a necklace such that it matches best with a parabola drawn on the blackboard would
quickly reveal that the shape of a chain hanging under gravity cannot be parabolic.
Experimenting with bouncing balls (cf., Section 2.8) would quickly lead to simplifica-
tions and assumptions in the mathematical modeling of the phenomenon. Quite often
one distinguishes subproblems of the given original situational model. For example,
in case of a bouncing ball problem, one might first study the motion of a dropped ball
by experimentation and mathematical modeling. Technology can be used for rapid
collection and analysis of data for these subproblems so that one can judge whether
one is on the right track and is able to formulate the real model in clear terms and to
come up with sensible conjectures and hypotheses.

Computer simulation may also help students come to grips with the problem or
task and distinguish between important and less relevant aspects, but in general ICT
plays in this step of the quantitative mathematical modeling process only a minor
role; strategic thinking and reliance on previous experience with modeling (e.g., seeing
the analogy with a model that one has explored before) and simple experiments seems
more important. An exception is when the situation problem itself is already ICT-rich:
Maurits (2007) presented an example from medical practice in which a doctor asked
a mathematician whether musculoskeletal ultrasound images of patients could also be
assessed in a quantitative manner. The translation of the question of the medical
specialist into a physico-mathematical problem boils in this case down to a search for
measurable concepts in an ICT-rich context.

Mathematizing
The process of mathematization is commonly considered a matter of doing a great deal
of brainwork and puzzling over problems with pencil and paper and a minor role for
ICT. In this phase, one is more concerned with the choice of the type and complexity
of the model. It may be true that ICT does not create a mathematical model, but
as Pierce and Stacey (2004) pointed out, tools like a computer algebra system may
help its user carry out routine manipulations and calculations, and create graphical
representations. Kaput and Shaffer (2002; see also Shaffer & Kaput, 1998), referred to
the processing power of new cognitive tools as the new representational infrastructure
of computational media that change the way people think and work.

In the example presented by Maurits (2007), three measures were determined on
the basis of descriptive statistics and mathematical filtering of data. The mathema-
tization process consisted here of contemplating on the desired functionality of these
measures. Generally speaking, ICT is often used to produce order out of huge data
sets and to put standard models to the test. Researchers build upon their knowledge
and experience with existing scientific methods. Blomhøj and Jensen (2003) presented
an example in which the mathematization of a pharmacological process of clearance
of a drug happened in reverse order: First a suitable regression formula in the form
of a sum of exponential functions was determined for a concentration-time profile and thereafter a multi-compartment model was developed that had the regression formula as its solution. This is an extreme example of ICT dominating the mathematization process. It is essentially an admission of weakness because data have only been described mathematically without any further explanation or reasoning; it is a kind of black-box, experimental modeling.

In general, the supportive use of ICT in the mathematization phase is not uncommon: A modeler uses calculators, spreadsheets, computer algebra systems, simulation programs, statistical softwares, and other programs for making computations with simple models. In this way, ICT may support the process of thinking and working toward a better understanding of the problem. Some reserve is appropriate here: There are plenty of examples of incompetent and meaningless use of ICT for developing mathematical models. Even in case of competent use of ICT there is always the danger of misleading results that may put a modeler on the wrong track. This is one of the reasons why mathematicians tend to have a strong reserve toward the use of ICT in the mathematization step of modeling and rely more on deep thinking and pencil and paper-based brainstorming to determine the basic characteristics of the model class to which the mathematical model under development is going to belong.

Working Mathematically
Mathematicians have in general a strong reserve toward using ICT for obtaining mathematical results. They prefer to first determine whether all relevant relationships and characteristics of the investigated real problem have been included properly in view of the modeling targets, and whether some properties of the outcomes of the mathematical modeling process or the solutions can be determined in advance, before doing computational work. Algebra may reduce the complexity of mathematical models: Parameter scaling, in which dimensional analysis is used to reduce the number of parameters, is a common technique applied by professional modelers to simplify a mathematical model and its analysis. In other words, mathematicians pay a lot of attention to the analysis of the mathematical model prior to the computational work and the attempts to find solutions. Broer (2007) made a plea in favor of such an approach in the teaching and learning of mathematical modeling.

Once the conceptual validation of the mathematical model has been done and one can move on to the implementation, the use of ICT is the rule rather than the exception. One of the professional competencies that an applied mathematician is expected to possess is ICT competency; that is, the ability to do computer programming, to carry out computer-based data handling such as processing of data, making meaningful graphical representations, and analyzing data (e.g., by regression analysis), and to judge whether the implemented methods have functioned correctly. ICT tools play in this phase of mathematical modeling a strong supportive role, allowing their users to carry out mathematical work for which algorithmic routines are available or can easily be implemented, to flexibly use multiple mathematical representations (mainly graphs and tables), and to make a computer model out of a mathematical model. In the spreadsheet-based classroom case studies about survival analysis and weather data handling (Section 2.6), students set up statistical models, underpinned their assumptions herein, and used the spreadsheet software only to remove the drudgery of data manipulation, to try out ideas, and to keep their focus on the modeling aspects of the assignments instead of on the computational work.
Chapter 4. Findings and Conclusions

A substantial factor for success of ICT-supported mathematical work is that the users of ICT tools already have sufficiently mastered the use of these tools in practice so that they can focus in their work on the mathematical progress made instead of dealing with ICT obstacles. This holds even more for students engaged in mathematical modeling: The cognitive load in the phase of working mathematically is only lowered and the students can only benefit from ICT tools when they already have built up sufficient knowledge and experience with the tools in a variety of contexts and already have developed a good sense of strengths and weaknesses of ICT tools and a good understanding of opportunities and threads of ICT use.

However, it should be noted that not only the students’ level of mastering a particular ICT tool plays a role: Also the facilities, restrictions, and user interface of an ICT tool have a strong influence on the usability of the tool and the way people work with it. For example, research of Löhner (2005) indicated that the choice of representations in computer modeling has a strong influence on the students’ construction of computer models and their reasoning during computer modeling. In other words, one may also expect that the design of a tool affects the way students and teachers work with it and benefit from this use, and that details of tool design matter much.

For example, the built-in facilities of a video analysis tool, its user interface, and its integration with other ICT tools, say for data representation and data analysis, determine to a large extent the usability of the video analysis tool in educational and professional practice. In the case study about graphical modeling of chemical kinetics (Section 3.3.1) I discussed the methodology, strengths, and weaknesses of the implementation of graphical, system dynamics-based software for modeling chemical kinetics and I proposed an extension of classical graphical modeling that would bring it closer to how chemists think about chemical reactions and that could make it easier for students to investigate chemical kinetics. In the case study about bouncing balls (Section 2.8) I exemplified how the introduction of discrete time event handling in system dynamics-based modeling according to the principle of software triggering could lead to a modeling tool that combines a classical system dynamics approach with event-based modeling for abruptly changing processes and that has a hybrid structure of which students can easily make use. In the modeling of the motion of a yoyo (Section 3.5.1) I exemplified a feature of the graphical modeling tool of Coach to (temporarily) hide detailed information about the implemented model. By creating open or closed subsystems of a graphical model, an instructor could focus on the overall structure of the model and zoom in on any subsystem for more details. In other words, the information could be shown at different levels of complexity.

Interpreting Mathematical Results

In the end, the modeling results obtained so far must be interpreted in terms of the real world context where one started the investigation. In this phase the descriptive quality of the mathematical model is more or less questioned. Two central questions are: (1) What do the model outcomes mean in the real context? and (2) How well do the available empirical data fit with the worked-out model? The second question certainly illustrates that this step in the mathematical modeling is closely connected with the comparison of experimental results with mathematical results. But even without direct comparison of these results, one can still interpret mathematical results in a qualitative sense as real results. For answering the questions one must at least employ a criterion of fit and find the best values for the model parameters, either
by trial and improvement or in a more systematic way by means of algebraic pre-
processing and regression analysis. One may, for example, do computer simulations
and analyze how changes of parameter values affect the numerical solutions and what
this means in the context of phenomenon investigated.

In the case study about the motion of a falling badminton shuttlecock (Section 3.5.2),
for example, it turned out that a linear drag model would never lead to a good
description of the experimental data: One could not get a decent description of both
position and velocity at the same time; the quadratic drag model worked much better
in this sense. Examples of algebraic pre-processing and regression analysis for pa-
rameter estimation with an eye on the real context can be found in the case studies
of mathematical modeling of sprinting in Section 3.5.5 (pp. 181–182), and upward
hopping in Section 3.5.6 (p. 190).

Validating Real Results
The model validation concerns all quality aspects, properties, and behaviors of a model.
Four central questions are: (1) Do the real results obtained from the model reasonably
and accurately describe or predict reality? (2) Do the real results explain or contribute
to understanding of reality? (3) Are the assumptions and simplifications in the model
adequate or must they be reviewed? and (4) Are the results of analyzing specific model
properties and behavior satisfactory?

If a computer model has been implemented, then one can run miscellaneous simu-
lations in order to explore the sensitivity of modeling results to small disturbances in
initial conditions or model parameter values. In case of stochastic models one can ex-
plode by simulation (e.g., Monte Carlo simulations) how uncertainty propagates from
input to output. But once more, ICT has in this phase only a supportive role: ICT
tools do not interpret results, only humans do.

The motion analysis of a yoyo (Section 3.5.1) provided a clear example of ICT
support in the interpretation of the real results in the context of the phenomenon: By
using trend functions as coordinate functions for a computed point and by displaying
this point in the video clip of a moving yoyo, the suggestion that this point was close
to the position of the axle during the unwinding phase of the yoyo was supported and
therefore the mathematical modeling was underpinned.

In the case study about mathematical modeling of sprinting (Section 3.5.5), the
wind-adapted Keller model and the parameter values that fitted data of the first 80 m
of Usain Bolt’s sprint at the Beijing Olympics 2008 were used to compute what sprint
time the athlete could have run if he had not decelerated in the last meters. This
model was also used to predict a 100 m sprint time of a female student on the basis
of her 25 m sprint and interpreted as somewhat unrealistic, hereby underpinning the
change of the mathematical model into the Tibshirani model for the same purpose.

In the profile project about the backward giant circle (Section 3.5.7), the student
researchers qualitatively analyzed their graphs and explored whether a gymnast can
return to handstand when his shoulder joint angle gets small during the movement.
They concluded that only a free hip swing to handstand on the high bar would be
possible; but this is not considered a giant circle. They also used mathematics and
physics knowledge about rotational motion to explain why certain techniques on the
high bar would be possible or impossible. This work showed a surprisingly high level
of validation of real results in the context of the phenomenon investigated.
Validation of the methods used may also mean that one compares several routes to real results. In the case study about vertical vertical jumps (Section 3.4.2), for example, the sensor-based analysis of the motion was compared with the video analysis. In the case study about bouncing balls (Section 2.8) sound-based experimental results were compared with results obtained by video analysis.

Presenting Findings
It goes without saying that ICT tools such as text processors and presentation tools can be used to report and present the final answer to the original problem. The digital format is in general better readable than hand-written reports and looks nicer, but the major advantage is that it allows inclusion of graphs, tabulated results, screen shots, and so forth, so that the conclusions can be supported by data representations that would be impossible without ICT support; see, for example, Figure 2.11 (p. 54), which is a picture taken from a student’s profile report about gait analysis.

Planning and Designing an Experiment
ICT tools themselves do not create an experimental design; what matters is mainly the knowledge and experience that an experimenter has with ICT-supported experiments when (s)he designs a new experiment. ICT can support the experimenter or a student researcher as a communication tool. A student, for example, may retrieve information about the real problem, the context of the practical investigation, and possibly related experiments from available CD-ROMs, DVDs, and Internet resources. Occasionally in the educational context, ICT is used to communicate with experts, as happened in the profile research project about bungee jumping (Section 3.5.4), or to do remote experiments (See, for example, Engelbarts, 2009). Video clips may illustrate how experiments can be set up or explain the functionality of measurement apparatus. Even a digital image of the equipment needed for carrying out a specific experiment or a picture of the experimental set-up in instructional materials may already give students a clearer view of what to do or what can be done in the laboratory. Simulated computer laboratory experiments, also known as virtual labs or web labs (See, for example in the context of general chemistry education, the Virtual Lab Simulation at www.chemcollective.org, or in the context of physics education, the software Interactive Physics at www.pasco.com) may prepare a student to do experiments or use instrumentation before they enter the real laboratory.

Planning and design of an experiment must be carefully done to increase the chance of success and useful results, but also to avoid dangerous circumstances. In the experiment of measuring the pupil reflex (Section 3.4.3), for example, one must realize that a bicycle lamp is strong enough for the experiment but will not cause damage to one’s eye. In case of measurement of muscle activity via electromyography (Section 3.4.1), an experimenter must know about the workings of the apparatus in the experimental set-up to understand and appreciate that safety rules have complicated the measurements. In the case study about modeling bouncing gaits (Section 3.5.6), students must realize that experiments of upward and forward hopping must be done under similar circumstance, that is, on the same motorized treadmill. In motion analysis activities, students must reflect on size, shape, color, and position of markers before recording the motion on video in order to make video measurement afterwards an easier job.

Conducting an Experiment and Data Handling
The ICT use at this stage can, and in my opinion should, reflect the innovation of ICT
use in experimental work by scientists and practitioners. Computers are in practice much used to collect, present, process, and analyze data. All case studies presented in Chapter 2 and 3 illustrated how ICT-supported practical investigations can be carried out by students and what affordances an integrated, multimedia, tool-based computer working environment can offer, ranging from measurements with sensors and video analysis to rather advanced data processing and data analysis. Some advantages are:

- The versatility of data logging methods enhances the diversity of experiments that can be done and supports a multi-methods approach. For example, the experimental study of standing vertical jumps (Section 3.4.2) by means of simultaneous measurement with a force plate and video recording of the motion, provided the opportunity to compare the experimental results of sensor-based measurement with those obtained by video analysis. The experiment of measuring the pupil light reflex (Section 3.4.3) illustrated what can be achieved by a combination of experimental methods.

- Real-time data logging gives students direct feedback in addition to the possibility of a replay of an experiment. Especially in motion analysis activities it is important that one can examine video clips and dynamically linked graphical representations as many times as needed to explore details. Prototypical examples presented in the case studies were the high speed video analysis of moving coins (Section 3.2.3) and gait analysis via electromyography (Section 3.4.1).

- If a data logging environment allows users to make predictions in the forms of graphs before doing the experiment, students can learn from comparing their predictions with actual measurements. This has been investigated in various studies of students’ understanding of motion graphs (See, for example, Kozhevnikov & Thornton, 2006; Nicolaou et al., 2007; Russell et al., 2003, 2004);

- The quality of a dataset obtained by means of sensors connected to a computer or via digital video and image measurements is in general higher than of a manually collected data set. For example, a titration curve is more easily obtained with an automated pH titration system than by means of a hands-on titration and is quality is in general also much better (cf., Section 3.3.2). The case studies about bouncing balls (Section 2.8), falling badminton shuttlecocks (Section 3.5.2), and falling chained objects (Section 3.5.4) illustrated that advancements in video technology affect the possibilities of experimental exploration of phenomena and the quality of the investigations.

- Repetition of experiments, possibly under different experimental settings, can be easily and rapidly realized. This also means that one can do some try outs for the purpose of getting used to and improving the experimental setting before a definite experiment is carried out. The profile research projects about bungee jumping (Section 3.5.4) and circling around the high bar (Section 3.5.7) illustrated that some student researchers reflect on the experimental settings and change the experimental set-up after some trials.

- Computer-based graphing helps students to create multiple representations of data. If these multiple representation are dynamically linked, it may help better understand the phenomenon under investigation. The analysis of muscle activity
via electromyography (Section 3.4.1), for example, would be impossible without
the possibility of simultaneous scanning of the video clip of the gait pattern and
the processed graphs of the EMG data.

- Numerical differentiation and integration, data smoothing, regression, and signal
analysis are computer-based methods for data handling that enhance the investiga-
tive work of students. Tidal analysis (Section 3.3.3) was an authentic context
illustrating advanced sinusoidal regression that would hardly be possible with-
out computers. Computer-based data handling methods support an experi-
mental modeling approach, but at the same time they are needed in mathematical
modeling based on fundamental science principles. With numerical differentia-
tion and integration, case studies like the quantitative mathematical modeling
of standing vertical jumps (Section 3.4.2) and sprinting (Section 3.5.5) would
not have been easily realized. In Newtonian mechanics, velocity is a more fun-
damental variable than position because Newton's second law concerns for most
systems the change of velocity in terms of forces acting a system. Problematic
is only that one is often interested in or can only measure the position of an
object. Data processing become inevitable.

v. Interpreting Experimental Results
In the end, the experimental results obtained so far must be interpreted in terms of the
real world context where one started the investigation. In computer-based inquiry, the
readily available and possibly dynamically linked multiple representations support the
interpretation of results. In the case studies that involved video analysis, for example,
the dynamic linking between the video clip and graphical representations of measured
or computed quantities helped the interpretation of results in the real context.

In the classroom experiments about human locomotion (Section 2.4), students
interpreted hip-knee cyclograms for various gait patterns and described the differences
between diagrams in terms of the differences between the gait patterns. They could
use the video clips for this purpose to enforce their arguments.

The student researchers who analyzed the backward giant circle on the high bar
(Section 3.5.7) used their video clips to give meaning to the created angle-time dia-
grams in the context of the gymnast's body motion. Naturally, it is only support for
the interpretation of results that ICT tools can provide; in the end, the researchers
have to do the job.

In a sense one can view comparison of experimental results with results found in
the literature also as a form of interpreting experimental results as real results because
the experimental methods in which these data have been obtained can differ much.
In the case study about tidal movement (Section 3.3.3), the results of harmonic ana-
lysis of tidal data were compared with literature data and the tidal components were
characterized as such. In the case study about bouncing balls (Section 2.8) and the
bouncing gait (Section 3.5.6), the experimental results for a table tennis ball and a
hopping person were compared with data from research papers, respectively. In the
case study about a falling badminton shuttlecock (Section 3.3.3), the experimental
results and the applied methods were compared with those of an experiment reported
about earlier by another student researcher and with those published in research pa-
pers. In all case studies, the comparison of own experimental results with outcomes
4.3. Aspects of Tool Design

of others were used to explore the validity and reliability of the results by checking if the claims fitted within earlier published work.

**Comparing Experimental and Mathematical Results**

Computer generated graphs or tables, either from measurements or mathematical modeling, make it easier to compare experimental results and results from mathematical modeling. The graphs can be visually compared and a difference graph can be computed to validate the accuracy of the results. In the ICT-supported study of acid-base titration curves (Section 3.3.2), and in particular in the practical investigation of analyzing the acids in a Cola drinks (pp. 140–142), students improved values of parameters in a computer model by trial and error until they considered the measured and modeled titration curves as close enough. Tables can be used to get precise information about differences between experimental and mathematical results and to further analyze the residual data.

Table 4.5 summarizes possible roles of ICT and other forms of technology in the model of quantitative mathematical modeling shown in Figure 4.1. But there are many other motives for using ICT in modeling; I list a few arguments:

- **Authenticity:** Educational use of ICT should reflect innovations in science practice on this terrain. The many motion analysis studies presented in this thesis illustrated what is possible in this respect.

- **Efficiency:** ICT helps students efficiently and effectively carry out routine manipulations and calculations, and create graphical representations so that they can keep their focus on the modeling aspects of the assignments instead of on the computational work. The examples of this thesis showed that this holds especially for data handling and visualization.

- **Motivation and fun:** Some students are more motivated when they can do ICT-supported inquiry-oriented activities in a real world context (See, for example, the screen shots in Figure 3.8 and Figure 3.13 on page 110 and 114, respectively).

- **Interactivity:** ICT and technology invite students to experiment and try out methods, in cases where they would otherwise keep a passive attitude. The case study about survival analysis (Section 2.6.1) exemplified this motive and the previous one.

- **Feedback:** ICT enables the realization of instantaneous and direct feedback to students. The dynamic linking between multiple representations plays a key role in this. Feedback from computer simulations may also enable students to examine mathematical models instead of just to assume or accept a given model.

- **Enrichment:** ICT and technology offer students opportunities to explore a variety of applications of mathematics and science, and helps them get a more realistic picture of the role of ICT in science and technology. Computer models and modeling is one of the areas that students can enter; motion analysis is another one explored in the many case studies presented in this thesis.

- **Access to advanced methods:** ICT enlarges the students’ repertoire of methods and techniques that can be used in inquiry-oriented activities. One of the aims of
Chapter 4. Findings and Conclusions

my research and development work was to explore what kind of integrated tools would support inquiry-oriented student activities. Thus I paid much attention to data processing and analysis methods such as data smoothing, numerical differentiation, and regression.

<table>
<thead>
<tr>
<th>Transition</th>
<th>Potential role of ICT and technology</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Understanding the task</td>
<td><strong>Problem orientation</strong>: observing videos, experimenting with computer simulations and animations; <strong>Information retrieval</strong>: using CD-ROMs, DVDs, or Internet; <strong>Communication</strong>: chat, email, VLE.</td>
</tr>
<tr>
<td>2. Simplifying and structuring</td>
<td><strong>Getting a strong grip on the situated problem</strong>: observing videos, experimenting with computer simulations and animations, data logging in small experiments (possibly for subproblems).</td>
</tr>
<tr>
<td>3. Mathematization</td>
<td><strong>Collecting one’s thoughts in mathematical representations</strong>: representing schematically and graphically with the help of software, making ICT-supported calculations for simple models.</td>
</tr>
<tr>
<td>4. Working mathematically</td>
<td><strong>Computer-based data handling</strong>: representing data in graphical and tabular form, and processing and analyzing data on a computer (e.g., numerical differentiation, data smoothing, regression analysis); <strong>Construction and simulation of computer models</strong>: using modeling tools and computer programming to solve mathematical models numerically.</td>
</tr>
<tr>
<td>5. Interpreting mathematical results</td>
<td><strong>Looking at results in the context of the phenomenon investigated</strong>: doing computer simulations and analyzing how changes of parameter values affect the numerical solutions.</td>
</tr>
<tr>
<td>6. Validating real results</td>
<td><strong>Making predictions</strong>: doing ICT-supported calculations and simulations for the purpose of predict results in the investigated context; <strong>Evaluating the model and methods used</strong>: validating computer-based methods, examining the quality of the computer model, suggesting adaptations of the model if necessary; <strong>Reasoning</strong>: enforcing arguments by means of computer-generated visualization and other ICT-based inscriptions.</td>
</tr>
<tr>
<td>7. Presenting findings</td>
<td><strong>Text processing and presentation</strong>: reporting via Internet, presentation software, or a text processor, with inclusion of multimedia components.</td>
</tr>
<tr>
<td>iii. Planning and designing an experiment</td>
<td><strong>Orientation on the experimental setting</strong>: retrieving information from Internet, watching video demonstrations, reading on-line help, considering pros and cons of experiments (remote experiments, web labs, video technology, sensor choice, and so forth).</td>
</tr>
<tr>
<td>iv. Conducting an experiment and data handling</td>
<td><strong>Data collection</strong>: computer-based measurement (sensor-based or measurement on video clips and digital images) and representation in graphical or tabular form; <strong>Data processing</strong>: applying mathematical formula to data, filtering of signals and data smoothing, computing numerical derivatives, numerical integration; <strong>Data analysis</strong>: regression and signal analysis.</td>
</tr>
<tr>
<td>v. Interpreting experimental results</td>
<td><strong>Looking at results in the context of the phenomenon investigated</strong>: changing experimental conditions and analyzing how these changes affect the results.</td>
</tr>
<tr>
<td>viii. Comparing experimental and mathematical results</td>
<td><strong>ICT-supported comparison</strong>: visual examination of computer-generated graphs to find similarities and differences between experimental and mathematical results.</td>
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</tbody>
</table>

Table 4.5: A non-exhaustive listing of potential roles of ICT and technology in transitions in the model of quantitative mathematical modeling presented in Section 4.1 (pp. 206–208); see Figure 4.1 and Table 4.1.
4.3.2 ICT-supported Mathematical Representations

I reflect on the main ICT-supported mathematical representations in quantitative mathematical modeling and on transitions between these representations. It is based on a representational framework, known as the Rule of Five, for thinking about the use of external representations in mathematics and science. It also forms a framework for thinking about possible roles of ICT in the process of handling such a variety of external representations.

The Rule of Five Framework of Multiple Representations

The theoretical rationale of tool integration in a versatile computer environment for inquiry-oriented mathematics and science education is that the use of multiple representations is crucial for deep understanding of real phenomena and that this process of understanding is promoted when learners are not distracted by technical burdens that could have been avoided by the provision of tools that work well together. This view can be underpinned by theoretical frameworks such as the Kaput-Goldin representational framework for mathematical cognition and learning (cf., Goldin, 2008; Goldin & Kaput, 1996; Kaput, 1992, 1994) and the so-called Rule of Five framework on multiple representations (cf., Dick & Edwards, 2008).

The representations used in an environment such as Coach are often not static entities, but dynamic elements that are linked so that a change in one representation affects the other representations. This is one reason to speak about integrated tools (There are more reasons, such as integration of different styles of tool use). The underlying ideas of having multiple, dynamically linked representations available in a student activity are:

- It illuminates the meaning of actions in one representation by exhibiting their consequences in another representation.
- The number of ways to come to a solution of a problem increases.
- A person’s understanding of a phenomenon, a problem, or a concept is refined the more representations (s)he can interact with.
- It supports the construction of deeper understanding when students relate those representations to identify strengths and weaknesses of particular representations and shared invariant features of all representations in use.

Kaput (1989, p. 179) phrased the last argument as follows: “The cognitive linking of representations creates a whole that is more than the sum of its parts.” In the SIMCALC project on the mathematics of change and variation, Kaput (1998) went one step further and proposed the use of multiple, dynamically linked descriptions of experientially real situations or phenomena, for example by cross-connecting a rate graph to a total amount formula. In COACH 6, such dynamic cross-connections between totals and rates descriptions in the context of kinematics are also supported. In order to avoid that students become too passive in the process of constructing a deep understanding by making cognitive links between various representation, Ainsworth (1999) suggested the strategy to provide full linking when students are new to the task and to fade this support as their knowledge and experience grows. I am not
a proponent of various implementations of the dynamic linking behavior in a learning environment because of the many experiences in an inquiry-oriented approach to mathematics, science and technology education that full linking is very useful in working with data, even for more experienced persons. It would be a waste of convenient tooling in investigative work. For example, how more difficult it would be to understand an electromyogram in a biomechanics study when the body motion is not simultaneously recorded and dynamically linked with the data graphs and related diagrams. Dynamic linking of representations is common in many professional software systems and in my opinion this should be reflected in computer learning environments that also serve as transitional professional softwares. As a positive side effect too, my experiences with practical investigations and student research projects are that the use of multiple, linked representations and descriptions of phenomena in a dynamic, interactive medium for investigation of real world problem situations makes mathematics and science more attractive, challenging, and interesting for students.

Like Kaput (1998), I am of opinion that, with the advent of the computer in mathematics and science education, the Rule of Four framework for a context-based approach to the mathematical concept of function, which advocates that this topic is treated numerically, graphically, symbolically, and verbally, must be extended by a new representation system, namely, that of a materialized phenomenon. This phenomenon can either be cybernetic—as with screen objects whose movement is controlled by a model, computer program, or mathematical functions—or physical, as with sensors and with devices linked to a computer where their motion is controlled by mathematical functions defined on the computer or by control programs. I also categorize video clips or digital images as materialized phenomena. Kaput placed the cybernetic and physical phenomena in the heart of mathematics education of change and variation. Below I give a more neutral description of the Rule of Five framework.

<table>
<thead>
<tr>
<th>from → to</th>
<th>cybernetic &amp; physical phenomena</th>
<th>situations, verbal descriptions</th>
<th>tables</th>
<th>graphs</th>
<th>formulae</th>
</tr>
</thead>
<tbody>
<tr>
<td>cybernetic &amp; physical phenomena</td>
<td>mouse interaction</td>
<td>describing</td>
<td>data collection &amp; generation</td>
<td>MBL/LBM</td>
<td>modeling</td>
</tr>
<tr>
<td>situations, verbal descriptions</td>
<td>programming, animation &amp; control</td>
<td>rewording</td>
<td>measuring</td>
<td>sketching</td>
<td>modeling</td>
</tr>
<tr>
<td>tables</td>
<td>animation &amp; control</td>
<td>reading</td>
<td>data transforming</td>
<td>plotting</td>
<td>fitting</td>
</tr>
<tr>
<td>graphs</td>
<td>re-enacting, LBM</td>
<td>interpretation</td>
<td>reading off</td>
<td>re-scaling &amp; smoothing</td>
<td>curve fitting</td>
</tr>
<tr>
<td>formulae</td>
<td>re-enacting, LBM</td>
<td>parameter recognition</td>
<td>computing</td>
<td>sketching</td>
<td>algebraic manipulation</td>
</tr>
</tbody>
</table>

Table 4.6: A table of translations and transformation processes for five representations.

Table 4.6 shows a 5×5 table of transitions between and transformation processes within the representation systems (in fact, this table is a non-exhaustive list of processes). It is an extended version of the 4×4 table of (Dick & Edwards, 2008, Figure 10.4), which was in turn an augmented version of the original version of Janvier (1987, p. 28). Two terms used in the above table need explanation: LBM is an acronym of ‘Lines Become Motion’ that is the reverse of MBL (Microcomputer-Based Laboratory, also known as data logging) and refers to the possibility to define functions graphi-
4.3. Aspects of Tool Design

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These authors went one step further and argued (p. 109) that “within dynamic media (e.g. Dynamic Geometry), the user (like the infant) can seek to explore the actions of their intentions as a reaction to the surrounding within which they act. The dynamic media co-acts with the user, so the user can guide or be guided by the software environment.” In their perspective, the knowledge of students also shapes the use of the representational infrastructure.

What Figure 4.7 clearly illustrates is that multiple representations of the concept of function offer teachers and students the freedom to choose various starting points or continuation points in approaching a mathematical problem that they consider feasible. For example, functions are in the COACH environment often described by lists of numerical values. This has to do with its origin in science education, where working with measurable physical quantities, which possibly bear a functional relationship, is an important issue. Values of quantities can be obtained in many ways: (1) through (real-time) measurements with sensor; (2) through measurements on a video clip or digital image; (3) by filling out a table manually or via special commands; (4) by importing data from a file (with automatic conversion of units) or copying from other software; (5) through a computer program or computer model; (6) read-off in a graph; (7) by sketching a graph in a diagram window and converting the sketch into a table; and (8) via a mathematical formula. In my opinion, students must get acquainted with all of these technology-supported methods in order to get a more complete picture of variables and functions, and of their practical value in mathematics, science and technology (cf., Heck, 2001).

4.3.3 Tables and Graphs
Tables and graphs are one of the most important representation systems in scientific inquiry. I briefly discuss both forms of mathematical representations to illustrate
4.3. Aspects of Tool Design

how many design choices one has to make and how details matter when one develops a versatile, tool-based computer environment for inquiry-oriented mathematics and science education. I give examples of some choices made in the development of the Coach hardware and software environment.

Tables

A table is used as a convenient and clear arrangement of data for representing measurement (with respect to some unit when it concerns a physical quantity) and counting (dimensionless), and for exploring and analyzing relationships and patterns within the data. The first purpose stresses the use of a table as a display notation and the second purpose is about the ease of user-directed actions on data for better understanding or modeling of real data.

In the Coach software, since its first release, the table window has been built with a column-oriented structure that makes actions on individual or linked columns of data as a whole, instead of actions at cell level or on whole rows, the elementary unit of action for a user. Derived quantities can be created by menu commands or by formulas in terms of already defined columns and/or connected quantities. For each physical quantity one can specify the unit used and the arithmetic precision, that is, the number of decimal places used to express a value. The physical set-up of an experiment, where sensors have been connected to specific channels of a measurement interface panel, determines strongly the mathematical representation, processing, and analysis of data.

Although Coach is column-oriented with respect to variables in tables, the system only allows the storage of one datum at a time in a measurement, a computer model, a computer program, or in an animation. The entry of a formula or a manipulation command leads to an action on entire columns. This is one of the differences of the table structure of Coach with a standard spreadsheet. Two other differences are worth mentioning. Firstly, in a standard spreadsheet, cells play the role of variable: The cell itself, being a spatial address on a sheet, corresponds to the concept of variable, whereas the content of the cell corresponds to the current value of the variable. Secondly, in a standard spreadsheet, a user has the choice to refer to a cell by value or by reference, and this determines whether cells are automatically updated or not when values are changed. Since version 5 of Coach, when values in a particular column of a table are changed, the corresponding values in all linked columns are automatically updated, no matter whether it concerns the same table window or a separate one. In early stages of development, in IP-COACH 4, semi-links were used and the student or teacher had to select the ‘Recompute command’ for this purpose. Studies of classroom use, and comments from teachers and developers of instructional materials led to the change in the design choice to automatic linking of columns. With this choice, the computer learning environment has a more dynamic, interactive, user-directed agent nature (in the sense of the Kaput-Goldin framework) than with semi-linking of columns. What this particular example of continuous development of the computer learning environment illustrates is that experiences gained from practical use of the environment may lead to design and implementation changes.

The column-orientation and automatic linking of columns of the table window in Coach are expected to help students overcome their seemingly natural tendency of
concentrating on individual measurements and considering a measured quantity as
a property of the object of study (cf., Hancock, Kaput, & Goldsmith, 1992), and
to let them divide their attention between local features of data (a specific value)
and global features (patterns or trends in lists of values). Local and global features
come together in one of the most central points in calculus with lists of function
values, namely, the rate of change of a quantity. Below, I briefly discuss the topic
of computing numerical derivatives because it gives a good impression of the many
choices designers of computer environments for learning and doing mathematics and
science in an inquiry approach have to make for letting the hardware and software
environment be most suitable for the envisioned users, who have various backgrounds
and wishes.

The envisioned computer learning environment must provide advanced methods for
calculating numerical derivatives, but it also must allow an approximation of a rate of
change based on rather simple principles like divided difference formulas. The essential
point is that more than one approach to numerical differentiation seems appropriate
in school practice. For an accessible road to basic understanding of numerical methods
applied to lists of values, a small set of elementary list operations such as a difference
operator and an accumulation operator suffice (cf., Spunde & Neidinger, 1999; Confrey
& Maloney, 2008). In case of doing practical investigations or research projects, when
the numerical differentiation method itself is not object of study anymore, a student
simply wants to use the most suitable method available, for example to minimize the
effect of noise in the measured signal. The table in Figure 4.8 illustrates the different
results obtained by different methods in COACH 6.3, applied to the function $y = x^4$
on the interval (1,3), which has been specified as a short list of values.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>dx</th>
<th>dy</th>
<th>dy/dx</th>
<th>y'</th>
<th>y'_sm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>1.00</td>
<td>-</td>
<td>-</td>
<td>3.63</td>
<td>3.99</td>
</tr>
<tr>
<td>2</td>
<td>2.00</td>
<td>2.07</td>
<td>0.20</td>
<td>1.07</td>
<td>5.37</td>
<td>7.10</td>
</tr>
<tr>
<td>3</td>
<td>3.00</td>
<td>3.94</td>
<td>0.20</td>
<td>1.77</td>
<td>8.44</td>
<td>11.20</td>
</tr>
<tr>
<td>4</td>
<td>4.00</td>
<td>6.55</td>
<td>0.20</td>
<td>2.71</td>
<td>13.56</td>
<td>16.64</td>
</tr>
<tr>
<td>5</td>
<td>5.00</td>
<td>10.30</td>
<td>0.20</td>
<td>3.94</td>
<td>19.72</td>
<td>23.62</td>
</tr>
<tr>
<td>6</td>
<td>6.00</td>
<td>16.00</td>
<td>0.20</td>
<td>5.50</td>
<td>27.51</td>
<td>32.12</td>
</tr>
<tr>
<td>7</td>
<td>7.00</td>
<td>23.43</td>
<td>0.20</td>
<td>7.43</td>
<td>37.13</td>
<td>42.94</td>
</tr>
<tr>
<td>8</td>
<td>8.00</td>
<td>33.18</td>
<td>0.20</td>
<td>9.75</td>
<td>48.36</td>
<td>56.68</td>
</tr>
<tr>
<td>9</td>
<td>9.00</td>
<td>45.70</td>
<td>0.20</td>
<td>12.52</td>
<td>62.60</td>
<td>70.72</td>
</tr>
<tr>
<td>10</td>
<td>10.00</td>
<td>61.47</td>
<td>0.20</td>
<td>15.77</td>
<td>78.84</td>
<td>88.26</td>
</tr>
<tr>
<td>11</td>
<td>11.00</td>
<td>81.00</td>
<td>0.20</td>
<td>19.53</td>
<td>97.67</td>
<td>107.68</td>
</tr>
</tbody>
</table>

Figure 4.8: Numerical derivatives of $y = x^4$ in the table window of COACH.

The very rough sampling of the function $y = x^4$ on the domain (1,3) illustrates the
differences in power of the three methods that have been applied in this example of
numerical differentiation. First the interval has been partitioned into 10 subintervals
of width 0.2 by filling 11 equidistant values within the specified range and then we
have evaluated the function at these sample points. Next, the backward differences
between consecutive points in a sample have been computed with the Delta function,
the name of which reflects the connection with Leibniz’s notation for infinitesimals
and derivatives: Delta($y_i$) = $y_i - y_{i-1}$. The backward differences of the $x$ and $y$
sample have been used to compute the list $dy/dx$ of difference quotients of the form
$(dy/dx)_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}$. In other words, the backward divided difference approximation
of the derivative of the function has been computed by elementary methods. It is well
known and one can read about it in every textbook on numerical analysis that methods
with higher-order accuracy, reduced endpoint error (i.e., less erratic behavior at the beginning and end of the computed first and second derivative data), and better noise-reduction exist. I derived three-point difference formulas from standard numerical analysis techniques (Fornberg, 1988, 1998) and implemented them in Coach 6 for the first and second derivatives of a set of non-equidistant, distinct data points. I decided not to use five-point difference formulas for the following two reasons: (1) It does not make much sense when only a few data points have been collected (e.g., in manual data collection or limited sampling of data with sensors, video clips, or digital images); (2) It enlarges the chance of rounding errors in the numerical computations. The second reason and practical experience also made me decide to apply the three-point formulas with alternating skipping and usage of data points in the computation of derivatives by divided differences for large data sets, when consequently the sample interval is very small and would cause noisy derivatives.

The result of applying a three-point difference formula for the first derivative is visible in the column labeled \( y' \) in Figure 4.8. As shown, these column values are much closer to the exact result of differentiation \( (y' = 4x^3) \) than the values obtained from backward divided differences \( (dy/dx) \). However, these finite difference formulas are sensitive to noise in the data. In many applications of mathematics and science, say for example in motion analysis, calculating kinematics data (i.e., calculating velocity and acceleration) from experimental displacement data is problematic because of noise on signals. Finite difference formulas for derivatives do not give nice results in many applications and this holds especially for second or higher order derivatives. More advance methods are needed and one approach is to first smooth the data and then compute numerical derivatives of the smoothed data instead of using the raw data for this purpose. In the case study of modeling sprinting (Section 3.5.5, pp. 179–181) I briefly explained a data smoothing technique that is also used for computing smooth numerical derivatives, namely, the generalized cross-validatory penalized quintic spline smoothing technique. This algorithm has been implemented in Coach 6 and the last column in table in Figure 4.8 shows the result of computing the derivative by this algorithm: the calculated values do not differ much from the values that one would obtain by means of the exact formula for the derivative.

The above example of numerical differentiation also illustrates the White Box/Gray Box/Black Box metaphor of interaction of users with technological tools in an educational context, put forward by Buchberger (1990) and elaborated by others (See, for example in the context of computer algebra in mathematics education, Macintyre & Forbes, 2002; Cedillo & Kieran, 2003). When elementary arithmetics is used to learn about numerical differentiation and to carry out simply computations of numerical derivatives, one speaks of white box interaction with ICT. The learner’s awareness of all mathematical details is then highly valued. The other extreme use of ICT is characterized as black box interaction: technology is just considered a tool for carrying out some task and its user hardly cares about the underlying mathematical algorithms or hardly feels the need to know about the internals of the tool. Somewhere in between is the gray box approach: Normally one speaks of this approach when a learner uses ICT tools in the black box sense during steps in the problem-solving process, but keeps this process under control by deciding what steps to make, reflecting on intermediate results, and by making strategic choices. I consider the use of more advanced finite difference methods for numerical differentiation as a form of gray box interaction when
the user knows about the basic underlying principles and the strengths and weaknesses of these methods.

I am of opinion that a versatile tool-based computer learning environment for mathematics and science education should support all three forms of interaction of its users with the provided ICT tools. I also recommend designers and developers of such learning environments to anticipate these different forms of interaction with ICT in education, for example by thinking about and providing facilities to set the mode of the students' use in accordance with their competencies. This has been done in Coach via the so-called Junior mode, in which the openness of the student work and the range of available tools is controlled by the author of the learning activities. This appreciation of three modes of interaction with ICT in education is also one of the reasons of paying in many case studies attention to the least squares method of peeling-off functions (Foss, 1969), which can be considered as a gray box approach, before using an advanced tool for regression analysis as a black box (e.g., in case of sinusoidal and exponential curve fitting).

Graphs

Visual representations play a key role in the use of a versatile, multimedia, computer learning and authoring environment for mathematics, science, and technology education. A picture of an experimental set-up often makes things much clearer for students than a long verbal or textual explanation. A movie or animation can illustrate a phenomenon or the working of a device much better than a long textual explanation. A data graph may reveal a trend in a data set better than a table of recorded quantities (cf., Zacks & Tversky, 1999; Shah & Freedman, 2010). These are just a few of the examples that support the view that one can hardly do without visualization in mathematics and science education. I discuss the work of Kosslyn (1989) on the classification of visual displays and the structure of a graph for the purpose of describing some aspects of the design of the graph window in Coach 6 environment.

Kosslyn's Classification of Visual Displays

There exists no universal classification scheme of visual displays, but Kosslyn (1989) distinguished four common types of visual displays that use symbols, which are marks that are interpreted in accordance with convention: graphs, charts, diagrams, and maps. These four types differ in terms of what information is communicated and what level of analogy is used for this purpose.

Graphs are the most constrained form, with at least two scales always being required and values being associated via a ‘paired with’ relation that is always symmetrical. They are also referred to as logical pictures (Schnotz, 2001; Tversky, 2001) because they have lost their pictorial characteristics, have no similarity with the designated object, resemble their referents in terms of the relationships between elements, and frequently represent a subject matter which cannot be perceived at all. Graphs use spatial characteristics to represent quantity: They represent greater quantities of the measured entity and greater numbers by higher position, longer lines, greater areas, or more of some other visual dimension on a two-dimensional surface. Thus a graph uses space to convey information and this distinguishes it from a table, which uses specific iconic representation.
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Charts specify discrete relations among discrete entities through lines, arrows, or relative positions that serve as links. Like graphs, charts are entirely conventional. Examples are tree charts, organizational charts, flow charts, a network diagram, and so on. The specified relations are not necessarily symmetric, but may be directed like in flow charts.

A diagram is a schematic picture of objects or entities and their relationship, but at a higher level of analogy than a chart. It consists of analogous elements, such as parts of an machine, as well as abstract analogous symbols, such as force vectors. Examples are how-to illustrations, figures in a manual, force diagrams, and so on. Unlike graphs and charts, parts of a diagram correspond to parts of some actual object or entity.

Unlike graphs and charts, maps are not entirely conventional: a part of a map corresponds non-arbitrarily to a part of the pictured territory. A map differs from a diagram in the sense that it is a nearly photo-realistic or more schematic picture of a physical, spatial environment, which shows the location of objects and the spatial relationship between them. Examples are physical maps, topographic maps, maps of census data, and so on. Diagrams and maps are more concrete in terms of their meaning than graphs and charts, and therefore often make more use of representational analogies.

In a Coach activity, all four aforementioned types of visual displays may be present; this is illustrated in the screen shot of a modeling activity in Figure 4.9.

![Figure 4.9: Tool windows illustrating the basic-level constituents of graphs.](image)

Cleveland (1984) noted that the majority of scientific visualizations consists of graphs, in particular two-variable graphs in which one variable is plotted against another, such as scatter plots, time series plots, and function plots. Later studies (e.g., Arsenault, Smith, & Beauchamp, 2006; Smith, Best, Stubbs, et al., 2000; Zacks, Levi, Tversky, & Schiano, 2002) showed that this has hardly changed. This explains the special attention given to data graphs in my research and development work. Henceforth I focus on graphs.
Structure of a Graph According to Kosslyn and the Implementation in Coach

Kosslyn (1989) distinguished four structural components of graphs, namely, background, framework, specifier, and labels. Annotations in the lower right diagram window of the screen shot in Figure 4.9 point at these basic-level constituents of a graph. Kosslyn used his structural scheme to analyze the information in charts and graphs and to diagnose syntactic, semantic and pragmatic problems with visual displays. I use it for discussing the design and implementation of the diagram window in the Coach working environment.

The background of a graph includes any coloring, shading, or picture over which the graph is superimposed. In a Coach activity, graphs may be present in the background and have appeared in one of the following ways:

- Graphs of previous measurements and of previous runs of a model or computer program stay visible (by default in a gray color) on the screen (See, for example, Figure 3.24, p. 128).

- A background graph of data of another Coach activity can be imported (in various display styles; see, for example, Figure 3.49, p. 159).

- A sketch of a function graph can be created with the mouse used as digital pencil. Contrary to graphs that have a connection with input (manual or from a sensor), a quantity (from a video, model, or computer program) or a formula, the data of previous measurements or computations and the data of an imported background graph cannot be directly manipulated. Their main purpose is for visual comparison of the current data in an activity with the ones created under previous circumstances. For example, the effect of a change of the value of a parameter in a computer model can be investigated by looking at plots of previous runs or at a data plot restored from another model. Some user support for importing background graphs is convenient: Because data restored from a file might have been recorded with different units or ranges of the independent variable, automatic conversion of standard units and a feature to translate the background graph horizontally make the comparison of data easier.

The framework of a graph (axes, scales, grids, . . .) represents the kinds of entities being related, but does not specify the particular information about them conveyed by the display. For example, the framework of a data graph provides information about what kinds of measurements are being used and what things are being measured, but the measured data are not part of the framework. The simplest framework of a data graph has an L-shape with one leg (the horizontal axis) standing for the things being measured and the other (vertical axis) providing information about the measurements being used. A framework often consists of two parts: (1) the outer framework consisting of the axes and extending to the edges of the graphical display; and (2) the inner framework (scales, data ranges, a grid or regular pattern, cross hairs in scanning mode) that serves to map points on the outer framework to points on the specifier of the display. In the lower right diagram window in Figure 4.9, the inner framework consists of a gray grid and (because of scanning mode) dashed cross hairs, which happen to be superimposed on the grid in the screen shot. 
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Indispensable elements of the inner framework are the scalings and the data ranges that may be freely chosen for both axes. Scaling in the graphical display is semi-automatic in Coach for reasons of user-friendliness: Default data ranges are pre-determined, but if the software ‘knows’ from measurement settings or modeling settings which values some variables actually take, then the computer environment adjusts the corresponding data ranges when a quantity is displayed graphically. Rescaling in the diagram window requires zooming. For this purpose, one can (1) select a ‘zoom-to-fit’ menu item or toolbar button to automatically rescale the axes to optimally display all data; or (2) one can use the mouse to select a rectangular area of the graph for having a closer look (zooming-in).

Cleveland and coworkers found in a series of research studies that the graphical perception of a two-variable line graph or a scatter graph is affected by: (1) the choice of the shape of the display (actually, only the aspect ratio of the viewport or plotting area matters); (2) the data rectangle (i.e., the rectangle that shows the maximum and minimum values along the horizontal and vertical axes in the world coordinate system in which the graph is defined or data generated), and the data window (i.e., the rectangular region of the world coordinate system that one wants to be displayed, for example after zooming); and (3) the relative scaling of the axes in the world coordinate system. The data window is displayed by a windowing transformation in the viewport of the computer screen via a transformation that still allows operations such as any scaling to be applied to the world coordinate definition of the graph. Actually, two aspect ratios are important in the display of data: (1) the aspect ratio of the viewport; and (2) the aspect ratio of the data window. The windowing transformation from the data window to the viewport determines the viewer’s impression of the graph. For example, Cleveland, Diaconis, and McGill (1982) experimentally verified that viewers consider variables on scatter graphs more highly correlated when the point-cloud size is decreased by increasing the scales of the axes. Cleveland and McGill (1987) found that the shape of a graph affects the interpretation of time series data and slope judgments of line graphs. They advised (See also, Cleveland, 1993a,b) to choose the shape of a line graph in such way that the most relevant physical slopes along the curve are about 45° with the horizontal. This process of choosing appropriate aspect ratios to enhance the perception of the orientations of line segments in the viewport is called ‘banking’ (Cleveland, 1993a). The following example of gait analysis illustrates the fact that a graph cannot be interpreted fully without taking into account the process of construction and in particular the scaling of the graph.

Figure 4.10 is a screen shot of a video analysis activity with Coach 6 in which the time series of the knee angle during normal walk at a speed of 5 km/hr has been plotted in three ways. The upper diagram window contains a scatter graph for which the shape parameter, defined as the ratio of the viewport’s height and width, is a small number. This type of banking reveals that the forward rotation of the hip in the leg’s swing phase is faster than the backward rotation during the stance phase. In the scatter graph in a lower diagram window it is not so easy to detect a pattern in the data; it looks more as a bunch of dots. Connecting consecutive data points in the time series by straight line segments (i.e., changing the specifier into a line graph) helps a viewer see the motion graph, but the upward and downward slopes do not show well the differences in hip motion between swing and stance phase. This example illustrates once again the finding of Leinhardt et al. (1990) that the issue of
scaling becomes more fundamental when using graphing technologies. The purpose of re-scaling and banking is to visualize a graph as a whole in a different sense and ICT supports this exploratory graphing approach. Practical examples like the given one and others made the designers of COACH drop the fixed layout of four visible tool windows in an activity and allow students, teachers, and authors to use as many tool windows as needed, in any wanted shape.

Figure 4.10: Effect of the shape of a two-variable on graph interpretation.

Dual-scaled axes in graphs enable the presentation of two related graphs in a single diagram with each graph having its own, most suitable scaling of the vertical axis used for the plotting. Another application of a second vertical axis in the outer framework of a graph is the following: The first vertical axis is replicated at the right side in order to help convey an accurate visual impression of the relation of two line plots at both sides of the horizontal range. Thus, the underpinning of this second application is rooted in knowledge about human perception.

Figure 4.11: Examples of dual-scaled axes in a graph.

Figure 4.11 illustrates both uses of a second vertical axis in the outer framework; both diagrams come from instructional materials concerning a mathematical analysis of human growth (Heck & Holleman, 2002a, 2004a). In the picture to the left, two related but different kinds of quantities, namely the average height of a boy and the growth speed, are shown together. One needs in this case a second vertical axes because a single vertical scaling would not work well for both curves together: the growth speed curve would then be too flat. In the diagram to the right, the inner framework and the two vertical axes support the reading and comparison of the human growth data displayed in the curves. It is easier to compare the height of the girls at early age and at adult age in a graph with vertical axes on both sides than in a graph
4.3. Aspects of Tool Design

with one vertical axis alone on one side. The results of the classroom study about human growth (Section 2.2) did not indicate the upper-level pre-university students had extra difficulty with understanding dual-scaled axes in a graph, compared with graph comprehension in case of a graph with one vertical axis.

The *specifier* conveys the particular information about the entities represented by the framework (often data values). Specifiers are the lines, bars, point symbols, or other marks that specify particular relations among the things represented within the framework. In Figure 4.9, four different specifiers are shown, namely, a vertical bar (in the meter window next to the animation window), a case value bar graph (height-time graph in which the data points are linked by vertical bars with the time axis), a line graph (height-time graph), and a scatter plot (velocity-time graph with open dots as markers). As a matter of fact, the meter window in COACH is not a graph because it only displays one value at a time. But its display looks like a graph according to Kosslyn’s terminology: A value can be represented by a horizontal or vertical value bar in a rectangular framework, or by pie in a quarter, half or full circular framework. In the COACH activity shown in Figure 4.9, the animation window (with an arrow indicating the distance of the ball to the ground), the meter window, and the case value bar graph have been added to support students’ understanding of the height-time line graph in the upper-right window. Research results of Ploetzner et al. (2008, 2009) indicated that supportive dynamic visualizations on the comprehension of position, velocity and acceleration graphs have a positive effect (especially on the learning of weaker students), provided that the students are given enough time and guidance to understand the dynamic visualizations, as well as how they are related to line graphs. These side conditions on learning circumstances and teaching strategies are in accordance with the principles of segmenting, guidance, and pre-training from models of multimedia learning.

In Figure 4.9, different specifiers where used in separate diagrams to support the learning of the line graph concept of a time series. However, quite often more than one data set is displayed in a single graph and then differences in specifiers are necessary to distinguish the data sets. Figure 4.12, which comes from a gait analysis study by students (See Section 2.4), shows a time series graph where three kinds of specifiers are combined. There is a point plot of measured knee angles during normal walk on a treadmill, a line plot of the best sinusoidal function fit of the data, and a case value bar graph showing the residue, that is, the differences between the measured data and the corresponding values of the function fit. The specifiers are deliberately different.

Figure 4.12: Sinusoidal approximation of the knee angle during normal walk.
Let me first discuss the plot of the data points connected with line segments. The discreteness of the measured data is expressed by plotting points. But, if the data points were drawn alone, it would be in this case less easy for the viewer to see to them as a whole or to notice a pattern. Two of the Gestalt laws of perceptual organization discovered by Gestalt psychologists (see, for example, Kaufman, 1974) play a role here, namely good continuity and proximity: Good continuity refers to the principle that marks which suggest a continuous line will tend to be grouped together; proximity means that marks near each other will tend to be grouped together. In this case, the marks are at certain time intervals a little bit too far away from each other to ensure that they are perceptually grouped together as an entity. By connecting the data points by straight line segments, the viewer is supported to perceive them as grouped together and (s)he is reminded to consider the knee angle as a continuous quantity, which happens to be measured at discrete times.

The line graph in Figure 4.12 is a regression curve that one can easily compare with the data graph. In the diagram the choice has been made to display the differences between data points and values of the regression function at corresponding times by vertical bars to emphasize that the conventional least squares function fit has been applied. The reader is reminded that the regression curve has been determined by minimizing the sum of squared vertical distances between the curves, not by minimizing the sum of squared shortest distances of the data points to the regression curve. What has been done here is nothing else than applying the structure mapping effect, which was distinguished in the Integrated Model of Text and Picture Comprehension proposed by Schnitz (2001, 2002, 2005). It recommends the following: if a subject matter can be visualized by different pictures in different ways that are informationally equivalent, use a picture with the form of visualization that is most appropriate for solving future tasks. Thus, a well-thought choice of specifiers lies behind the visual display shown in Figure 4.12. One is recommended to pay ample attention to such issues in teaching and learning the use of visual displays.

Labels of displayed variables are placed near the axes of a graph. Other labels may occur in a graph as further explanation or for attracting someone's attention to features of the graph. The title of the graph is itself a kind of label.

Taking the rather complicated structure of a graph and the many aspects that play a role in reading, interpretation, and construction of graphs into account, it comes as no surprise that graph comprehension is an ability that takes a long learning curve and requires exposure to a variety of contexts in which graphical representations play an important role. In computer graphing, details of the design of the graphing tool matter and affect the instrumental genesis.

4.3.4 Data Manipulation, Processing, and Analysis

Once raw data have been collected, they must be often manipulated and processed in inquiry activities before they can be analyzed. I briefly discuss these three types of user actions on data in the context of the design and implementation of a versatile, tool-based computer environment for inquiry-oriented mathematics and science education.
4.3. Aspects of Tool Design

Data Manipulation

One may expect in a tool-based computer learning environment like COACH and Vernier's LOGGERPro basic facilities for adding, copying, inserting, deleting, importing, exporting, and sorting data in a table. The transition between a tabular and a symbolic representation is manifest when a column in a table is filled by a mathematical formula. Any change in the formula has an immediate effect on the corresponding tabular values. Tabulated quantities can be connected to communication channels of data-gathering tool windows such as a measurement panel or a video analysis window. Ideally, the linking of table entries with other tool windows is quite direct and dynamic, in accordance with the signalling principle of multimedia learning: For example, highlighting a row entry in a table of measured positions in a digital image highlights in COACH the point in the data image window. The linking goes beyond the representation of the raw data: When a numerical derivative has been computed and tabulated for a particular quantity, a change of the values of this quantity is immediately reflected in a change of the values of the numerical derivative. Another example of such dynamic linking of descriptions is the direct connection between raw data and smoothed data: Every change in a value of raw data leads to an immediate change of the smoothed data. In case of penalized spline smoothing, which is a method that uses all data, this can be time-consuming in cases of large data sets. Thus, the resources needed for continuous updating of representations in a computer learning and working environment must not be underestimated and the availability of appropriate mathematical methods that operate locally instead of globally on given data is desirable.

As Kaput (1992) pointed out, the built-in constraint-and-support structures have a great influence on the nature of the interactivity of a computer environment. An example in which the user's support of the interactive system is very large comes in COACH from the linking of the table window with other tool windows. The table window and the diagram window are actually tightened to each other in COACH: every table has a graphical representation and vice versa, to every Cartesian graph belongs a table structure. The table window and the diagram window are synchronized, share a number of settings (e.g., their name, visibility of quantities, and numerical precision), and have many actions on data in common (e.g., the creation and editing of the underlying data, and the processing and analysis tools). A change in one of the representations has an immediate effect on the other representation. This forms the basis of the educational use of multiple representations. This parallel use of representations on a computer (or graphing calculator) allows switching between representations by minimal activation. The learning and working process consists of shuttling between several representations and investigating the influence of changes of one representation in the others. In the context of computer algebra, this is referred to as the window shuttle strategy (Heugl, Klinger, & Lechner, 1996). It also means that one selects for any data handling procedure a specific representation that makes the action most easily accomplished: Sometimes visual data selection or removal is convenient (See, Heck & Dorenbos, 2007, for an example in the context of non-invasive measurement of the peripheral arterial blood pressure in human with a cuff pressure sensor though the so-called oscillometric method); in other cases, it is easier to apply a selection procedure to a list of values.
Data Processing

The case studies in Chapter 2 and 3 confirm that the following data processing tools are indispensable in a computer environment for inquiry-oriented mathematics and science education: tools for data smoothing, numerical differentiation, and numerical integration. The last data processing tool (and a related tool for determining the area under a curve) may seem a bit unnecessary at first sight, but the case study about standing vertical jumps (Section 3.4.2) illustrated the relevance of the availability of this tool for solving kinematics problems in which the impulse-momentum relationship \[ p_{\text{final}} - p_{\text{initial}} = \int F \, dt \] (with force \( F \) and linear impulse \( p \)) provides a good solution strategy. Another example of using the numerical integration facility in a physics context is the following treatment of a magnet falling through a solenoid (See also, Amrani & Paradis, 2005). The Faraday-Henry law of electromagnetic induction states that the induced electromotive force (EMF) is proportional to the rate of change of the magnetic flux (\( \Phi \)) through the solenoid: \( \text{EMF}(t) = -\frac{d\Phi(t)}{dt} \). The extremal value of the magnetic flux depends on the velocity of the magnet. Thus, the recorded EMF must be integrated as shown in Figure 4.13.

![Figure 4.13: The graph of the recorded EMF and of the area under the curve.](image)

In my research and development work I have also paid much attention to data smoothing and numerical differentiation. The case study about weather data (Section 2.6.2) illustrated that vwo-3 students at gymnasium level were able to understand the mathematics of moving average in the context of smoothing sharp fluctuations in weather observations so that a trend becomes more visible. Spline smoothing was applied in many case studies where numerical derivatives had to be computed (e.g., in the motion analysis of a falling badminton shuttlecock and the mathematical modeling of sprinting in Section 3.5.2 and 3.5.5, respectively). I have argued that a generalized cross-validatory penalized quintic spline-smoothing based algorithm (Ramsey & Silverman, 2005) for computing numerical derivatives is appropriate; for details, the reader is referred to Section 3.5.5 (pp. 179–181) and to references (Green & Silverman, 1994; Heckman & Ramsay, 2000; Ramsay & Silverman, 2005).

Regression and Signal Analysis

At the level of reading between and beyond the data, one of the most important underpinning statistical principles is covariation, that is, the principle that two variables may vary in a predictable way. The correspondence of variation of two variables
depends on the data types of the variable: Moritz (2004) distinguished logical, numerical, and statistical covariation. Logical covariation is expressed by a logical statement such as, for example, “if an object is in harmonic oscillation, then its position-time curve can be described by a sinusoidal function,” or a logical expression such as “at \( x = x_0 \), the function \( f(x) \) increases = NOT\( (f'(x_0) \leq 0) \).” A mathematical equation such as \( y = x^2 \) expresses a numerical covariation between real-number variables \( x \) and \( y \). Numerical covariance may be represented as a function graph. Statistical covariation refers to the correspondence of variation of two statistical variables that vary along numerical scales, such as the correspondence between human height and weight. Bivariate statistical covariation may be represented in a scatter graph using a Cartesian coordinate system that shows the correspondence of the ordination of each variable. The correspondence is sometimes made more explicit by a regression curve, but this supposes a dependence of one variable on the other. The more general term statistical association is also used in statistics literature to include associations between categorical variables (commonly represented in contingency tables), but I focus on numerical and statistical covariation, which only involves numerical data and assumes a potential functional relationship. These types of covariation not only allow a mathematical description of data, but also a prediction of values of one variable using values of the other variable.

The majority of data obtained via data logging typically show the time dependence of one or more variables. So, a description of a trend or pattern in the behavior of such a variable relates it, implicitly or explicitly, to the time variable. The description of data is frequently connected with changes, rates of change, growth, and decay. An implicit relation normally takes the form of (1) a logical statement such as “Temperature goes up when the heating system is turned on” and “A major growth spurt normally occurs at the time of puberty;” (2) a judgment such as “When the ball bounces it rapidly changes direction of motion;” or (3) a qualitative observation such as “It is getting hotter” and “position is alternatingly left and right.” An explicit relation usually takes the form of a functional relationship such as “The orbit of the moving object is a parabola” (qualitative) and “the distance traveled in a free fall equals \( \frac{1}{2}gt^2 \), where \( g \) is the constant of acceleration and \( t \) is the elapsed time after release” (algebraic). The shape of the data graph gives information about the relationship between the variables, at least when convenient data ranges and scaling have been used.

The type of association can be justified by using increasing, decreasing, or constant trend of points in a scatter plot. For example, a linear relationship can be recognized in various ways:

- Data points seem to lie on a straight line.
- The gradient is the same at all places on the graph.
- Changes in the variables occur at a constant rate.
- For a given increment in one variable, the other variable always increases or decreases by the same amount, independent of the magnitude of either variable.

A nonlinear relationship can be recognized as:

- Data points seem to lie on a non-straight curve.
Chapter 4. Findings and Conclusions

- The gradient varies across different parts of the graph.
- One variable changes more rapidly than the other.

Each of the above descriptions of linear and nonlinear relationships is testable using software tools: In case of a linearity test, for example, a graphical slope tool may convince a student that the gradient at any data point is more or less the same and that the data points are always close to a particular straight line. Rates of change in variables can be numerically computed, tabulated, or graphed. Incremental changes of variables can be tabulated and compared. I consider the exploration of a relationship between variables as an important element of inquiry-oriented mathematics and science education. Therefore, I am of opinion that secondary school students should learn in mathematics lessons to reason about covariation, acquire in this context a good level of graph and data sense, and practice data analysis skills in practical investigations.

Henceforth I focus on regression modeling, also known as curve fitting, its role in data analysis and mathematical modeling, and what kind of functionality a user of a versatile computer environment for inquiry-oriented mathematics and science education may expect.

Regression as Part of Data Analysis and Mathematical Modeling

As Moritz (2004) pointed out, mathematical modeling to fit data to a specific functional equation is one of the processes in statistical reasoning about covariation. Other processes are the translations and transformations among raw numerical data, graphical representations, and verbal statements about covariation and causal association. They are all elements of the Rule of Five framework discussed before (pp. 261–264) and the role of technology in the translation processes was illustrated in Figure 4.7. In this figure, curve fitting was the label used to get from data presented in a table or graph to a symbolic representation.

Regression modeling served the following two purposes in the quantitative mathematical modeling activities presented in previous chapters: (1) modeling functional dependencies in experimental data and in stochastic situations; and (2) finding initial estimates for parameters in a mathematical model. The application of regression modeling to determine good initial estimates of parameters for use in computer models was illustrated in the case studies about modeling human running via a kinematic approach (Section 3.5.5, pp. 181–182) and about modeling upward hopping (Section 3.5.6, p. 190). The first purpose, fitting experimental data, was illustrated by many examples of practical work.

Supported by ICT tools, researchers use curve fitting as the main method of experimental modeling. In this type of quantitative mathematical modeling, a simple mathematical formula is sought that describes the data reasonably well, but it does not provide any explanation or underpinning of the model. It is in my opinion important that students understand this intrinsic weakness of experimental modeling. In a human gait analysis study (Section 2.4) it may be very impressive to mathematically describe motion data by models of sums of two sine functions, but it becomes more interesting when a possible link is made with a double pendulum model of the human leg. Regression analysis of the motion of human kangaroo jumping (Section 3.5.6) may provide a good component-wise description of motion data, but the planar inverted spring-mass model of the leg gives probably more insight in the motion during stance...
phase. One may fit data of various hanging chains and main spans of bridges (Section 2.3), and discover that they are often well described by a catenary and parabolic function, respectively. However, one has to check this over and over again for any shape that one encounters. By use of fundamental principles of physics, one can reason about the mathematical shape of an object once, understand that one actually investigates a prototypical example, and then apply one’s findings to every other copy of the prototypical example.

It is important that students have a good understanding of the distinction between fitting to a linear model versus a nonlinear model. Calling the dependent variable $y$ and the independent one $x$, a general representation of a linear model $\mu(x)$ can be given: 

$$y = \mu(x; \gamma_0, \gamma_1, \ldots, \gamma_n) = \sum_{k=0}^{n} \gamma_k \phi_k(x),$$

where $\gamma_k$ are the parameters to be fit and $\phi_k$ are the basic (possibly nonlinear) functions. By far the most common choice of basic functions are monomials, leading to a straight-line fit ($y = \gamma_0 + \gamma_1 x$), a quadratic fit ($y = \gamma_0 + \gamma_1 x + \gamma_2 x^2$), and more generally, an $n$th order polynomial fit ($y = \gamma_0 + \gamma_1 x + \cdots + \gamma_n x^n$). The least squares method of minimizing the sum of squares of residuals (i.e., minimizing the sum of squares of differences between experimental and modeled data) is the most popular linear regression method because it can be solved analytically. The larger the sum of squares of residuals, the less well the model fits the experimental data. This works under the assumption that there is no error in the independent variable (often time) and that each data point carries equal weight, that is, each data point has exactly the same actual (not relative) error associated with it. Otherwise, it makes sense in curve fitting to give the least amount of weight to data points that are the least reliable or to extend the method of least squares to the general case of both variables subject to experimental or measurement error (See, for example, Glaister, 2001). If one designs a computer learning environment for secondary education and assumes that data logging is done with reliable sensors, a least squares linear regression method without weights seems satisfactory.

More important for students than fitting to data with experimental errors is to get some basic understanding of nonlinear curve fitting, that is, about models $\mu(x)$ that cannot be expressed as a linear combination of the parameters. Students must learn that in nonlinear curve fitting no analytic solution is possible, unless the associated nonlinear equations can be solved analytically or when there exist a transformation that turns the nonlinear model into a linear model. For example, the nonlinear exponential fit $y = ae^{bx}$ can be transformed into a linear fit by applying the logarithm on both side: $\ln y = \ln(a) + bx$. It is noted that I ignore here that random errors in $y$ that were of equal size, become unequal by this transformation (cf., Glaister, 2007). The bi-exponential fit $y = ae^{bx} + ce^{dx}$ is a nonlinear fit that cannot be linearized.

When no closed-form expression for the best-fitting parameters is available, a numerical optimization algorithm is applied to find best-fitting parameters by iterative improvement of values of parameters. A popular numerical optimization algorithm, implemented in many ICT tools including COACH, is the Levenberg-Marquardt method (Marquardt, 1963). Good initial values of the parameters must be supplied to this numerical algorithm by any of the following three methods: (1) trial-and-improvement; (2) educated guessing; or (3) a parameter estimation algorithm.

In case of a bi-exponential fit, the graphical method of peeling-off functions (Foss, 1969) sometimes provides good initial values. More advanced methods for fitting linear combinations of exponents with the purpose of finding good initial estimates for a
nonlinear regression method exist (See, for example, Holmström & Petterson, 2002): There are methods based on partial geometric sums, linear difference equations (Prony algorithms), and differential equations (assuming that numerical derivatives of data are available or can be easily computed). For other families of nonlinear regression models, the graphical method of peeling-off functions and specialized parameter estimation techniques apply as well. In my development work, I paid special attention to sinusoidal regression. I presented examples of the method of peeling-off functions in the case studies about human gait analysis (Section 2.4) and tidal movement (Section 3.3.3). In the presentation of the latter case study (pp. 147–149) I also outlined the design choices that I made for tools to carry out harmonic analysis of signals.

Functionality of a Regression Tool

Any regression tool should give its user access to a large set of commonly used regression models and offer the opportunity to (1) view the graph of the model function and the scatter plot of the data in one diagram, and (2) to compute and graph the residual data. In this way, one can visually inspect the quality of a regression model, look for patterns in the residuals or residual bias. The graphical method of peeling-off functions (Foss, 1969) relies on the use of visualization and curve fitting of residual data. I refer to the examples given in the case studies about human locomotion (Section 2.4, p. 47), video analysis of spring board diving (104), tidal analysis (Section 3.3.3, p. 145), and motion of a yoyo (Section 3.5.1, p. 163).

Ideal in a regression tool would be the affordance of dynamic change of a parameter of the regression model so that one could immediately see its effect on the graph of the model and on the graph of the residuals. With this in mind, a manual mode of curve fitting, in which a user translates and re-scales the graph of the model function by mouse actions, has been implemented since version 5 of COACH; see Figure 2.2 (p. 33). The interaction model is based on the idea that many popular regression models can be written as $\mu(x) = af(bx + c) + d$, where $a$, $b$, $c$, and $d$ are constants, $a \neq 0$, $b \neq 0$, and $f$ is a simple mathematical function (e.g., $f(x) = x^2$, $e^x$, $1/x$, $\sin x$, etc.). Translation of the graph of the function changes $c$ and $d$; if one pins the graph down in a point, that is, if one fixes the values of $c$ and $d$, then one is still left with two degrees of freedom, namely, in the parameters $a$ and $b$, which relate to vertical and horizontal scaling of the graph, respectively. While keeping an eye on the data points in the background, one can determine by these two types of interaction with the model graph a nice curve fit with respect to data shown in the background. The parameter values change in accordance with the interactions. Finally one can use the current parameter values as initial values for a numerical optimization routine such as the Levenberg-Marquardt method. This interaction model was first field-tested in the classroom case study about human growth (Section 2.2) and the conclusion was that students had no difficulty with it. The students also found it natural that one could zoom in on a selection of data points and apply regression to this subset of data only.

The experiences with the function fit window in the classroom case studies, in particular in the studies where sinusoidal and exponential regression analysis was needed, caused two more changes in the regression tool of COACH:

1. In addition to manual estimation of initial parameter values for the Levenberg-Marquardt method, one can also ask the software to use a built-in algorithm for this purpose.
2. One can always decide to keep some parameters unchanged while applying the Levenberg-Marquardt method.

The first option was to improve the ease and quality of curve fitting: For example, when parameters of nonlinear models have been obtained in the background by transformation to a linear model, then one can still apply the Levenberg-Marquardt method and try to improve the intermediate result. The second option was introduced because quite often a regression model with specific predetermined parameter values is needed. One reason for doing this could be that one wants to verify that a symmetry can be preserved in curve fitting. In the screen shot shown in Figure 4.14, for example, the coordinate system has been positioned in the image window such that in theory the measured suspension points of the hanging weights should be on a parabola of type $y = ax^2$. The function fit window reveals that when certain parameters are kept zero, invoking the Levenberg-Marquardt method by pressing the Refine button leads to a nice parabolic fit. By the way, the Estimate button is used for asking the software to generate good initial values.

![Figure 4.14: Curve fitting with constraints on certain parameters.](image)

### 4.3.5 Video Analysis

The application of video analysis in education is to a large extent technology-pushed. In physics education in the Netherlands, it already started in 1992 with the development of interactive video in IP-Coach 3 under DOS and with laser disc technology that was marketed in that period. Despite high expectations (see, for example, Glover, Graham, & Macdonald, 1989; Ross, 1991; Zollman & Fuller, 1994) and positive experiences in educational research (see, for example, Ellermeijer, Landheer, & Molenaar, 1996; Escalada & Zollman, 1997; Laws & Pfister, 1998; Zollman & Brungardt, 1995) it never caught on in the Netherlands, nor in other countries. The main reasons were that (1) the laser disc technology (i.e., the apparatus and the creation of educational video discs) was too expensive for a school budget; (2) it was a replay-only video technology; and (3) the user interaction was not easy and obvious for many teachers and students.

The last two drawbacks were more or less removed at the end of the nineties, when the creation of digital video by individuals became possible by digital camcorders and by digitization of analog video recordings (e.g., from television broadcasts). This allowed video recordings of own experiments. At the same time, with the introduction of the desktop metaphor for using a personal computer and the mouse as the main interaction device, the user interface of software changed dramatically from text-based to
nonlinguistic (iconic, GUI) modes of interaction. Point-and-click and dragging methods of interaction increased flexibility in structuring interaction with the computer. This type of interaction was actually what was needed for making video analysis easier for teachers and students. Several educational video analysis softwares were developed; I mention only two popular softwares that are still available: VideoPoint (Luetzelschwab, Laws, & Gile, 2006) and Measurement in Motion (Cappo & Darling, 1996). The time was also ripe for developing a Windows version of IP-COACH that would include a video analysis tool. This development of COACH 5 started in 1998 and coincided with the early phase of my research and development work. In this section I outline my contributions to the design and implementation of the digital image and video analysis tool window of COACH. I also give a short impression of how design principles of multimedia learning and cognitive load theory were applied.

The Development of the Digital Image and Video Analysis Tool

Because of the years of experience of the multidisciplinary team of developers, researchers, and teachers with the laser disc technology and the early video tool in IP-COACH, the design process could simple start with a short description of the desired functionality and primarily concern interaction and user interface design. The wish list could be as concrete as statements listed below (Fragment taken from an early email correspondence in 1998 between members of the design team):

“In the Video Measurement Window, a movie or a sequence of images can be viewed, but it is also possible to measure on homemade videos and on sequences of images (converted to measurable frames).

1. In addition to measuring one point in one frame, it is possible to make a whole series of measurements in one frame.
2. The students have to calibrate horizontally and vertically.
3. Each frame has a reference point (origin), but it may differ on frames.
4. You can see the series of frames on a frame bar.
5. You can select frames (e.g., 10 measurements in 200 frames).
6. . . .”

An early draft version of the video measurement window is shown in Figure 4.15. This figure illustrates the idea of a video tool window that not only allows a student and teacher to replay and step through a video, but also allows its user to define a coordinate system inside the video frame and to measure coordinates of points inside the video frame. It is one of the many pictures from discussion papers of members of the multidisciplinary design team to specify the functionality of the tool. It differs from actual implementations of the video tool in COACH, even from early versions (See, for example, the screen shot in Figure 4.16).

Figure 4.15 is also a bit misleading because it does not illustrate that, in essence, one is designing a computer environment with linked multiple representations such as the motion event recorded in digital video format, a table of coordinates for measured positions of points and derived quantities like velocity and acceleration, and graphs of kinematic quantities. These representations are dynamically linked: In video measurement mode, an additional measurement in the video clip automatically leads to an
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Figure 4.15: Early draft of the design of the video measurement window for Coach 5

additional row in the associated tables and to an update of the graphs. In normal video
display mode, when one changes the position of a measured point in a particular frame
by mouse dragging, the corresponding entries in the tables and graphs are updated.
In scanning mode, pointing at a graph or a table entry automatically shows the cor-
responding video frame, and selecting a particular frame highlights the corresponding
points in diagrams and the corresponding table entries. This makes scrubbing, that
is, manually advancing or reversing a clip, an effective means to precisely identify and
mark interesting events in the video clip and to relate them with graphical features.

The following basic features of a video analysis tool are visible in Figure 4.15 and
in the screen shot shown in Figure 4.16:

- Position vs. time data can be collected on a movie by clicking on the location of
  the items of interest for each frame in the video clip; a sight helps improve the
  accuracy of the measurement.

- A frame bar shows which frames in the video clip have been selected for mea-
  surement and allows the user to scroll through the video frames.

- The coordinate system can be transformed: it can be rotated, mirrored, and
  re-scaled (In the screen shot of the basketball throw, the coordinate system has
  been reflected in the vertical axis, so that the positive horizontal axis is directed
to the left).

- A user can calibrate in horizontal and vertical direction separately.

- A digital ruler and protractor allows measurements of distances and angles.
  Angles can be measured in radians and degrees; the latter option is, for example,
  important for applications in biomechanics.
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- The orbit of the point of interest measured can be displayed in each frame, with the point measured in the particular frame being highlighted. This allows the determination of the angle of release in the example of basketball shooting.

- One can overlay (the orbit of) a calculated point.

- Collected data can be tabulated, plotted, and used for further data analysis.

There are many other choices for the video tool to be made (but invisible in the screen shot): Amongst others, one has to make decisions about the supported video formats (Audio Video Interleave [avi] format, QuickTime [mov] format, etc.), supported compression-decompression methods (Cinepak, MPEG, or other codecs), and supported video adjustments (brightness, contrast, rotation, horizontal or vertical reflection, resizing, clipping, deinterlacing, etc.), and one must consider the inclusion of a polar coordinate system, a moving coordinate systems, different time settings (e.g., use time from video clip or set the time at a designated frame), change of replay speed, annotation possibilities, and a variety of methods for selecting or removing video frames for measurement. Details in the design choices make or break the usability of the video tool, which to a large extent can only be established by usability studies and field experiments.

Figure 4.16: Screenshot of a video analysis activity in Coach 5.0: basketball shooting. (Picture taken from [Heck, 2000a, 2002a]).

The cyclic process of design, implementation, and formative testing of a video tool prototype in conjunction with local evaluation characterized the first years of my research and development on the topic of video analysis by students. The classroom
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case studies about image analysis of real-world objects (Section 2.3), human locomotion (Section 2.4), and motion analysis by students at pre-vocational secondary school level (Section 2.5) served, amongst other things, to gain insight in the needs of secondary school students for carrying out video-based inquiry activities and to test the usability and scope of (prototypical) implementations of a video analysis tool.

In the first mentioned case study, I explicitly investigated whether a video analysis tool could easily be used as to analyze a digital image by replicating it into a sequence of equal images, or that a new tool or special settings dedicated toward image analysis are needed. As explained in Section 3.2.1, the conclusion was that this rather unusual application of a video analysis tool for image measurement caused some problems in practice: It resulted in more work and discomfort for students, teachers, and authors of activities. Identified drawbacks were:

- Some of the tools for movies do not make sense for an image measurement. Think of time recordings, time calibration, moving origin, and playing speed.
- The linking between graphical and table representations and the image measurement is somewhat different. In image measurement, the ordering of marking points may be irrelevant and may not be connected with time.
- One cannot simply use a single picture; one must always duplicate it. This is computer space and memory consuming.

The conclusion was quickly drawn that indeed a dedicated digital image analysis tool was needed in Coach. As a matter of fact, it turned out to be possible (1) to mimic as much as possible the tools from the video analysis environment; (2) to remove or disable irrelevant tools from data video; and (3) to include common formats of digital images (bitmap, jpeg, etc.) and simple modifications of digital images like change of brightness, contrast, and orientation (rotation and reflection), all for the purpose of making the measurement process easier.

In the case studies about human locomotion (Section 2.4), and motion analysis by students at pre-vocational secondary school level (Section 2.5), the research focus was more on the quality of students’ work when they used the video analysis tool of Coach. One of the things explored was the kind of obstacles students encountered in practical investigations when they recorded self-chosen motion events and used the recorded video clips for further analysis in the software environment. The students worked in the experiments with webcams, that is, with low-cost video cameras that feed their images in real time to a computer. Around the year 2000, the development of webcam technology (e.g., the CMOS and CCD image sensor technology) and the availability of affordable, powerful notebook computers had progressed so much that low-cost webcams could reliably record video clips at a frame rate of thirty frames per second in a sufficiently high resolution (i.e., exceeding a minimum resolution of $320 \times 240$ for standard movies). In the mentioned classroom experiments, students used the video software VirtualDub (Lee, 2011) to capture a motion and to edit the recorded movie with the goal to make it more suitable for video analysis (e.g., through file compression). Hereafter they imported the processed movie into the video analysis tool. Especially in the classroom study with vmbo tl-3 students (Section 2.5), it turned out that, although most students were in the end successful in their inquiry work and obtained good results, there were many technical problems with video activities to be
resolved: For example, it took too much time to do this kind of work with a whole class during a lesson at school, students had to put extra effort in learning to effectively use new software for capturing and editing videos, and students forgot to use video compression and consequently had difficulties with file handling. The main conclusion was that capturing and editing of videos should be incorporated into the computer learning environment, which actually comes as no surprise when the STOLE concept, acronym for Scientific and Technical Open Learning Environment (Ellermeijer, 1988) and outlined in Section 1.2 (pp. 13–14), is taken as basic design principle.

Video capture in the COACH environment was designed and implemented in version 6. This does not mean that import of external video clips was abandoned. On the contrary, the use of video clips recorded by students with their smart phones and of movies obtained from video-sharing websites has expanded in recent years. Video capture is only an additional feature to make the video-based inquiry work (e.g., the video analysis part of the bouncing ball study presented in Section 2.8) less demanding for students and better realizable in a classroom situation. Actually, the most important reason for inclusion of video capture in a versatile tool-based computer environment for mathematics and science education is that it allows synchronized video recording and data logging. The case studies about gait analysis via electromyography (Section 3.4.1), motion analysis of standing vertical jumps (Section 3.4.2), and the exploration of the pupil light reflex (Section 3.4.3) illustrated the educational potential of combining video with data logging and control of experiments.

A recent video technology push that influenced my research and development work was the introduction of high speed technology at consumer level. I explored its educational potential in various field experiments. Topics were:

- a detailed video analysis of bouncing balls (Section 2.8),
- a detailed video analysis of moving coins (Section 3.2.3),
- an examination of various drag models of a falling badminton shuttlecock by analysis of high speed video data (Section 3.5.2),
- the mathematical modeling of the motion of a falling chain and validation of the model by means of high speed video data (Section 3.5.4),
- a video analysis of the start of a sprint (Section 3.5.5),
- the mathematical modeling of bouncing gaits and validation of the models by means of high speed video data (Section 3.5.6), and
- a motion analysis of the backward giant circle on the high bar (Section 3.5.7).

When high speed video analysis comes into play, automated point tracking in video analysis becomes indispensable. But also in case of analysis of normal speed videos, it is in education often convenient that one can quickly collect data on a video clip. When video capture via a webcam and automated point tracking can be combined in a computer learning environment, then the webcam would become just another sensor attached to the computer and the identified contributions of data logging to graph sense will probably also hold for video-based laboratory. I refer to Section 3.2.3 for details about the implementation of automated point tracking in COACH 6.
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The opposite of high speed video analysis also exists and has educational potential: the data collection in the case study about the decay of beer foam (Section 3.5.3) was, for example, done via a webcam capturing one frame per four seconds. This was an illustration of usage of low speed video technology, which is popular in fields like astronomy and biology.

The last feature of the digital image and video analysis tool in COACH that I would like to mention in this section is correction of perspective distortion. It is important because quite often the camera cannot be oriented fronto-parallel to the plane of interest, with the consequence that Euclidean properties such as length, angles, and parallelism of lines are distorted. Computer vision provides methods for correcting this perspective distortion, that is, for transforming the digital image such that the geometrical relationships are those which would be seen had the photograph of the plane of interest been taken with the camera fronto-parallel to the plane. This process is called (image) rectification. In Section 3.2.2 I explained how rectification has been implemented in COACH 6. A rectified video clip can be exported and other video settings like the brightness, contrast, orientation, and the selection of frames can be taken into account in the creation of a new video clip.

A Glimpse of Design Principles Used

Research-based design principles of multimedia learning (Mayer, 2005, 2009) and cognitive load theory (Kalyuga, 2009; Plass et al., 2010; Sweller, 2005) have been applied in the design of the video analysis tool. For example, the coherence principle, which states that people learn better when interesting but extraneous material is excluded rather than included in instruction, has been applied in the design of dialog windows. This has been done in the sense that not all interactivity elements are visible at once, but only when related options have been checked and there is a need to show the interactivity elements. In a video measurement activity, only after choosing the option to select frames manually, the various ways to specify the selection of frames become visible in the dialog window. Figure 4.17 shows the dialog windows in the default format (‘Use all frames’), for selection of individual frames, and in formats in which a fixed step sized sequence of frames can be specified.

Figure 4.17: Screen shots of various formats of the frames selection dialog window.
The spatial contiguity principle, which states that people learn better when words and pictures are presented near rather than far from each other on the page or screen, favors the direct annotation of a particular video frame and the logical grouping of menu items in a video tool (such as grouping menu items for specifying measurement settings, items for video adjustments of video clips, and items for measuring physical quantities in individual frames).

The signalling principle, which states that people learn better when cues that highlight the organization of the essential material are added, comes to the fore in a video analysis activity when a video frame, a table entry, and a point in a graph are dynamically linked in the so-called scanning mode (See, for example, the screen shot in Figure 2.25, p. 87). In case of a data image activity, the corresponding measured point would be highlighted.

The personalization principle, which emphasizes the use of personalized style rather than formal style, proposes that social cues affect multimedia learning through their potential of increasing the learner’s motivation to make sense of the instructional material or the learner’s interest in the subject, so that (s)he is more keen on understanding the lesson material deeply. In the context of doing a video analysis activity, Heck and Uylings (2006) pointed out that appealing videos which contain persons in motion (such as the motion of a toddler in a playground [Figure 2.13, p. 58]) increase the students’ motivation to learn and are more effective than video clips that are undoubtedly and formally oriented towards physics, such as movies of falling balls, collision of balls, or motion of objects on an air track. The learner’s interest is certainly increased when (s)he is visible in a self-recorded video clip, doing an experiment. The theoretical explanation for this effect of personalization is straightforward and worded by Mayer, Fennell, Farmer, and Campbell (2004, p. 391) as follows: “Using the self as a reference point increases the learner’s interest, which in turn encourages the learner to use available cognitive capacity for active cognitive processing of the incoming information during learning.” But even when no person is visible in a video recording, personalization effects can be achieved by requiring that a student is responsible for doing the video recording of an experiment, with the extra bonus of the student’s engagement that (s)he may understand better what was going on.

4.3.6 Graphical System Dynamics-Based Modeling

Computer-based modeling is in the STOLE concept (Section 1.2, pp. 13–14) one of the computer activities for practical investigations and problem solving. Bliss and Ogborn (1989) distinguished between two types of modeling tools for exploratory learning: (1) exploratory tools, which represent expert models; and (2) expressive tools, which allow learners to build their own models and to test their own hypotheses. A quote:

“Exploratory tools allow learners to investigate views of a given domain, which are different from theirs. Expressive tools permit pupils to represent their own models of a domain and in this way reflect upon and explore their own mental model. In the former learners are developing models based on the assumptions of others and in the latter they are modeling their own assumptions.”

A versatile computer learning environment designed and implemented in accordance with the STOLE concept is also integrated in the sense that it combines these two
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types of modeling tools. Examples of exploratory tools are pre-defined measurements with fully prepared data analysis and simulations or animations of given models. Programming, modeling, and animation editors are examples of expressive modeling tools, at least when students use them to adapt or create computational programs, models, and animations. In this section I focus on one particular type of modeling tool, namely, a graphical system dynamics-based modeling tool. I outline my contributions to the design and implementation of this type of modeling tool in Coach 6.

A Historical Perspective

I go with seven-league strides through the history of computer modeling with Coach. The software contains since its first release two types of modeling tools: (1) a tool in which a student writes computer programs using a language dedicated, but not restricted, to the creation of control programs; and (2) a dynamic modeling tool. Dynamic modeling concerns the construction and use of a computer model of a dynamic phenomenon, that is, a phenomenon that can be mathematically described by variables that change over time (discretely or continuously). Typically, the mathematical description is a difference equation or a differential equation.

Dynamic modeling tools can be roughly classified by the way they allow their users to construct models: text-based or graphically. In physics education, Ogborn and Wong (1984) introduced the so-called Dynamic Modelling System (DMS), which was an environment for programming and simulating computer models that compute the evolution of a system step by step. Users of DMS expressed their ideas about the steps in the form of equations and the program helped specify initial values, carry out iterations, store calculated values in tables, and plot graphs. The text-based mode of the modeling tool of IP-Coach (Van Bart & Mulder, 1990) and the simulation software VU-Dynamo (Van Blokland & Kok, 1987) were environments for physics and mathematics education in the Netherlands, respectively, that were based on these ideas. A screen shot of a text-based IP-Coach 4 model of the motion of a pendulum is shown in Figure 4.18: When time $t$ increases by $\Delta t$, a new position $x$ is calculated based on Newton's second law of motion. This modeling mode hardly differs from the text-base modeling in Coach 6, shown in Figure 2.4 (p. 42) and Figure 2.31 (p. 96).

![Figure 4.18: Screen shot of a text-based Coach 4 model of the motion of a pendulum.](image-url)
In order to minimize the programming skills needed for construction of computer models and to promote a diagram-based approach to computer modeling, a graphical interface was developed in COACH 4. Herein a model consisted of variables that change over time, constants, and auxiliary variables, each represented by a separate icon and connected to each other by relation arrows. The software took care of automatically generating the corresponding text-based model. A screen shot of a diagram-based model of the motion of a pendulum is shown in Figure 4.19.

![Figure 4.19: Screen shot of a diagram-based COACH 4 model.](image_url)

Research (e.g., Löhner, 2005) seems to indicate that graphical modeling offers students an easier-to-use and richer framework for understanding the structure of a dynamic model in comparison with text-based modeling, and that it allows students to build more complex models of high quality, because they can concentrate on qualitative specifications during initial stages of the modeling process and do quantitative, formula-based specifications at later stages. But this cannot be taken for granted: The fact that the diagram-based modeling tool of versions 4 and 5 of COACH did not catch on in Dutch physics education shows that more factors come into play.

The design team of COACH 6 compared the diagram-based modeling tool of version 5 with much more popular graphical system dynamics-based modeling tools such as STELLA (Steed, 1992) to identify strengths and weaknesses of both types of tools. It is important to realize that these tools are based on principles that may look similar, but differ on essential points. In the dynamic modeling environment of COACH 5, the basic principle was that the graphical model represents a computer model that calculates the evolution of a system step by step. In each iteration, variables are updated and the icons in the graphical model determine what lines of computer code are generated. A variable \( var \) that changes over time by \( dvar \) in a time step \( dt \) was represented by one icon that contained two components, as shown on the left-hand side of Figure 4.20. The generated line of code for this icon is \( var = var + dvar \times dt \). In mathematical terms, the icon represents both a quantity that depends on time and its differential. However, the idea of using one icon for the combination of two variables is stretched when relation arrows, which help specify dependencies between variables, can only come and go to one of the two parts of the icon. This may confuse novice modelers. In a graphical, system dynamics-based modeling tool like STELLA, these two types of variables are graphically represented by two different icons in order to make clear that one distinguishes a state variable from its rate of change; see the picture on the right-hand side of Figure 4.20. Although the two diagrams in Figure 4.20 are
similar, there is a big change in the creation of the diagrams and probably also in the thinking behind it. In comparison with graphical system dynamics-based modeling, the cognitive load of the diagram-based modeling implemented in COACH 5 is definitely enlarged because a modeler is forced to simultaneously think about two types of variables and (s)he cannot easily divide attention to these elements in the thinking process. This may be one reason why teachers and students were less attracted to graphical modeling than to text-based modeling.

Figure 4.20: Basic elements of a diagram-based model in COACH 5 (left-hand side) and of a graphical, system dynamics-based model in STELLA (right-hand side).

In many practical cases, the diagram-based modeling approach of COACH 5 led to more complicated and less understandable graphical models than the graphical system dynamics-based approach with a tool like STELLA. Figure 4.21 shows the graphical models of radioactive mother-daughter decay in both modeling tools. The STELLA model on the right-hand side shows the relations between variables in a clear way: In the stock/flow language one can see in the picture that what flows out from the Radium stock, flows into the Polonium stock, and that Polonium decays further. The COACH 5 model is more complicated and more difficult to understand.

Figure 4.21: Graphical models of radioactive mother-daughter decay: a diagram-based model in COACH 5 (left-hand side) and a graphical, system dynamics-based model in STELLA (right-hand side).

The distinction between state variable and rate of change in graphical, system dynamics-based modeling becomes problematic when quantities are playing both roles. This always happens, for instance, in Newtonian mechanics: Velocity is on the one hand a rate of change, because it is the derivative of position, and on the other hand a state variable of which the rate of change equals acceleration. One must either use different names for two icons that represent the same quantity, or consider velocity in two ways, namely, as the rate of change of position and as the quotient of momentum and mass (cf., Doerr, 1996). In the diagram-based model of COACH 5, this problem of overloading the role of a quantity as state variable and rate of change does not occur.

Figure 4.22 illustrates on the right-hand side the complication of graphically modeling kinematics problems via a system dynamics approach and shows on the left-hand side the solution implemented in COACH 6, namely, the feature of identification of variables. This means that it is possible to relate the icon of a state variable, an auxiliary variable, or a constant with the icon of a rate of change, and to identify the linked variable with the one that stands for the rate of change. Internally in the generated computer code, a local name for the rate of change is present, but the modeler is not forced to think in terms of this variable and keep recalling its real meaning.
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After evaluation of various prototypical implementations of a graphical system dynamics-based approach, the design team of COACH 6 decided to adopt this modeling approach, but also to add a third modeling mode, namely, equations-based modeling. The main reasons were the following:

1. Equations-based modeling is a compromise between having all mathematical equations hidden behind icons in a graphical model and having each and every detail visible in computer code. In equations mode, the mathematical equations that drive the simulations are always visible; see the equations-based model of the shape of the main span of the Golden Bridge shown in Figure 3.7 (p. 109) and the equivalent graphical model in Figure 3.6 (p. 108).

2. In a graphical model, one has no control over the order in which variables are updated in the generated computer code. This order of updating is irrelevant for almost all applications, but in special cases it is important: Then, in equations mode, one can reorder the mathematical equations displayed on the screen and influence the order of updating of variables. I used this feature, for example, in the modeling of a stochastic walk in a bounded area via event handling (Heck & Bruidgrom, 2007): The events for imminent crossings of boundaries needed to be handled after the handling of the event that determined the direction of the next step in the random walk.

Another extension of a traditional graphical system dynamics-based modeling tool explored by the development team was the possibility to define an auxiliary variable by means of a table or a graph. Figure 4.23 shows a user interface design considered during the design process: A special dialog window allows the user to fill out a table or to sketch a graph. Finally it was decided to use an already available table or graph in the COACH activity for this purpose. This illustrates the application of the
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‘Keep It Simple and Smart’ (KISS) design principle, which promotes minimization of the complexity of working with a computer learning environment by not adding continuously new elements to the human-computer interface. In this particular case, a table window already exists and offers the possibility of manual filling and import (or copy) of data from external resources. The graph window already exists and offers the possibility to sketch a graph.

![Design of a graph/table editor for the modeling tool.](image)

Figure 4.23: Design of a graph/table editor for the modeling tool.

New Elements of Graphical System Dynamics-Based Modeling

In my research and development work I contributed to the following aspects of the modeling tool in Coach 6:

- The choice of algorithms for solving differential equations numerically.
- The design and implementation of hybrid systems modeling.
- The extension of the graphical modeling tool with a process icon.
- The design and implementation of a model-based animation tool.

Because it must be able to transform each graphical model in Coach into a text-based computer model in the Coach language, fixed-time step methods such as the Euler method, and the 2nd and 4th order Runge-Kutta methods were selected for implementation in the computer learning environment. Runge-Kutta methods were included to avoid as much as possible the necessity to choose small time steps for accurate numerical solution. But this is only the minimal set of numerical algorithms that must be available in any modeling tool. As discussed in the case study about modeling chemical kinetics (Section 3.3.1, p. 135), more advanced numerical methods with variable time steps are convenient and needed for practical investigations.
Hybrid systems modeling, which combines a classical system dynamics approach with event-based modeling for processes that change abruptly, can be realized entirely within a system dynamics-based model (see, for example, Levin & Levin, 2002), but it is too complicated for use at secondary school level. For this purpose, an event has been implemented in Coach 6 according to the principle of software triggering: As soon as some particular condition is fulfilled, a once-only action on state variables is put in running and only at the point that this condition is not satisfied anymore, then the event can occur anew. I refer to the case study about bouncing balls (Section 2.8) for more details about event-handling in Coach. Other examples of event-based modeling were discussed in the case studies about chemical kinetics (Section 3.3.1), the motion of a yoyo (Section 3.5.1), bouncing gaits (Section 3.5.6), and alcohol metabolism (Heck, 2007a,b).

In the case study about modeling chemical kinetics graphically (Section 3.3.1), I proposed an extension of the classical graphical system dynamics-based modeling, namely a process element iconified by the Erlenmeyer flask icon, to overcome the principle of flow balance and to simplify graphical modeling of chemical kinetics. This new graphical element was also convenient in graphical modeling of quantitative pharmacological processes (see, for example, Figure 3.29, p. 135) and the decay of beer foam in a glass (Section 3.5.3, Figures 3.62 and 3.63, p. 173).

Model-based animation is possible in Coach 6: I refer to a description of the animation tool to Section 3.1 (cf., Kędziierska et al., 2009). I only presented examples of animation in this thesis (such as an animation of a bouncing ball on an oscillating platform [Figure 2.32, p. 97], an animation for exploring the free fall of a chained object [Figure 3.67, p. 178], and an animation of a vertical spring-mass system [Figure 3.78, p. 191]). I do not discuss the design and implementation of the animation tool.

4.4 Concluding Remarks

In this final section I take stock of my research and development work. I summarize the main outcomes, answer the driving questions, and reflect on the set-up of the study and implications for future work.

4.4.1 Main Outcomes

The general purpose of my research and development work in the last decade was to improve the contribution of ICT to inquiry-oriented mathematics and science education at secondary school level. In particular, I aimed to contribute to

- the development of an open activity-based computer working environment that offers its users a versatile set of integrated tools for the study of natural phenomena, mathematics, science and technology; and

- perspectives on the role of ICT in quantitative mathematical modeling by secondary students.

More specifically, I aimed at contributing to the development of an integrated computer learning environment by making recommendations about the functional design on the basis of classroom experiments and sample activities, and by exploration and analysis
4.4. Concluding Remarks

of educational needs and possibilities, especially regarding inquiry activities. The second purpose of the classroom experiments was to explore how ICT can contribute to the realization of students’ practical work that resembles applied mathematics and science practice.

Development of a Versatile Computer Environment for Mathematics and Science Education

The open activity-based computer working environment mentioned in the introductory paragraph was materialized in a specific computer learning environment, namely, COACH (Heck, Kędzierska, & Ellermeijer, 2009). This hardware and software environment was used to learn lessons from developing ICT tools that are integrated in an open, activity-based, multimedia authoring environment for mathematics and science education. The main outcomes of my development work in collaboration with other members of the multidisciplinary development team were the design, redesign, and implementation of tools for digital image and video analysis, computer modeling and animation, and data handling. The usability and scope of all tools, either used separately or in combination with each other, were tested in classroom case studies, field experiments, and usability studies presented in Chapter 2 and 3. The tool design was explained in detail in Section 4.3.

Highlights of the Tool Design

Highlights of the digital image and video analysis tool, which mainly originated from my work were: video capture with webcams and high speed cameras, video editing, automated point tracking, correction of perspective distortion, and digital image analysis. New elements of computer-based modeling, simulation, and animation coming from my activities were: numerical algorithms, graphical system dynamics-based modeling, an extension toward easy event-based modeling, an extension of graphical modeling with a process icon, and model-based animation. Data handling was redesigned by improvements of tools for data smoothing, numerical differentiation, regression analysis, and signal analysis.

Driving Forces

The improvements of the COACH hardware and software environment are concrete deliverables, the development of which was driven by classroom experience, educational research, curriculum change, and technology. Below I spend some words on these driving forces and illustrate them in the context of my work.

The classroom case studies presented in Chapter 2 served to test the usability of particular (prototypical) tools and the need for improvements. In the case study about human locomotion (Section 2.4), for example, I tested the first implementation of the video analysis tool and I explored whether upper-level pre-university students were able to effectively work with the tool in a practical investigation and whether they liked to work with the software. In the case study about video analysis at pre-vocational level (Section 2.5) I explored whether vmbo-tl3 students could get fruitful first experiences with inquiry work through video analysis. One of the outcomes was the need for a built-in video capture facility. One of the results of the case study about image analysis of shapes of main spans of bridges and hanging chains (Section 2.3.1) was that one cannot effectively use a video analysis tool for this purpose, under the assumption that one can take measurements on a video clip with equal frames, but that one really benefits from
a dedicated image analysis tool. When high speed video technology was used in the case study about bouncing balls (Section 2.8), automated point tracking was evaluated in practice. In the case study about quantitative pharmacology (Section 2.7) I explored whether upper-level pre-university students could be quickly introduced in an e-class setting to computer modeling and in particular learn to work with a graphical system dynamics-based modeling tool.

The driving forces behind innovations in mathematics, science, and technology (MST) education from educational research take many forms. Some of the underpinning views are:

- Learning is a constructive and situated process that only takes place when the learner is actively engaged. In a constructivist view on teaching and learning (cf., Steffe & Gale, 1995), knowledge is in fact constructed in the mind of the learner and not transferred from teacher to student. So, the main task of the teacher is to create conditions under which an active attitude leads to learning.

- Learning is a social process. Competencies with regard to collaboration, discussion, sharing of ideas and results, reporting, and so on, are important in the students preparation for a knowledge society. The task of the teacher is to organize sessions where the social processes take place and can be practiced.

- Learning becomes more relevant when it is done in a context. Authentic learning, interpreted here narrowly as working on real-world, complex problems, with the goal to come to grips with phenomena through scientific methods, is generally considered to motivate students and to lead to better understanding. Thus, the task of the teacher is to give students the opportunity to engage inside and outside school in authentic learning activities (See, for example, Reeves et al., 2002).

- Learning mathematics, science and technology includes not only acquiring theoretical knowledge, but also doing and learning about MST (cf., Hodson, 2008, 2009). This requires a contemporary outlook on what is going on in MST, not only the issues of decades or centuries ago. Since tools play a crucial role in current research and development, this should be reflected in education.

- There are many reasons why learning to use multiple representations is relevant in education (cf., Ainsworth, 1999, 2006, 2008), but one of them is that mathematicians and scientists use multiple representations in their work as well.

In all of the listed views, ICT is expected to contribute to the improvement of teaching and learning. In my work I envisioned a scenario of teachers and students using a set of tools for inquiry-based study of natural and mathematical phenomena. This set of tools was supposed to be integrated in one open environment designed for a broad educational setting. Openness means that it is

- a flexible, customizable, multi-purpose system;

- an environment for solving open problems that need definition, set-up, exploration, data processing and analysis, mathematical modeling, and so on, that is, primarily a cognitive tool;
4.4. Concluding Remarks

- as much as possible free of didactic context or principles, that is, it is less considered as a pedagogical tool, but more as a tool for doing mathematics and science. What really matters is that the environment is suitable for inquiry-oriented student activities, practical investigations, and research projects by secondary school students.

Results from educational research played a role in the tool design. Research-based principles of multimedia learning (Mayer, 2005, 2009) and cognitive load theory (Kalyuga, 2009; Plass et al., 2010; Sweller, 2005) were applied in the design of tools; see, for example, the discussion about dialog windows in the video analysis tool (pp. 287–288). Research-based frameworks about the use of multiple external representations (See, for example, Ainsworth, 1999, 2006, 2008; Goldin, 2008; Goldin & Kaput, 1996; Kaput, 1992, 1994), graph and data sense (Curcio, 1981, 187, 2010; Shaughnessy, 2007; Shaughnessy et al., 1996), symbol sense (Arcavi, 1994, 2005; Drijvers, 2003, 2006, 2011; Pierce & Stacey, 2004; Zorn, 2002), and a framework for practical work in science and scientific literacy through concepts of evidence and argumentation (Gott & Duggan, 2007; Gott, Duggan, & Roberts, 2003) were used in Chapter 4 to analyze the students’ work in the classroom case studies presented in Chapter 2, the field experiments presented in Chapter 3, and some profile research projects. Because of the exploratory nature of the case studies and the diversity of the participants, subject matter, and ICT use, many of the classroom outcomes cannot be strongly and thoroughly supported by quantitative evidence. Nevertheless they give a good impression of the abilities of the students who participated in these studies regarding quantitative mathematical modeling. Details of findings and conclusions can be found in Section 4.2 and are summarized later in this section when I discuss the findings about the modeling abilities of the participating students.

As explained in Section 1.1, the educational context and in particular the curriculum changes in the Second Stage and Renewed Second Stage had a large impact on my research and development work. The introduction of a problem-oriented approach in mathematics education focusing on applications and mathematical modeling as well as the growing interest in inquiry-based science education made me focus on quantitative mathematical modeling using ICT; that is, on inquiry activities in which students explore mathematical models with the support of ICT tools in order to come to grips with natural phenomena and to interpret real data. It was no coincidence that the redesign of the modeling tool in Coach 6 overlapped with the growth of interest in computer modeling in Dutch secondary education. I restricted my attention to practical investigations and profile research projects, which became part of the curriculum when the Second Stage was introduced.

ICT-supported inquiry-oriented mathematics and science education is to a certain extent technology-pushed. I illustrated in Section 4.3.5 that this very much holds for digital image and video analysis: The introduction of webcams, smart phones, and low-cost high speed cameras enabled students to record everyday phenomena at normal speed and in slow motion, and then do motion analysis in much the same way as scientists and practitioners do. But also the increased power of notebook computer and advances in sensor technology made it possible to let students do experiments and collect real data that could not have been done before and reflect the way scientist and practitioners work at present. The powerful tools for acquisition, processing, and analysis of data, and the tool for modeling, simulation, and animation, have
proven themselves as very helpful to bridge the gap between abstract mathematics and science concepts on the one side and real-world contexts on the other side. The case studies about human locomotion (Sections 2.4 and 3.4.1), sports motion analysis (Sections 3.5.5 and 3.5.7), and the decay of beer foam in a glass (Sections 3.5.3) served as showcase examples.

From the curriculum implementation point of view, a computer learning environment that teachers and students can use frequently and throughout the curriculum has many benefits. It means for them that it makes sense to become familiar with the environment, and a learning path from simple ICT use up to the usage in advanced projects can be developed. The integration of tools enables them to move data to other parts of the environment and compare for instance in the same graph data obtained from measurements with data from modeling. This was illustrated many of the case studies presented in Chapters 2 and 3.

On the other hand, such a versatile tool might become complicated. It has to be possible to adapt the complexity and the features needed to the level of the student. In Coach, since the introduction of the so-called Junior version, an author has the facility to create tailor-made activities. This activity-based approach also seems necessary because designers of computer learning environments are seriously confronted with differences between scientific practices. For example, the meaning of variable is variable in mathematics and science (Ellermeijer & Heck, 2002; Heck, 2001; see also the summary in this chapter, pp. 233–237). This cannot be ignored in software development. Contexts for graphing may differ from one discipline to another (Ellermeijer & Heck, 2002; see also the summary in this chapter, pp. 222–223). However, the biggest challenge lies in the differences in language and representational conventions between mathematics and science. In one way or another designers must cope with these issues when they try to develop a general purpose learning environment. This is a very complex task and despite the progress in the understanding of the relevant issues and advances in computer and software technology, it is still an open question whether one can simultaneously meet so many requirements for a versatile computer leaning environment. In an activity-based approach one can at least use the language of the target discipline and use external representations that can be understood by the target group of students.

Regarding the suitability of the Coach environment for practical investigations by secondary school students, the main conclusions were that, in general, students

- quickly familiarized with the computer environment,
- had no insuperable problems with video measurement and working with graphs,
- were in an e-class setting able to master the basics of computer modeling rather quickly and at an appropriate level, and
- liked to do practical investigations with the ICT tools.

It seems that the student benefited from the consistent design following from the STOLE concept, acronym for Scientific and Technical Open Learning Environment (Ellermeijer, 1988), outlined in Section 1.2 (pp. 13–14). In this concept, a hardware and software environment was envisioned in which tools for essential elements of doing investigative work are integrated in a single system that supports students’ learning in
4.4. Concluding Remarks

an inquiry-oriented approach of mathematics and science education. The effectiveness and acceptance of the versatile tool-based computer environment, interpreted as the findings that the goals of using the ICT tools in inquiry-oriented mathematics and science education were met and that the students showed willingness to use the tools, respectively, were observed in the classroom case studies and evidenced in the profile research projects about gait analysis via electromyography (Section 3.4.1), bungee jumping (Section 3.5.4), circling around the high bar (Section 3.5.7), and in several projects about human growth, which have not been discussed in this thesis.

The Quantitative Mathematical Modeling Abilities of Students

In Section 4.2 I reflected on the outcomes of the exploratory case studies presented in the previous chapters concerning aspects of scientific inquiry and authenticity in the practical investigations, field experiments, and usability studies. I used for this purpose a framework for quantitative mathematical modeling (Figure 4.1, p. 206) that combined an empirical inquiry cycle with the mathematical modeling cycle of Blum and Leiß (2005). Below I summarize the main findings on the basis of the same selection of aspects of scientific inquiry and authenticity.

Design and Conduct of Experiments, and Basic Data Handling

The main conclusions regarding digital image and video analysis activities (summarized in Section 2.4 and 2.5) were the following:

- Most participants in the classroom case studies were sufficiently able to carry out sub-processes of empirical modeling on request.
- Not only upper-level pre-university students, but also students in pre-vocational secondary education were able to get an impression of what it takes to do scientific inquiry and to develop inquiry abilities by carrying out a small investigation task at their own educational level using digital video technology.
- The radius of action of video analysis, that is, the range of situations in which a person is able to activate his or her video analysis competency, seems large. Students who learned and practiced video analysis in one situation (e.g., gait analysis) seemed to have no difficulty in applying it in other situations.

The surprisingly quick uptake of video analysis technology and motion analysis by secondary school students does not mean that there were no comments on the quality of the students’ work and on the support level of the ICT tools, but at least there were no insuperable obstacles or quality issues that are difficult to improve.

Graph Sense

The students who participated in the classroom case studies had difficulties at all levels of graph comprehension in the framework of Curcio (1981, 1987, 2010) and Shaughnessy (2007), except the first level of reading the data. At the level of reading between and beyond data, students had difficulties in interpreting and reasoning with unfamiliar graphs and graphs of derived quantities in terms of the real-world context from which the graphs originated. In video analysis activities, many a student did not autonomously use the video scrubbing technique to link graph features with motion events, as if they had forgotten about this feature of the computer learning environment. Reading behind data is difficult, but not unreachable for pre-university
Chapter 4. Findings and Conclusions

students. The spreadsheet-based case studies about survival analysis (Section 2.6.1) and data handling of weather data (Section 2.6.2) illustrated this. Understanding, interpretation, evaluation, and manipulation of data played a large role in the student activities and the participants in the case studies performed surprisingly well (or at least better than expected) and they very much enjoyed the inquiry activities.

Data Sense
The focus of many practical investigations was on getting a global view of data. In exploratory data analysis, global understanding of data commonly refers to the ability to search for, recognize, describe, and explain general patterns in a set of data (change over time, trends) by observation and intuition, or by statistical techniques such as regression analysis. The classroom case studies, especially the ones about human growth (Section 2.2) and mathematical modeling of shapes (Section 2.3), revealed that the students had informal ideas about association and correlation of quantities, but did not have many formal concepts and skills connected to these notions.

Students understood the idea of a best line fit of data, but only based this on visual judgment in data graphs and not on criteria of goodness of fit. They were readily willing to accept other regressions formulas, but kept a strong preference for linear curve fitting, even in cases where a nonlinear approach would be more appropriate.

In the classroom activities about gait analysis (Section 2.4) and quantitative pharmacology (Section 2.7) students were introduced to the least squares method of peeling-off functions (Foss, 1969) for finding appropriate regression curves that are formulated as a sum of two or more mathematical functions. They had no difficulty in following the procedural steps. When this technique was applied to distinguish between a clearly separated global trend and a superimposed function, the students had no difficulty in relating the summands in the regression formula to components of the described phenomenon. In cases where it is less obvious that a global trend and a superimposed function can be separated, students seemed to neglect the possibility of a regression model consisting of a sum of mathematical functions.

I also noted in some classroom case studies that students tended to stick to a global view in data fitting and did not autonomously considered the idea of taking a component-wise view in data fitting. It was as if they did not realize that they could make use of the feature of the regression tool of COACH to zoom in on part of the data set for applying regression analysis to the particular data selection. Anyway, I noticed a tendency in the students’ actions and reports to forget that they could split data in parts that could be modeled separately.

Symbol Sense
In the classroom case studies it was often observed in class and noticed in students’ written reports that the students had rather weak algebraic skills and lacked confidence in using mathematical formulas. Illustrative were the difficulties of

- the vwo-5 students in the case study about bouncing balls (Section 2.8) with writing down a formula for the percentaged loss of energy at a bounce of the table tennis ball (p. 95), despite their Mathematics B background in the Nature & Technology profile, and

- the vwo-5 students from the Culture & Society stream in the case study about survival analysis (Section 2.6.1) with pencil-and-paper based algebraic manipu-
lation and entering of formulas in an Excel sheet (p. 67), leading to a behavior of avoiding the use of mathematical formulas.

In both cases, I had the strong impression that the so-called algebraic expectation of the students, which is the thinking that allows a student to monitor working with mathematical formulas, was underdeveloped, hindered them in their work, and led in some cases to a behavior of guessing a formula without giving it much thought.

Representational Fluency
One cannot close one’s eyes for students’ difficulties associated with using multiple representations. The cognitive load is definitely enlarged when multiple representations come into play and it has been reported in many research studies (cf., Ainsworth, 2006, 2008; Ploetzner et al., 2008, 2009) that students find retrieving information from representations, moving between and within representations, and coming up with appropriate representations difficult. The students who participated in the classroom case studies were no exception. For example, in video activities many of the students did not spontaneously and autonomously use the replay option or the video scrubbing technique to find more details about the recorded motion and the connection of events in the video clip with features in the graphical displays. Only when they were advised to do this, they would follow the suggestion without much difficulty. Lack of graph sense and representational fluency seemed to hinder students in extracting all information that was intrinsically available in several linked representations (such as in hip-knee cyclograms of recorded human gaits) and in evaluating the quality of their experimental work (such as the quality of the experimental set-up).

Instructional Design
The structured and guided inquiry approach that I used in most classroom case studies worked reasonably well, but suffered from known difficulties with this approach if not enough moments of reflection on the practical work as a whole class activity are organized by the teacher. The biggest problem was the following: Students successfully carried out the closed tasks in an activity without really reflecting on the reasons of doing these tasks, their place in the inquiry route, or their links with the main research question in the activity, even though the authors had put great effort in positioning the tasks in the inquiry cycle and linking them with the research question for which an answer was sought.

I designed modeling activities in which students could go several times through the modeling cycle while investigating the same phenomenon, starting with simple models first and then improving them by making small changes or adding details. Positive experiences and satisfactory results with this instructional approach were obtained in the classroom experiments in which students explored mathematical models of alcohol metabolism (Section 2.7) and explored the motion of bouncing balls (Section 2.8). This approach of exploring various models of the same phenomenon was also expected to contribute the students’ development of a critical attitude and a feel for the parsimony principle that guides modeling, which means that simple models are in general preferred and are developed and validated first to arrive at feasible models of low complexity.

I explored four types of instruction for teaching effective use of ICT tools in an inquiry-oriented activity. There were structured inquiry activities of the following kind:
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1. The teacher gave a quick introduction into tool use and the students hereafter worked with instructional materials that contained detailed information about how to use the ICT tools.

2. The teacher gave a thorough classroom introduction into tool use with students making their own notes, which they could consult when they carried out their practical investigative work.

3. The inquiry route was outlined in the instructional materials, but these materials only contained minimal explanations or hints about the tool use, because the students were expected to have already acquired the necessary ICT skills. Students who needed more support or brushing up of their ICT skills could use an enclosed short manual.

4. Instructions for learning to work with software and demonstrations of worked-out examples were given through screencasts, and students were additionally supported by communication tools offered in an e-class setting.

Although all instructional designs worked out quite well in practice, it was found that:

- Detailed instructional notes had two drawbacks: (1) students reported that explanations were sometimes too vague for them; and (2) even simple mistakes in the explanation of a tool occasionally led to confusion and blockages because students rightly assumed that they had to do exactly what was written down. Too much detail in instruction is error-prone.

- Making useful notes of a teacher-led demonstration of tool use is something that students must learn to do.

- When students were already sufficiently acquainted with an ICT tool (e.g., a spreadsheet program), the use of optional software guides seemed to work well.

- Screencasting was appreciated by all students in the e-class setting and they quickly learned in this way the basics of using a graphical system dynamics-based modeling tool. Screencasts seemed to work much faster and better than user manuals, written instruction, or help pages.

Authenticity

I mainly interpreted the authentic nature of the practical investigations as the opportunity for students to work on real-world problems, with the goal to come to grips with phenomena through scientific methods. Students worked directly with high-quality data in much the same way as scientists and practitioners do. One of the aims was that the tool use reflected innovation in mathematics, science, and technology. Many times I guided the students in their use of the same theoretical framework, nomenclature, research methods, and techniques as professionals. Students found it difficult to adopt the nomenclature when they could not personally relate to the subject, but in general they got a good impression of how ICT tools can be used in practical investigations and research projects, and how they are used by scientists and practitioners. Motion analysis studies seemed to be most promising and attractive for students to get in touch with real science.
The Role of ICT in Quantitative Mathematical Modeling

I aimed to contribute to the perspectives on the role of ICT in quantitative mathematical modeling by secondary students. I also used for this purpose a framework for quantitative mathematical modeling (Figure 4.1, p. 206) that combined an empirical inquiry cycle with the mathematical modeling cycle of Blum and Leiß (2005). Although this framework contains no reference to ICT or other forms of technology and in essence each transition in the cycles is unaffected by the availability of technology, because a tool only supports its user and does not take over the real job to be done, this does not mean that ICT and technology play no role. For each transition in the model of quantitative mathematical modeling I listed in Table 4.5 (p. 260) options of meaningful use ICT and other forms of technology. Details and an additional listing of possible roles of ICT regarding authenticity, efficiency, motivation, interactivity, feedback, enrichment, and access to advanced methods, can be found in this chapter on pages 251–260.

The possible roles of ICT in the process of handling various external representations in the so-called Rule of Five framework of multiple representations (See Table 4.6, p. 262) were summarized in Figure 4.7 (p. 264). It is part of the theoretical rationale of tool integration in a versatile computer environment for inquiry-oriented mathematics and science education: The use of multiple representations is crucial for deep understanding of real phenomena and this process of understanding is promoted when learners are not distracted by technical burdens that could have been avoided by the provision of tools that work well together. This view can be underpinned by theoretical frameworks such as the Kaput-Goldin representational framework for mathematical cognition and learning (cf., Goldin, 2008; Goldin & Kaput, 1996; Kaput, 1992, 1994) and the Rule of Five framework on multiple representations (cf., Dick & Edwards, 2008).

The representations used in an environment such as COACH are often not static entities, but dynamic elements that are linked so that a change in one representation affects the other representations. This is one reason to speak about integrated tools. The underlying ideas of having multiple, dynamically linked representations available in a student activity are:

- It illuminates the meaning of actions in one representation by exhibiting their consequences in another representation.
- The number of ways to come to a solution of a problem increases.
- A person’s understanding of a phenomenon, a problem, or a concept is refined the more representations (s)he can interact with.
- It supports the construction of deeper understanding when students relate those representations to identify strengths and weaknesses of particular representations and shared invariant features of all representations in use.

4.4.2 Answers to the Driving Questions

The scope of my study was limited to pedagogical and software design perspectives on ICT use in inquiry activities in which students develop mathematical and scientific literacy. Driving questions were:
Chapter 4. Findings and Conclusions

- How can the use of ICT and in particular of an integrated computer learning environment contribute to the realization of challenging, cross-disciplinary practical work of good quality, in which pre-university students can work with real data, apply mathematical methods and techniques in concrete problem situations, improve their mathematical and scientific knowledge and skills, and increase their mathematical and scientific literacy?

- What integrated tools should the computer learning environment provide for inquiry-oriented mathematics and science education? What are the requirements for the computer learning environment from a mathematical point of view and do they link up with requirements coming from science fields?

In other words, the two main results at which I aimed in my study were: (1) better understanding of how, why, and to what extent ICT tools can support students in their learning and practice of scientific inquiry; and (2) more insight in what it takes to develop an integrated computer learning environment for learning mathematics and science in the context of inquiry-oriented approach, the usability of which is explored within educational practice. In both explored perspectives, I used a specific computer learning environment, namely, COACH (Heck, Kedzierska, & Ellermeijer, 2009), to learn lessons from developing ICT tools that are integrated in an open, activity-based, multimedia authoring environment for mathematics and science education, and to learn lessons from exploring their usability in specific practical investigations for upper secondary students, in sample activities, and in usability studies.

The analysis framework that I used for answering the above driving questions was outlined in Section 4.1. It depended much on the kind of scientific inquiry by students that I had in mind and my view on how students can best learn to do inquiry and develop research abilities. In my research study, I focused on ICT-supported quantitative mathematical modeling activities, that is, on activities in which students explore mathematical models with the support of ICT tools in order to come to grips with natural phenomena and to interpret real data, which they often collect themselves.

In most classroom case studies, I opted for a structured and guided inquiry approach, interpreted here as instruction in which students are first directed in their work—mostly in order to brush up required knowledge and skills, to make them familiar with ICT tools needed in the investigation, or to direct the pathway of inquiry—and in which students at the later stage of the investigation can choose an optional activity or carry out a more open task in which they have to make own decisions on how to proceed. From a research point of view, because of the explorative nature of my study, I explored how the designed instructional materials and the ICT tools functioned in school practice, whether learning goals were achieved, what obstacles students encountered in their work, what recommendations for improvement of the instructional materials and the ICT tools could be made, and so forth. This contributes to answering the driving questions, although it is acknowledged that these questions are subject to a degree of ‘interpretative flexibility’ surrounds all learning technologies (cf., Hennessy, 2006; John & Wheeler, 2008; Keengwe, Onchwari, & Wachira, 2008; Osborne & Hennessy, 2003; Ruthven, Hennessy, & Deane, 2008), that is, each learning technology has different meanings and interpretations for various stakeholders.
4.4. Concluding Remarks

Question 1. How can the use of ICT and in particular of an integrated computer learning environment contribute to the realization of challenging, cross-disciplinary practical work of good quality, in which pre-university students can work with real data, apply mathematical methods and techniques in concrete problem situations, improve their mathematical and scientific knowledge and skills, and increase their mathematical and scientific literacy?

It is necessary first to explain a few elements of the question. For example, what is meant by challenging, cross-disciplinary practical work? My point of view has been that it concerns student investigations that have the following characteristics:

- Students work on a (preferably self-chosen) rather challenging, ill-defined or ill-structured, open-ended problem that is rooted in a real life situation instead of a more abstract or ideal situation.

- Students do not follow some standard recipes, but they examine their problem from different perspectives, using a variety of resources and high-order skills. Think, for example, of research abilities such as choosing a manageable problem, formulating a good research question, structuring work, and so forth.

- A broad range of competencies is required to make the project a success. Think of making good use of ICT for information gathering, data acquisition, data processing and analysis, problem-solving, and reporting.

- The students’ work is open-ended in the sense that there exist multiple methods or approaches to obtain many possible or even competing results. The student researchers actually decide if the investigation is finished for whatever reason.

- It offers students the opportunity to be in contact with contemporary, cross-disciplinary research and to learn about the nature of mathematics and science.

- Students disclose their own understanding through a portfolio or a polished product like a report, paper, or presentation.

A project is called cross-disciplinary when more than one discipline contributes in an essential way to the process of coming to an understanding of the problem situation. The investigations presented in this thesis are mainly rooted in applied mathematics, biology and physics. The term cross-disciplinary is used, and not a term like interdisciplinary, to emphasize that all disciplines are required to get satisfactory results: the whole is more than the sum of the parts.

The main role of ICT in investigative work can be summarized as the change of the computer into an instrument that allows students to collect real-time data of good quality, to construct and use computer models of dynamic systems in much the same way as professionals do, and to compare results from experiments, models, and theory. Furthermore, students can develop and practice through the activities their research abilities, and be in contact with experts in the field of study. The fact that they must apply their knowledge of mathematics and science in a meaningful way in a concrete context (hopefully) leads at the same time to deepening and consolidation of this knowledge. Through this kind of practical investigation students practice the following important research abilities:
• Formulate good research questions that guide the work.
• Design and implement an experiment for collection of relevant data.
• Apply mathematical knowledge and techniques, and science concepts in new situations.
• Construct, test, evaluate, and improve computer models, and have insight in their role in science.
• Interpret and theoretically underpin results.
• Collaborate with others in an investigation task and reflect on the work.

ICT plays an important role in enabling students to carry out investigations at a high level of quality. It also brings the real world into mathematics and science education in an attractive way. It makes it possible to involve students in similar activities to what scientists, engineers, and practitioners engage in and thus lead to authentic mathematics, science, and technology learning, in which various higher-order thinking skills like problem solving, critical thinking, creativity, and connecting contexts with fundamental concepts in mathematics and science are highly valued (See, for example, Chinn & Malhotra, 2002; Edelson, Gordon, & Pea, 1999; Edelson & Reiser, 2006; Roth et al., 2008).

The case studies presented in this thesis illustrated that ICT tools help bridge the gap between school mathematics and science on the one side and their real-world application on the other side. Especially, the motion analysis studies illustrated that, upper-level pre-university students, when supported by a suitable versatile computer environment, can work directly with high-quality, real-time data about human body motion in much the same way movement scientists and practitioners do. In such inquiry activities, students can practice mathematical knowledge and skills such as graph comprehension, numerical differentiation and integration, data processing and analysis, regression, and so forth. They can also develop the critical attitude that is necessary for successful modeling of natural phenomena. For this it is very important that the students can compare the results of computer models with real data, preferably collected in an earlier measurement activity. Confrontation of a model with reality turns modeling not only into a fun way of learning, but it also makes it exciting, challenging, and concrete work. I actually consider the student-driven experimental design, the modeling process, the underlying thinking processes, and the discussions with peers during the research as more important in the students’ work than the obtained results. But it is of course joyful when experiment, model, and theory are in good agreement. Students all seemed to be attracted by this kind of practical work.

Question 2. What integrated tools should the computer learning environment provide for inquiry-oriented mathematics and science education? What are the requirements for the computer learning environment from a mathematical point of view and do they link up with requirements coming from science fields?

The envisioned versatile, tool-based computer learning environment for inquiry-oriented mathematics and science education was based on the STOLE concept, which was an acronym for Scientific and Technical Open Learning Environment (Ellermeijer, 1988).
4.4. Concluding Remarks

I refer to Section 1.2 for an outline of this concept. The STOLE concept was realized in COACH. A one-sentence description of this learning and multimedia authoring environment is as follows: COACH is a single, activity-based, open computer working environment that is designed for the educational setting and that offers a versatile set of integrated tools for the study of natural phenomena, mathematics, and technology. It is meant to aid students in collecting, processing, and analyzing various types of data, to provide visualization and analysis tools, and to offer opportunities for creating computational models and animations so that experimental results can be compared with results from mathematical models. I refer to Section 3.1 for an overview of the supported activity types. Its suitability for practical investigations, research projects, and design studies carried out by secondary school students was illustrated by the many case studies and field experiments presented in Chapter 2 and 3. The complexity of the design of such a versatile tool-based learning environment and the many difficult decisions that the multidisciplinary development team had to make in order to cope with requirements from various fields and with various ability levels of students using the computer environment was discussed in the last four subsections of Section 4.3. One is constantly balancing between the wish to extend the environment with more advanced tools useful for inquiry activities (and pushed by advancements in technology) and the need to reduce tool complexity by adjusting tools and user interaction so that they are appropriate for mathematics and science education at various levels. In the end, the usability of included tools can only be validated in education practice and not determined from behind the designer’s desk.

As noted before, a big challenge lies in the differences in language and representational conventions between mathematics and science. This is very complex task and despite the progress in the understanding of the relevant issues and advances in computer and software technology, it is still an open question whether one can simultaneously meet so many requirements for a versatile computer leaning environment. An activity-based approach for the design of a computer learning environment for inquiry-oriented mathematics and science education seems a nice way-out because one can then at least use the language of the target discipline and use external representations that can be understood by the target group of students. This does not take away the burden of transfer of knowledge and skills across the disciplines, but it is already expected to help promote transfer when students can use the same computer environment for learning mathematics and science in an inquiry-oriented approach.

4.4.3 Reflection on the Presented and Future Work

This thesis is about my research and development work in the last decade that aimed at improving the contribution of information and communication technology (ICT) to inquiry-oriented mathematics and science education at secondary school level.

I focussed in my development work on the design and implementation of an integrated computer environment for learning mathematics and science in an inquiry-oriented approach. Much of this was realized in the environment named COACH (Heck, Kędzierska, & Ellermeijer, 2009). In section 1.4 I advocated a case-based design of educational software. This case-based design approach may give the wrong impression of a somewhat unstructured development process that is not guided by ideas about mathematics and science education and the role of ICT herein, but only progresses by the
method of trial and improvement in a rather unpredictable direction. In Section 4.3 I tried to take away this alternative conception of case-based design of educational hard- and software. Essential are an embedding of the work in educational practice and a consistent approach of research-based design, development, and implementation of a computer learning environment, sustained over a long period of time. See, for example, Roschelle et al. (2008a,b) and Ellermeijer (2004) for discussion of these facets in the context of the SimCalc project and the COACH project, respectively. These computer learning environments were developed in a continuous, ongoing process that probably only ends when they reach the ‘state of retirement’ and fade away (This happens to almost all software because of changes in technology).

A possible danger for case-based design is that technology changes more rapidly than one can really cope with in educational software development. For example, low-cost high speed video technology came available as a surprise and it was only because of educational benefits of automated point tracking in videos at normal speed that COACH was ready for this innovation. Thus, it is important to keep an eye for the research base of the educational design and development work. The STOLE concept helps in this respect because it emphasizes the needs of teachers and students in mathematics, science, and technology education. One of its grounding principles is that the computer learning environment should reflect innovation in mathematics, science, and technology, and not the appearance of technological gadgets in society.

In my study, research and development were intertwined. Focus was on students’ working with real data and on the design of supportive tools. The intention was to bridge the gap between mathematics and science education at school on the one side and modern research carried out by professionals on the other side, through provision of a suitable computer learning environment for inquiry-oriented mathematics and science education. The case studies presented in this thesis illustrate what can be achieved: The gap is in certain areas, for example movement science, not as big as expected and can be bridged. The validity of this statement was increased by the number of case studies done and, more importantly, by my collaboration in the classroom case studies with experienced mathematics and physics teachers from two schools. These teachers participated in the instructional design and taught the lessons in their classes. In other words, I benefited in my research from the teachers’ experience and expertise, and I could more or less separate teaching and researching. Naturally, this collaboration as well as the fact that the schools were well-equipped for computer-based inquiry work by students influenced the research outcomes. But I am of opinion that innovations in education are better tried out first under conditions that seem optimal, before testing them broadly under regular conditions, for example with less experienced teachers, or at schools with less good facilities. In exploratory case studies, this selection of special schools and teachers allows full exploration of the potential of innovations. Naturally, when an innovation seems promising, it must be evaluated more broadly under conditions regularly found in educational practice. Thus, a natural next step would be to involve other mathematics and science teachers and do the case studies at several schools. Professionalisation of teachers could be part of this future work. Research could be oriented toward implementation issues at school level of ICT-supported inquiry-oriented mathematics and science education.

A single ready-to-use framework for my research and development work was not available (or at least was not found). The approach in my study was therefore to select
elements from several frameworks for research and development in ICT-rich mathematics and science education that seemed promising for application and adaptation. Design research, case-based design of educational software, frameworks on using multiple representations, frameworks on evaluating inquiry activities of students, and models of modeling were the main sources of inspiration. I look back at this eclectic approach in my work as an inevitable and fruitful research and design methodology.

Looking at design research aspects of my study, I characterized it in Section 1.4 as type I design research in the typology of Richey and Nelson (1996; see also, Richey, Klein, & Nelson, 2004, p. 1102): “The product development process used in a particular situation is described and analyzed, and the final product is evaluated.” Within the distinction made by Van den Akker (1996, p. 6) between formative and reconstructive studies, my research work belonged to the first category and, like in many formative design research studies, the roles of designer and researcher coincided within the development context. In Van den Akker’s labeling (1999, p. 6), my work would be considered an explorative design study: I did not aim at statements of general nature, at developing theories, and at generating empirically grounded understanding of students’ inquiry processes. Instead I aimed at clarifying the design problem-in-context and at generating tentative design ideas, thus obtaining results that were context and product specific and that directed the development work. An explorative design study can easily be misinterpreted as less scientific work, but I am of opinion that this kind of breadth-first studies must precede or must be carried out alongside in-depth research studies aiming at the generation of theories.

From design research I took the perspective that it involves design of an intervention or experiment in the real word, that the output of my research must have practical value to real world users, and that teachers are involved in the research. I did three types of case studies:

1. Classroom research studies, in which students did practical investigations on the basis of specially designed instructional materials.
2. Field experiments, in which ICT innovations were tried out on a small scale and not necessarily in the classroom.
3. Usability studies, in which I evaluated the potential of a specific ICT tool or a set of integrated tools in a particular subject or domain, leading to a set of sample activities.

My exploratory case studies served many goals:

- They were meant to gain insight in the needs of secondary school students for doing authentic inquiry work.
- They helped me specify requirements for an integrated computer learning environment from a mathematical point of view.
- They served to test the usability and scope of (prototypical) implementations of particular tools for collecting, processing, and analyzing data.
- They gave an impression of the potential of ICT regarding the realization of challenging, cross-disciplinary practical work in which secondary school students
Chapter 4. Findings and Conclusions

were engaged in activities such as experimenting, data collection, and data analysis in much the same way as scientists and practitioners.

In other words, I did not focus in the exploratory case studies on a single or central research question, I did not include a control group for effect comparison, and I did not set up the research activities to have the characteristics of a (quasi-)experimental design in which evidence of instructional effects is collected. Nonetheless, the case studies were a source of considerable qualitative and quantitative data about ICT-supported student inquiry in practice and about the validity of the learning environment design and prototype functionality in diverse real-world contexts. The framework of quantitative mathematical modeling consisting of an experiment and modeling cycle helped me design instructional pathways and analyze the students' work in the case studies. I am of opinion that the goals served by the case studies were met.

The intended limitations of my research work automatically suggest possible directions for future research and development. One could design and do educational research on learning trajectories for introducing quantitative mathematical modeling at secondary level. My former AMSTEL colleague Onne van Buuren (2010, 2011) is carrying out such a research study in physics education: He is designing a learning route that starts with computer modeling activities for students in lower education and continues until the end of upper level education. Ormel (2010) and Westra (2008) explored a problem posing approach to introduce computer-based modeling in upper-level physics education and biology education, respectively. One could also go into another research direction and investigate in detail single steps in the quantitative mathematical modeling process. Schaap and colleagues (2011a,b), for example, started with a study of the translation from a problem situation to a mathematical formula, that is, a study of the process of mathematization. Thirdly, one could focus on the potential of ICT for improvement of learning: One could, for example, investigate how modern video based laboratory can help students at primary and secondary level develop graph comprehension. In sum, there are many directions for further research on quantitative mathematical modeling.

The same holds for development work. Tools could be improved and extended. For example, the current educational video analysis tools could be fine-tuned for motion analysis by inclusion of a feature to compare and synchronize two videos, so that one can more easily compare two video clips. More advanced computer vision techniques could be applied to correct perspective distortion, to handle movements and zooming of the camera in video clips, to carry out automated camera calibration, to offer three-dimensional video measurements, to provide to provide real-time automated point tracking, and so forth. Measurement in video clips is currently in most educational video analysis softwares restricted to position of points. However, one can envision extensions toward measurements of angle, circumference, and area, as well as counting of objects. Multidisciplinary development teams of a computer learning environment must continuously watch for opportunities originating from technological advancements.