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Publication date
2010

Document Version
Final published version

Published in
Acta Didactica Napocensia

Link to publication

Citation for published version (APA):
Heck, A., & Ellermeijer, T. (2010). Mathematics assistants: Meeting the needs of secondary school physics education. *Acta Didactica Napocensia*, *3*(2), 17-34. http://adn.teaching.ro/v3n2a3.htm

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ACTA DIDACTICA NAPOCENSIA
Volume 3, Number 2, 2010

MATHEMATICS ASSISTANTS: MEETING THE NEEDS OF SECONDARY SCHOOL PHYSICS EDUCATION

André Heck, Ton Ellermeijer

Abstract: Coach is an activity-based, open computer environment for learning and doing mathematics, science, and technology in an inquiry approach, developed in the last twenty-five years at the AMSTEL Institute of the University of Amsterdam. It offers a versatile set of integrated tools for data collection, data analysis, modelling and simulation, and for multimedia authoring of activities. In this paper, we present the STOLE concept that underpins the design and implementation of systems like Coach. It is an example of how members from the physics education research community came to convergence on tools for doing investigative work and achieved integration of tools. Special attention goes further to the mathematical requirements of such a learning environment and to the computer support of various representations of one and the same phenomenon or scientific concept. We also discuss one of the most complicating factors in the implementation of an integrated learning environment for mathematics and science, namely that mathematical concepts are not always used the same in these fields. Differences between the use of variables, functions, and graphs in mathematics and physics are briefly discussed, and consequences for the design of a general-purpose learning environment are addressed.

Key words: secondary physics education, information and communication technology, Coach, tool design

1. Introduction

The driving forces behind innovations in mathematics, science, and technology (MST) education take many forms. Some of the underpinning views are:

- Learning is a constructive and situated process that only takes place when the learner is actively engaged. In a constructivist view on teaching and learning (cf. Steffe & Gale, 1995), knowledge is in fact constructed in the mind of the learner and not transferred from teacher to student. So, the main task of the teacher is to create conditions under which an active attitude leads to learning.
- Learning is a social process. Competencies with regard to collaboration, discussion, sharing of ideas and results, reporting, and so on, are important in the students' preparation for a knowledge society. The task of the teacher is to organize sessions where the social processes take place and can be practiced.
- Learning becomes more relevant when it is done in a context. Authentic learning, interpreted here narrowly as working on real-world, complex problems, with the goal to come to grips with phenomena through scientific methods, is generally considered to motivate students and to lead to better understanding. Thus, the task of the teacher is to give students to opportunity to engage inside and outside school in authentic learning activities that match some of Reeves *et al*'s (2002) ten characteristics that constitute authentic tasks.
- Learning mathematics, science and technology includes not only acquiring theoretical knowledge, but also doing and learning about MST (cf. Hodson, 2008, 2009). This requires a contemporary outlook on what is going on in MST, not only the issues of decades or centuries ago. Since tools play a crucial role in current research and development, this should be reflected in education.

This paper is an expanded version of the presentation given to the second Computer Algebra and Dynamic Geometry Systems in Mathematics Education (CADGME) conference at the University of Linz, Austria, in July 2009.

Published: 1 June 2010.

In all of the listed views, information and communication technology (ICT) is expected to contribute to the improvement of teaching and learning. The development of ICT for MST education is mainly driven by a combination of educational research, curriculum development and technological development. We envision a scenario of teachers and students using a set of tools for inquiry-based study of natural and mathematical phenomena. This set of tools is integrated in one open environment designed for a broad educational setting. Openness means that it is

- a flexible, customisable, multi-purpose system;
- an environment for solving open problems that need definition, set-up, exploration, data processing and analysis, mathematical modelling, and so on, that is, primarily a cognitive tool;
- as much as possible free of didactic context or principles, that is, it is less considered as a pedagogical tool, but more as a tool for doing mathematics and science.

This computer learning environment does not only exist in the minds of software designers, but it has already been realized to a large extent in the Coach learning and authoring environment (Heck *et al*, 2009), which is the result of more than two decades of sustained research and development work at the AMSTEL Institute of the University of Amsterdam to improve MST education.

In this paper we address the design rationale of an integrated set of tools for an inquiry-based approach to MST education, illustrate it with the realisation with Coach, and we discuss challenges that designers of computer learning environments face. Regarding the design, we address the following research question: What are the requirements for an integrated computer learning environment for MST education and do requirements coming from various science fields link up with each other? The contribution of this paper to answering this question is design related and theoretical, not empirical. Results on how certain requirements have been realised in Coach and results on the benefits of the designed tools come mainly from recent explorative design research studies of the authors.

2. The STOLE concept

At the introduction of the computer in science education it was predominantly seen as an aid for practical work and investigations in school lab, so for data logging and computer modelling. Initially this meant the use of dedicated, stand-alone programs, but soon the demand for an integration of tools arose. At the beginning of the nineties, it became possible to develop educational software and hardware for more general use because of increased insights in the potential of educational software, progress in standardization in computer technology, the growing number of computers in schools, and the technical improvement of the personal computers in terms of speed, memory, and operating system. This caused a rethinking of how ICT could facilitate inquiry activities in science and a reconsideration of the design of computer environments for science education. Of great importance for the development of the Coach environment and other products was an initiative of stakeholders that led to the introduction of the STOLE concept, which is an acronym for Scientific and Technical Open Learning Environment. This concept was further developed and underpins the design and implementation of computer learning environments like Coach until today.

In short, it started with a vision of a hard- and software environment in which tools for measuring, data processing, and modelling are integrated in a single system that supports students' learning in an inquiry-based approach of science education. STOLE focused on essential elements of doing investigative work, which includes the following activities:

- setting up and controlling experiments;
- collecting data from experiments;
- displaying measurement data graphically and analysing data;
- retrieving information and making hypotheses;
- proposing, constructing, and testing computer models.

Other central ideas in the STOLE concept were that

- students were expected to be actively engaged in realistic investigations using the environment, while the teacher would facilitate the learning process;

- the environment should reflect innovation in science itself and be suitable for a wide range of science experiments, serving many science topics and many teaching and learning approaches, but it should be all by itself content-free (that is, the user should be able to add contents);
- the system would be an integrated collection of tools that could be frequently used during science lessons and practical investigations. This goal was also based on the idea that the learning curve of becoming accustomed to and proficient with the hard- and software environment should be as smooth as possible and follow the 'learn once/use frequently' philosophy.

Software tools selected for STOLE were grouped into functional modules that combined instruments needed by a student at a certain stage of a scientific investigation. The student was central in this concept and it should be possible to adapt the environment to the student's level and to the science curriculum in which the student and his/her teacher participate. The criteria for tool selection were:

- all the relevant tools (functions) for practical investigations must be included;
- tools must be selected for functional use in investigations;
- the environment must be transparent for the data in all the modules in the software package. Users should not have to make conversions because of the format of data;
- it must be possible to exchange data, models, and information between students within and outside the school. The environment must also work with data coming from other resources;
- if a module in STOLE cannot be constructed, then it must be possible to work with other software packages and exchange data;
- it must be possible for publishers to deliver templates, data, models, and information from science textbooks that can be processed by students.

IP-Coach 4, released in 1993, was one of the first implementation of the STOLE concept that was used at large scale in science education. The name of the software environment reflects its purpose: IP stands for 'interface program', which refers to interfacing of sensors to the computer, and Coach refers to coaching and support of learning. It consisted of a shell program from which it was possible to choose different modules. The functionality of these modules corresponded roughly with activities in the several stages of investigations. The modules were grouped into tool environments:

- the measuring environment (data logging and calibration of sensors);
- the data processing environment (processing and analysing of data, spreadsheet calculations);
- the modelling environment (construction of computer models and analysis of simulation results);
- the control environment (steering and control of actuators)
- the authoring environment (setting the hard-/software to the need of the teacher/student).

The introduction of the desktop metaphor for using a computer as a multimedia environment with a multitude of linked representations and the introduction of the mouse as the main interaction device drastically changed the way of working and learning with computers. Around the same time, a new vision on practical work in science education was arising that promoted practical work and research projects in which students are engaged in activities that resemble those of 'real' scientists (cf. Gott & Duggan, 1995; Wellington, 1998; Woolnough, 2000). The changed technological and pedagogical circumstances asked for a revision of STOLE. In one sentence of pragmatic nature, the main question was how the computer could contribute to learning science, doing science, and learning about science.

The original STOLE concept did not offer much for the design phase of student-directed practical work and research projects. In this phase a student-researcher will need information: (s)he must analyse the scientific problem, simulate a model, or look up information about work of other 'researchers.' Thus, the computer is more than only a tool to collect and analyse data; it must also give access to information resources and allow the display of information in various formats. The resources may be supplied on a CD-ROM or through Internet and they can be in various formats (sound, picture, video, digital image, hyperlinked text, and so on). The display of information and the inquiry nature of students' activities ask for multiple linked representations.

The application area of the computer environment was envisioned to become larger than science investigations: The field of technology seemed appropriate for students undertaking design projects in which control of models is realized through computer models and programmable microworlds. Thus, the role of data logging with sensors connected to the computer became less prominent than before. Focus was more on learning science and technology by practical work and by doing authentic investigations or design activities. It was envisioned that it should be possible to fine-tune the whole cycle of doing investigations and design work. This meant that a teacher should be able to design a sequence of activities for a particular investigation or design, and to organize these activities in a project to structure the lesson materials (experiments) for his/her students. This steering of the learning process by adapting or authoring student activities and bringing them together in a project is more important in lower secondary education than in upper secondary education. At lower secondary level, a teacher may not want to put the burden of selecting appropriate displays of information to his/her students and (s)he may want to provide information in an informal, qualitative, more visual and/or playful way. This is the reason that authoring of multimedia-based activities becomes important too. At higher secondary level, students are even expected to author activities themselves: from scratch or by selecting tools independently from prior choices made in an activity. In authoring mode one can:

- insert instructions, notes and text along with images, animations and video clips;
- add links to relevant Internet pages;
- predefine experiments and/or prepare a programming environment;
- set the movies to collect video data;
- create dynamical models and animations.

An author of an activity decides about the mode of the students' use. The modes may range from a restricted setting, in which students only have access to a few necessary, unchangeable controls and information displays, through a semi-open setting, in which a student can work freely with resources that have been preselected by the teacher, up to a fully open mode that allows students to freely use an open set of tools. These modes with different levels of openness and adaptability offer the opportunity of a suitable learning path for every student to become knowledgeable, skilled, and proficient in using the computer for scientific and technological purposes.

The renewed and extended STOLE concept of an integrated, tool-based environment suitable for an inquiry-based approach to science and technology learning, in which students and teachers work with multimedia-based activities and create such activities themselves by using a variety of resources, can be looked upon as a predecessor of the idea of working in a virtual learning environment. In the next section we look briefly at the realisation of this concept in Coach 6 (Heck *et al*, 2009).

3. Realisation of the STOLE concept in Coach

A one-sentence description of the Coach learning and multimedia authoring environment is as follows:

> Coach is a single, activity-based, open computer working environment that is designed for the educational setting and that offers a versatile set of integrated tools for the study of natural phenomena, mathematics, and technology.

A closer look at the elements of this description reveals that the environment is meant to

- aid students in collecting, processing, and analysing various types of data, to provide visualization and analysis tools, and to offer opportunities for creating computational models and animations;
- be universal and applicable in several curricula, in various instruction models, and at many levels of education, and be adjustable by teachers to their students' abilities;
- transform mathematics and science lessons into learning activities in which students are deeply engaged in building up and practicing knowledge, experience, and skills;
- change the computer into an instrument that allows students to explore real-world phenomena, helps them develop deep understanding of mathematical, scientific, and technological concepts, and supports communication of students' ideas and results;
- involve students in similar activities to what 'real' scientists and engineers engage in and thus lead to authentic mathematics, science, and technology learning, in which various higher-order thinking skills like problem solving, critical thinking, creativity, and connecting contexts with funda-

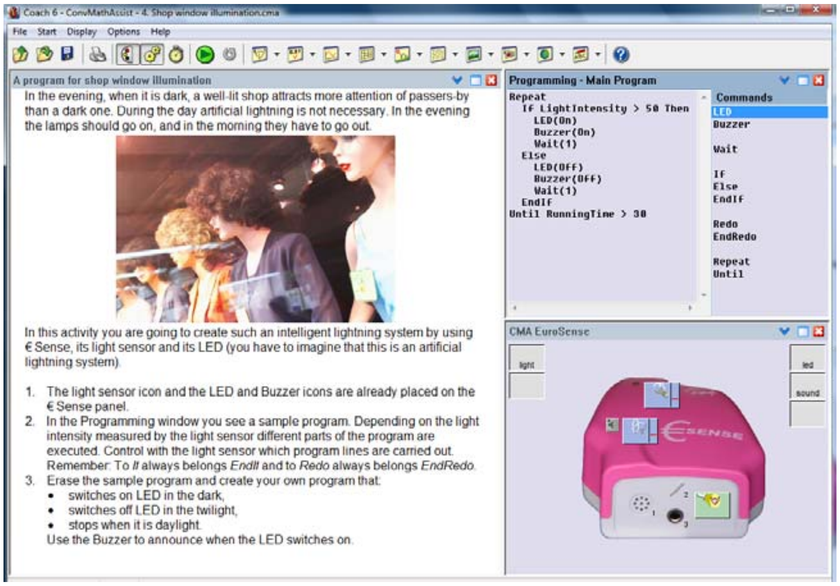
mental concepts in math and science are highly valued (cf. Chinn & Malhotra, 2002; Edelson & Reiser, 2006; Roth *et al*, 2008).

Coach activities are mostly based around the selected tool for collecting, generating, processing, or analysing data. Teachers can use ready-made activities or author new activities, and they can organise them in projects to structure the lesson materials (experiments) for their students. Activities typically contain components of various types:

- texts with explanations and/or instructions of activities;
- pictures to illustrate experiments, equipment, and/or context situations;
- video clips or digital images to illustrate phenomena or to use for measurements;
- representation of measured data and computed results as graphs, tables, meters, or digital values;
- models (textual, equations-based, or graphical) to describe and simulate phenomena;
- programs to control devices and to do mathematical computations;
- animations to dynamically represent and interact with models of phenomena;
- links to Internet sites and other external resources for students.

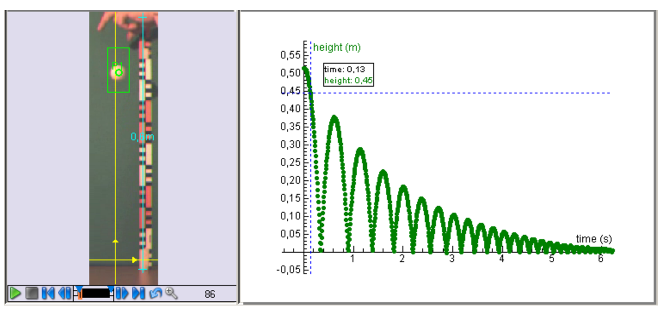
Three examples are given below, but at this point it is good to realize that all kinds of activities are supported in a single computer environment and not in a suite of separate programs. The main advantages of having a single environment instead of a bunch of special purpose software packages are of course that students and teachers only need to familiarise themselves with one environment, in which components are geared with each other, and that they can grow into their roles of skilled users of the system during their learning and teaching. A learn-once-use-often philosophy of educational tools is realisable and students may experience the connections between different school subjects through the use of a single environment instead of a grab bag of disconnected tools. Another advantage of a single environment compared to a software suite is the possibility to easily combine different tools in one activity. We refer to (Heck & van Dongen, 2008) for a description of an investigation of muscle activity during human gait through synchronous electromyographic measurement and video capture, and to (Heck & Bruidegom, 2007) for an example of simultaneous use of measurement with sensors, control of a device, and video capture in the context of the pupil light reflex of the human eye. The screen shot of a Coach activity in picture 1 shows a design activity at primary school level, which combines sensor-based measurement of light intensity with control of actuators (a lamp and a buzzer) through an interface with the computer (the €Sense interface).

Students can gain through measurement and control activities understanding of designing and doing experiments, and of the role of technology in daily life. Experiments are quite easily set up, with a variety of interfaces supported and a large library of calibrated sensors and actuators available. Experimenters can select an appropriate measurement method and a useful measurement setting. The control tool window offers several modes of programming with varying levels of difficulty. They are used to create and execute programs for automated measurements (for example, an automated pH titration system in which a titrator is controlled while measuring with a pH sensor), for manipulation of measurement data (for example, converting voltage signal from a sound sensor to decibels), for control of systems such as LEGO® models, and for programming any phenomenon (mathematical, scientific, natural, artificial, or whatsoever), such as the intelligent lightning system in picture 1. Control activities give students the opportunity to create physical artefacts, such as vehicles, line-followers and robots, and program them with interesting behaviour. In this way a design project becomes a fun project and at the same time students learn a lot about the difficulties that technicians and engineers have to overcome in similar work.



Picture 1. *Screen shot of a Coach activity in which a student needs to design and implement a lightning system.*

Picture 2, taken from (Heck *et al*, 2009), illustrates automated point tracking in the context of a bouncing table tennis ball, recorded with a high-speed camera at a frame rate of 150 fps. For each point of interest (here the centre of the ball), the user specifies at the start of the tracking process a template around this point that will henceforth be automatically matched in subsequent video frames. Matching takes place in a certain moving search area, the size of which is also user-definable. In picture 2, the search area around the currently measured point (P_1) is visible as a small rectangle.

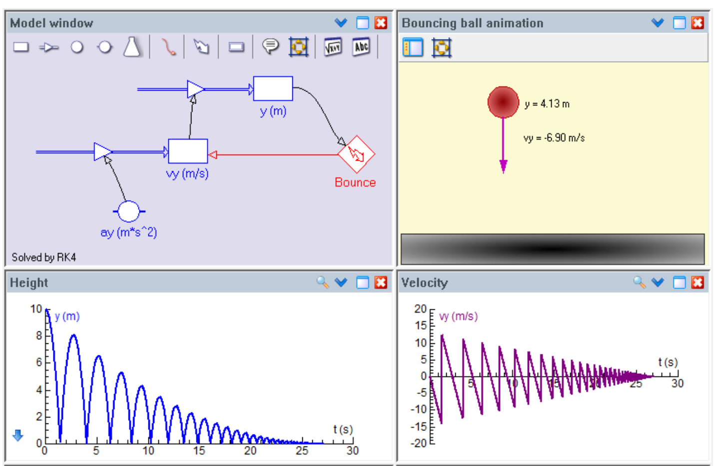


Picture 2. *Video measurement of a bouncing table tennis ball via point tracking.*

The video clip on which one measures and the corresponding mathematical representations such as graphs and tables are always synchronized in Coach. This means that pointing at a graph or a table entry automatically shows the corresponding video frame and that selecting a particular frame highlights the corresponding points in diagrams, when scanning mode is on. This makes scrubbing, that is, manually advancing or reversing a clip, an effective means to precisely identify and mark interesting events in the video clip and to relate them with graphical features. This supports students to transition between graphs and physical events and is probably the main reasons for the reported improvements in graph interpretation skills due to video analysis activities.

Another important feature of video analysis activities is that they mimic a technique often used in science, for example in movement science. This means that students can act in research projects on body motion like real movement scientists: they record video clips of a motion in which they are interested with a webcam or digital video camcorder, they measure on these movies and analyse the collected data. We refer to (Heck & Holleman, 2003; Heck & van Dongen, 2008; Heck & Ellermeijer, 2009; Heck *et al*, 2010) for illustrative examples, which show that data video and image measurement have great potential for studying everyday scenes of motion and for linking mathematics and science with the real world. The user must decide in video activities about many things, like how to make the best video recording, how to enable calibration, what and how many points to measure, which coordinate system to use, in which frames of the video to measure, and so on.

The modelling tool in Coach allows students to create and run numerical models, and to compare modelling results with experimental data. A text-based, equations-based, and graphical editor is provided. The first editing mode is programming in a language that is dedicated to mathematics, science, and technology education. The last two editing modes support a system dynamics approach that is the basis of a graphical aggregate-focused software such as STELLA (www.iseesystems.com) and Powersim (www.powersim.no). This type of modeling involves quantities (knowns as levels or stocks) that change in time by inflows and outflows. Physical flow and information flow determine the system's behaviour over time. Information flow is best understood as an indication of dependencies or influences between variables. The variables can be levels, flows, parameters, and auxiliary variables. These relations are made explicit as mathematical formulas and graphical or tabular relations. STELLA, Powersim and Coach provide a graphical representation in which users can express their ideas about the behaviour of a dynamical system; these ideas are then converted into more formal mathematical representations. The upper-left panel in picture 3 is an example of a graphical model of a bouncing ball in which only gravity is taken into account. It illustrates that the level/flow metaphor of a dynamical system should not be taken literally. On the one hand, the graphical model can be considered as a representation at a conceptual level of the system dynamics, where relation arrows indicate dependencies between quantities. On the other hand, the graphical model represents a computer model, which provides in many cases an iterative numerical solution of a system of differential equations. Computer modeling extends the set of realistic problems that can be solved by students without the need of sophisticated mathematics beyond their educational level. Examples are models of yoyos (Heck & Uylings, 2005), falling and bouncing objects (Heck & Uylings, 2010; Heck *et al*, 2009, 2010), alcohol metabolism (Heck, 2007), and running by athletes and students (Heck & Ellermeijer, 2009).



Picture 3. *A graphical model and animation of the bouncing ball.*

Picture 3 is a screen shot displaying a graphical model that implements the bouncing ball, graphs obtained by a simulation run, and an animation built from the model. The mathematical model can be improved in many ways (Heck *et al*, 2009) and this is an interesting practical investigation for students. We are of opinion that by looking at various models of the same phenomenon, a critical attitude of students is promoted and the importance of theoretical underpinning of a mathematical model comes to the fore. Furthermore, by linking modelling directly with experimental work, students get the important message that ideally there exists a synergy between theoretical and empirical scientific work. Just as empirical science cannot do without theory, theoretical sciences benefit from empirical work.

4. Multiple representations

Key feature of the Coach environment is that the tools for data collection, processing and analysis of data, modelling and simulation, and so on, are integrated in one system. This is far more than just a matter of technology. For us, the theoretical rationale of tool-integration is that the use of multiple external representations is crucial for deep understanding of real phenomena and that this process of understanding is promoted when learners are not distracted by technical burdens that could have been avoided by the provision of tools that work well together. This view can be underpinned by theoretical frameworks such as the Kaput-Goldin representational framework for mathematical cognition and learning (cf. Kaput, 1992, 1994; Goldin, 2008; Goldin & Kaput, 1996) and the 'Rule of Five' framework on multiple representations (cf. Dick & Edwards, 2008).

The representations used in a Coach activity are often not static entities, but dynamic elements that are linked so that a change in one representation affects the other representations. This is one reason to speak about integrated tools (There are more reasons, such as integration of different styles of tool use). The underlying idea of having multiple, linked tools available in an activity is that the building of conceptual knowledge gains from the use of multiple, linked representations of mathematical, scientific, and physical objects, and that the number of ways to come to a solution of a problem increases. Note that we assume here that the students have already learned to use conventional mathematical representations like graphs and tables, and that they are already familiar with built-in data processing and analysis tools. The environment primarily provides facilities to utilize these representations in a study of a phenomenon, to become proficient in using representations and tools, and to learn about the strength of using multiple representations in practice. This reflects that in scientific practice heavy use of multiple representations is made for describing a phenomenon. Mathematicians and scientists often use multiple representations to study problems and to investigate phenomena. They do this because

- different kinds of information can be conveyed with specific types of representations (for example, phenomena with simulations, animations, or video);
- interaction with multiple representations supports various ideas and processes in problem solving;
- use of multiple representations promotes deeper, abstract, and general understanding.

In practice, mathematicians and scientists select those representations that they feel comfortable with and that match best with their working style. In our opinion, this must also be reflected in mathematics and science education: the main idea is that multiple representations act to enrich the activities from which a student gains experience and understanding, and that they serve as a language with which the student organises and reorganises experiences about mathematical and scientific phenomena and concepts, hopefully giving the student the opportunity to use representations that match best with his/her learning style. Students learn that a single representation system does not suffice for problem solving and modelling in the most interesting cases, simply because it cannot cover all aspects of a mathematical/scientific phenomenon or concept. In other words, multiple representations can support learning by allowing for complementary information or complementary roles. For example, tables make information explicit, allow quick and accurate read off (of single values), and facilitate pattern recognition. In diagrams, a lot of information can be grouped together such that certain aspects of a phenomenon (for example, linearity/non-linearity, acceleration/deceleration/constant velocity, etc.) can be quickly recognized. A mathematical formula is a compact, but precise way of describing a quantitative relationship between variables.

In addition to the complementary roles of multiple representations, students learn that multiple representations can offer a source of referential accuracy by providing redundancy and that one representation can constrain interpretation of another. For example, in the animation example shown in picture 3, the animation of the bouncing ball (with the in reality invisible velocity vector) on the right-hand side can constrain the understanding of the velocity-time diagram and help the student understand that positive and negative velocities mean upward and downward directions of motion, respectively. We believe that students' understanding of instructional content can grow when combinations of representations are used, and that multiple representation can support the construction of deeper understanding when students relate those representations to identify strengths and weaknesses of particular representations and shared invariant features of all representations in use. Furthermore, we hope and expect that a learner who moves from one notation system to another, while comparing and (re) organizing experiences, finds regularities between the two systems that enable him or her to use one representation system to build a stronger concept in another representation system. In general, a person's understanding of a phenomenon, a problem, or a concept is refined the more representations (s)he can interact with. Being able to move flexibly across representations and perspectives when the task warrants it, knowing or identifying strengths/weaknesses and differences/similarities of various external representation systems, and thoughtful decision making about which representation to turn to next during a problem solving or modelling activity are personal abilities that must be learned, practised, and maintained in an inquiry-based teaching and learning approach to mathematics and science. Practical investigations and student research projects are in our opinion crucial in giving students first-hand experiences, provided that these students have already learned the basics of the external representation systems, and contribute to consolidation and solidification of the so-called representational fluency of students. Here we have adopted the comprehensive definition of Sandoval *et al* (2000, p. 6):

> "We view *representational fluency* as being able to interpret and construct various disciplinary representations, and to be able to move between representations appropriately. This includes knowing what particular representations are able to illustrate or explain, and to be able to use representations as justifications for other claims. This also includes an ability to link multiple representations in meaningful ways."

Links between several representations help the learner especially when all aspects of a complex idea cannot be adequately represented with a single system, which is often the case in learning and doing mathematics and science, and when the meanings of actions in one representation system can be illuminated by exhibiting their consequences in another representation. This is also why Coach provides so many linked representation systems, which are referred to as 'tool windows,' because they contain not only 'display notations' (graph, table of data, …), but also 'action notations' (formula, command line, action menu, computer programs, …).

How strong our motivation for using multiple representations in mathematics and science education may be, it does not mean that we close our eyes for difficulties associated with using multiple representations. The cognitive load is definitely enlarged when multiple representations come into play and it has been reported in many research studies (cf. Ainsworth, 2006, 2008) that learners find retrieving information from representations, moving between and within representations, and coming up with appropriate representations difficult. But we believe that teachers can guide and support their students in learning to read and use information from representations and to work effectively with multiple representations. We also concur with Kaput (1992, pp. 533-543) that computer technology, through the dynamic linking of representations and immediate feedback, can assist students in their learning process from concrete experiences to ever more abstract objects and relationships of more advanced mathematics and science, and can support visualisation and experimentation with aspects of investigated phenomena. Ainsworth (2008) summarised a number of heuristics that could be used to guide design of effective multi-representational systems [Between brackets we place labels of the connected principle(s) of multimedia learning listed by Mayer (2009)]:

- minimize the number of representations employed and avoid too similar representations (the coherence and redundancy principle);
- carefully assess the skills and experiences of the intended learners in order to decide on support of constraining representations to stop misinterpretation of unfamiliar representations, and to avoid unnecessary constraining representations (pre-training principle);
- select an ordering and sequencing of representations that maximizes their benefits by allowing learners to gain knowledge and confidence with fewer representations before introducing more (segmenting principle);
- consider extra support like help files, instructional movies, exercises, and placement of related representations close to one another on the computer screen, to help learners overcome the cognitive tasks associated with learning with multiple representations (guided activity principle, worked-out example principle, segmenting principle, modality principles, navigation principles, spatial and temporal contiguity principle);

- consider what pedagogical functions the multi-representational system is designed to support: avoid unnecessary time and effort spent by learners on linking between representation (for example, by not making representations co-present or by not automatically linking representations), make sure that learners understand the representations that constrains interpretation and do not overburden learners by making the task of mapping between representations too complex (coherence principle, split-attention principle, spatial and temporal contiguity principles).

The authoring facilities of the Coach environment that allow teachers to adjust activities to their students' level (for example, on the one hand to incorporate guidance or on the other hand to handle the prior knowledge effect), the consistent (semi-)automatic linking between representations within the environment, its default suggestion of not using more than four tool windows at the same time on the computer screen, the pedagogical organisation of activities in projects, and the user control to pace the presentation of the instructional materials (the pacing principle) reveal design choices that are in line with the above recommendations.

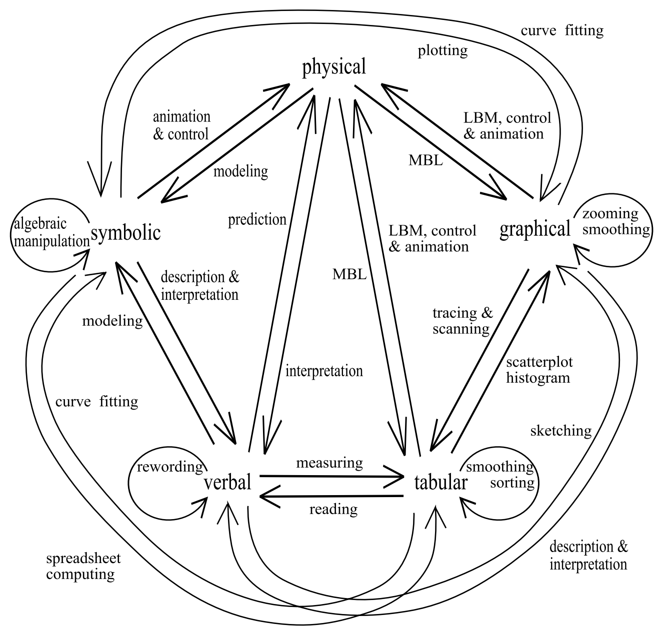
Like Kaput (1998), we are of opinion that, with the advent of the computer in mathematics and science education, the 'Rule of Four' for a context-based approach to the mathematical concept of function, which advocates that this topic is treated numerically, graphically, symbolically, and verbally, must be extended by a new representation system, namely, that of a materialised phenomenon. This phenomenon can either be cybernetic – as with screen objects whose movement is controlled by a model, computer program, or mathematical functions – or physical, as with sensors and with devices linked to a computer where their motion is controlled by mathematical functions defined on the computer or by control programs. We also categorise video clips or digital images as materialised phenomena. Kaput placed the cybernetic and physical phenomena in the heart of mathematics education of change and variation. Below we give a more neutral description of the 'Rule of Five.'

\ *to* \ *from* \	*cybernetic & physical phenomena*	*situations, verbal description*	*tables*	*graphs*	*formulae*
cybernetic & physical phenomena	mouse interaction	describing	data dollection & generation	MBL / CBL	modelling
situations, verbal description	programming, animation & control	rewording	measuring	sketching	modeling
tables	animation & control	reading	data transforming	Plotting	curve fitting
graphs	re-enacting, LBM	interpretation	reading off	rescaling & smoothing	curve fitting
formulae	re-enacting, LBM	parameter recognition	computing	sketching	algebraic manipulation

Picture 4. *A table of translation and transformation processes for external representations.*

Picture 4 shows a 5×5 table of transitions between and transformation processes within the representation systems (in fact, the table is a non-exhaustive list of processes). It is an extended version of the 4×4 table of (Dick & Edwards, 2008, Figure 10.4), which was in turn an augmented version of the original version of Janvier (1987, p. 28). Two terms used in the above table need explanation: LBM is an acronym of 'Lines Become Motion' that is the reverse of MBL (Micromputer-Based Laboratory) and refers to the possibility to define functions graphically or algebraically and then drive physical phenomena, including cars of tracks (Nemirowsky *et al*, 1998). In other words, LBM is all about generating phenomena as opposed to modelling phenomena. In the context of the Coach working environment this is called a control activity and the generation of a phenomenon is done through (microworld) programming. For virtual phenomena (animations), this can be controlled by graphical and tabular data as well. To understand why we use the term 're-enacting' in the translation from graphical representations to phenomena, one must first recall what the term 'enactive' means. Bruner (1966) distinguished the following three modes of mental representation: (1) 'sensori-motoric,' also called 'enactive;' (2) 'iconic;' and (3) 'symbolic.' The first mode is representation by action, for example in mathematics, by working with Dienes blocks or Cuisenaire rods. In other words, in enactive mode one learns from concrete objects and devices, and gradually moves to more genuine mathematical notations. In LBM the opposite road is taken: one moves from a mathematical representation to an observable motion of concrete objects. Since motion graphs often originate from motions that can be experienced enactively, we called the reverse process of generating motion from graphs 're-enacting'.

Keep in mind that the 'Rule of Five' only deals with the main external representations for the concept of function and not with all thinkable representations. In addition, different topics in mathematics and science may use different external representations such as pictorial representations, concrete or virtual manipulatives, aural representations, gestural representations, and others. In the context in which the Coach working environment is mostly used, this restriction to or strong focus on external representations related to the concept of function is in our opinion justified.



Picture 5. *The 'Rule of Five' and technology-based representational transformations.*

Picture 5 illustrates how technology, for example, a computer algebra system, a symbolic calculator, or a working environment like Coach, can aid to representational translations and transformations when investigating problems that involve the mathematics of change and variation (Note that picture 5 does not show an exhaustive list of processes). Many of the transformations are today carried out through technology outside human minds, autonomously, and in some cases without making visible, intermediate steps: a graphing calculator can plot a function, a sensor connected through a measurement panel with a computer may be used to collect data in an easy way, computer algebra systems can do formula manipulations. This does not mean that the processes are not under control of humans; only, the computations occur outside human minds. Kaput & Shaffer (2002) refer to this as the new representational infrastructure of 'computational media,' via which we have entered a new stage of cognitive development, after episodic (ape-like), mimetic (physical-action-based), mythic (spoken), and theoretical (written) stages, and leading to a virtual culture.

What picture 5 clearly illustrates is that multiple representations of the concept of function offer teachers and students the freedom to choose various starting points or continuation points in approaching a mathematical problem that they consider feasible. For example, functions are in the Coach environment often described by lists of numerical values. This has to do with its origin in science education, where working with measurable, physical quantities, which possibly bear a functional relationship, is an important issue. Values of quantities can be obtained in many ways: (1) through (real-time) measurements with sensor; (2) through measurements on a video clip or digital image; (3) by filling out a table manually; (4) by importing data from a file or copying from other software; (5) through a computer program; (6) read-off in a graph; and (7) via a mathematical formula. In our opinion, students must get acquainted with all of these technology-supported methods in order to get a more complete picture of variables and functions, and of their practical value in mathematics, science and technology.

5. The language of mathematics and science: how to cope with differences in tool design?

Developers of a versatile computer learning environment that offers integrated tools for mathematics, science and technlogy are faced with the following two difficult questions: How to deal with

1. the versatility of mathematical language and mathematical notation, and in particular, how to deal with the variability of the concept of variable in mathematics and science?
2. the differences in language between mathematics and science?

The interested reader is referred to (Heck, 2001; Ellermeijer & Heck, 2002) for a thorough discussion about these issues. Here we only briefly discuss some differences in use of variables, function, and graphs in mathematics and physics.

The meaning of variable is variable in mathematics

The first point to make is that in mathematics, the concept of variable is one of the most important algebraic ideas. But what makes it hard for secondary school students to understand this concept is that the term ‘variable’ has various meanings, depending on the situation in which it is used (cf. Schoenfeld & Arcavi, 1988). Even if letters are used for numbers only, different roles of letters in the algebraic context can be distinguished (Kücheman, 1981; Usiskin, 1988). It may be

- an *indeterminate*, in statements like $a^2 - 9 = (a-3)(a+3)$.
- an *unknown*, in equations such as $a + b = 7$.
- a *known number* like π.
- a *variable (generalized) number*, e.g., in $x \in \mathrm{N}$ and in differences like $f(a+1) - f(a)$.
- a *computable number* like A in the formula $A = 2\pi r$.
- a *placeholder*, e.g., in function definitions $f: x \mapsto 2x+1$ or $f(x) = 2x+1$.
- a *parameter*, e.g., as a label in the function definition $f_p(x) = p \cdot x$ to distinguish several cases.
- an *abbreviation* like $V = \{1,2,3\}$.

The fact that it is difficult, if not impossible to rigorously define the term ‘variable’ does not mean that one cannot classify the various appearances of variables in mathematics. Ursini and Trigueros (1997) distinguished three main uses of variables: variable as unknown, variable as general number, and variables in a functional relationship. This resembles the distinction of the three uses of variable made by Freudenthal (1983), which is also applicable to parameters:

- as a polyvalent name, i.e., a name for an object than can take a multitude of values. In the statement that for any real number x we have that $x^2 > 0$, the x does not have a particular value. Solving the equation $x^2 - 2 = 0$ means the x for which it is true. A priori x is indeterminate, a posteriori x can take two values;
- as a placeholder, which denotes the places in an expression where the same object is meant. In $f(x) = x^2$ the name x occurs twice, but the symbol is irrelevant: $f(\alpha) = \alpha^2$ is the same function;
- as a variable object, i.e., a symbol for an object with varying value. The object can be a physical quantity such as time, position, and temperature, or an economic quantity such as price, capital, and income. A variable object may be related with others. One speaks of independent variables, whose values one is free to choose, and of dependent variables, whose values one can compute given the values of the independent variables. The roles of independent and dependent variables are often not fixed during a computation. For example, studying the motion of a sprinter, one may

on the one hand consider acceleration as a function of time, but on the other hand describe it as a function of the velocity of the sprinter. One of the big ideas in calculus, and in mathematics in general, is the freedom of choosing independent and dependent variables.

Another obstacle for many a learner is that mathematical notation has many implicit aspects connected with the context in which mathematical expressions is used. For example, what does the symbolism $a(x+e)$ mean? Which of the following meanings would you choose?

- a generalized number $a \times (x+e)$. By the way, does *e* stand for the base of the natural logarithm?
- the function *a* applied to $x+e$ (or do you care that *a*, used as a function, is usually not in italics).
- a function in *x* with parameters *a* and *e.*
- a function in two or more indeterminates.
- the instruction *a* applied to the argument $x+e$.

Because of your training and experience, you probably answered that you could not make a choice without knowing the mathematical context or the wording used about the expression. However, for a secondary school student it takes time and practice to get used to the fact that a variable actually gets meaning in mathematics through its use (as indeterminate, as unknown, as parameter, etc.), through its domain of values, and through the context in which it is used. By the word ‘context’ in the last sentence we also mean the context of ‘doing school mathematics,’ which has its own conventions. For example, the word ‘formula’ has a special meaning in school mathematics and the role of the letters in the <u>formula</u> $y = x^2$ is not the same as in the <u>equation</u> $y - x^2 = 0$. The words ‘formula’ and ‘equation’ are used to distinguish between the case of a functional relationship between the isolated variable *y* that depends the other variable and the case of a more general relationship between unknowns. For students it is important to clearly distinguish between these different notions. A mathematician or scientist, however, is much used to ambiguous notations and to applying the same algebraic symbolism for many purposes: $y = x^2$ may stand for an equation, a function definition, an abbreviation of the expression x^2 , as well as for the process of computing the value of *y* from the value of *x*.

The meaning of variable is variable in physics

In physics, a variable is most often used as a name for a quantity that can vary (often with respect to time) and that in many cases can be measured. Picture 6 lists some of the essential differences in terminology and notational systems between physics and mathematics. We refer to (Ellermeijer & Heck, 2002) for more details.

Variables in computers

In general, a computer variable

- stores a numerical value or points to an object;
- may play different roles in a statement, for example in the assignment i := i + 1;
- may have a special, non-mathematical meaning, for example, a reference to previous results;
- obey unusual manipulation rules, e.g., ordering of commands or automatic simplification;
- is often a finite representation of a variable in the mathematical sense. The most complicated representation concerns the concept of variable object. In many a computer environment this is either a finite indexed list of values or an algorithm expressed in finite terms.

A user of mathematical and scientific software must be aware of these differences between computer variables and variables as they are used in mathematics and science. The differences are not obvious at all. In Heck (2001) recommendations were made to teachers interested in using computer algebra systems in their instructions.

Mathematics	*Physics*
Generalized arithmetic with dimensionless variables is used.	Quantity arithmetic with its own rules and use of units is dominant.
Names of variables are free to choose and changing names in an expression does not change its meaning	A variable is often related with some physics concept and its name is an abbreviation of this notion.
Irrational numbers like $\sqrt{2}$, π and e are important; floating-point numbers are without accuracy: $1.0 = 1.000$	In measurements, only natural numbers and floating-point numbers with accuracy occur: $1.0 \neq 1.000$.
There is a strong focus on special properties of functions, e.g., on asymptotic behaviour.	Properties and assumptions may rule out parts of mathematical interest.
Words like 'big,' 'small,' and 'negligible' have little meaning.	A small change of a quantity Q is also a quantity ΔQ with its own arithmetic.

Picture 6. *Some essential differences between the use of variables in mathematics and physics.*

Different contexts for graphing

Why do we ask pupils to make graphs? The answer to this question differs from discipline to discipline, but the reasons for using graphs are commonly divided into two classes: Analysis and communication (Friel *et al*, 2001). For example, a physics teacher may say that the graph is simply a means to an end: plotting graphs helps to interpret measured data. A diagram gives an overview of the measured data and from its shape one may get a clue about the possible relationship between the physical quantities in which one is interested. In order to better see or verify these relationships all kinds of scaling of graphs are at hand, such as (semi-)logarithmic and double-logarithmic plots. Derived quantities can be introduced to make the relationship clearer. For science, one could certainly say that graphing is not a context-independent skill. Rather, competencies with respect to graph interpretation are highly contextual and are a function of the scientists' familiarity with the phenomena to which a graph pertains and their understanding and familiarity with representation practices. The contrast with mathematics cannot be bigger: Here, drawing the graph of a function has not much to do with finding a relationship between quantities. In most cases, the function is already given by a formula or a table of function values, and has nothing to do with a real world context. Other different contexts for graphing in mathematics and physics are listed in picture 7.

	Mathematics	*Physics*
Graph	Represents a single object, viz., a function. Main purpose is to give a single view of various aspects of a given function.	Represents a relationship between two quantities. Main purpose is to explore or to present the relationship between quantities.
Axes	Dimensionless numbers are represented. Scaling is by default linear.	Values of quantities are expressed in a unit. Scaling is a matter of choice and may be non-linear.
Origin	(0,0) is the fixed position.	Arbitrary position.
Plot range	In principle infinite	Determined by the ranges of the quantities.
Slope/ Gradient	Dimensionless number having a geometric interpretation only.	Represents the change of a quantity w.r.t. another and is again a quantity with a unit.

Picture 7. *Different contexts for graphing in mathematics and physics.*

Graphs in mathematics and physics do not only differ in their construction or purpose; also reading of graphs is different and this is reflected in the language that is commonly used. Picture 8 lists some differences in language.

Mathematics	*Physics*
Tangent line, slope.	Gradient, steepness.
Origin (of coordinate system).	Zero (of a quantity).
Domain and range (of a function).	Range (of quantity values).
Graph of $y(x)$, *x*-*y* graph.	*v*-*t* diagram, *x*-*t* diagram.
Set of 2-tuples (*x*,*y*)	Plot of *y* vs. *x*

Picture 8. *Different language in mathematics and physics.*

Consequences for a versatile learning environment

All these differences do not only contribute to understanding of the problems of transfer between disciplines, but it also clarifies why the design and implementation of general purpose computer working environments is complex and challenging. Details count in the design of a suitable tool environment. It is still not clear whether a single environment that meets both the requirements of mathematics and science education can be completely realised. Anyway, the international trends are certainly in the direction of combining and linking various tool components in one environment. For example, the current version of the dynamic mathematics program GeoGebra (www.geogebra.org) combines the functionality of an algebraic tool, a geometry tool, and a spreadsheet. Efforts are made in adding proving capabilities to dynamic geometry system (Janičić & Quaresma. 2007; Abánades *et al*, 2007) and to building computer algebra systems on top of proof assistants (Kaliszyk & Wiedijk, 2007).

6. Conclusion

The concept of a Scientific and Technical Open Environment (STOLE concept) began about 20 years ago, was renewed and turns out to provide to this day a useful framework for the design and implementation of computer learning environments for inquiry-oriented mathematics, science and technology education. Some of the identified key features were discussed in this article in more detail: (i) a set of integrated tools for working with data, exemplified by its realisation in the activity-based, inquiry-oriented Coach learning environment; (ii) the theoretical framework of multiple dynamically linked digital representations, which encompasses benefits and drawbacks of representational multiplicity; and (iii) the complexity of the design of a versatile tool-based learning environment regarding the variable use of variables and other discipline-specific practices. Our main conclusions are:

(i) The required tools in a versatile, inquiry-oriented computer learning environment for mathematics, science and technology education can be directly linked with identified inquiry processes such as problem orientation and data acquisition (experiments with data logging or video measurement), data processing and analysis with the purpose of evaluating hypotheses and mathematical models (differentiation and integration, function fit, signal analysis,…), and setting up and evaluating computer models;

(ii) One of the key advantages of computer technology is the simultaneous use of multiple digital representations. We concur with (Ainsworth, 2006) that multiple representations can complement, constrain, and help to construct understanding of a particular phenomenon. We have suggested that the 'Rule of Five' and the 5×5 table of technology-based, representational transformations provide a fruitful framework for teachers and researchers to design and analyse students' activities that involve a multitude of mathematical representations. At the same time, a good advice is not to close one's eyes for potential drawbacks of representational multiplicity such as cognitive overload, insufficient learner's familiarity with representational components and processes, and undesired multimedia effects. We are of opinion that an activity-based computer learning environment that facilitates teachers to adapt or create (sequences of) activities in accordance to their students' level of experience and that allows application of basic principles of multimedia learning (Mayer, 2009) can help to overcome the drawbacks of representational multiplicity;

(iii) All kinds of demands for a versatile learning environment for mathematics, science and technology education hold. Amongst other things, we find it important that the computer environment reflects innovation in science itself, helps to bridge the gap between the real-world context and the more abstract mathematics and science, provides an integrated set of tools, is open to teacher and students, and offers multimedia authoring facilities for the creation of tailor-made activities and students' reports on practical investigation. But designers of computer learning environments are seriously confronted with differences between scientific practices. For example, the meaning of variable is variable in mathematics and science. This cannot be ignored in software development. Contexts for graphing may differ from one discipline to another. However, the biggest challenge lies in the differences in language and representational conventions between mathematics and science. In one way or another designers must cope with these issues when they try to develop a general purpose learning environment. This is very complex task and despite the progress in the understanding of the relevant issues and advances in computer and software technology, it is still an open question whether one can simultaneously meet so many requirements for a versatile computer leaning environment. However, successes of current systems like Coach in educational practice and improvements in tool-based software engineering give hope for the future.

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