On the Distribution of Discounted Loss Reserves

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Different factors can delay the payment process of claims. Therefore insurers need to keep reserves. This article gives a brief overview of the statistical models involved in loss reserving and explains how the theory of comonotonicity and convex order (see Kaas et al. (2000) and Dhaene et al. (2002 a,b)) can be used to determine accurate approximations for the distribution of the discounted loss reserve. 1

1. Introduction

Claims originating in a particular year often cannot be finalized in that year. For instance, long legal procedures are the rule with liability insurance or several compensations need to be paid, as in disability insurance. An important problem for the insurer is the determination of the provision for claims already incurred but not yet reported (hence IBNR) or not fully paid.

Let the random variable \( Y_{ij} \) with \( i, j = 1, ..., t \) denote the total claim figure for year of origin \( i \) and development year \( j \), meaning that the claims were paid in calendar year \( i + j - 1 \). Traditionally, these data are represented in a run-off triangle (see Table 1 for an example). The upper part of such a triangle (\((i, j)\) combinations with \( i + j \leq t + 1 \)) is used to construct estimates for future payments. The purpose of loss reserving techniques is to complete the run-off triangle to a square or even to a rectangle if estimates are required pertaining to development years for which there are no data at hand.

Loss reserving deals with the determination of the uncertain present value of an unknown amount of future payments. The actuary can make use of a broad range of techniques to determine the expected value of future, yet unpaid, losses. Section 2 provides the reader with a brief overview of statistical models that are often used for this purpose. Since the reserve is a provision for future payments, we believe the estimated loss reserve should reflect the time value of money and consider the discounted loss reserve in this text. Loss reserving is a prediction process. Therefore it is necessary to be able to estimate the variability of claims reserves and ideally to be able to estimate their full distribution. In general, it is very tough or even impossible to determine the cdf (cumulative distribution function) of the discounted reserve analytically, because the discounted reserve will be a sum of strongly dependent random variables. In section 3 an elegant way out is presented by calculating upper and lower bounds (in the sense of convex order) for the discounted loss reserve, with an easily computable cdf. The article ends with an example that illustrates the usefulness and accuracy of the presented techniques.

2. Statistical models for loss reserving

The first statistical approach to the IBNR problem goes back to Verbeek (1972). Taylor (1977) introduced the separation method and emphasized the special character of the “third dimension” in a run-off triangle, namely the calendar year direction. De Vylder and Goovaerts (1979) suggested the simultaneous treatment of the three directions in the triangle: year of origin (horizontal), development (vertical) and calendar year (diagonal). The rest of this section will focus on two classes of statistical models that are nowadays widely used for loss reserving: lognormal models on one hand and generalized linear models on the other hand.

Lognormal regression models

Let \( Y = (Y_{i1}, Y_{i2}, ..., Y_{it}, ..., Y_{nt}) \), the lognormal regression model can then be denoted as

\[
\tilde{Y} = \ln(Y_{ij}) = \alpha_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2 I) \tag{1}
\]

with \( X \) the design matrix of the regression model, \( \beta \) the \( p \times 1 \) vector of unknown regression parameters and \( \varepsilon \) the vector containing the error terms. Year of origin, year of development and calendar year act as possible explanatory variables for \( Y_{ij} \) (and thus for \( Z_{ij} \)). Various choices are possible for the regression parameters, for instance:

- the stochastic chain-ladder model as introduced by Kremer (1982):

\[
Z_{ij} = \ln(\gamma_{ij}) = \gamma_i + \beta_j + \varepsilon_{ij} \tag{2}
\]

(with \( \gamma_i \) the parameter for each year of origin \( i \) and \( \beta_j \) for each development year \( j \)).

- the probabilistic trend family (PTF) as suggested in Barnett & Zehnwirth (1998)

\[
Z_{ij} = \ln(Y_{ij}) = \gamma_i + \sum_{k=1}^{t} \beta_k + \sum_{k=1}^{t} \gamma_i + \varepsilon_{ij} \tag{3}
\]

where \( \gamma \) denotes the calendar year effect.

Maximum likelihood techniques provide estimates of the unknown regression parameters. The unknown variance
$\sigma^2$ is estimated by the residual sum of squares divided by the degrees of freedom (i.e., number of observations minus the number of parameters to be estimated). Residual plots should be checked to verify the fit of the model and hypothesis tests lead to a reduction of the number of regression parameters.

Following Doray (1996) the total IBNR reserve becomes

$$\text{IBNR reserve} = \sum_{i=2}^{t} \sum_{j=i+1}^{t} e^{(\mathbf{R} \hat{\beta})_y + \epsilon_y}$$

with $\mathbf{R}$ the design matrix for the complete square. As mentioned in the introduction, the discounted reserve will be considered here. Assume that the reserve will be invested such that it generates a stochastic return $Y_j$ in year $j$ ($i=1, \ldots, t-1$). This implies that an amount of 1 at time $i$ will become $e^{\delta_{1}}$ at time $j$. The discount factor for a payment of 1 at time $i$ is then given by $e^{\delta_{1}}$. The return vector $(Y_1, \ldots, Y_{t-1})$ is assumed to have a multivariate normal distribution, independent of $\epsilon$ from (1). More specifically, the random variable $Y(i) = Y_1 + \cdots + Y_i$ is assumed to satisfy

$$Y(i) = (\mu - \delta_{1})i + \delta B(i)$$

where $B(i)$ is the standard Brownian motion and $\mu$ is a constant force of interest. The discounted IBNR reserve now becomes

$$S = \sum_{i=2}^{t} \sum_{j=i+1}^{t} e^{(\mathbf{R} \hat{\beta})_y + \epsilon_y} \times \exp \left( - (\mu - \delta_{1} / 2) (i+j-t-1) - \delta B(i+j-t-1) \right)$$

Hereby it is well known that

$$\epsilon_y \sim \text{i.i.d} \ N(0, \sigma^2)$$

$$\mathbf{R} \hat{\beta}_y \sim \mathcal{N}((\mathbf{R} \hat{\beta})_y, \sigma^2 (\mathbf{R}(\mathbf{X}^{T})^{-1} \mathbf{R})_y)$$

$$B(i) \sim \mathcal{N}(0, i).$$

Lognormal models like those in (2) and (3) are widely used in actuarial practice. Though, at the same time normality, constant variance and additivity of systematic effects should hold. In an attempt to fulfill these conditions, the models described above use the logarithmic transformed data. This not only requires that the available data are strictly positive, predictions on the logarithmic scale should be transformed back correctly. This is a far from simple task and can lead to totally unusable results. The class of Generalized Linear Models (GLMs) generalizes the classical linear regression model in a natural way and gets around some important drawbacks of the lognormal model.

Generalized Linear Models

McCullagh & Nelder (1992) give a general introduction to generalized linear models. A first important feature of GLM's is that they extend the framework of normal linear regression models to the class of distributions from the exponential family. The random variables $Y_{ij}$ are assumed to be independent with a density function of the form

$$f_{\eta}(y_{ij}) = \exp \left( \frac{y_{ij} \theta_{ij} - \psi'(\theta_{ij}) + c(y_{ij}, \phi)}{\phi} \right)$$

whereby $\psi(.)$ and $\phi(.)$ are known functions, $\theta$ is the natural and $\phi$ the scale parameter. The following two well-known relations can easily be proved for distributions from the exponential family

$$\mu_{ij} = E[Y_{ij}] = \psi'(\theta_{ij}) \text{ and } \text{Var}[Y_{ij}] = \phi \psi''(\theta_{ij}) = \phi V(\mu_{ij})$$

where the primes denote derivatives with respect to $\theta$ and $V(.)$ is called the variance function. This function captures the relationship, if any exists, between the mean and variance of $Y_{ij}$. Possible distributions to work with in loss reserving include the normal, Poisson, gamma and inverse Gaussian distribution.

A second important feature of GLMs is that they try to get away from the idea of transforming the data. Instead of a transformed data vector, a transformation of the mean is modelled as a linear function of some predictors

$$g(\mu_{ij}) = \eta_{ij} = (\mathbf{R} \hat{\beta})_y, \quad i, j = 1, \ldots, t.$$

Hereby $\bar{\beta} = (\beta_1, \ldots, \beta_p)'$ contains the model parameters and $\mathbf{R}$ is the design matrix. $g$ is called the link function and $\eta_{ij}$ the $ij$th element of the so-called linear predictor. Various choices (see e.g. (2) and (3)) are possible for the linear predictor.

Maximum likelihood equations can be derived for the regression parameters and solved numerically. Tests for model development to determine whether some predictor variables may be dropped from the model can be conducted using partial deviances. Two measures for the goodness-of-fit of a given generalized linear model are the scaled deviance and Pearson’s chi-squared statistic. As in general linear models, residual plots (e.g. Pearson residuals versus explanatory variables) should be checked to verify the fit of the GLM.

An insurer is interested in the aggregated value

$$\sum_{i=1}^{t} \sum_{j=i+1}^{t} Y_{ij}.$$ In a GLM framework this IBNR reserve will be predicted by

$$\text{IBNR reserve} = \sum_{i=2}^{t} \sum_{j=i+1}^{t} \hat{\mu}_{ij}$$

with $\hat{\mu}_{ij} = g^{-1}(\mathbf{R} \hat{\beta})_y$ for a given link function $g$. In the same way as before, a discounting process is added to the reserve, thus we consider

$$S = \sum_{i=2}^{t} \sum_{j=i+1}^{t} \hat{\mu}_{ij} e^{Y(i+j-1)}$$

$$= \sum_{i=2}^{t} \sum_{j=i+1}^{t} \hat{\mu}_{ij} \exp(- (\mu - \delta_{1} / 2) (i+j-t-1) - \delta B(i+j-t-1))$$
whereby
\[
\left[ \hat{\mu} - \tilde{\mu} \right] \overset{d}{\rightarrow} N(0, \Sigma(\hat{\mu}))
\]

\[
B(1) \sim N(0, 1)
\]

Hereby \(d\) denotes convergence in distribution, \(\hat{\mu} = g^{-1}(\tilde{\tilde{y}})\) is the maximum likelihood estimator of \(\mu\) and \(\Sigma(\hat{\mu})\) is the asymptotic variance-covariance matrix of \(\hat{\mu}\).

Loss reserving is a predictive process; given the upper part of the run-off triangle, the analyst tries to predict the lower part. It is necessary to be able to estimate the variability of claims reserves and, ideally, to be able to estimate a full distribution of possible outcomes from which risk measures can be calculated. In any realistic model for the return process, \(S\) in (6) and (12) will be a sum of strongly dependent random variables. Therefore, one cannot rely on traditional risk theory and, in general, it becomes very hard or even impossible to compute the cdf of \(S\) analytically. Goovaerts & Redant (1999) derived an upper and lower bound for the IBNR reserve, in the sense of stop-loss ordering. The next section uses the theory of comonotonic risks to provide us with a lower bound. Finally, if we combine both ideas, we end up with an improved upper bound.

3. Upper and lower bounds for discounted IBNR reserve

In this section it is explained how the ideas of Kaas et al (2000) and Dhaene et al (2002 a,b) lead to upper and lower bounds for the discounted IBNR reserve \(S\) (see (6) and (12)). These bounds rely on convex order and comonotonicity, which will be introduced below. For a profound discussion the interested reader is referred to the above mentioned papers and to Hoedemakers et al (2003, 2004).

Convex order and comonotonicity

The notion "less favourable" or "more dangerous" will be defined by convex ordering, namely.

**Definition 1** (Convex order) Consider two random variables \(X\) and \(Y\). Then \(X\) precedes \(Y\) in the convex order sense, written \(X \leq_{cx} Y\), if and only if \(E[X] = E[Y]\) and \(E[(X - d)\cdot Y] \leq E[(Y - d)\cdot X]\) for all real \(d\).

\(X \leq_{cx} Y\) implies that extreme values are more likely for \(Y\) than for \(X\) and thus \(Y\) is "more dangerous" than \(X\).

**Definition 2** (Comonotonicity) A random vector \(\vec{X} = (X_1, ..., X_n)\) is said to be comonotone if any of the following conditions hold (with \(F_j\) the cdf of \(X_j\)):
1. For the \(n\)-variate cdf we have
\[
F_{\vec{X}}(\vec{x}) = \min \{F_1(x_1), F_2(x_2), ..., F_n(x_n)\}, \quad \forall \vec{x} \in \mathbb{R}^n;
\]
2. There exists a random variable \(Z\) and non-decreasing functions \(g_1, g_2, ..., g_n : \mathbb{R} \rightarrow \mathbb{R}\) such that \((\vec{x} \mapsto g(i)z)\) provides us with a lower bound. Finally, if we combine both ideas, we end up with an improved upper bound. These ideas are formalized in the following theorem from Kaas et al (2000) and Dhaene et al (2002a).

**Theorem 1** Consider an arbitrary sum of random variables, \(V = X_1 + X_2 + ... + X_n\)

and define the related stochastic quantities
\[
V' = E[X_1|Z] + E[X_2|Z] + ... + E[X_n|Z]
\]
\[
V'_u = F_{X_1|Z}(U) + F_{X_2|Z}(U) + ... + F_{X_n|Z}(U)
\]
\[
V_u = F_{X_1}(U) + F_{X_2}(U) + ... + F_{X_n}(U)
\]

with \(U\) a uniform random variable on (0,1), and with \(Z\) an arbitrary random variable, independent of \(U\). The following relations then hold
\[
V_i \leq_{cx} V \leq_{cx} V'_u \leq_{cx} V_u.
\]

Hereby the random variable \(Z\) represents some additional information available concerning the stochastic nature of \(X_1, ..., X_n\). In Hoedemakers et al (2003) the results in Theorem 1 are extended from ordinary sums of variables to sums of scalar products of independent random variables. This extension allows to derive a lower (say \(S_u\)) and upper (say \(S_u\)) bound for the discounted IBNR reserve \(S\) as in (6) and (12). The interested reader can find the explicit expressions for \(S_u\), \(S_u\) and \(S_u\) in Hoedemakers et al (2004) (for GLMs).

Using the results from Kaas et al (2000) and Dhaene et al (2002a), Hoedemakers et al (2003, 2004) explain how the cdf of \(S_u\), \(S_u\) and \(S_u\) can be derived explicitly. These three (easily computable) approximations for the cdf of \(S\) provide an analyst with very useful information. Suppose e.g. that an
The insurer wants to set the initial reserve as $F_{S}^{-1}(0.95)$. Instead of relying on (very time consuming!) Monte Carlo simulations from $S$, one computes $F_{S}^{-1}(0.95)$, $F_{S}^{-1}(0.05)$, $F_{S}^{-1}(0.95)$ and $F_{S}^{-1}(0.05)$. As such a band of possible values for the quantile of interest is obtained. This provides more reliable information than just a point estimate obtained from a number of simulations.

4. Numerical illustration

This section illustrates the accuracy of the proposed bounds ($S$, $S'$) for the discounted IBNR reserve. We compare the cdf of the bounds with the empirical cdf of $S$, obtained via Monte Carlo simulation. Consider the data in Table 1 which are modelled by a quasi gamma GLM with logarithmic link and predictor structure as in (2). Estimates of the regression parameters, together with their standard errors, can be obtained with a standard statistical software package like SAS or S-Plus. The stochastic returns $Y_j$ are assumed to be i.i.d. (independent and identically distributed) and $N(\mu, \delta^2)$ distributed with $\mu = 0.08$ and $\delta = 0.11$.

Figure 1 and Table 2 show that the convex bounds approximate the real discounted reserve very well.

5. Conclusion

In this contribution statistical models for loss reserving are discussed. Focus lies on the cdf of the discounted reserve.

The determination of this cdf is very urgent (because it provides much more information than just a point estimate) but cannot be done analytically. Convex bounds for the discounted reserve are presented which have a simpler structure and an easily computable cdf.

Table 1: Run-off triangle with non-cumulative claim figures

<table>
<thead>
<tr>
<th>year</th>
<th>$F_{S}$</th>
<th>$F_{S}'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>102,356</td>
<td>85,934</td>
</tr>
<tr>
<td>3</td>
<td>462,847</td>
<td>387,251</td>
</tr>
<tr>
<td>4</td>
<td>619,090</td>
<td>503,187</td>
</tr>
<tr>
<td>5</td>
<td>1,042,181</td>
<td>842,092</td>
</tr>
<tr>
<td>6</td>
<td>1,432,744</td>
<td>1,142,369</td>
</tr>
<tr>
<td>7</td>
<td>2,286,615</td>
<td>1,815,836</td>
</tr>
<tr>
<td>8</td>
<td>3,590,200</td>
<td>2,864,235</td>
</tr>
<tr>
<td>9</td>
<td>4,197,088</td>
<td>3,312,169</td>
</tr>
<tr>
<td>10</td>
<td>4,197,710</td>
<td>3,264,577</td>
</tr>
</tbody>
</table>

| total | 17,888,702 | 14,217,631 |

Table 2: 95th percentiles, means and standard deviations of the distributions of $S$ and $S'$ vs. $S$.

The convex bounds approximate the real discounted reserve very well.

Figure 1: The cdf's of the lower bound $S$ (dotted line), the upper bound $S'$ (dashed line) vs. the distribution of the discounted IBNR reserve $S$ approximated by extensive simulation (solid line) for the run-off triangle in Table 1.
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Notes

References