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ON A GENERIC CLASS OF LÉVY-DRIVEN VACATION MODELS

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This article analyzes a generic class of queuing systems with server vacation. The special feature of the models considered is that the duration of the vacations explicitly depends on the buffer content evolution during the previous active period (i.e., the time elapsed since the previous vacation). During both active periods and vacations, the buffer content evolves as a Lévy process. For two specific classes of models, the Laplace–Stieltjes transform of the buffer content distribution at switching epochs between successive vacations and active periods, and in steady state, is derived.

1. INTRODUCTION

Motivated by problems arising in the areas of computer, communication, and production systems, over the past decades substantial attention has been paid to the analysis of *vacation models*. These are queuing models in which the server alternates between active and passive modes; passive periods (i.e., periods in which the server does not work) are referred to as "vacations". It is commonly assumed that the durations of the vacations constitute a sequence of independent and identically distributed random

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variables, where this sequence is also assumed to be independent of the past evolution of the queuing system. For extensive surveys on this type of vacation queues we refer to [3,4,10,11] and the book by Takagi [9]. For a related article and further references, see [5].

In [2] we have departed from the independence assumptions mentioned earlier and explicitly model *positive correlation* between subsequent active and passive periods. Notice that in many situations there is such a positive correlation; an example is provided by a queue in a two-queue polling system, in which the intervals in which the other queue are served are identified with server vacations. In the setup of [2], it is assumed that the buffer content during both the active periods and the passive periods evolves as a Lévy process (which is assumed to have no negative jumps during the active periods and to be a subordinator during the vacations). We refer to [2] for additional references on vacation models with some form of dependence. See also [7], in which another form of correlation between active periods and vacations is considered, with special attention for the interesting phenomenon of a possible explosion of an infinite number of active periods and vacation periods in a finite time interval.

In the present article we analyze a model similar to the one in [2], but we allow the nature of the dependence between the active and passive periods to be more general. To explain the model, we first consider the following elementary system. Focus on the start of an active period; let τ be the time until the buffer is empty and let T be an independent exponential clock. Then we sample the length of the next vacation as a random variable S_1 if T is smaller than τ and as S_2 otherwise. This model is highly flexible; it, for instance, covers a broad range of possible correlations (both positive and negative) between the active and passive periods, as opposed to the classical models in which these were assumed to be independent.

In this article we study two generalizations of the above model. In the first, which we will refer to as Model 1 and which is treated in Section 2, we have d exponential clocks (whose means are not necessarily identical). If the set of clocks that is still active at time τ is $A \subseteq \{1, \ldots, d\}$, then the duration of the next vacation is distributed as a random variable S_A. In Model 2, which is addressed in Section 3, there is just one clock, but this clock has a phase-type distribution. The distribution of the next vacation then depends on whether the clock is still active at τ.

For both models, we compute the Laplace–Stieltjes transform of the steady-state storage level. We do so by first analyzing the storage level at "switch epochs", i.e., epochs at which there is a transition from the passive to the active mode. Then we use an "averaging procedure" to translate the results for the embedded epochs into time-average results. We are also able to explicitly characterize the correlation between subsequent active and passive periods.

2. MODEL 1: VACATION DETERMINED BY MULTIPLE EXPONENTIAL CLOCKS

Consider the following model of a storage system that alternates between service periods (active periods) and vacations (passive periods). Its dynamics are described as follows.

- Suppose the storage level is $z \geq 0$ at the beginning of a service period. Then the storage level evolves as a Lévy process $X(\cdot)$ (which is assumed to have no negative jumps). It has a drift downwards, that is, $\varrho_X := -\mathbb{E}[X(1)]$ is positive (and finite). Observe that the process will actually attain level 0, say at time $\tau \equiv \tau(z)$:

$$\tau(z) := \inf\{t \geq 0 : z + X(t) = 0\}. \tag{1}$$

  Let us introduce some additional notation. The *Laplace exponent* of $X(\cdot)$ is given by $\varphi(\alpha) := \log \mathbb{E}[\exp(-\alpha X(1))]$ and, hence, $\mathbb{E}[\exp(-\alpha X(t))] = e^{\varphi(\alpha)t}$ and $\varrho_X = \varphi'(0)$. Additionally, $\psi(\cdot)$ denotes the inverse of $\varphi(\cdot)$. It is well known that for any Lévy process that has no negative jumps, $\tau(\cdot)$ is a Lévy process itself, with Laplace exponent $-\psi(\alpha)$; that is,

$$\mathbb{E}[e^{-\alpha\tau(z)}] = e^{-\psi(\alpha)z} \tag{2}$$

  (see, for instance, [8, Thm. 46.3]). Notice that, for $\alpha \geq 0$, $\psi(\alpha)$ is uniquely defined as the inverse of $\varphi(\cdot)$, as $\varphi(\cdot)$ increases on $[0, \infty)$.

- At the beginning of the service period, we initiate $d \in \mathbb{N}$ independent exponential clocks, say, $T_1, \ldots, T_d$; define $\mathscr{N}_d := \{1, \ldots, d\}$. Let $\mathscr{A}(t)$ be the set of clocks that are still active at time $t \geq 0$:

$$\mathscr{A}(t) := \{i \in \mathscr{N}_d : T_i > t\}. \tag{3}$$

  For any $A \subseteq \mathscr{N}_d$ we define a nonnegative random variable S_A with Laplace–Stieltjes transform (LST)

$$\beta_A(\alpha) := \mathbb{E}[e^{-\alpha S_A}], \tag{4}$$

  with mean $\mu_A = -\beta_A'(0) < \infty$. To ensure a nontrivial model, it is assumed that $\mu_{\mathscr{N}_d} > 0$, although it could be the case that $\mu_A = 0$ for *strict* subsets A of $\mathscr{N}_d$.

  If, at the end of the service period the set of active clocks is A (i.e., $\mathscr{A}(\tau(z)) = A$), then the next vacation is distributed as S_A. During this vacation, traffic is generated according to a nondecreasing Lévy process (i.e., a *subordinator*) $Y(\cdot)$ with Laplace exponent $-\eta(\alpha) := \log \mathbb{E}[\exp(-\alpha Y(1))]$. The corresponding (positive) drift is denoted by $\varrho_Y := \eta'(0) = \mathbb{E}[Y(1)] < \infty$.

  In the sequel, we refer to a consecutive active period and passive period as a "cycle". Let Z_n be the storage level at the end of the n-th cycle. It is straightforward to derive that after the first cycle the storage level has a LST (in the sequel, $\mathbb{E}_z$ and $\mathbb{P}_z$ denote means and probabilities, respectively, conditional on the process starting in z at time 0):

$$\mathbb{E}_z[e^{-\alpha Z_1}] = \sum_{A \subseteq \mathscr{N}_d} \mathbb{P}_z(\mathscr{A}(\tau) = A)\beta_A(\eta(\alpha)). \tag{5}$$

- Then a new service period starts (i.e., the Lévy process $X(\cdot)$ becomes active again), etc.

Remark 1: A special case of the above model results by taking $X(\cdot)$ a compound Poisson process minus a positive drift and $Y(\cdot)$ a (e.g., the same) compound Poisson process but without the drift. This is an $M/G/1$ queue with vacations; the length of a vacation depends on the length of the preceding busy period.

Now that we have defined the dynamics of the process, we subsequently (1) indicate how the probabilities $p_z(A) := \mathbb{P}_z(\mathscr{A}(\tau) = A)$, for $z \geq 0$, $A \subseteq \mathscr{N}_d$, can be computed; (2) determine the steady-state distribution of the storage level at the "switching epochs", (i.e., epochs that the process switches from passive periods to active periods); (3) study the correlation between consecutive active and passive periods; and (4) determine the steady-state distribution of the storage level in continuous time.

1. *Reflections on the computation of the* $p_z(A)$. We first observe that $\mathbb{P}_0(\tau = 0) = 1$ so that $p_0(\mathscr{N}_d) = 1$ and, consequently, $p_0(A) = 0$ for all strict subsets A of $\mathscr{N}_d$. Now, consider the case $z > 0$. Then, using $\mathbb{E}(\mathbb{E}(U \mid V)) = \mathbb{E}U$,

$$p_z(A) = \mathbb{E}_z\left[\prod_{i\in A} 1_{\{\tau < T_i\}} \prod_{j\notin A} 1_{\{\tau \geq T_j\}}\right] = \mathbb{E}_z\left[\prod_{i\in A} e^{-\lambda_i\tau} \prod_{j\notin A}(1 - e^{-\lambda_j\tau})\right], \tag{6}$$

where an empty product is one. For conciseness, we introduce for $A \subseteq \mathscr{N}_d$, the "aggregate rate" $\Lambda(A) := \sum_{i\in A}\lambda_i$, where an empty sum is zero. Observe that

$$\begin{aligned}\prod_{i\in A} e^{-\lambda_i\tau} \prod_{j\notin A}(1 - e^{-\lambda_j\tau}) &= \sum_{k=0}^{n-|A|}(-1)^k \sum_{\{B:\, B\supseteq A \text{ and } |B|=|A|+k\}} e^{-\Lambda(B)\tau}\\ &= \sum_{\{B:\, B\supseteq A\}} (-1)^{|B|-|A|} e^{-\Lambda(B)\tau}.\end{aligned} \tag{7}$$

Taking the expected values in (7), in conjunction with (2), now gives that

$$p_z(A) = \sum_{\{B:\, B\supseteq A\}} (-1)^{|B|-|A|} e^{-\psi(\Lambda(B))z}. \tag{8}$$

In particular,

$$p_z(\mathscr{N}_d) = e^{-\psi(\Lambda(\mathscr{N}_d))z}. \tag{9}$$

As an aside, we mention that it is easy to check that (8) and (9) remain valid for $z = 0$. In particular, for A being a strict subset of $\mathscr{N}_d$, and with $A^c := \mathscr{N}_d \setminus A$,

$$p_0(A) = \sum_{\{B:\, B\supseteq A\}} (-1)^{|B|-|A|} = \sum_{\{B':\, B'\subseteq A^c\}} (-1)^{|B'|} = 0, \tag{10}$$

as the number of subsets of A^c is $2^{|A^c|}$, where half of them are odd and the other half even. Additionally, evidently (9) is in agreement with $p_0(\mathcal{N}_d) = 1$.

2. *Steady-state distribution of the storage level at the switching epochs.* We now consider the steady-state distribution of the Markov process $(Z_n)_{n\in\mathbb{N}}$. Clearly, such a stationary distribution, to which we associate the random variable Z, exists (independent of the initial state Z_0): It is evident that the process is regenerative and the mean regeneration time is finite. Denote by $\zeta(\cdot)$ the LST of the steady-state distribution Z of $(Z_n)_{n\in\mathbb{N}}$. Then, from (5) and (8), for $\alpha \geq 0$,

$$\begin{aligned}\zeta(\alpha) &= \sum_{A\subseteq B\subseteq \mathcal{N}_d} (-1)^{|B|-|A|}\zeta(\psi(\Lambda(B)))\beta_A(\eta(\alpha)) \\ &= \beta_\varnothing(\eta(\alpha)) + \sum_{\substack{B\subseteq\mathcal{N}_d\\ B\neq\varnothing}}\left(\sum_{A\subseteq B}(-1)^{|A|}\beta_A(\eta(\alpha))\right)(-1)^{|B|}\zeta(\psi(\Lambda(B))). \qquad (11)\end{aligned}$$

Later we will use the identity that is obtained after inserting $\alpha = 0$; we then find

$$0 = \sum_{\substack{B\subseteq\mathcal{N}_d\\ B\neq\varnothing}}\left(\sum_{A\subseteq B}(-1)^{|A|}\right)(-1)^{|B|}\zeta(\psi(\Lambda(B))). \qquad (12)$$

We remark that the last equation is also obvious from the fact that, for any nonempty B, $\sum_{A\subseteq B}(-1)^{|A|} = 0$ (cf. (10)).

From (11) we observe that if we know, for any $B \subseteq \mathcal{N}_d$, the value of $\zeta(\psi(\Lambda(B)))$, then we have identified the LST $\zeta(\cdot)$ of Z. These constants can be found as follows. For each $B' \subseteq \mathcal{N}_d$, we substitute $\alpha := \psi(\Lambda(B'))$. In this way, one obtains a system of linear equations defining the missing constants. We note that it is possible that for two or more different sets B', the values of $\Lambda(B')$ are the same. In this case, the number of unknowns is reduced, but the number of equations is reduced accordingly.

Example: Consider the simplest case (i.e., $d = 1$). Then the above formulas yield that

$$\zeta(\alpha) = \beta_\varnothing(\eta(\alpha)) - [\beta_\varnothing(\eta(\alpha)) - \beta_{\{1\}}(\eta(\alpha))]\zeta(\psi(\lambda_1)) \qquad (13)$$

so that

$$\zeta(\psi(\lambda_1)) = \frac{\beta_\varnothing(\eta(\psi(\lambda_1)))}{\beta_\varnothing(\eta(\psi(\lambda_1))) + 1 - \beta_{\{1\}}(\eta(\psi(\lambda_1)))} \qquad (14)$$

and, thus,

$$\zeta(\alpha) = \frac{\beta_\varnothing(\eta(\psi(\lambda_1)))\beta_{\{1\}}(\eta(\alpha)) + (1-\beta_{\{1\}}(\eta(\psi(\lambda_1))))\beta_\varnothing(\eta(\alpha))}{\beta_\varnothing(\eta(\psi(\lambda_1))) + 1 - \beta_{\{1\}}(\eta(\psi(\lambda_1)))}. \qquad (15)$$

When $\mu_\varnothing = 0$, and thus $\beta_\varnothing(\cdot) = 1$, we have

$$\zeta(\alpha) = \frac{\beta_{\{1\}}(\eta(\alpha)) + (1-\beta_{\{1\}}(\eta(\psi(\lambda_1))))}{2-\beta_{\{1\}}(\eta(\psi(\lambda_1)))}. \qquad (16)$$

3. *Correlation between consecutive active and passive periods.* We note that, due to the standard identity $\mathbb{C}\mathrm{ov}(U,V) = \mathbb{C}\mathrm{ov}(U, \mathbb{E}[V|U])$, for any given initial level $z \geq 0$, the covariance between the durations of the active and passive periods is

$$\mathbb{C}\mathrm{ov}_z\left(\tau, \sum_{A \subseteq \mathcal{N}_d} 1_{\{\mathscr{A}(\tau)=A\}}\mu_A\right) = \sum_{A \subseteq \mathcal{N}_d} \mu_A \mathbb{C}\mathrm{ov}_z\big(\tau, 1_{\{\mathscr{A}(\tau)=A\}}\big). \qquad \mathbf{(17)}$$

This covariance can be computed in a rather straightforward manner. To this end, first observe that it follows immediately from (2) that

$$\mathbb{E}_z[\tau e^{-\lambda\tau}] = -\frac{d}{d\lambda}\mathbb{E}_z[e^{-\lambda\tau}] = z\psi'(\lambda)e^{-\psi(\lambda)z}, \qquad \mathbf{(18)}$$

and as $\mathbb{E}_z[\tau] = z/\varrho_X = z\psi'(0)$, we have that

$$\mathbb{C}\mathrm{ov}(\tau, e^{-\lambda\tau}) = -ze^{-\psi(\lambda)z}(\psi'(0) - \psi'(\lambda)), \qquad \mathbf{(19)}$$

which is *strictly* negative when $X(\cdot)$ is not a linear drift (as $\varphi(\cdot)$ is then strictly convex and, thus, $\psi(\cdot)$ is strictly concave); the covariance is zero when $X(\cdot)$ is a linear drift.

Relying on (7) and (19), we can now further determine the terms in the sum in the right-hand side of (17). It is readily checked that, again using $\mathbb{C}\mathrm{ov}(U,V) = \mathbb{C}\mathrm{ov}(U, \mathbb{E}[V|U])$,

$$\begin{aligned}\mathbb{C}\mathrm{ov}_z\big(\tau, 1_{\{\mathscr{A}(\tau)=A\}}\big) &= \sum_{\{B:\, B \supseteq A\}} (-1)^{|B|-|A|}\ \mathbb{C}\mathrm{ov}_z(\tau, e^{-\Lambda(B)\tau}) \\ &= -z \sum_{\{B:\, B \supseteq A\}} (-1)^{|B|-|A|} e^{-\psi(\Lambda(B))z}(\psi'(0) - \psi'(\Lambda(B))) \qquad \mathbf{(20)}\end{aligned}$$

and, thus, the covariance (17) between the active period and the next passive period is given by

$$\begin{aligned}& -z \sum_{A \subseteq B \subseteq \mathcal{N}_d} \mu_A (-1)^{|B|-|A|} e^{-\psi(\Lambda(B))z}(\psi'(0) - \psi'(\Lambda(B))) \\ &= -z \sum_{B \subseteq \mathcal{N}_d} \left(\sum_{A \subseteq B} (-1)^{|A|}\mu_A\right)(-1)^{|B|} e^{-\psi(\Lambda(B))z}(\psi'(0) - \psi'(\Lambda(B))) \\ &= -z \sum_{\substack{B \subseteq \mathcal{N}_d \\ B \neq \varnothing}} \left(\sum_{A \subseteq B} (-1)^{|A|}\mu_A\right)(-1)^{|B|} e^{-\psi(\Lambda(B))z}(\psi'(0) - \psi'(\Lambda(B))). \qquad \mathbf{(21)}\end{aligned}$$

This expression reveals that if, for each B, $\sum_{A \subset B}(-1)^{|A|}\mu_A$ is positive when $|B|$ is even (odd) and negative when $|B|$ is odd (even), then the covariance will be negative (positive, respectively).

For example, in the case $d = 1$, we have that for $B = \{1\}$, $\sum_{A \subset B} (-1)^{|A|} \mu_A = \mu_\varnothing - \mu_{\{1\}}$. In this particular case, the covariance is

$$-z(\mu_\varnothing - \mu_{\{1\}}) e^{-\psi(\lambda_1)z} (\psi'(0) - \psi'(\lambda_1)). \tag{22}$$

4. *Steady-state distribution of the storage level in continuous time.* In order to compute the stationary distribution for the continuous time process, we look at a single cycle that begins with the stationary distribution of the embedded Markov process $(Z_n)_{n \in \mathbb{N}}$. Then

$$\begin{aligned} \mathbb{E}\left[\int_0^\tau e^{-\alpha X(t)}\, dt\right] &= \int_0^\infty \mathbb{E}\left[\int_0^{\tau(z)} e^{-\alpha X(t)}\, dt\right] d\mathbb{P}(Z \le z) \\ &= \int_0^\infty \mathbb{E}\left[\int_0^{\tau(z)} e^{-\varphi(\alpha) t}\, dt\right] d\mathbb{P}(Z \le z) \\ &= \int_0^\infty \mathbb{E}\left[\frac{1 - e^{-\varphi(\alpha)\tau(z)}}{\varphi(\alpha)}\right] d\mathbb{P}(Z \le z) \\ &= \int_0^\infty \left(\frac{1 - e^{-\alpha z}}{\varphi(\alpha)}\right) d\mathbb{P}(Z \le z) \\ &= \frac{1 - \zeta(\alpha)}{\varphi(\alpha)}. \end{aligned} \tag{23}$$

In the last line, we have used (2) and $\psi(\varphi(\alpha)) = \alpha$.

Let V be a random variable with the same distribution as a typical passive period. This yields

$$\mathbb{E}\left[\int_0^V e^{-\alpha Y(t)}\, dt\right] = \mathbb{E}\left[\int_0^V e^{-\eta(\alpha) t}\, dt\right] = \frac{1 - \mathbb{E}[e^{-\eta(\alpha) V}]}{\eta(\alpha)} = \frac{1 - \mathbb{E}[e^{-\alpha Y(V)}]}{\eta(\alpha)}. \tag{24}$$

Since Z_1 is distributed like $Y(V)$, and using that the process $(Z_n)_{n \in \mathbb{N}}$ is stationary, it immediately follows that

$$\mathbb{E}\left[\int_0^V e^{-\alpha Y(t)}\, dt\right] = \frac{1 - \zeta(\alpha)}{\eta(\alpha)}. \tag{25}$$

Conclude that the time-stationary distribution of the process (i.e., in continuous time) is given by

$$\frac{\mathbb{E}\left[\int_0^\tau e^{-\alpha X(t)}\, dt\right] + \mathbb{E}\left[\int_0^V e^{-\alpha Y(t)}\, dt\right]}{\mathbb{E}[\tau] + \mathbb{E}[V]} = \frac{(1 - \zeta(\alpha))(1/\varphi(\alpha) + 1/\eta(\alpha))}{-\zeta'(0)(1/\varphi'(0) + 1/\eta'(0))}. \tag{26}$$

Using (11) and (12), this expression can be further simplified, as follows. Denote $\beta_A^e(\alpha) := (1 - \beta_A(\alpha))/(\mu_A \alpha)$ when $\mu_A > 0$ and $\beta_A^e(\alpha) := 1$ when $\mu_A = 0$. Let

$$c_A := \mu_A \sum_{\{B : A \subseteq B \subseteq \mathcal{N}_d\}} (-1)^{|B| - |A|} \zeta(\psi(\Lambda(B))) = \mu_A p_Z(A). \tag{27}$$

We have from (11) and (12) that

$$\frac{1-\zeta(\alpha)}{-\zeta'(0)} = \frac{\eta(\alpha)\sum_{A\subseteq\mathcal{N}_d} c_A\beta_A^e(\eta(\alpha))}{\eta'(0)\sum_{A\subseteq\mathcal{N}_d} c_A}. \tag{28}$$

This implies that (26) simplifies to

$$\frac{\sum_{A\subseteq\mathcal{N}_d} c_A\beta_A^e(\eta(\alpha))}{\sum_{A\subseteq\mathcal{N}_d} c_A}\cdot\frac{1+\eta(\alpha)/\varphi(\alpha)}{1+\eta'(0)/\varphi'(0)}. \tag{29}$$

Remark 2: It can be easily checked that for $d=1$, the resulting LST is

$$\frac{\mu_{\{1\}}\beta_\varnothing(\eta(\psi(\lambda_1)))\beta_{\{1\}}^e(\eta(\alpha)) + \mu_\varnothing(1-\beta_{\{1\}}(\eta(\psi(\lambda_1))))\beta_\varnothing^e(\eta(\alpha))}{\mu_{\{1\}}\beta_\varnothing(\eta(\psi(\lambda_1))) + \mu_\varnothing(1-\beta_{\{1\}}(\eta(\psi(\lambda_1))))}$$
$$\times\frac{1+\eta(\alpha)/\varphi(\alpha)}{1+\eta'(0)/\varphi'(0)}. \tag{30}$$

If we assume in addition that $\mu_\varnothing = 0$ (i.e., there is no vacation when the active period is too long), then this LST further simplifies to

$$\beta_{\{1\}}^e(\eta(\alpha))\frac{1+\eta(\alpha)/\varphi(\alpha)}{1+\eta'(0)/\varphi'(0)}. \tag{31}$$

At first sight, it might seem that this is a mistake, as it does not depend on λ_1. However, some thought reveals that if $\tau \geq T_1$, then there is no vacation. This means that during the next cycle, the process starts from zero and, hence, the hitting time of zero is zero, and thus is majorized by the next exponential and, as a consequence, a vacation (with LST $\beta_{\{1\}}(\cdot)$) will begin. Therefore, the process is identical to one in which *no* comparison with exponentials is made and in which we have just active periods followed by independent vacations with LST $\beta_{\{1\}}(\cdot)$. A related model was considered in [6]. The same phenomenon will occur if $\mu_A = 0$ for all A that are strict subsets of $\mathcal{N}_d$ and $\mu_{\mathcal{N}_d} > 0$.

Remark 3: A special case of the model with $d=1$ is the following. Assume that the length of a cycle is $\max(\tau, T_\lambda)$, where $T_\lambda \sim \exp(\lambda)$ and is independent of all other stochastic quantities involved in the system. If $\tau < T_\lambda$, then the length of the vacation is $T_\lambda - \tau$, otherwise it is zero. This model is indeed equivalent to the above model with the choices $d=1, \lambda_1=\lambda, \mathbb{P}(S_\varnothing \leq t) = 1_{[0,\infty)}(t)$, and $\mathbb{P}(S_{\{1\}} \leq t) = (1-e^{-\lambda t})1_{[0,\infty)}(t)$ (i.e., with $\beta_\varnothing(\alpha)=1$ and $\beta_{\{1\}}(\alpha) = \beta_{\{1\}}^e(\alpha) = \lambda/(\lambda+\alpha)$).

Remark 4: When $\varphi(\alpha) = r\alpha - \eta(\alpha)$, (i.e., when during active periods there is a deterministic outflow at rate r and during vacations there is none), the LST of the last term in (29) is given by

$$\frac{1+\eta(\alpha)/\varphi(\alpha)}{1+\eta'(0)/\varphi'(0)} = \frac{r\alpha/\varphi(\alpha)}{r/\varphi'(0)} = \frac{\alpha\varphi'(0)}{\varphi(\alpha)}, \tag{32}$$

which is simply the LST of the steady-state storage level of a reflected Lévy process with exponent φ [1, Ch. IX]. This leads to the decomposition result discovered in [6] for a model in which the length of the vacation is independent of the previous active period but, on the other hand, is a general stopping time with respect to some filtration (not necessarily the one generated by $Y(\cdot)$).

Remark 5: Let $F_A(\cdot) := \mathbb{P}(S_A \leq \cdot)$. Whenever a vacation is zero, it is as if we are immediately starting another vacation with distribution

$$\frac{F_{\mathcal{N}_d}(t) - F_{\mathcal{N}_d}(0)}{1 - F_{\mathcal{N}_d}(0)}. \tag{33}$$

Therefore, without loss of generality we can assume that $F_A(0) = 0$ and, in particular, that $\mu_A > 0$ for all A. For example, if $F_A(0) > 0$, then when replacing $F_A(t)$ with

$$G_A(t) := F_A(t) - F_A(0) + F_A(0)\frac{F_{\mathcal{N}_d}(t) - F_{\mathcal{N}_d}(0)}{1 - F_{\mathcal{N}_d}(0)}, \tag{34}$$

the continuous-time process remains the same, but $G_A(0) = 0$ (it evidently has an impact on the embedded process $(Z_n)_{n\in\mathbb{N}}$, though).

3. MODEL 2: VACATION DETERMINED BY A PHASE-TYPE CLOCK

This model is similar to the one in Section 2, in that the storage level is determined by alternating active and passive periods. Additionally, as earlier the buffer level at the beginning of the active period has impact on the duration of the vacation. Suppose that at the beginning of the active period that the buffer level was z and let τ be the time it takes the system to empty. Now, consider a "clock" random variable T that has a *phase-type* distribution. If $T \leq \tau$, the next vacation period lasts for a random time with LST $\beta_1(\cdot)$, and if $T > \tau$, it lasts for a random time with LST $\beta_2(\cdot)$. Define $p_z(1) := \mathbb{P}_z(T \leq \tau)$ and $p_z(2) := \mathbb{P}_z(T > \tau)$. Similar to Section 2, we first (1) compute the probabilities $p_z(1)$ and $p_z(2)$, then (2) determine the storage-level distribution at "switch epochs" (i.e., as earlier, epochs at which the active period starts), then (3) analyze the correlation between subsequent active and passive periods, and, finally, (4) determine the steady-state distribution at an arbitrary point in time.

1. *Reflections on the computation of the $p_z(i)$, for $i = 1, 2$.* Any phase-type distribution is characterized by a d-dimensional ($d \in \mathbb{N}$) vector γ and a $d \times d$ matrix Q. Here, γ is the "initial distribution", and Q is such that there is a generator matrix $\bar{Q}$ that can be written as

$$\bar{Q} = \begin{pmatrix} Q & q \\ \vec{0}^{\mathrm{T}} & 0 \end{pmatrix}, \tag{35}$$

with q chosen such that the rows sum to zero (here, $\vec{0}$ is a d-dimensional vector with all its entries equal to 1, and the superscript T denotes transpose). The last row (containing just zeros) indicates that state $d + 1$ is absorbing; notice also that the vector q is necessarily componentwise nonnegative. From [1, Prop. III.4.1], we immediately have that

$$\mathbb{P}(T \geq t) = \gamma e^{Qt}\vec{1}.$$

Let $\lambda_1, \ldots, \lambda_d$ denote the eigenvalues of Q and let $\Lambda := \mathrm{diag}\{\lambda_1, \ldots, \lambda_d\}$; all eigenvalues have a negative real part [1, Sect. II.4d]. Assume for the moment that all eigenvalues of Q are simple, so that we have, for some matrix S,

$$\begin{aligned} p_z(2) = \int_0^\infty \mathbb{P}(T > t)\, d\mathbb{P}_z(\tau \leq t) &= \int_0^\infty \gamma e^{Qt}\vec{1}\, d\mathbb{P}_z(\tau \leq t) \\ &= \int_0^\infty \gamma S e^{\Lambda t} S^{-1}\vec{1}\, d\mathbb{P}_z(\tau \leq t) \\ &= \sum_{i=1}^d \delta_i e^{-\psi(\lambda_i)z}\,, \end{aligned} \tag{36}$$

for suitable coefficients $\delta_1, \ldots, \delta_d$. The last equality follows by combining the fact that $\gamma S e^{\Lambda t} S^{-1}\vec{1}$ is some weighted sum of terms $e^{-\lambda_i t}$ with (2) (as an aside, we remark that if λ_i and λ_j are complex conjugates, then so are $e^{-\psi(\lambda_i)z}$ and $e^{-\psi(\lambda_j)z}$). In many important cases, however, the eigenvalues of Q are *not* simple. If, for instance, T is Erlang(d, λ) distributed, it is easily verified that the eigenvalue λ has multiplicity d. In general, one could say that for any T having a phase-type distribution, there are constants δ_{ij} such that

$$\mathbb{P}(T > t) = \sum_{i=1}^k \sum_{j=1}^{m_i} \delta_{ij} t^{j-1} e^{-\lambda_i t}; \tag{37}$$

here, the λ_i are, as earlier, the eigenvalues of Q, $k \in \{1, \ldots, d\}$ is the number of distinct eigenvalues, and m_i is the multiplicity of the ith eigenvalue (so that $\sum_{i=1}^k m_i = d$). It is now readily verified that

$$p_z(2) = \mathbb{P}_z(T > \tau) = \sum_{i=1}^k \sum_{j=1}^{m_i} \delta_{ij} \mathbb{E}_z[\tau^{j-1} e^{-\lambda_i \tau}], \tag{38}$$

which can be rewritten as

$$\sum_{i=1}^k \sum_{j=1}^{m_i} \delta_{ij}\, (-1)^{j-1} \left(\frac{d^{j-1}}{d\lambda^{j-1}} e^{-\psi(\lambda)z} \bigg|_{\lambda=\lambda_i} \right). \tag{39}$$

It now follows that we can identify coefficients $\bar{\delta}_{ij}$ such that

$$p_z(2) = \sum_{i=1}^k \sum_{j=1}^{m_i} \bar{\delta}_{ij} z^{j-1} e^{-\psi(\lambda_i)z}. \tag{40}$$

Example: Consider the case that T has an Erlang$(2, \lambda)$ distribution; we have $k = 1$, $m_1 = 2$, and $\lambda_1 = \lambda$. It is straightforward to obtain that $p_z(2) = e^{-\psi(\lambda)z}(1 + \lambda\psi'(\lambda)z)$. In other words, in this case we have that $\bar{\delta}_{11} = 1$ and $\bar{\delta}_{12} = \lambda\psi'(\lambda)$.

2. *Steady-state distribution of the storage level at the switching epochs.* Let Z denote the steady-state storage level at switching epochs. We immediately find the following relation for the LST $\zeta(\cdot)$ of Z:

$$\zeta(\alpha) = \mathbb{E}[p_Z(1)]\beta_1(\eta(\alpha)) + \mathbb{E}[p_Z(2)]\beta_2(\eta(\alpha)). \tag{41}$$

Using the expressions for $p_z(i)$ derived earlier, we find

$$\begin{aligned}
\zeta(\alpha) &= \mathbb{E}\left[1 - \sum_{i=1}^{k}\sum_{j=1}^{m_i}\bar{\delta}_{ij}Z^{j-1}e^{-\psi(\lambda_i)Z}\right]\beta_1(\eta(\alpha)) \\
&\quad + \mathbb{E}\left[\sum_{i=1}^{k}\sum_{j=1}^{m_i}\bar{\delta}_{ij}Z^{j-1}e^{-\psi(\lambda_i)Z}\right]\beta_2(\eta(\alpha)) \\
&= \left(1 - \sum_{i=1}^{k}\sum_{j=1}^{m_i}\bar{\delta}_{ij}(-1)^{j-1}\zeta^{(j-1)}(\psi(\lambda_i))\right)\beta_1(\eta(\alpha)) \\
&\quad + \left(\sum_{i=1}^{k}\sum_{j=1}^{m_i}\bar{\delta}_{ij}(-1)^{j-1}\zeta^{(j-1)}(\psi(\lambda_i))\right)\beta_2(\eta(\alpha)).
\end{aligned} \tag{42}$$

The d constants $\zeta^{(j-1)}(\psi(\lambda_i))$ can be identified by inserting $\alpha = \psi(\lambda_i)$ (for $i = 1, \dots, k$) into the $(j-1)$th derivative of the above equation (for $j = 1$ to m_i).

3. *Correlation between consecutive active and passive periods.* The covariance between active and passive periods is, with $\mu_i := -\beta_i'(0)$, for an initial level $z \geq 0$, given by

$$\mu_1 \,\mathbb{C}\mathrm{ov}_z\big(\tau, 1_{\{T\leq\tau\}}\big) + \mu_2 \,\mathbb{C}\mathrm{ov}_z\big(\tau, 1_{\{T>\tau\}}\big). \tag{43}$$

Elementary algebra, and relying on the results for $p_z(i)$, yields that this equals

$$\begin{aligned}
&(\mu_1 - \mu_2)\sum_{i=1}^{k}\sum_{j=1}^{m_i}\delta_{ij}\,\mathbb{C}\mathrm{ov}_z(\tau, \tau^{j-1}e^{-\lambda_i\tau}) \\
&\quad = (\mu_1 - \mu_2)\sum_{i=1}^{k}\sum_{j=1}^{m_i}\delta_{ij}(\mathbb{E}_z[\tau^j e^{-\lambda_i\tau}] - \mathbb{E}_z[\tau]\mathbb{E}_z[\tau^{j-1}e^{-\lambda_i\tau}]).
\end{aligned} \tag{44}$$

The latter expression can be further evaluated by differentiating $\mathbb{E}_z[e^{-\lambda_i\tau}] = e^{-\psi(\lambda_i)z}$.

4. *Steady-state distribution of the storage level in continuous time.* This can be done by mimicking the arguments used for Model 1; in particular, (23)–(26) remain valid.

Remark 6: It is clear that we can combine the models of this and the previous section; then there are d clocks, each having a phase-type distribution, and the vacation has distribution S_A if the set of clocks still active at τ is A. It is easily seen that the probabilities $p_z(A)$ are mixtures of terms of the type $z^j e^{-\beta z}$, for $j \in \{0, 1, \ldots\}$ and $\beta > 0$, and therefore the analysis of the previous sections carries over to this more general model.

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