A developmental research on introducing the quantum mechanics formalism at university level
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Chapter 4

Problem Analysis


From February through May 2005, two courses on quantum mechanics have been closely observed to identify problems students may have when learning quantum mechanics. These observations were done in the first phase of a PhD project on the teaching and learning of introductory quantum mechanics. The observations show that students seem to have difficulties with the Copenhagen interpretation of quantum mechanics. They do not master necessary mathematical skills, and have difficulties with some classical concepts that are needed to learn quantum mechanics. Deterministic thinking is prevalent throughout the course. Possible research questions for the next phase of the research are considered.

4.1 Introduction

This research is the result of the first phase of a PhD research project on the learning of quantum mechanics by first year university students. For this research two courses on quantum mechanics will be closely followed, one for physics students, and the other for chemistry students. Both courses give an introduction on quantum mechanics. In the physics curriculum two follow-up courses on quantum mechanics are given, in the second and third year. Only the first two courses are compulsory. The chemistry students have one follow-up course, which is not compulsory.

In the Netherlands this is the first moment students come into contact with quantum mechanics. At secondary school, some subjects that relate to

\footnote{Minor adjustments have been made: correction of spelling, unclear formulations, and synchronization of the terminology with the terminology used in other chapters.}
quantum mechanics are discussed, but in a semi-classical way (Bohr’s model of the atom), and on a basic level. Some schools spend more time on “modern” physics. In the Netherlands the project called “Moderne Natuurkunde” (Modern Physics) is an example of such an initiative.

These introductory courses are thus a logical place to look at difficulties students might have when learning quantum mechanics, since it is their first serious encounter with the subject. The research aims of the PhD research are as follows:

- Introduce teachers to relevant literature on quantum mechanics education,
- Learn what students find difficult in quantum mechanics courses,
- Understand why they find this difficult, and
- Use this understanding to improve education in quantum mechanics.

Research has already been done on the learning of quantum mechanics. We expect that for some of the problems that are found, insights from research can be applied. Where applicable, we want to introduce these insights to the teachers, who may use it to improve their teaching.

This paper reports on a first analysis done in the primary stage of the research. In this primary phase, the two courses have been closely observed. In the next section the method that has been used is explained and motivated. Next the results are presented, categorized by theme. In the section “Discussion”, the results are related to other research, and some comments are given on the results and methodology. In the conclusion possible interesting points for the next stages of the research will be considered.

### 4.2 Methodology

The research that will be conducted, might be called developmental research. This means that the researcher plays an active role in teaching, as well as in developing the education. This is done in collaboration with the “regular” teachers of the courses. Hopefully this method will safeguard implementation, or adoption of the findings of the research.

In developmental research, two main stages can be defined: the diagnostic, and the developmental stage. In this first stage, the researcher participates in current practice, without pushing his views. The goal is to identify problems, or difficulties, and to develop hypotheses based on these observations. In the developmental stage, the researcher will develop education in collaboration with the teachers to test the hypotheses from the first stage. In the developmental stage the development, implementation, and evaluation can be

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2More information in Dutch can be found on the following website: [http://www.phys.uu.nl/~wwwpmn/](http://www.phys.uu.nl/~wwwpmn/)
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repeated several times. In this way, the hypotheses can be adjusted according to results from teaching. This paper reports on the first stage of this research.

4.2.1 Participatory observations

The researcher had different roles during lectures and tutorial sessions. During lectures, the researcher was merely observing, sitting amongst the students and taking notes. These notes consisted of all questions students asked, a concise description of the contents of the lecture and notable events during the lecture.

The tutorial sessions were led by a teacher. The researcher was sometimes only observing (e.g. during plenary discussions), but most of the time he took on the role of second teacher, answering questions from students and helping them to solve problems. These participatory observations enabled the researcher to experience the difficulties students have when solving problems and a teacher has when helping students. By joining the educational process, the researcher is able to directly ask students questions to test mini-hypotheses.

To closely monitor the process tape recordings were made during all lectures and tutorial sessions. In tutorial sessions, both the regular teacher and the researcher carried a sound recorder. Afterwards these recordings were used to make a description of the lecture, or tutorial session. Notable observations were transcribed.

4.2.2 Questionnaire

Before and after instruction students were given the questionnaire as reproduced in Appendix 4.A. The questionnaire before instruction was meant to obtain a global image of the background students had in quantum mechanics and what they expected of the course (whether they were motivated or not). The questionnaire after instruction was meant to see how students appreciated the course and how much time they had spent studying.

Besides the open ended questions on students background and interests, statements were given on quantum mechanics to determine their conceptions of quantum mechanics. These questions were taken from an article by Müller and Wiesner (2002) and were translated first to Dutch, for use in the research, and then to English as presented in the appendix. Before instruction only six questions on the atom were given (part B of the pre-instruction questionnaire in Appendix 4.A.1). After instruction the full questionnaire was given (parts B-E of the post-instruction questionnaire in Appendix 4.A.2). The reason this was done, is twofold. Firstly, it was expected that most students already knew something about the atom, but not about the other subjects in the questionnaire. Secondly, the size of the questionnaire was kept to a minimum, to get as high a response as possible.

Each of the statements on quantum mechanics had a weight factor of ±2, or ±1, to indicate to what extent it agreed with (positive factor), or disagreed with (negative factor) quantum mechanics. Each question could be answered with a number from 1 (strongly disagree) to 5 (strongly agree). From all these
answers an index $C$ was calculated, ranging from $C = -100$ (not quantum mechanical thinking), to $C = 100$ (quantum mechanical thinking).

The first six questions were given before and after instruction, making it possible to see if instruction had changed the way students think about the atom. In addition the full set of questions from Müller and Wiesner (2002), given after instruction, made it possible to determine how students thought of a wider range about quantum mechanical subjects.

The questionnaire was held on-line.\(^3\) Students were invited through email to fill the questionnaire.

## 4.3 Results

### 4.3.1 Observations

In this section several notable observations from both courses (Quantum Chemistry and Quantum Physics) are discussed, ordered by theme. When an observation is specific for one of the courses, this is mentioned with the abbreviations QC for Quantum Chemistry and QP for Quantum Physics. Some of the discussions are transcribed. The following abbreviations are used: T for teacher of either lecture, or tutorial session, O for observer (the author of this article) and $S_n$ for the $n$-th student in a discussion. Square brackets are used to add clarifying comments to transcribed discussions, or to indicate that part of the discussion has been left out. All transcriptions were translated from Dutch.

To have an idea in what context these observations were made and because the setting of the two courses was different, we will first discuss some general impressions.

**Quantum Physics** The first lecture was attended by approximately 60 students, the last by 23 students. On average 40 students attended the lectures. 46 students enlisted for the exam at the end of the course. During lectures there was little interaction between students and the teacher: most of the time the teacher was lecturing and students could ask questions. On average four questions were asked each lecture (ranging from 0-11). During lectures many of the students sitting at the back were busy doing other things: reading newspapers, doing exercises for other courses, reading in the textbook for this course and chatting with each other. Most of this was not noticed by the teacher, or it did not distract him from teaching. Students sitting at the front were attentive and asked most of the questions. Overall the students gave a calm impression.

Tutorial sessions were given in three separate groups, each attended by about 15 students. Observations were made mostly in one and the same group. These sessions mostly started with a short introduction by the teacher on problems to be solved in this session. After this introduction, students

\(^3\)Using PHP Surveyor, an open source survey program, see [http://www.limesurvey.org](http://www.limesurvey.org) for more information.
started solving problems and the teacher walked around the classroom to answer questions. Depending on the difficulty of the problems, the teacher would give a short explanation of the answers. Sometimes questions were discussed plenary.

Each week students had to hand in selected problems that were graded. The overall grade weighed one third in the exam result, provided that their exam grade was higher than 5 out of 10, and the grade for homework was higher than the exam result. Most of the students answered these problems, but apart from this activity students did not seem to spend much time on this subject.

The textbook used was Griffiths (2005). A workbook with exercises and problems was used in tutorial sessions. The workbook was written by the teachers of the tutorial sessions.

The course started with a historical motivation for quantum mechanics, and proceeded with the first two chapters of Griffiths (2005). These chapters discuss subjects like: the Schrödinger equation, (Born’s) statistical interpretation, normalization, the uncertainty principle and the time-independent Schrödinger equation which is solved for several potentials (for instance the harmonic oscillator).

Quantum Chemistry 15 students enrolled for the Quantum Chemistry course, four of them attended it for the second time. Attendance ranged from 6 to 13 students, with an average of 10. There was much interaction between the teacher and the students. The teacher started each lecture by asking the students about matters discussed in the previous lecture. He also stimulated them to ask questions if things from the previous lecture were still unclear. The lecturer used transparencies in his talk. When answering students’ questions the blackboard was often used.

To help students prepare for a lecture, the teacher provided questions which could be easily answered by browsing through the textbook. Very few students actually answered these questions. Most of them did not prepare for the lectures, or tutorial sessions at all.

Because of the small number of students, tutorial sessions were given to the group as a whole, by two teachers. At the start of a session, students directly started working on problems and exercises. The teachers walked around answering questions and helping students solving problems. Sometimes (parts) of problems were worked out in front of the blackboard by one of the teachers.

In four of the tutorial sessions students worked on a project about the molecule rhodopsin. They used a numerical software program called Spartan to determine the properties of this molecule. Students had to write a report on this work, which was graded.

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4 Rhodopsin is a protein complex that plays a role in the detection of light by humans and other vertebrates.

5 Spartan is developed by Wavefunction and available on [http://www.wavefun.com](http://www.wavefun.com)
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The textbook used was Atkins and de Paula (2002). In tutorial sessions students worked on selected exercises and problems from the textbook and extra problems written by the teachers. Students had access to a Blackboard site from which they could download the sheets used in lectures, problem sets, preparatory questions for lectures and a schedule for the course.

This course discusses chapters 11-14 of Atkins and de Paula (2002), which in short, contain the following subjects: principles of quantum mechanics, time-independent Schrödinger equation, solutions for several potentials (particle in a box, harmonic oscillator, rotation on a sphere), atomic and molecular structure.

Quantum versus classical mechanics

There are three roles classical mechanics might play when learning quantum mechanics:

- Certain concepts from classical mechanics might be necessary for learning quantum mechanics.
- Strange, or surprising results might attract attention by explicit comparison to what is classically expected.
- Concepts from classical mechanics might hinder the understanding of quantum mechanics.

Several observations related to these roles are discussed in this section.

The concept of energy in general and potential energy in particular proves to play an important role in quantum mechanics. The Schrödinger equation explicitly makes use of potential energy, defining the interaction between a particle and its environment. In most applications, solutions to the Schrödinger equation are sought for a certain potential: infinite square well, finite square well, step potential, etc. Knowing that students have difficulties with the concept of potential energy, the teacher of the QP course discussed its definition at the beginning of the course. The following discussion about (total) energy and potential energy took place in the 12th week, during a tutorial session.

S1 Can the energy be smaller than zero? That is not possible, right?
T If the potential is smaller than zero, then it is possible. In chapter five [of the workbook] we have seen that the energy should always be greater than the minimum of the potential.
S1 But the potential itself cannot be smaller than zero.
T It is possible, because you can always add a constant [to the potential].
S2 But then you can have negative energy.
T That is possible, why not?

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6Blackboard is an online learning environment. See http://www.blackboard.com for more information.
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S1 But that is not possible. If you have a particle with a certain mass, then its energy is larger than zero.

T [...] You can always add a constant to the energy, that will not physically alter the problem at hand. So when describing the hydrogen atom, an electron round a nucleus, the potential at infinity is set equal to zero. Then the energy on the nucleus itself is minus infinity. Because the particle [electron] wants to be near the nucleus, the energy decreases when nearing the nucleus. Because the energy is infinite there, you cannot set it equal to zero. That will not work, so you will have to choose something different. Therefore we choose to set the energy equal to zero at infinity. So then the energy is negative. So you can always do this.

S2 Yeah well, the particle does not really have negative energy. That is a somewhat vague notion.

T Look, the kinetic energy is always positive. This is always a positive quantity. You mention the mass, this also represents a certain energy, this is also positive. But you can always choose the potential as you like. When I have an apple resting on a table, then I can say it has energy zero. If it then rolls off the table, it has a lower energy. But you might as well say that the potential energy on the ground is zero. You can choose this as you like. It will not change Newton’s laws.

The two students in this discussion have fundamental difficulties with concepts like (total) energy and potential energy. It seems that they use an implicit definition for energy, and perhaps for potential energy, that disables them from understanding the explanation of the teacher. It is unlikely that they are convinced that energy can have a negative value, after having this discussion. This discussion emphasizes that students have no full understanding of concepts like energy and potential energy, as used in classical mechanics. Because these concepts are so important for quantum mechanics, we may expect that these students also have difficulties understanding quantum mechanics.

In the tunneling phenomenon we see another kind of relation between classical and quantum mechanics. Tunneling is often explained by considering a potential barrier of height $V = V_0$, located between $x = 0$ and $x = a$ and a particle with energy $E < V_0$. Left of the potential barrier, this particle is described by the wave function $\psi(x) = Ae^{ikx} + Be^{-ikx}$, where $k = \sqrt{2mE/\hbar^2}$ and $A$ ($B$) is the amplitude of the incoming (reflected) wave. The Schrödinger equation is solved for this potential and boundary conditions at $x = 0$ and $x = a$ are imposed. It then follows that the particle has a probability of being transmitted through the potential barrier.

To appreciate this result, students have to know what to expect classically. For this the teacher in the QC course compares the quantum mechanical result with the throwing of a ball against a wall. Classically the ball will bounce back, but quantum mechanically, the “ball” has a chance to tunnel through the wall. A student then asks if it is possible for this particle to fly back through the hole once it is on the other side of the barrier. Using the word “hole” shows that...
this student does not understand what is happening: there is no hole. When asked about tunneling, students are able to mention the effect that there is a chance to find a particle on the other side of a barrier. However, they are unable to define the problem in terms of potential, and some seem to think there is a hole in the barrier through which the particle is able to travel. The meaning of tunneling can only be explained by referring to what is expected classically. The word “tunneling” even suggests this.

Classical conceptions can also be a hindrance. After four weeks a student in the QP course asks what $\psi(x)$ and $\Psi(x, t)$ are. This student is able to recite Born’s statistical interpretation (see Section 4.3.1), but still has the feeling he does not understand what its meaning is. Later in this discussion it becomes clear that this student holds a deterministic view of the world. He thinks it should be possible to give a function that describes where the particle is at a certain time. When he is told this is inherently impossible, he is truly appalled. Part of this discussion follows here.

O $\psi$ is the wave function. Do you know... Is it possible to measure $\psi$?
S Yeah, well, actually that is what I do not know.
O But what do you think? What have you heard the last couple of weeks?
S I know you can integrate this function and determine the probability.
O OK, but how does this work in detail, the relation between $\psi$ and the probability?
S The probability between $a$ and $b$ is the integral of the modulus squared of $\psi$ between $a$ and $b$.
O Yes, perfect. So this, $\psi$ in itself, this is not of any practical use. But for a specific system, for instance a particle in a box, if you solve the differential equation and you find $\psi$, then with $\psi$ you can calculate the probability.
S Is this the only thing you can do with it?
O Unfortunately this is the only thing you can do with it.
S $\psi$ in itself does not have any physical meaning?
O No, because uh... not directly, so to speak. I say this because the square of the modulus will give you the probability, and this is something you can measure. You also see, this is what the teacher told in last week’s lecture, that $\psi$ has a phase, it is a complex number [...] and this gives rise to phenomena such as interference.

[... part of the discussion left out ...]
S But why can’t we just determine where this particle is? Doesn’t it follow a certain trajectory, which it follows around the nucleus?
O You should think of what you mean when you say “doesn’t follow a certain trajectory”. ... What do you mean by that?

$^7$O is not very precise here: the measurements approach the probability distribution.
S Well, that it, that you can give a function to describe where the particle is at a certain time.

O OK, precisely, that is a very precise way of saying it, because this means that your system, or your world, is deterministic, right?

S Yes.

O Because you know where [the particle] is.

S Yes, precisely. Is this not the case, then?

O No.

S But, how can you know? But they cannot find a relation, a formula, a function [to describe this]?

O No.

S But are they sure that... Maybe they will find it some day. Right, or not? Or is this impossible, have they proven that this is impossible?

O Well, that is a very meaningful question. You wonder if it is possible that some day an equation will be found that does give us a [deterministic] relation between $x$ and $t$?

S Yes, that is what I ask myself. I find it very hard to accept that one way or the other it coincidentally happens, the movement of this particle. I find that improbable.

This discussion shows there is a difference between knowing and understanding. This student thinks it should be possible to give a function which describes the trajectory of the particle (line 26). He is, however, aware of Born’s statistical interpretation, and he is able to give its definition (line 7). Because of this the teacher might not even notice that this student holds a deterministic view. Most of the problems that are discussed in tutorial sessions do not directly test this. But teachers will find it unsatisfactory when students cling to a classical, deterministic view. Moreover, it could well be that students have difficulties solving problems because of having a deterministic view, when an indeterministic view is needed to interpret the problem correctly.

The first of these observations shows that students need to know certain concepts such as potential energy, used in classical mechanics, before they can learn quantum mechanics. The second observation shows that students need to understand what would be expected classically in a certain situation in order to appreciate the implications of quantum theory. Teachers often compare quantum mechanics with classical mechanics. For these comparisons to work, teachers need to make sure students understand the classical counterpart. Besides being useful, or necessary, concepts from classical mechanics can also hinder students learning quantum mechanics, as is shown in the last observation. From this we see that the relation between classical and quantum mechanics is a very subtle one.
Difficulties with Heisenberg’s uncertainty relation

In both courses the Heisenberg uncertainty relation is “defined” as the impossibility to measure position and momentum simultaneously with arbitrary accuracy: measuring position (momentum) with high accuracy makes it impossible to measure momentum (position) with high accuracy. Some students seem to think that the measurement itself causes the outcome to become uncertain. They interpret this as an error in the measurement. When asked what the uncertainty relation is, students in the QC course repeat the definition of the teacher and add that the measurement influences the system. They feel, nevertheless, that it should be possible to measure both position and momentum with arbitrary accuracy. When asked if it is possible to predict the position and momentum of a particle, a student answers affirmatively, but he adds that it probably contradicts Heisenberg’s uncertainty relation. These students hold a hybrid view of quantum mechanics: they know about the uncertainty relation and can explain what is means, but they feel that the world should be deterministic. We also see this in the discussion on page 74, where the student knows perfectly well what Born’s statistical interpretation of the wave function is, but does not want to accept its consequences.

The definition of the uncertainty relation as given in the textbooks does not mention the simultaneous measurement of position and momentum. Griffiths (2005) defines the uncertainty principle in terms of the (statistical) spread in position and momentum and then notes:

Like position measurements, momentum measurements yield precise answers — the “spread” here refers to the fact that measurements on identical systems do not yield consistent results. You can, if you want, prepare a system such that repeated position measurements will be very close together (by making \( \Psi \) a localized “spike”), but you will pay a price: Momentum measurements on this state will be widely scattered. Or you can prepare a system with a reproducible momentum (by making \( \Psi \) a long sinusoidal wave), but in that case position measurements will be widely scattered. [p. 19]

The teacher in the QP course tries to explain that the uncertainty relation is not because of an error in measurement, but he does relate the principle to what can be measured:

You should realize, that this uncertainty relation has nothing to do with the uncertainty in measurement. Measurements can be made as precise as possible, you are never able to perform a measurement that yields a more precise answer than this value. This is a fundamental lower bound for the uncertainty in measurements you can perform.

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8 As this is translated from Dutch, there is some difficulty with the meaning of the words used. Uncertainty may here be read as the error in measurements.
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The teacher tries, in a complicated way, to explain that the uncertainty relation is an intrinsic property of nature and the lower bound has nothing to do with our (in)ability in measurements. As will be shown in Section 4.3.1 the viewpoint of the teacher follows logically from the Copenhagen interpretation and is in agreement with the statement from the textbook. Viewed from the statistical ensemble interpretation [Ballentine, 1970], there is a subtle difference and the position of the teacher contradicts that of the textbook.

It is important that students understand Heisenberg’s uncertainty relation, because it shows plainly that quantum theory is not deterministic. We have seen that students think the uncertainty relation is a consequence of the measurement process. This could have been caused by the definition of the uncertainty relation as an impossibility to simultaneously measure momentum and position with arbitrary precision.

**Interpretation of quantum mechanics**

Both courses explain Born’s statistical interpretation of the wave function, where the probability $P$ of finding a particle between $a$ and $b$ at a time $t$ is given by:

$$P_{a,b}(t) = \int_a^b dx |\Psi(x,t)|^2.$$  \hfill (4.1)

The probability density for the position of a particle at time $t$ is then given by $\rho(t) = |\Psi(x,t)|^2$. The QC course does not spend much more time on interpretation and students do not seem to be too interested. This contrary to the QP course. The teacher tries to discuss some of the more difficult conceptual issues, such as what happens to the wave function upon measurement, the double slit thought experiment as described by [Feynman et al., 1965] and the completeness of quantum theory. Students are also more interested in these issues. Especially in the first weeks of instruction, seemingly concrete questions from students about exercises end in discussions about interpretational issues. An example of this is the discussion with a student on page 74.

Students seem to have difficulties with the Copenhagen interpretation. In the Copenhagen interpretation it is said that the wave function is a complete description of an individual system. The teacher discusses the measurement problem and the Copenhagen interpretation as follows:

...before the measurement you only have the wave function. You should not think this is a sort of incomplete description of what is going on, no this is what describes the particle.

He then explicitly notes that the orthodox, or Copenhagen interpretation, as described in [Griffiths, 2005] will be used.

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9Quantum theory can be divided in a mathematical formalism, and an interpretation of this formalism. The interpretation relates the formalism to empirical reality. One such interpretation is the Copenhagen interpretation, which is discussed in the course.
One of the questions asked by a student during a lecture, is how the probability density can be found for one particle. The teacher explains that the measurement has to be repeated on identically prepared systems. When repeated often enough, the outcomes of these measurements give the probability distribution.

From this answer it can be seen that the teacher makes a very subtle distinction between interpretation and measurement. A distinction that is commonly made by proponents of the Copenhagen interpretation. In the Copenhagen interpretation the wave function provides an exhaustive description of the properties of the particle: the wave function is identified with the individual particle. When performing a measurement, it is not possible to say anything about the wave function: multiple measurements are needed on many particles.

In the statistical ensemble interpretation as described by Ballentine (1970), the assumption in the Copenhagen interpretation that the wave function is a complete description of the particle, is left out. There a wave function “provides a description of certain statistical properties of an ensemble of similarly prepared systems” (Ballentine 1970, p. 360). This has consequences for the meaning of the Heisenberg uncertainty relation, as mentioned in Section 4.3.1. The Heisenberg relation, as derived from the postulates of quantum mechanics, reads:

\[ \sigma_x \sigma_p \geq \frac{\hbar}{2}, \]  

(4.2)

with \( \sigma_{x,p} \) the standard deviation for respectively the position \( x \) and momentum \( p \). In the Copenhagen interpretation it makes sense to say that \( \sigma_{x,p} \) is a property of the particle itself, because the wave function holds the information for \( \sigma_{x,p} \). In the statistical ensemble interpretation such a claim cannot be made: the wave function only contains information on an ensemble, and as such, \( \sigma_{x,p} \) only has a meaning for the ensemble as a whole. Therefore, in the Copenhagen interpretation it follows that the Heisenberg uncertainty relation prohibits simultaneous measurement of both position and momentum with high accuracy. In the statistical ensemble interpretation, this does not follow from relation (4.2). There an ensemble of similarly prepared systems, cannot have a spread in \( x \) and \( p \), such that relation (4.2) is violated.

The Copenhagen interpretation, unlike the statistical ensemble interpretation, also contains the projection postulate, often called “collapse of the wave function”. Loosely stated, the idea of the collapse of the wave function is that upon measuring the position of a particle, yielding a position \( x = a \), the wave function is sharply peaked around \( x = a \), such that a successive measurement will also return \( x = a \) for the position of the particle. This projection postulate 10 Ballentine (1970) speaks of the Statistical Interpretation and notes that it should not be confused with the more general usage of this term. For clarity I will speak of the statistical ensemble interpretation, when I refer to this Statistical Interpretation and Born’s statistical interpretation, to refer to the postulate by Born that the square of the wave function gives a probability density.

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is discussed in one of the lectures of the QP course. The following discussion is part of a tutorial session, led by the author of this article.

S1  When you have one single measurement, then the particle... so for instance, you have two of these slits, where it goes through, then it always remains a peak when you conduct the measurement repeatedly.

O   Shortly afterwards?

S1  Yes, but I wonder how shortly after it is and what it depends on? If it is less shortly, will it then become such a spread that slowly expands? In what way will it evolve in to a Gaussian distribution?

O   Does anybody have an answer to that?

S2  Actually I don’t understand it completely.

O   […] You [S1] say: suppose we have a certain experiment, the particle flies through the apparatus, and you measure its position. If you repeat this measurement shortly afterwards, you will find the particle on the same place. There where you have found it before. And you say: this wave function will be sharply peaked around...

S1  Yes, it [the wave function] will really be like that, and if you wait some time, in what way does it change, because in the end you will get a Gaussian distribution. This is what the textbook says, but nothing is told about the intermediate states and in what way this depends on time.

S3  But it will slowly become more shallow, this peak will become lower, the area...

S4  The interference pattern is gone, so then you can never uh... At least, you will always keep the same distribution that slowly becomes constant.

O   But suppose that you measure in short succession. Let’s say that by short, I’m just making this up, we mean one second. Suppose that you measure ten seconds after the first measurement, will you then find the same peak?

S7  But you will always find a peak when you measure? Because if you measure a particle, then its wave function is always...

After transcribing this discussion and rereading it, it became apparent that the students, as well as the teacher (author) use unclear language. By identifying the wave function with the particle, it is tempting to say that “a peak is measured”. Student and teacher fail to distinguish between an individual particle having a wave function (Copenhagen) and the measurement of such a wave function, which is not possible for one particle. This could be the result of the subtle difference between the meaning of the wave function in the Copenhagen interpretation and how the probability distribution is measured.

Another problem in this discussion is that two topics are discussed simultaneously: measuring the position of a particle, and the thought experiment where electrons pass a double slit [Feynman et al. 1965]. In this experiment,
the electrons that have passed the double slit create an interference pattern on a screen behind the double slit. This interference pattern will disappear if one determines through which slit a single electron has passed. Later on, the discussion focuses on this thought experiment. An excerpt of this part of the discussion follows.

O [The particles] are emitted one at a time, so each instant, only one particle is present in the apparatus. This was the experiment the lecturer described. […]

S7 But when you have one particle, is it then possible to have an interference pattern, only you cannot measure it? Because, if you measure it, then you have only one data point, but then it still has a chance of 50% to come through one of the two holes [slits]? And these waves still have interference? If you would draw the wave function for a longer time?

O Yes, OK, for a longer time. But then you would need more particles in order to see it [the interference pattern]. Does anybody understand this? If you have sent one particle through, then for this particular particle, the same conditions and circumstances hold as for all the other particles. But one particle does not give you an interference pattern. For that you will need to send more particles.

S7 But you do have an interference pattern, only you cannot measure it? Or is that not true? A particle is a wave before you have measured it?

S9 I think that you cannot say if it is there or not, because you have not yet measured it.

The remark student S9 makes in line 17 reminds us of the the title of Mermin (1985): “Is the moon there when nobody looks?” It touches upon one of the most difficult aspects of the interpretation of quantum mechanics. Again in this discussion we see, that students have difficulties with the Copenhagen interpretation. Student S7 identifies the wave function with the particle, and thinks that you will still get an interference pattern with one particle in the apparatus.

Students might benefit from explicitly discussing different interpretations of quantum mechanics, and making clear what is to be considered interpretation, and what formalism. With formalism being the “rock-bottom” of quantum theory. Students should also know how quantum theory is used in experimental practice. In particular, what is meant by “measurement”.

Constructing a mental picture

Students in both courses have difficulties understanding what the meaning is of certain formula’s, calculations and concepts. Some examples are given below.

When discussing the nodes (i.e. the zeros) in a probability distribution for a particle, a student in the QC course asks: “If you translate that to a tangible idea, where is the chance zero…? For instance: an electron describes a
wave round the nucleus, then where is [the chance] zero?” (author’s emphasis). This student explicitly asks the teacher for a more insightful explanation. Furthermore, in the way this student phrases his question, it can be seen that he holds a deterministic view. The teacher does not notice this. Of course this is not the point the student wants to make, and it is rather subtle to notice. It does however show that students might hold wrong conceptions on quantum mechanics, which show themselves in the way they ask questions.

Questions students ask can also give an indication of what they have a need for. Below are some questions that express this:

- “This $\hbar$ is sort of mysterious, what does it do in front of the term for kinetic energy [in the Schrödinger equation]?”
- “Why can’t the momentum operator operate on the square of the wave function [e.g. $\hat{p}|\psi|^2$ instead of $\psi^*\hat{p}\psi$]?”
- “We have seen that a spiked function is found [upon measurement]. So in practice, will you never find a certain other function?”
- “I notice that the function before and after the potential barrier, has a $i = \sqrt{-1}$ in the exponent of $e$, which is missing in the function in the barrier. I suppose this means something?”
- “Has it been deduced that it [the particle] can pass the [potential] barrier? Or is it an assumption?”
- “Is there any physical relevance that you can write the wave function as: $\psi = \psi_1 + i\psi_2$?”
- “We have never done anything with [the probability current], apart from the fact that we derived it. Where do you use it for?”
- “Where does the Schrödinger equation come from?”
- “What is the difference between the particle in a box and the harmonic oscillator?”
- “What do the nodes [in a wave function] represent?”
- “Why does [the wave function of an electron] have to be totally antisymmetric?”

These questions show that students want to create a mental picture of what is taught. Teachers could take advantage of this “demand”, so to speak. It would be interesting to know if students need these kind of questions answered, before they are able to move on in their learning process.
Instructional approach

This section discusses some observations concerning the style of instruction of influence on learning quantum mechanics. Not all of these observations are specific to quantum mechanics, though.

In lecture ten of the QP course, the students are introduced to the free particle. They learn that it is not possible to have a free particle with definite energy. This is due to the fact that it is not possible to normalize the wave function of such a particle \( \psi(x) = A e^{\pm ikx} \). To accommodate this problem, the free particle is described as a superposition of wave functions with different wave number \( k \). Three weeks later tunneling is discussed. A particle with energy \( E \), incident on a potential barrier of height \( V_0 > E \) and width \( a \), has a chance of being found on the right side of this barrier. Classically this would not have been possible.

The wave function that is used in the tunneling problem has the same shortcoming as that of the free particle: it is not normalizable. With the discussion of the free particle, this was an argument not to accept such a wave function, but instead to create wave packets. Because there is a (classical) problem in the discussion of tunneling, students might use the same argument: such a wave function is not possible, so also tunneling might not be possible. In fact most students do not notice that the wave function is not normalizable. Which shows that the explanation of the free particle is not yet fully understood. One student that did notice the problem, had difficulties doing an exercise about tunneling.

Usage of graphs in instruction should be done with care. It can easily lead to confusion, as the two following examples show.

In one of the tutorial sessions in the QP course, the teacher discusses one of the problems in the workbook. Two wave packets, \( \psi_{1,2}(x) \), are considered, one with a Gaussian wave number distribution, \( \phi_1(k) \), centered around \( k = 0 \), and one, \( \phi_2(k) \), centered around \( k = k_0 > 0 \). The wave packets themselves are Gaussian, and centered around \( x = 0 \). The graphs of \( \phi_{1,2}(k) \) are drawn upon the blackboard. Students are then asked what will happen to the two wave packets, as time progresses. One of the students answers that the first distribution, centered around \( k = 0 \), will remain centered around \( x = 0 \), but the second will move to the right. Another student disagrees: he thinks both remain where they are. A discussion between these two students follows.

All this time, both students and teacher refer only to the wave number distributions \( \phi_1(k) \) and \( \phi_2(k) \), drawn upon the blackboard. But in the discussion, both the wave packet, and the wave number distributions are used. It appears that the student that disagrees, was only talking about the wave number distribution. This is not strange, since each time the teacher asked what would happen to a wave packet, he pointed at its corresponding wave number distribution. All this time, both students actually agreed, there was only confusion about the graphs used.

In the QC course graphs showing the electron density distribution, \( \rho(x) = |\psi(x)|^2 \), for a certain wave function are often used, for instance to explain
bonding of atoms in a molecule. These density distributions show the region in space where $\rho(x) > \rho_0$, and what the sign is of $\psi(x)$. After having seen these graphs for several lectures, one of the students asks what the plus and minus represent. None of the other students can answer this question. It appears that all this time, students did not know the meaning of these graphs. It can be questioned whether students have understood the explanations where these graphs were used.

Graphs can be used very easily by teachers, without students fully understanding their meaning. Because it can take some time before students and teacher realize that graphs are unclear, it is important that graphs are introduced properly.

**Mathematical skills**

Although the mathematics needed in both courses is not sophisticated, students already seem to have difficulties with very basic manipulations. Especially in the QC course students encounter most difficulties with the mathematics in doing exercises and solving problems during tutorial sessions and not with the quantum mechanics involved. That is not to say they do not find the quantum mechanics difficult: they do not see through the mathematics and do not get to the quantum mechanical issues.

At the start of the course, students in the QC course indicated they were concerned with the level of mathematics that would be used. During the first semester all chemistry and physics students have had to follow an introductory course in calculus. Even the chemistry students who completed this course successfully, had difficulties with mathematics. Unfortunately, passing for the calculus course does not seem to give any guarantee that students will not encounter any difficulties with the mathematics used in the quantum mechanics course. A lack of mathematical skills proves to be an impediment when learning quantum mechanics.

### 4.3.2 Questionnaire

Because of the low number of students participating in the Quantum Chemistry course (13), of which only half completed the questionnaire (6), we only discuss the results of the questionnaire given to the students in the Quantum Physics course. The questionnaires are given in Appendix 4.A.

The questionnaire before instruction consisted of two parts. In the first part the students were asked about their knowledge on quantum mechanics and their expectations of the course. The second part consisted of six statements about the atom and were taken from a questionnaire by Müller and Wiesner (2002). After instruction the questionnaire consisted of three parts. In the first part students were asked about their opinion of the course. The second part was the full questionnaire from Müller and Wiesner (2002), including the six questions given before instruction. In the last part students were asked how
many hours they had studied and how many lectures and tutorial sessions they had missed.

As explained in Section 4.2.2 an index $C$ was calculated from the students answers on the statements. An index value of $C = 100$ corresponds to quantum mechanic thinking and a value of $C = -100$ corresponds to thinking contradictory to quantum mechanics. Figure 4.1 shows the spread of concept indexes $C$ for the complete set of statements, after instruction. All answers of students that signed up for the final exam are included in this graph. Figure 4.2 compares the answers given before and after on the six questions about the atom. For the last figure, only answers from the students that completed both pre- and post-instruction questionnaires and who had signed up for the exam were used.

![Figure 4.1: Number of students in a given range of the conception index $C$ after instruction ($N = 24$, $\bar{C} = 29$, $SD = 23$).](image1)

![Figure 4.2: Conception index for questions on the atom before (left) and after (right) instruction ($N = 16$, pre: $\bar{C} = 29$, $SD = 34$, post: $\bar{C} = 39$, $SD = 40$).](image2)

From Figure 4.1 we see that students’ conceptions after instruction tend to be in accordance with quantum mechanics, although there is a large spread. From the comparison of the answers students gave to the statements on the atom (Figure 4.2), we see that $C$ increases: students have a more quantum mechanical view about the atom. However, we also see that the spread is much larger.

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4.4 Discussion

The results presented in Section 4.3.1 show some important difficulties students have when learning quantum mechanics. Some of these observations have already been reported by others. In this section we will try to relate the observations given in this paper to existing work, as known to us. The same structure is used as in Section 4.3.1.

4.4.1 Literature

Quantum versus classical mechanics  Steinberg, Wittmann, Bao, and Redish (1999) give a short overview of difficulties students have when learning quantum mechanics, related to classical mechanics. In a talk Edward F. Redish, Bao Lei, and Pratibha Jolly report on difficulties students have with energy in quantum mechanics, although they do not mention the conception that energy cannot be negative.

Fischler and Lichtfeldt (1992) report extensively on the relation between classical and quantum mechanics. Of special interest is their remark:

... a concept which prevents the students from attempting to understand the phenomena of quantum physics in terms of classical physics, will have to proceed from the following basic decisions:

- One should avoid reference to classical physics.
- ...

Difficulties with Heisenberg’s uncertainty relation  Styer (1996), Müller and Wiesner (2002) and Johnston, Crawford, and Fletcher (1998) also report that students think Heisenberg’s uncertainty relation has something to do with an error in measurement.

The article by Styer (1996) is an interesting starting point for further research, as it gives an extensive list of common misconceptions regarding quantum mechanics.

Interpretation of quantum mechanics  The article from Müller and Wiesner (2002) introduced us to the statistical ensemble interpretation from Ballentine (1970). The authors used this interpretation in their instruction, and motivate their choice as follows:

In our view, this interpretation provides a clear and comprehensible way of talking about quantum phenomena. Similarly, the idea of state preparation helps one to construct a conceptual framework that serves as a basis for deeper understanding.

\footnote{Available from internet: http://www.physics.umd.edu/rgroups/ripe/perg/papers/redish/talks/quantum/aapt97qe.htm}
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Constructing a mental picture Cataloglu and Robinett (2002) have developed an assessment instrument, the Quantum Mechanical Visualization Instrument (QMVI), to test the conceptual and visualization understanding of quantum mechanics. They were interested in the development students make through their undergraduate career.

Somewhat related to this topic is the article by Johnston et al. (1998). They report on a study to determine in which ways certain quantum mechanical ideas are conceptualized by students. They conclude that “the learning that has been observed might be described as ‘surface’, . . . And indeed we have found little evidence in this research of a ‘deep’ approach to learning.”.

Amongst other things, Singh (2001) has found that students in advanced quantum mechanics courses have difficulties with conceptual understanding of quantum mechanics and time development.

Instructional approach Slightly related to what is written in Section 4.3.1 is the article from Zollman, Rebello, and Hogg (2002). They describe a “hands-on approach to learning and teaching quantum mechanics”.

Mathematical Skills The results from the aforementioned article by Johnston et al. (1998) are also relevant for the problem reported that students “do not see through the mathematics”. In their conclusion they remark:

When asked directly why quantum mechanics is difficult most students answer something to the effect: “It’s all mathematics”. Our conclusions suggest this means that the mental models they are working with are tenuous constructs, extended far beyond the point where they are buttressed by perceived relationships with other, better understood concepts.

Although we have not observed the difficulty Bao and Redish (2002) report on, we would like to mention their article. They have found that students have problems understanding probabilistic interpretations of physical systems, and see this as a prerequisite to learning quantum mechanics.

4.5 Conclusion

The observations show clearly, what we already knew: quantum mechanics is a difficult subject. It is striking however that after instruction, students still seem to hold onto a deterministic world view. Also, we have the impression that students do not have consistent ideas on quantum mechanics. It is questionable whether students master the basics of quantum mechanics. The questionnaire on conceptions does show a preference for quantum mechanical thinking, but we do see however that students have contradictory ideas about quantum mechanics.

The relation between classical and quantum mechanics is subtle. Sometimes classical concepts seem to be necessary, sometimes useful, and sometimes
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harmful. It still is a question whether the teachers know of these different relationships, and if they do, how they use this knowledge in their teaching.

Heisenberg’s uncertainty relation is one of the clearest results from quantum theory that shows that our classical view has important shortcomings. The definition of the uncertainty relation has caused students to think that the uncertainty relation is a consequence of measurement error. Some students showed a hybrid understanding of the relation.

The interpretation of quantum mechanics seems to be an underlying mechanism for multiple observed problems. The misinterpretation of Heisenberg’s uncertainty relation is one of these problems. Students also seem to have difficulties with the Copenhagen interpretation. In particular, the projection postulate, and the postulate that the wave function is an extensive description of the particle.

Students also have difficulties visualizing quantum mechanics, and understanding the meaning of formula’s and concepts. In this respect, emphasizing the experimental basis of quantum mechanics might be helpful.

Finally, students had difficulties with the mathematics that was used in these courses. As a result, the students in the Quantum Chemistry course were not always able to see the quantum mechanical content of exercises and problems.

From the observations, and the discussion, it seems that few, or no research has been done on the influence that the various interpretations of quantum mechanics could have on learning quantum mechanics. Interpretation of quantum mechanics might be considered a part of the language we use to interpret, or give meaning to the formalism of quantum theory. In that sense, the following question is relevant:

How does the language, used by teachers and students in first year quantum mechanics courses, affect students’ learning of quantum mechanics?

More concrete sub-questions are:

1. What impact does the use of the Statistical Interpretation of quantum mechanics have on the understanding of quantum mechanics by first year physics/chemistry students?

2. What concepts from classical theories are helpful for learning quantum mechanics?

3. What influence does the use of Bohr’s model of the atom have on the learning of quantum mechanics?

These last two questions might be partially answered by the literature on this subject.

For the next stage of the research, changes will have to be made to the courses as they are given currently. It is therefore important to have full
cooperation from the teachers of both courses. This will be a challenge for the next couple of months. Discussing subjects like the interpretation of quantum mechanics, can be a delicate subject. We do not want have a discussion of what is the right, and what is the wrong interpretation. This is an ongoing debate, that is not the subject of this research.

Besides cooperation of the teachers, we will also need cooperation of the students. This means we need to motivate the students to spend more (or at least enough) time on these courses. From the questionnaire, as well as from observations, we have seen that students spend little time on this subject. If we want an intervention to have success, we must make sure students will give our new instruction a fair chance. Of course, a question remains how effective current instruction would have been, if students would have spent more time studying.

From the observations we have seen students have difficulties with basic mathematical manipulations. Especially the chemistry students. If we want to do any interesting research, we have to find ways to help students overcome their difficulties with mathematics. This can be done in several ways. We could construct pre-tests students can do in advance of the lectures, to test if they master the necessary mathematical skills needed for a certain lecture. For more complex problems they have to solve, we could try to use mathematical software\textsuperscript{12}. This way we can let the students focus on the quantum mechanics, involved in a problem. For this topic we might want to look at the following additional question:

4. How can the use of mathematical software help chemistry students to focus on the quantum mechanics instead of on the mathematics?

\textsuperscript{12}For instance Mathematica: \url{http://www.wolfram.com/products/mathematica/}
4.A Questionnaire

Before and after instruction the students were given a questionnaire. The English versions are reprinted below. Appendix 4.A.1 shows the pre-instruction questionnaire, Appendix 4.A.2 the post-instruction questionnaire. The questions on student conceptions of quantum mechanics (group B and groups B-E respectively) were translated from German (Müller & Wiesner, 2002) to Dutch and then to English. The questionnaire was held in digital form and thus the lay-out was slightly different than displayed here. The weights of the questions are given in square brackets. A negative weight corresponds to a classical question, a positive weight to a quantum question.

4.A.1 Pre-instruction questionnaire

A) General questions

Below are some open ended questions concerning your background and your (possible) knowledge of quantum mechanics.

1 What is your highest level of education?
2 What is your age?
3 What do you expect from this course?
4 Have you already had education in quantum mechanics? If so, where?
5 Where, apart from other education, have you learned something about quantum mechanics?
6 Please describe in one sentence what quantum mechanics is about.
7 What is the most important result of quantum mechanics you know? Please explain what this result means.
8 Which physical concepts do you identify with quantum mechanics?

B) Quantum mechanics

Below are statements about quantum mechanics. Please indicate to what extent you agree, or disagree (i.e. strongly disagree, disagree, neutral, agree, strongly agree).

1 The structure of an atom is similar to that of our solar system (planets revolving around the sun). [-2]
2 Electrons in an atom are spread out clouds of electrical charge, surrounding the nucleus. [1]
3 Electrons move in distinct orbits at high speed around the nucleus. [-2]
4 You cannot say that electrons in an atom follow a specific trajectory. [2]
5 The energy of an atom can take any value. [-2]
6 You cannot determine the position of an electron in an atom with high precision, because the electron is very small and moving fast. [-1]
4.A.2 Post-instruction questionnaire

A) General questions
Below are some question on how you experienced the course.

1 What part of the course did you find most difficult?
2 What have you missed in this course?
3 What did you find most interesting/fun/remarkable in this course and why?

B) Visualization of the atom
Below are statements about quantum mechanics. Please indicate to what extent you agree, or disagree (i.e. strongly disagree, disagree, neutral, agree, strongly agree).

1 The structure of an atom is similar to that of our solar system (planets revolving around the sun). [-2]
2 Electrons in an atom are spread out clouds of electrical charge, surrounding the nucleus. [1]
3 Electrons move in distinct orbits at high speed around the nucleus. [-2]
4 You cannot say that electrons in an atom follow a specific trajectory. [2]
5 The energy of an atom can take any value. [-2]
6 You cannot determine the position of an electron in an atom with high precision, because the electron is very small and moving fast. [-1]

C) Double slit experiment
Below are statements about quantum mechanics. Please indicate to what extent you agree, or disagree (i.e. strongly disagree, disagree, neutral, agree, strongly agree).

1 In the double slit experiment, the electron behaves like a particle and a wave, it is neither. [1]
2 When the electron in the double slit experiment approaches the screen, it follows a distinct trajectory, although I cannot determine it. [-1]
3 The electron follows a distinct trajectory, whether I observe it or not. [-1]
4 It is not possible to say something about the behavior of a single electron in the double slit experiment. I can only say something about the statistical behavior of many identically prepared electrons. [2]
5 The position of the electron at a specific moment between the source and the screen is fundamentally undetermined. [2]
6 The position of the electron at a specific moment between the source and the screen is not fundamentally undetermined, but unknown to the experimenter. [-2]
7 No one can predict with certainty where the next electron will hit the screen. [2]
8 The precise position, where one will find a specific electron, is fixed by the preparation of the initial state. [-2]
9 The wave function prescribes the distribution of electrons on the screen. [1]
10 The initial state of the electrons (preparation) fixes the probability of finding an electron on a specific position on the screen. [1]

D) **Interpretation**

Below are statements about quantum mechanics. Please indicate to what extent you agree, or disagree (i.e. strongly disagree, disagree, neutral, agree, strongly agree).

1 In quantum mechanics, it is possible that a quantum object does not possess properties that are classically well defined, such as position or energy. [2]
2 Whenever the initial conditions are known with high enough precision, it is possible to predict where on a screen an electron will hit. [-2]
3 Whenever the initial conditions are known with high enough precision, it is possible to classically predict the outcome of a roll of a dice. [1]

E) **Uncertainty**

Below are statements about quantum mechanics. Please indicate to what extent you agree, or disagree (i.e. strongly disagree, disagree, neutral, agree, strongly agree).

1 Classical objects can in principle simultaneously have a position and momentum. [1]
2 Quantum objects can in principle simultaneously have a position and momentum. [-2]
3 The more precise the position of a quantum object is determined, the less precise the momentum is determined. [1]
4 The less precise the momentum of a quantum object is determined, the more precise the position is determined. [-1]
5 When an ensemble of quantum objects is prepared in a state with small spread in position ($\Delta x$), then the spread in momentum ($\Delta p$) will become larger. [1]
6 You cannot simultaneously determine the position and momentum of a quantum object with arbitrary accuracy. [1]
7 When the position of an electron is measured with high accuracy, a successive measurement of momentum can only yield an inaccurate result. [-1]
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8 The uncertainty relation gives a bound from below for the accuracy with which the momentum of a quantum object can be determined. [-1]

9 The $\Delta x$ in the uncertainty relation is a measure for the spread in a statistical distribution of positions from an ensemble of quantum objects. [1]

10 Because of the uncertainty relation it is not possible to say that electrons in an atom follow a certain trajectory. [1]

F) Conclusion

1 How many lectures did you miss?
2 How many tutorial sessions did you miss?
3 Approximately how many hours did you study each week apart from lectures and tutorial sessions?

References


