A developmental research on introducing the quantum mechanics formalism at university level
Koopman, L.

Citation for published version (APA):
Koopman, L. (2011). A developmental research on introducing the quantum mechanics formalism at university level
Chapter 6

Designing an Introduction to Quantum Mechanics

A problem when introducing quantum mechanics is that it is difficult to make productive use of students’ prior knowledge. New concepts are needed in quantum mechanics for which students lack the necessary experiences. We describe a design to introduce quantum mechanics to first year university physics and chemistry students using a guided discovery approach. The design is based on a developmental research conducted at our university. The main design principle was to start from experiments that our students could interpret with their prior knowledge. Next, students are guided to formulate hypotheses to explain the experiments. These hypotheses together are a motivation to formulate quantum theory. Furthermore, based on the hypotheses a choice can be made for the axioms of quantum theory. This article describes the rationale of the design, describes experiences from the classroom, and indicates how Physics Education Research (PER) literature has played a role in designing this introduction.

6.1 Introduction

There are many descriptions of difficulties students experience when learning quantum mechanics at different levels of education (Fischler & Lichtfeldt, 1992b; Styer, 1996; Johnston, Crawford, & Fletcher, 1998; Steinberg, Wittmann, Bao, & Redish, 1999; Singh, 2001; Bao & Redish, 2002; Olsen, 2002; Koopman & Ellermeijer, 2005; Singh, 2008b). Many initiatives exist that try to overcome these learning difficulties (Fischler & Lichtfeldt, 1992b; Petri & Niedderer, 1998; Zollman, Rebbelo, & Hogg, 2002; Kalkanis, Hadzidaki, & Stavrou, 2003; McKagan, Perkins, & Wieman, 2008; McKagan, Perkins, Dubson, et al., 2008; Singh, 2008a; McKagan, Handley, Perkins, & Wieman, 2009). A central issue seems to be how to motivate the need for quantum theory. More generally, educational research has shown the importance to take into account students’ prior knowledge (Ausubel, Novak, & Hanesian, 1978).

In introductory text books on quantum mechanics, we roughly see two
One approach tries to follow (more or less) the historical development of quantum mechanics, starting with what is now called the old quantum mechanics (Messiah, 1961; Liboff, 1980; Eisberg & Resnick, 1985; Gasiorowicz, 1996; Le Bellac, 2006). The approach starts with the axioms and rules of quantum mechanics and lets students first work with the theory. The assumption is that a discussion of the philosophical issues of quantum theory should come later (Griffiths, 2005).

Neither approach adequately takes into account students’ prior knowledge. Although the historical approach tries to give a motivation for quantum theory, it does so by discussing a lot of classical physics that is equally new to students, and perhaps as difficult as quantum theory. The axiomatic approach begins without this extra ballast; students start with the new (quantum) theory straight away. But it might be confusing why this new theory is needed, what the meaning is of the newly introduced terms, and what relation these terms have to the concepts students have learned earlier. In between these approaches are textbooks that more carefully try to work towards quantum theory by considering what rules this new theory should satisfy (Shankar, 1980; Gottfried, 1966). However, to be able to follow their argument, a student must already think like a physicist.

Our goal is to design an introduction to quantum mechanics that builds upon students’ prior knowledge as much as possible, and thereby take into account the fact that students are not yet physicists and do not yet think like them. This approach is in part inspired by Zollman et al. (2002) and Müller and Wiesner (2002). The design presented here is based on research conducted in two university level, introductory courses on quantum mechanics at the University of Amsterdam: one for physics students and one for chemistry students. In the Netherlands, physics and chemistry students have followed physics and chemistry respectively throughout their high school career, comparable to the level of Advanced Placement Physics or Chemistry in the U.S. This does not mean that the transition from high school to university is an easy one. For instance, students in the Netherlands who start a science study experience difficulties with the level of mathematics (Koopman, Brouwer, Heck, & Buma, 2008). Furthermore, students have to get used to a new way of thinking, learning, and working.

Most of the design is a result of work done in the chemistry course. Together with the lecturer we have analyzed what problems students experience when learning this subject. Based on this analysis, and available PER literature, adjustments to the course have been made. After each experimental round, the results were analyzed. Based on the findings, a new design was made for a next experimental round. The design presented was developed in four such experimental rounds. This approach is called ‘developmental research’ (Gravemeijer, 1994).

Besides the design described in this article, adjustments were made to the quantum chemistry course that focused on improving students’ study strategies. The difficulties many students experience with the mathematics involved
Calculate the first and second derivative with respect to $x$ of the following functions:

a) $\sin kx$;
b) $\cos kx$;
c) $e^{kx}$;
d) $e^{kx}$;
e) $e^{ikx}$.

Figure 6.1: Sample remedial mathematics question, given before the fourth meeting described in Section 6.3.4.

in this subject were found to be caused by a lack of mathematical fluency and poor transfer from the preparatory calculus course. To help students improve in these respects, a remedial mathematics program was set up. Before each lecture, students made an online mathematics test with which they could practice their mathematics skills. The online system that was used provided students feedback. The test could be repeated, each time presenting the student with slightly different questions. An example of such a mathematics question is given in Figure 6.1. The remedial mathematics program is described in more detail in a separate publication (Koopman et al., 2008). To refresh students’ knowledge of the physics needed in the lectures, a set of online multiple choice questions was prepared for each lecture. Shortly before the start of the lecture, the lecturer would review the answers and could adjust his lecture depending on the answers given. This method is called Just-in-Time Teaching (Novak, Gavrin, Christian, & Patterson, 1999). An example of a question used is given in Figure 6.2. Finally, for each lecture a diagnostic test was made available with which students could assess whether they had understood the lecture (Figure 6.3).

In the next section we describe the principles used in this design, and in Section 6.3 we sketch the design itself, focusing on the (physical) reasoning we expect of our students. Teaching experiences are given to illustrate how students worked with the designed material.

### 6.2 Design principles

The design is based on the following principles:

1. As much as possible take into account students’ prior knowledge that is relevant when learning quantum mechanics.
Which of the following is Newton’s second law?

a) \( p = mv \);

b) \( P = F \Delta t \);

c) \( F = ma \);

d) \( F_{\text{action}} = -F_{\text{reaction}} \).

Figure 6.2: Sample JiTT question, given before the fourth meeting described in Section 6.3.4.

What is the difference between the displacement and the amplitude of a wave? Explain using the double-slit experiment for water why these concepts are useful.

Figure 6.3: Sample question from the diagnostic test given to students after the second meeting described in Section 6.3.2.
2. Start from experimental results that are meaningful to our students. Guide students to formulate hypotheses to explain the phenomena they are confronted with. After making a motivated selection, these hypotheses can be seen as rules to which a new mechanics should adhere.

3. Use relevant results from PER literature.

Following Lijnse and Klaasen, to implement the first two principles we use the problem posing approach in organizing the teaching–learning sequence (Lijnse & Klaassen 2004). As argued by Klaassen (1996), the problem posing approach is based on the idea that for students to “meaningfully engage in an activity there should be a sense in which they know why they are doing it” (p. 87). Furthermore, “they will have to develop some sense of purpose for going to study events they have never witnessed or paid attention to” (p. 87). Science education should make pupils want to add to their “existing conceptual resources, experiential base and belief system . . . in a way that leads to a proper understanding of science” (Klaassen, 1996, p. 106). The problem posing approach aims at all the above.

The characteristics of a teaching–learning sequence following the problem posing approach can be defined by a didactical structure. This didactical structure defines the following:

- The physics content (knowledge) of each step;
- The motive for students to study certain knowledge content.

With motive, we mean the logic that leads from one step to the next from the student’s perspective. This motive can be formulated in the form of a question that a student is likely to pose at the particular point in the teaching–learning sequence. The didactical structure for the designed introduction is given in Figure 6.4 and further explained in this article.

The teaching approach we use may also be qualified as guided discovery learning. It is discovery learning in the sense that students are expected to formulate the key characteristics of the phenomena we confront them with. However, research has shown that pure discovery learning is ineffective and that some guidance is needed (Mayer 2004, Kirschner, Sweller, & Clark 2006). Furthermore, the sequence presented here is designed to let students acknowledge that a new theory is needed and give them an idea what characteristics this new theory should have. After this point, we expect that it is more efficient to proceed with a more traditional (i.e. verbal reception learning) approach. But even then it is possible to follow the problem posing approach. In any case, the education should be meaningful.

6.3 Design of the studio course introduction

The design consists of four topics, presented in the next four sections: quantization, matter waves, probability, and ‘towards a new theory’. The design
Chapter 6. Designing an Introduction to Quantum Mechanics

Quantization

- Photoelectric effect
  - Light comes in packets with energy \( E = hf = \hbar \omega \)
  - Light has a particle-like property

- Matter waves
  - If light has particle-like properties, might electrons conversely have wave-like properties? A characteristic of waves is that they interfere. First study double-slit interference for waves.

- Probability
  - Towards a new theory
    - Photoelectric effect: light comes in packets with energy \( E = hf = \hbar \omega \).
    - Matter waves: light has a particle-like property.
    - Probability: towards a new theory.

- Hydrogen spectrum
  - The hydrogen spectrum has a regular structure. We may find a function that describes this pattern. From this description, we can formulate hypotheses to explain the pattern: existence of energy levels.

- De Broglie postulate
  - Electron diffraction on graphite:
    - Newton rings;
    - Postulate for electron.

- Probability
  - Classical probabilities
    - First try to describe a classical system with probabilities:
      - for continuous variable \( x \) we need probability density \( \rho(x) \);
      - \( \rho(x) = \lim_{h \to 0} \frac{f(x, x + h)}{h} / \hbar \);
      - \( P(a, b) = \int_a^b \rho(x) \, dx \)

- Probability
  - Which-way experiment
    - Path information destroys interference pattern:
      - No superposition: electron passes one slit;
      - Superposition: electron "passes" both slits.

- Operators
  - For each observable we need an operator. Idea of eigenvalues as possible measurement outcomes.

- Expectation values
  - A superposition has multiple possible measurement outcomes, each with a certain probability: expectation value.

- Towards a new theory
  - An equation for \( \Psi \)
    - Search for an equation analogues to Newton's second law, satisfying:
      1) Solution for free electron (assumption from double-slit experiment);
      2) superposition principle (linearity);
      3) \( \Psi \) with wave number \( k \) describes electron with momentum \( p = \hbar k \);
      4) \( \Psi \) with angular frequency \( \omega \) describes electron with total energy \( E = \hbar \omega \).

- We know how to describe electrons with \( \Psi \), but how do we find \( \Psi \) for a given situation?

- How do we measure a superposition?

- How do we find the momentum, energy, etc. from an arbitrary \( \Psi \)?

Figure 6.4: Didactical structure of the introduction. The rectangles show the physics content (or knowledge) which is connected by motives shown in gray, rounded rectangles. The dashed rectangles are activities that have not been tested.
is a proposal based on the result of several experimental rounds during an introductory course on quantum chemistry (chemistry majors). Some of the four parts were also tried out in an introductory course on quantum mechanics (physics majors). In addition, relevant PER literature has been consulted. To be explicit: the design has never been tried out in the form in which it is presented here.

During the years 2007-2009, the quantum chemistry course started with four sessions (of two hours each) consisting of the experimental introduction described here. After these four sessions, the lecturer would continue with the formalism of quantum mechanics and next its application to quantum chemistry. Over these years the number of students that actively took part in the course varied from 27 to 32.

In describing the design, we focus on the reasoning expected of our students. We estimate that the sequence can be carried out in approximately eight meetings of two hours each. The end point is the Schrödinger equation, after which instruction can be given to work with the formalism of quantum mechanics. After each part of the design some anecdotes are given under the heading “Teaching experiences” to illustrate how students work with this material. These results are based on general experiences from the classroom, as well as on audio recordings that were made of three student groups (two students each) that were followed during each experimental round.

6.3.1 Quantization

Quantization is a key concept in quantum mechanics. We have students explore this concept in the context of two phenomena: the photoelectric effect and the spectrum of hydrogen.

Photoelectric effect

The discussion of the photoelectric effect leads students to argue that classical mechanics falls short in explaining it, and encourages them to formulate an alternative hypothesis that can explain it. The hypothesis we are aiming at, is of course that light consists of photons. It is, however, not our intention to use the photoelectric effect to prove the existence of photons.

We make use of a demonstration experiment\(^1\) to qualitatively show the effect, and an applet\(^2\) that shows electrons moving in a photoelectric cell, for chosen settings of light intensity, light frequency, and potential difference.

---

\(^1\)The demo consists of an electroscope with a zinc plate attached to it and a mercury lamp. The zinc plate can be charged by rubbing a plastic rod against a piece of cloth. The electroscope makes this visible. When the light of the mercury lamp is directed on the zinc plate, the charge ‘leaks’ away. Repeating this procedure with a glass plate between the lamp and the zinc plate, nothing happens. Students now that you cannot tan behind glass, as ultraviolet light will not penetrate the glass. We conclude that the ultraviolet light is needed to free the charge of the zinc plate.

\(^2\)The photoelectric applet from PhET (http://phet.colorado.edu/) was used (McKagan, Perkins, Dubson, et al. 2008).
over the photoelectric cell’s electrodes (Figure 6.5). Working with the applet, students figure out how they can ‘measure’ the kinetic energy $E_{\text{kin}}$ of electrons emitted by the metal surface. Students find that the electrons are emitted with a distribution of kinetic energies and that only the kinetic energy of the fastest electrons can be determined. They next plot these values as a function of the frequency $f$ of the incident light beam. This results in the (well known) relation:

$$E_{k,\text{max}} = af + b,$$

where $a > 0$ and $b < 0$ are parameters to be determined. We ask students to interpret this relation based on the demonstration they have seen and decide whether it accounts for the observed effect. First of all, the relation falsifies the idea students might have that the electrons’ kinetic energy depends on the intensity of the light beam. This thus also falsifies the idea that the emission of electrons is determined by the intensity of the light beam. The intensity only influences the number of electrons that is emitted. Secondly, the frequency should be above some threshold value, as was concluded from the demonstration experiment. Finally, students give arguments whether the term $af$ can be seen as the energy corresponding to a quantum of light. Because the relation expresses the energy of one electron, it is expected students will see $af$ as an amount, or quantum, of the energy taken from the electromagnetic

---

3For a given light intensity and frequency, the electric current can be stopped by applying a voltage. This ‘stopping potential’ is able to stop the most energetic electrons that emerge from the metal. From this stopping voltage the kinetic energy of the fastest electrons can be inferred. With this method it is not possible to ‘measure’ the kinetic energy of slower electrons.
field. One electron picks up an energy amount of exactly $a_f$, not some factor times this amount.

The photoelectric effect applet enables students to repeat their ‘experiment’ with different kinds of materials (e.g. sodium, zinc, copper). For each they can determine the parameters $a$, and $b$, from Eq. (6.1). They will note that the parameter $b$ depends on the material used, but $a$ remains more or less constant, with a value of: $a = 6.6 \times 10^{-34}$ J s, which is Planck’s constant $h$. At this point students are introduced to the idea that the energy of a quantum of light, now called a photon, equals $E = h f$. Because light can be seen both as wave (as students are used to), and particle phenomenon (as follows from the photoelectric effect) we pose the hypothesis that for matter, ordinarily seen as particle phenomenon, this relation also holds. This hypothesis is tested in the assignment on the double-slit experiment (Section 6.3.2).

The prerequisites for this assignment are few. First of all, students need to acknowledge that the electron as particle exists and that a certain amount of energy is needed to free it from the atom. Furthermore, a basic understanding of energy conservation and the relation between electric potential and potential energy is needed to calculate the kinetic energy of the electrons.

**Teaching experiences** Many of our students are somewhat familiar with the photoelectric effect as an argument for the quantization of light. Still, students do not find it easy to explain how the kinetic energy of the electrons might be measured. In part this seems to be caused by the applet, because it suggests that individual electrons can be observed. This gives some students the impression they can use the electron’s position to determine its kinetic energy. After more emphasis has been put on the fact that the applet is a model and that the individual electrons will not be visible as in the applet, students realize that the minimum applied voltage needed to stop the current can be used to determine the kinetic energy of the fastest electrons. With help of the applet they can then determine the relation between the frequency of the incident light beam and the maximum kinetic energy of the electrons. Students determine the slope of the graph and recognize the term $h f$ as the energy of a single photon.

**Hydrogen spectrum**

Now that students have learned that light can be thought of as made up of photons, energy packets each with energy $hf$, they can look more closely at the structure of the atom. The assignment described here guides students in formulating postulates that can explain the spectrum of hydrogen. A more detailed discussion can be found in Koopman, Kaper, and Ellermeijer (2006).

In secondary education the Bohr model of the atom is used to teach students the structure of the atom. Fischler and Lichtfeldt argue that the discussion of semi-classical models, such as the Bohr model, should be avoided because they induce misconceptions (Fischler & Lichtfeldt 1992b, 1992a). How-
ever, McKagan, Perkins, and Wieman (2008) have found that such models can be taught effectively by comparing and contrasting them, and thus letting students distinguish the strengths and the shortcomings of various models. We do not focus on the Bohr model (planetary analogy, quantized orbits). Our aim is to have students formulate a minimum set of postulates that explain the hydrogen spectrum, and to have them find an expression for the energy of an electron bound in an hydrogen atom. These postulates may then be generalized to other atoms.

In the assignment students are given the spectral data of hydrogen and are asked to make a graph of the spectral lines and describe the characteristics of the graph. What is of course striking is the fact that only at certain wave lengths light is emitted, and that the lines appear in a regular pattern. The graph shows multiple series with the same structure: spectral lines that are closer with decreasing frequency. The pattern suggests a mathematical description which students try to find by fitting the data using a computer. Because of the structure of the pattern, we introduce two parameters: one to enumerate the series, say $m$, and one to enumerate the spectral lines within a certain series, say $n$. The series can best be enumerated starting from the highest frequency, as there appears to be a maximum value above which no more series occur. To enumerate the spectral lines within a series it should be noted that as the frequency increases the spectral lines get closer together. Because it is not clear whether there is an end point here, the lines within a series might best be enumerated from low to high frequencies. We can now try to find a function that fits a given series. This is repeated for the other series and the result is summarized in one expression, giving the frequency of the spectral lines as a function of the two parameters $m$ (i.e. the series) and $n$ (i.e. the line within series $m$):

$$f(n, m) = 32.88 \times 10^{14} \text{ Hz} \left( \frac{1}{m^2} - \frac{1}{(m + n)^2} \right). \quad (6.2)$$

Based on our findings from the photoelectric effect we can argue that the light emitted by hydrogen also comes in photons. Because there we found that the photons’ energy is given by $E = hf$, relation (6.2) can be converted to the energy of the photons emitted. Students are reminded of energy conservation and are asked to give an explanation of the difference in Eq. (6.2). We can understand this difference as a transition between two energy states of the hydrogen atom. This difference is quantized. We can hypothesize that the energy states themselves are quantized as well, which is why we would like to call them energy levels.

\footnote{The graph can for instance be generated by plotting the relative intensity of the lines as a function of their wavelength. This data is available from the NIST Atomic Spectra Database: \url{http://physics.nist.gov/PhysRefData/ASD/lines_form.html}}

\footnote{This highest frequency corresponds to an electron falling back to the ground state from the highest possible excited (bound) state. This corresponds to the ionization energy of hydrogen. The hydrogen ion would give us a continuous spectrum.}
By carefully analyzing experimental data, students reach the conclusion that the hydrogen atom can only have specific energies, and that transitions between these levels emit (absorb) light with an energy equal to the energy lost (gained) by the atom. This approach also avoids observed difficulties students have with the Bohr model: they often think the energy of the emitted photons equals the energy level of the atom (Zollman et al., 2002; Koopman et al., 2006). One objection to teach the Bohr model is its notion of electrons revolving in (classical) orbits. In our description, no such orbits were used. Of course, it remains a question why the hydrogen atom (or the system electron–proton) can only adopt fixed energy states. Furthermore, students will see that the modeling they do is a result of reasoning based on the experimental data available.

**Teaching experiences** This assignment was carried out only in the introductory course on quantum mechanics for first year physics students (Koopman et al., 2006). Students had no difficulties describing the characteristics of the hydrogen spectrum and most students could give a motivated choice for the numbering of spectral lines. Finding a fitting function became more difficult. Here students needed more guidance in finding a suitable function to fit the data with. In particular, students are already satisfied with a function of the form $\frac{1}{n}$. After stimulating students to try other functions to improve the fit and reminding them that enumeration has an arbitrary starting point, students found a good fit and could then easily summarize their findings in one function, resulting in Eq. (6.2). Based on the expression for $f(n, m)$, students were able to give an expression for the energies of the hypothesized levels in the hydrogen atom.

### 6.3.2 Matter waves

After studying the photoelectric effect, we hypothesized that because light has particle-like properties, the relation $E = hf$ also holds for matter (elementary particles). It is thus expected that matter also exhibits wave-like properties. Interference in a double-slit experiment is often used by physics teachers to motivate a discussion of the wave-like properties of matter. An early example can be found in Feynman, Leighton, and Sands (1965). Analysis of the double-slit experiment can help students formulate an hypothesis for the wave function to describe the behavior of electrons. A more detailed description and analysis of student discussions can be found in Koopman, Kaper, and Ellermeijer (2009).

**Double-slit experiment**

Students first study the interference of water waves in a double-slit set-up by describing the interference pattern mathematically. The waves emerging from
the two slits are each approximated by a cylindrical wave. The displacement of these waves are described by the real part of the following complex function:

\[ y = y_0 e^{i(kr - \omega t)}, \]  

with \( y_0 \) the amplitude, \( k \) the wave number, \( \omega \) the angular frequency and \( r \) the distance from the slit. We choose to let students work with this complex function, because it simplifies the addition of two waves and because the complex wave is needed later on. After studying interfering water waves, students should distinguish between displacement and amplitude and use the term 'wave' precisely.

Next, they compare the interference patterns of water and light and try to develop a description of the light pattern analogous to that of water interference. To do this, they have to assume that light can be described by a wave. This idea is not at all new to our students. However, they have never had to worry about the relation between the displacement of the light wave, and the observed intensity on the screen. The main point is that the displacement of the interfering water waves is time dependent, whereas locations of constructive and destructive interference, as they appear on the screen, are independent of time and fixed in space. For water it is possible to investigate this, because the water waves themselves are observable.

For light the distinction between amplitude and displacement is not straightforward. When this distinction has become clear from studying the water waves, students compare water and light to electrons. To explain why an electron interference pattern is visible, students are expected to hypothesize that in some way electrons are to be described by interfering waves. From this we can conclude that the collective behavior of electrons in the double-slit experiment (i.e. the pattern on the screen) can be described by interfering waves. However, it is still not clear how these waves relate to individual electrons.

**Teaching experiences** In high school double-slit interference is discussed, with the emphasis on calculating the points of constructive, and total destructive interference. Here we are interested in the displacement, and the relation with the amplitude of the interfering waves. Students experience most difficulties with the mathematical description of the interference pattern in case of water. In part these difficulties can be traced to a lack of understanding basic concepts of wave physics (Wittmann, Steinberg, & Redish 1999). On the other hand, the analogy with light and electrons is for most students ‘trivial’, but most students seem to miss the point that the wave hypothesized in case of electrons is a new concept whose similarity to water waves is only partial. For instance, in case of electrons we do not know (yet) which quantity (if any) is oscillating. Based on their work, students are able to introduce a wave function to describe the electron interference pattern. Although rudimentary, they

---

6In the case of water waves it is more accurate to speak of circular waves, because it is a surface that oscillates.
assume that the wave function relates to the probability of finding the electron at a certain position on the screen. However, many students identify an electron with a single cylindrical wave. Therefore, seeing that two such ‘waves’ are needed for interference, students reason that at least two electrons are needed: electrons interfere with each other (sic). This view is to be adjusted in the next assignment.

**Interference pattern build-up**

When describing electrons by interfering waves, students might wonder what the electrons are interfering with. Many appear to think electrons interfere with each other. This idea can be tested by performing the double-slit experiment with single electrons, as has been carried out by [Tonomura, Endo, Matsuda, Kawasaki, and Ezawa (1989)](tonomura-endomatsudakawasaki1989). An assignment was designed in which students carefully study the results presented in the Tonomura article.

First students have to convince themselves that, as claimed in the article, at any moment at most one electron is in the apparatus. Because the electrons are accelerated by a known electric potential of \( V_0 = 50 \, \text{kV} \), students can calculate their final speed, ignoring any relativistic effects.\(^7\) A similar calculation was needed in the photoelectric effect assignment to calculate the electrons’ kinetic energy. The electrons are thus found to be accelerated to \( v = 1.3 \times 10^5 \, \text{km/s} \). Each second one thousand electrons are emitted. Their average separation thus would be \( d = 130 \, \text{km} \) if they could move unhindered through a vacuum. However, the apparatus can be assumed to have a dimension in the order of one meter. It is thus safe to say that at most one electron is in the device: electrons cannot interfere with each other. The conclusion should thus be that one electron has to be described by two cylindrical waves emerging from the two slits. The superposition principle, used to calculate the amplitude of the interfering waves, can now be given a new meaning. Each of the cylindrical waves was assumed to describe an electron emerging from a slit. Now we have two superimposed waves describing one electron. We do not know through which slit the electron traveled; up to now we can only say that the superposition describes the potentiality that the electron will pass one of the two slits. For students a natural question to ask is: “Through which slit did the electron go?” This question will be addressed further on in the design.

**Teaching experiences** Students have no difficulties determining that there is only one electron in the apparatus, but they do find this conclusion disconcerting. They infer that this means that electrons do not interfere with each other. The waves interfere, and this wave tells with what probability an electron will hit the screen at a certain position. Although not all students use this formulation, most express the probabilistic aspect of the wave function. What remains difficult is that the wave function is understood to describe one

---

\(^7\)Students have heard about special relativity, and are told that a classical treatment is a good approximation. The error in doing so is approximately 7%.
individual electron. Many students still seem to view each of the cylindrical waves as describing one electron. This is a weak point in the design we would like to fine-tune.

**De Broglie postulate**

If electrons should be described by a wave function to account for their behavior in the double-slit interference experiment, then what is the relation between the wave length and frequency of this new wave and the particle-like properties we are so much accustomed to? In the double-slit experiment the wave length appeared to be an important parameter to describe the interference fringes. On the other hand, in the single electron double-slit experiment, we describe the electrons before entering the double-slit as classical particles having momentum. In general, students are used to momentum as a property of (classical) particles. The question now is what the relation is between the momentum and the wave length of the electrons? This relation is formulated by the de Broglie postulate. De Broglie arrived at his postulate on theoretical grounds ([Broglie 1924/2004](#)). Because our students do not have this same theoretical background, we use another way in which our students might formulate this relation between momentum and wave length.

An elegant demonstration experiment that can be used for this purpose is the diffraction of electrons on graphite in a set-up where the accelerating voltage, and thus the momentum of the electrons, can be varied. This way the relation between electron momentum and interference pattern properties can be studied. This leads to the result that the relation between the wave length of the matter wave, and the momentum of the electron is given by: \( p \propto \frac{1}{\lambda} \). Up to a constant factor, this is just the de Broglie relation for momentum and wave length: \( p = h/\lambda \). This relation is needed when searching for a new theoretical framework, described in Section 6.3.4.

**Teaching experiences** The demo was presented to the whole group, and with use of multiple choice questions, students determined the relation between momentum and wave length. With the set of relations between the physical quantities in place, students did not have much trouble reaching the intended conclusion.

**6.3.3 Probability**

From the double-slit experiment for electrons and the build-up of the interference pattern by single electrons, students conclude that the electron wave function only seems to describe the probability with which an electron will hit the screen at a certain location. At this point two main questions can be asked. First, what is the relation between the wave function and this probability? Second, as mentioned before, a natural question for students to ask when studying the interference pattern build-up is through which slit the electron
6.3. Design of the studio course introduction

goes. These two questions play a central role in the following assignments. We start with the latter question.

**Which-way experiment**

It puzzles students that an interference pattern also emerges when electrons are fired one by one at the double slit. Students spontaneously try to formulate hypotheses that can account for this behavior. One such idea is that the electron somehow ‘splits up’, passes through both slits, and recombines to give rise to the interference pattern. Furthermore, thinking of the electron as a classical particle, students wonder through which slit the electron goes. This is the objective of a so-called which-way experiment. As Feynman describes so vividly in “QED, the strange theory of light and matter”, if we detect through which slit the electrons pass, the interference pattern disappears (Feynman 2006).

From discussing the which-way experiment it should become clear that a superposition of two cylindrical waves describes a single electron. As seen before, the superposition can account for the interference pattern. However, a consequence of this description is that we do not know which slit the electron has passed through. Conversely, an electron of which we know through which slit it goes is described by a single cylindrical wave. But a single cylindrical wave is not able to describe interference. Thus, the results of the which-way experiment are in line with our description of electrons.

**Teaching experiences**  In our experiments there was not enough time to let students explore such an experiment themselves. In one experimental round we have explained this experiment to the whole group. The results were not satisfactory. Students had difficulties following the argumentation. Because the outcome is so strange, students might benefit from an activity in which they can explore this experiment themselves and are guided to think over what the consequences are. From our last experimental round, we have found that such an activity is needed to let students fine tune their conception of the wave function as introduced in the double-slit experiment.

**Superposition revisited**

The superposition needed to explain interference has a strange consequence in the sense that we no longer now beforehand which slit the electron passes through. We have a 50%-50% change of finding the electron in either of the two slits. Thus, the superposition has a special meaning when we consider measurements. To address this we let students study the Stern-Gerlach experiment (Gerlach & Stern 1922). We choose this experiment because it enables us to discuss what happens when measuring a physical quantity (or observable) for a state which is a superposition. Furthermore, as explained below, the concept of spin can be introduced, which is needed further on in the quantum chemistry course.
The Stern-Gerlach experiment is another illustration of a quantized system, but this time it is the observable angular momentum that is quantized. In the original set-up of the experiment, Stern and Gerlach used a beam of silver atoms that were sent through an inhomogeneous magnetic field. Because the silver atom has a magnetic moment, atoms will be deflected up or down by this magnetic field. The initial orientation of an atom will determine the location where it will hit the screen. Classically the dipoles can have any orientation in space so the magnetic field will spread them out into a continuous broad band where they strike the screen. The outcome of the Stern-Gerlach experiment, however, is very different. Two separate narrow bands appear where the silver atoms hit the screen. This suggests that the silver atoms can have only two orientations in the magnetic field: either up, or down. Our students are familiar with magnetic dipoles and their behavior in homogeneous and inhomogeneous magnetic fields. They are unfamiliar with the idea that an atom may behave as magnetic dipole. However, they do not need to fully understand this in order to understand the Stern-Gerlach experiment. Based on their prior knowledge, our students are able to follow what is going on. In the activity described below, our students are guided to formulate a hypothesis that explains the experimental outcomes of the Stern-Gerlach experiment.

The SPINS applet [Schroeder & Moore, 1993] enables students to explore what will be the outcome of such a sequence of measurements with these devices. They will see another illustration of superposition. When the first device selects particles with up spin (in the $z$-direction), and the next measures the spin in the $x$-direction, students will find that half of the particles now have spin up in this new direction, while the other half have spin down.

Now how can we account for that? First of all, we should not that we only want to describe the spin of the particles, not their position. We will denote the spin state with $\chi$. Furthermore, the fact that we only measure up or down, can be explained by assuming this spin state function describes two possible outcomes. In case of the double-slit experiment we have seen something similar. There the superposition of the two cylindrical waves described an electron that emerges from one of the two slits. Thus these two added waves should be reinterpreted as two possibilities. We might call these
possibilities *eigenstates*: accepted measurement outcomes. In case of spin, we might symbolically denote this situation as:

$$\chi = \uparrow_z + \downarrow_z,$$

(6.4)

where the $z$ subscript denotes the orientation of the Stern-Gerlach magnet. The up-arrow corresponds to an up measurement, the down arrow to a down measurement. (We have ignored the normalization constant the above equation.)

What about the finding that the up particles from a $z$-measurement appear to be again evenly distributed in up or down in the $x$-direction? If we assume that our wave function should also describe these possibilities, we must write this as:

$$\chi = \uparrow_x + \downarrow_x.$$  

(6.5)

But because this state should be equal to the up-state in the $z$-direction, we have (again, ignoring normalization):

$$\chi = \uparrow_z = \uparrow_x + \downarrow_x.$$  

(6.6)

Thus, the eigenstate (possible measurement outcome) $\uparrow_z$ can be written as a superposition of eigenstates in the $x$-direction. This gives a new meaning to superposition we have not found in classical mechanics.

Apart from motivating the need to reinterpret the notion of superposition when talking about measurements, using the Stern-Gerlach experiment has another advantage. We have found that students associate each of the cylindrical waves in the description of the double-slit experiment as an electron. The Stern-Gerlach experiment gives the opportunity to revise this idea.

**Teaching experiences**  As mentioned at the beginning of this section, due to lack of time, we have only been able to introduce the Stern-Gerlach experiment, without letting students work with the designed activity. Our experiences are that students understand the experimental set-up. Students are able to reason how a magnetic dipole will behave when it is passed through a Stern-Gerlach magnet. They also correctly describe how classical magnetic dipoles give rise to a continuous broad band.

**Classical probabilities**  

One of the conclusions of the previous assignments is that the electron wave function is able to describe the statistical aspect of the interference pattern. We first focus on how a physical system can be described probabilistically. In the next assignment we can consider how electrons can be described probabilistically and what the relation is between the wave function and the probability of finding an electron.

[Bao and Redish (2002)] have identified difficulties in learning quantum mechanics originating from a poor understanding of probability. They recommend
students examine a pendulum as a classical system and describe it in a probabilistic way. For this purpose we designed an activity in which the classical system was simulated by an applet of a mass on a string [Belloni, Christian, and Cox (2006)]. The applet shows a mass swinging back and forth on a string. If the swinging mass is stopped at random times, with what probabilities will it be found at various locations? Of course, where the mass moves slowly, it spends more time. These are the locations where it is found most probably. Students can see this by imagining a video is made for a certain number of oscillations of the mass. On each video frame, the mass is found at a certain location. The student can make a histogram of the number of frames belonging to a certain interval. The higher the bars in the histogram, the higher the probability of finding the mass in that particular interval.

Students can manipulate the swinging mass in the applet, and a table and histogram of data are generated automatically. In the activity, we ask students to calculate the probability of finding the mass at a certain location. For a given number of intervals, this probability corresponds to a certain region, not to a precise location. How can we make the location more precise? Students increase the number of intervals and recalculate the probability. They will notice that this probability decreases, and in the limit of an infinite number of intervals goes to zero. This is thus not a good measure for probability at a certain location. Students repeat their calculations, now dividing the probability by the interval length. This value remains (more or less) constant and it is a good measure for the probability at a certain location. This new quantity is called the probability density. Using this new concept, students calculate the probability of finding the mass in a given region, an arbitrary region, and in the whole range through which the mass moves.

Based on this work it is now straightforward to formulate the relation between the number of electrons found in a certain region (the intensity $I$), and the electron wave function. The quantity $|\Psi|^2$ plays the same role as the probability density introduced in the assignment on the swinging mass.

**Teaching experiences** After working with the applet, most students make the connection between the velocity of a classical particle, and the probability of finding it in a certain interval. They also see the use of introducing a probability density for a continuous observable (such as the location). Finally, students come up with a mathematical relation between the probability of finding a classical particle in a certain interval and the probability density associated with the particle. When students are reminded of the relation between wave function and electron density, they are able to use this to formulate what we call the Born statistical interpretation:

$$P(a, b) = \int_a^b |\Psi|^2 \, dx,$$

where $P(a, b)$ is the probability of finding the electron between $x = a$ and $x = b$.  

124
6.3. Design of the studio course introduction

**Diffraction at a straight edge**

Zeilinger *et al.* have carried out an experiment in which cold neutrons are diffracted at a straight edge (Zeilinger, Gaehler, Shull, & Treimer, 1982; Gähler, Klein, & Zeilinger, 1981). The results they present might enable students to formulate a functional dependence between the wave function and the probability of finding an electron in a certain region. For sake of simplicity, we assume electrons will behave similarly in such an experiment.

When electrons are diffracted at a straight edge, the pattern of electrons as measured behind the edge shows interference fringes (Figure 6.6). Because we have assumed that electrons are to be described by a wave, we might in principle calculate the amplitude of this wave behind the edge. Performing this actual calculation is beyond the scope of our course. However, we can think of the opening next to the edge as made up of infinitely many, infinitely small ‘slits’. Each slit emits a cylindrical wave as hypothesized in the double-slit assignment. These waves add up (superposition principle). Then from symmetry considerations, one can argue what the amplitude will be of the interfering electron waves along the screen in the middle of the set-up ($x = 0$), far behind the edge ($x \ll 0$), and far away from the edge ($x \gg 0$), as presented in Table 6.1. We can now compare these values to the number of electrons actually measured, as given in Figure 6.6.

![Figure 6.6: Number of electrons measured behind the straight edge, along the x-axis. The error bars represent counting statistics. Graph adapted from Zeilinger et al. (1982).](image)

Table 6.1 compares the expected amplitude to the measured electron intensity $I$, the number of electrons measured in during a certain time, in an interval of $x$. A quadratic relation between amplitude and electron intensity is able to describe these values. Because in the double-slit assignment the electron wave function was expressed as a complex function, this relation can be formulated as: $I \propto |\Psi|^2$. Of course, there is still a problem: $\Psi$ is a function of the continuous variable $x$, whereas $I$ is the number of electrons found in a certain interval of $x$. To find a better relation between the two, we let students first look at classical probabilities and probability densities.
Chapter 6. Designing an Introduction to Quantum Mechanics

Table 6.1: Amplitude for the hypothesized electron wave behind the straight edge ($A$), compared to the measured electron intensities ($I$), in the center of the set-up ($x = 0$), far behind the straight edge ($x \ll 0$), and far from the edge ($x \gg 0$). $A_0$ and $I_0$ are reference values for the amplitude and intensity without the straight edge.

<table>
<thead>
<tr>
<th></th>
<th>$x \ll 0$</th>
<th>$x = 0$</th>
<th>$x \gg 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A/A_0$</td>
<td>0</td>
<td>1/2</td>
<td>1</td>
</tr>
<tr>
<td>$I/I_0$</td>
<td>0</td>
<td>0.25</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Teaching experiences  In finding the amplitude of the hypothesized waves, behind the straight edge, students were expected to use the superposition principle, also used in the double-slit assignment. However, it appeared difficult to transfer this method from the double-slit assignment to the current assignment. Furthermore, students were unable to generalize from the two slits in the double-slit assignment to the infinitely many, infinitely small ‘slits’ in the straight edge assignment. Therefore, most part of this reasoning was done in a teacher-led discussion. This helped students find the expected amplitudes. Comparing the theoretical amplitudes to the experimental values, students easily come up with the quadratic relation between the wave function $\Psi$, and the electron intensity $I$.

6.3.4 Towards a new theory

The main concept introduced thus far is the wave function $\Psi$. Students should have an idea what its meaning is, and how it relates to experiment: a probabilistic interpretation is formulated. Based on the de Broglie postulate we also know that momentum is associated with a certain wave length of this wave function. What is lacking is a prescription that tells us how we can determine $\Psi$ for a given physical system. Students explore this in an assignment that ultimately leads to the Schrödinger equation.

This assignment is the most difficult one. First of all because of the mathematics involved. Secondly, because students are expected to reason how we can explain the findings of the previous assignments. This requires inductive reasoning. The mathematics, however, has all been discussed in the preparatory Calculus course and was repeated in the remedial mathematics assignments. Moreover, the assignment guides students by taking small steps.

The approach described here is based on Eisberg and Resnick (1985). Just as $F = m\ddot{x}(t)$ in classical physics is a differential equation with which we can determine $x(t)$, we would like to find a (differential) equation with which we can find $\Psi$ for a given situation. It should be emphasized that students will not derive this equation, as in deducing it. This is not possible. The point is that we are looking for a new formal system, because we have acknowledged that classical mechanics falls short in explaining electron interference. Based
on the results from the previous activities, this differential equation should meet the following criteria:

1. With no forces acting, we should find the solution:

\[ \Psi(x, t) = u_0 \exp \left[ i(kx - \omega t) \right]. \tag{6.8} \]

This is based on the wave function used to describe the electrons in the double-slit experiment, equation (6.3).\[ ^8 \]

2. If \( \Psi_1 \) and \( \Psi_2 \) are separate solutions, \( a\Psi_1 + b\Psi_2 \) should be a solution as well (linearity). This is the superposition principle. Students have already assumed it to account for the interference observed in the double-slit experiment.

3. If \( \Psi \) has a wave number \( k \) (i.e. wave length \( \lambda = 2\pi/k \)), it should describe a particle of which we can say it has a momentum \( p = \hbar k \). This follows from the demonstration experiment in Section 6.3.2.

4. Likewise, if \( \Psi \) has an angular frequency \( \omega \) (i.e. frequency \( f = \omega/2\pi \)), it should describe a particle with total energy \( E = \hbar \omega \). We here generalize the result from the photoelectric effect to particles with mass.

The problem now is choosing a starting point. We present students the choice to start either from Newton’s second law \( F = m\ddot{x}(t) \), or from an expression for total energy (a concept needed for energy conservation). We expect students to give a motivated choice based on their findings in the previous activities. A description in terms of \( x(t) \) is not consistent with the interpretation we give \( \Psi \): a trajectory cannot be assigned to our quantum particle. Therefore, Newton’s second law is not a good starting point.\[ ^9 \] We do, however, want to preserve at least from classical physics the notion of total energy.

Our starting point is thus:

\[ E = E_{\text{kin}} + V. \tag{6.9} \]

We start without a force (criterion one). This means \( V \) is a constant, and we might as well start with \( V = 0 \). The total energy of our system then is: \( E = p^2/2m \). Rewriting this using wave quantities, we find:

\[ \hbar \omega = \frac{\hbar^2 k^2}{2m}. \tag{6.10} \]

We now want to introduce \( \Psi \) in Eq. (6.10). We can do this by using our known solution, Eq. (6.8), and try to ‘extract’ \( k \) and \( \omega \) from it. This is analogous to

\[ ^8 \]We generalize that description from a radius \( r \) to a one-dimensional system with coordinate \( x \). Furthermore, we assume that the electrons move freely between the double-slit and the screen, without forces acting on them.

\[ ^9 \]In his 1925 article, Heisenberg actually did start from \( F = ma \), by redefining \( x \).
how we would find the speed in classical mechanics, by differentiating $x(t)$. This results in:

$$\omega = i \frac{1}{\Psi(x,t)} \frac{\partial \Psi(x,t)}{\partial t}, \quad (6.11)$$

$$k^2 = - \frac{1}{\Psi(x,t)} \frac{\partial^2 \Psi(x,t)}{\partial x^2}. \quad (6.12)$$

If we substitute these expressions for $\omega$ and $k^2$ in Eq. (6.10), we find:

$$\frac{i \hbar}{\Psi(x,t)} \frac{\partial \Psi(x,t)}{\partial t} = - \frac{\hbar^2}{\Psi(x,t) 2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2}. \quad (6.13)$$

Now if $V = V_0 \neq 0$, the total energy changes and repeating the steps above we will find the following:

$$i \hbar \frac{\partial \Psi(x,t)}{\partial t} = - \frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V_0 \Psi(x,t). \quad (6.14)$$

This equation satisfies the four criteria we started with. However, it does not tell us anything new, because solutions to this equation are already known to us.

The big step comes by hypothesizing that the differential equation found thus far, should also hold when the potential is some function of $x$, instead of just some constant $V_0$. We make the substitution $V_0 \rightarrow V(x)$ and find Schrödinger’s equation:

$$i \hbar \frac{\partial \Psi(x,t)}{\partial t} = - \frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x) \Psi(x,t). \quad (6.15)$$

Note that here we have silently replaced the $\Psi$ from Eq. (6.8) by a general $\Psi$.

Students might wonder whether a complex function to describe the electron is really necessary. This can be checked easily: by substituting the real part of expression (6.8), we can readily see this is not a solution. However, what if we would have started with the real part in finding a differential equation, in the first place? Then the approach followed here, taking derivatives of $\Psi$ to express $\omega$ and $k$ would break down. Although this would not make a difference for $k$, $\omega$ will have to be expressed in terms of the square root of $\Psi$ and the second derivative of $\Psi$ with respect to time. We would fail to meet criterion two: our differential equation is not linear and a superposition is not a solution anymore.

The Schrödinger equation is thus obtained by a creative process, making careful choices based on the experiences we have given students. There are of course many more ways to arrive at this result\textsuperscript{10} and it cannot be said that either approach is correct. We feel this is an important message to our students. What still has to be discussed is the normalization of $\Psi$. Without using

\textsuperscript{10}Schrödinger’s approach was based on the principle of least action for instance.
6.3. Design of the studio course introduction

this word, the probability density in the assignment on classical probabilities was already normalized. Also, it is important that $\Psi$, as it evolves as dictated by Schrödinger equation, remains normalized at later times (conservation of probability). It is also interesting to say something about the classical limit: when will we notice that particles should be described by a wave?

**Teaching experiences** Students find it difficult to choose between the two possible starting points for a differential equation for $\Psi$: Newton’s second law, or an expression for the total energy. They know that the idea of a trajectory for a quantum particle, as expressed by $x(t)$, contradicts the idea of a wave function they have introduced. However the assignment stated that we would try to find a solution for $\Psi$ in case of forces acting on the particle. Although attention has been given to the relation between force and potential energy, students get confused by this remark, making it difficult for them to choose a starting point. We here see that prior knowledge is important, particularly in cases where we introduce uncertainties to students. Some students were more confident that Newton’s second law is not suitable, and chose the relation for total energy.

Continuing with the total energy, the task is to introduce as many ‘wave quantities’ as possible, i.e. angular frequency $\omega$ and wave number $k$. Most students remember the de Broglie relation $p = \hbar k$. However, some do not come up with the Einstein relation $E = \hbar \omega$ and try to find a relation they do know that involves the frequency $f$ (in stead of $\omega$). They associate the frequency with light, and continue with $f = c/\lambda$. This is a dead end, and feedback of the teacher is needed to help students come up with $E = \hbar \omega$. What students find difficult here is the assumption that the Einstein-de Broglie relations hold for light, as well as for matter.

Taking the first and second derivatives with respect to position and time of the wave function, Eq. (6.8), consumes a lot of time, although students have assigned practice with them in preparation for this particular meeting. Some students get stuck at this point. At least one of the student groups finds the correct derivatives and uses them to rewrite the energy relation. With these derivatives, this group actually is able to set up the Schrödinger equation. All observed groups had difficulties applying what was learned earlier as criteria to find the equation for $\Psi$.

One of the last questions in this assignment was whether the differential equation found this way is derived. Students do seem to think they have derived the Schrödinger equation. Up to this point in their education, students have not been taught to ‘derive’ in this sense. It is thus not strange that they think they have given a derivation. It does give the lecturer an opportunity to explain what the difference is between a derivation within some system and the setting up of such a logical system by wisely choosing axioms and rules. In this assignment the inductive steps are explicit. It gives students a little insight in what it means to formulate a physical theory. Many students often believe that a theory already exists, whereas it can be seen as a product of
6.4 Conclusion and discussion

The sequence described here ends with the formulation of the Schrödinger equation. Up to this point we have been able to follow the four design principles laid out at the beginning of this article. First of all, students have shown to have enough background knowledge to understand the chosen experiments (principle 1). Furthermore, students have shown to be able to formulate many of the hypotheses by studying these experiments (principle 2). Finally, throughout the design we have indicated where we made use of available PER literature (principle 3).

This is by no means the ultimate and only possible route to the Schrödinger equation. Furthermore, we have made choices in selecting the experiments we have discussed with our students. For instance, we might also want to discuss the Franck–Hertz experiment, or the Compton–effect. However, time is limited and choices have to be made. It is already our experience that considerable time needs to be reserved just for discussing the setup of an experiment.

It is wonderful to see students making reasoned choices in setting up the Schrödinger equation. However, we can hardly call this a theory. For instance, we still do not know how to express all the physical quantities we know from classical mechanics. We find it difficult to imagine how a guided discovery approach, as followed up to here, could be prolonged from this point onwards. Therefore we find it natural to stop the guided discovery approach here, and let the traditional axiomatic teaching take over. Our hypothesis is that after this introduction, students will appreciate an axiomatic discussion better. In our research we have not been able to test this hypothesis. We have also not tested the effectiveness of our approach, because we do not consider the design as complete, or finished. Moreover, testing the effectiveness would require more students than we have available in our department. It would only be possible if other (Dutch) universities would be involved.

It would be interesting to further investigate the effect of this introduction on students’ conceptual understanding and problem solving abilities in the remainder of our course. We do have qualitative results on these effects. Our findings are that the integration of the introduction in the remainder of the course plays an important role in the effects. If there is little reference to what has been done in the introduction, students will see it as ‘something else’, and there is little transfer. Furthermore, it should be clear to students that it is important for them to develop their conceptual understanding of quantum theory. In the current situation there is emphasis on number crunching and manipulation of symbols and expressions. Students see quantum theory as ‘enabling them to do calculations’. While it is a great merit of quantum theory that it enables us to do such accurate calculations, we surely want students to also appreciate, and, at least basically, understand this theory.
References


