A developmental research on introducing the quantum mechanics formalism at university level
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Chapter 7

An Analogical Reasoning
to let Students Hypothesize
the Wave Function

The double-slit experiment for electrons is often discussed in introductory
courses on quantum mechanics to explain how electrons can behave as waves.
In analogy with double-slit interference of light, or water waves, it is motivated
that electrons too behave as a wave. However, research on the educational use
of the double-slit experiment and the role it can play in introducing the wave
function is sparse. This article applies research on analogical thinking and
the implementation of analogies in teaching to design a teaching sequence in
which students are guided to introduce the concept wave function themselves.
The teaching design implements an analogical reasoning using a three-phase
design (Kaper & Goedhart, 2005). Four introductory studio course meetings
were developed for the first year course Quantum Chemistry, part of which are
two assignments that implement the three-phase analogical reasoning. Three
groups of two students each were audio recorded and their discussions ana-
yzed to determine whether (a) the three phases of analogical reasoning occur
as expected, (b) a new term wave function is introduced by students, and (c)
whether this new term is acceptable from a scientific viewpoint. This study
shows that the three phases of the analogical reasoning are not passed through
sequentially. Furthermore, students successfully hypothesize a wave function
to explain the behavior of electrons. Their interpretation of this term varies,
but is mainly consistent with the scientific term, although it needs refinement.

For chemistry students Quantum Chemistry is not the most popular course
in the curriculum. Most students have chosen chemistry as a major subject
not because of the physics involved. This is a challenge for those lecturing the
subject.

In a first experimental round of this research learning difficulties were iden-
tified of students following the first year course Quantum Chemistry at the
University of Amsterdam. Students appeared to have major difficulties with
the mathematics involved. Before any possible learning difficulties related
to the subject might be addressed, it was considered wise to first tackle the
problems students experienced with the mathematics. The next experimental round an existing remedial mathematics program was extended to the Quantum Chemistry course, with the desired effect (Koopman, Brouwer, Heck, & Buma [2008]). Also an effort was made to assist and stimulate students as much as possible in their studying. Before each lecture students were encouraged to make an online test to prepare for the lecture (warming-up questions). Based on the test results, the lecturer could decide how much time he would spend on various subjects. This method is called Just-in-Time teaching (Novak, Gavrin, Christian, & Patterson [1999]). In addition, students were given a self-test after each lecture, with which they could acquire a bonus to their final grade (cooling-down questions).

These interventions had the desired effect on students’ mathematical skills and study strategies. Unfortunately, many students still failed to pass the course. Our conclusion was that the remaining difficulties students had, would have to originate from the subject matter. Some of the difficulties were found to originate from students’ (lack of) pre-knowledge on classical physics. Concepts such as potential energy and angular momentum also play an important role in quantum mechanics (Steinberg, Wittmann, Bao, & Redish [1999]; Koopman & Ellermeijer [2005]). However, students seemed to experience most difficulties with fundamental quantum concepts such as the wave function, operators, and expectation values (Bao & Redish [2002]; Koopman & Ellermeijer [2005]). For instance, students do not see the need for a wave function and often interpret it as the trajectory of an electron. Problematic in introducing concepts such as the wave function is that they have little relation to concepts already known to students. Moreover, these concepts are part of a logical system making it difficult to teach them isolated from the system to which they belong. We hypothesize that students will benefit from an introduction to quantum mechanics where they construct (hypothesize) quantum concepts themselves, starting from phenomena that are meaningful to them (i.e. without relying on quantum theory). In other words, we propose a guided discovery approach to introduce (the fundamentals of) quantum mechanics.

To implement this idea, where students hypothesize quantum concepts, four introductory meetings were designed that should help students work their way up to quantum theory. In these introductory meetings we start as much as possible from observations that students are able to interpret with their prior knowledge, without referring to concepts from the quantum theory we are teaching (avoiding vicious circles). In this article we will focus on the teaching design that aims at letting students hypothesize the wave function as a way to describe the behavior of electrons. We propose that this can be done using an analogy between double-slit interference experiments for water waves, light, and electrons. By aiming at an analogical reasoning we let students hypothesize the wave function as a means to explain the interference pattern of electrons.

Research on analogical reasoning focuses on either of the following topics:

1. Cognitive processes in analogical thinking (e.g. Gentner [1983]):
2. Providing schemes of implementation for use in education (e.g. Treagust, Harrison, & Venville [1998];
3. Using analogies in specific domains to overcome learning difficulties (e.g. Paatz, Ryder, Schwedes, & Scott [2004]).

This research uses results of the first two categories to contribute to the third category. Many introductory textbooks on quantum mechanics discuss the double-slit experiment and compare outcomes in different domains (see for example Feynman, Leighton, & Sands [1965]). However, research on the usage of this analogy in physics education is hard to find. With this article we hope to contribute to the understanding of this analogy, its usage in education and explore its possible role in the teaching and learning of quantum mechanics.

7.1 Theoretical Framework: Analogical Reasoning

In education analogies are often found to be useful to explain unfamiliar, new content, i.e. the target domain, using content already known to students, i.e. the base domain (Duit [1991], Wong [1993], Treagust et al. [1998], Coll, France, & Taylor [2005]). When using analogies in education, it is important to know what influences their effectiveness. Holyoak and Thagard [1989] propose three constraints to the strength of an analogical mapping:

- **pragmatic** Analogical thinking is sensitive to the purpose for which the analogy is being used.
- **semantic** Terms are being used in the two analogs that have related meanings.
- **structural** The two analogs use similar configurations of objects (Thagard [1992], p. 538, paraphrased).

Thagard [1992] summarizes that “ideally, a source analog should have great semantic similarity, structural correspondence, and pragmatic relevance to the target” (p. 539). However, when retrieving a potential base domain from memory, the most important constraint is semantic similarity. When base and target domain of the analogy are presented to students and only the mapping between them has to be determined, then their structural correspondence is the most important constraint. As we shall explain below, the latter situation applies to our experiment: we present students with both base and target domain. Therefore, the most important constraint is the structural correspondence of base and target domain.

The structural correspondence of an analogy is well described by Gentner’s structure-mapping theory (Gentner [1983]). Base and target domains are expressed as a set of objects, attributes, and relations (between objects and other relations), forming a structure. An analogy defines a mapping of the base domain structure onto the target domain structure. Relations are preserved as
much as possible, whereas attributes of objects are discarded. However, not all relations are mapped when considering a concrete analogy: choices are made. In particular, a relation that itself is connected to other relations by a higher order relation is to be favored over an isolated relation. Gentner calls this the \textit{systematicity principle}: It expresses the fact that we value connected knowledge more than isolated facts.

Gentner’s description of analogy might especially be interesting from a teachers point of view; knowing what the structure of the analogy is and how it can be used to teach certain concepts. Both Gentner’s and Holyoak and Thagard’s theory give constraints to the analogy used and its expected effect in education. What is needed in addition, is a description how to organize a learning process in which analogical reasoning is expected to occur. Kaper and Goedhart (2005) argue that an analogy can be used productively in education when organizing the learning process in three phases, on which we will base our teaching design. These three phases of the analogical reasoning are defined as:

0. Learners study base and target domains separately.
1. Homonymous relations in the description of both domains are noticed. Mapping relations from base to target domain is expected to motivate mapping of corresponding terms in these relations. A term from base domain and its mapped term in the target domain each refer to different concepts.
2. Some terms or relations in the base domain are noticed as not having corresponding terms or relations in the target domain. Learners hypothesize corresponding terms and relations.
3. The usefulness of new terms or relations is evaluated against experience. (pp. 298–299, paraphrased)

The hypothesized terms or relations from phase 2 make it possible to map higher order relations, which play an important role in analogy, as argued by Gentner (the systematicity principle, explained above). Because an analogical reasoning is an adventure where it cannot be known in advance in how far it will be successful, we regard it as essential that students are aware of the hypothetical status of the terms created in step 2. Such terms stand for hitherto unobserved entities. We call them \textit{model terms} and propose that students’ use of these terms should have these characteristics (Kaper & Goedhart 2002):

- Model terms are used to explain and predict observations.
- Use of model terms is tentative, i.e. subject to evaluation and change.

Without these characteristics, we will conclude that the new terms are not used as model terms. In other words: students are not sufficiently aware of their hypothetical status.

In many analogies used in education, both base and target domain are classical systems. Neither learning theories here described, give restrictions
to the nature of the domains in the mapping. In Gentner’s view an analogy is simply more powerful when more higher order relations are mapped from base to target domain, discarding attributes. Thagard (1992) does observe that chemistry teachers seem to favor analogies where base domain is some everyday life situation, familiar to students, whereas physics teachers use other physical systems as base domain. One might think that the analogy used in this research is a special one, as the base domain is a classical system, whereas the target domain is a quantum system. However, it is a principle characteristic of analogy to (systematically) compare two systems from completely different domains. As with all analogies, it is to be expected that the analogy used in this research will have its limitations when mapping terms and relations to the quantum domain. These limitations are accounted for in the third phase of the analogical reasoning.

There does exist some evidence against mixing classical and quantum systems. To teach atomic structure, the Bohr model is often used. To arrive at the Bohr model, an analogy is used in which the atom is compared to a solar system (clearly a classical system). Fischler and Lichtfeldt (1992) have found the use of the Bohr model to be problematic, in particular because it refers to classical mechanics. More recently, however, McKagan, Perkins, and Wierman (2008) have shown under what conditions the Bohr model can be used effectively to teach atomic structure. They have found that it is important to compare and contrast different models, and thus letting students focus on the features, as well as the limitations of various models used. Furthermore, it is stressed that linkage between these models influences the success of their use in education. This is in line with the approach of Frederiksen and White (2002) to teach a certain topic (e.g. electricity) by constructing a chain of linked models, increasing in abstractness. Similarly, Clement and Brown have found the concept of bridging analogies to be useful in this context (Brown & Clement, 1989; Brown, 1992, 1993; Clement, 1993). In particular, Brown (1992) has found that a bridging analogy is more effective than using examples when students hold initial misconceptions. One of the reasons for this difference is that a bridging analogy helps students construct a mechanistic model that explains the workings of the target domain. The three phases of analogical reasoning, as explained above and used in this research, are organized such that they incorporate two important ideas: systematically compare base and target domain (linkage), and focus on the (possible) limitations of the analogy.

7.2 Research Questions

The main research question is:

Can the quantum mechanical concept wave function be introduced by students using an analogical reasoning starting from classical base domains, prompted by a teaching approach as structured by the phases 0-3?
This main question has three subquestions:

1. Can the phases 0-3 of the analogical reasoning be discerned in the student discussions?

2. Is a model term wave function introduced by the students in the target domains?

It is very well possible that students introduce a model term wave function that does not have a correct interpretation when viewed from a current scientific viewpoint. Therefore, students need to evaluate their hypothesized term wave function for electrons, which is done in phase 3 of the analogical reasoning. After this evaluation, the question remains:

3. What interpretation do students give this hypothetical wave? How does this interpretation compare to the scientific term wave function?

7.3 Methodology

7.3.1 Context

The research was carried out in an existing course, Quantum Chemistry, given in the second semester of the first year chemistry study. The experiment consisted of four introductory, studio course meetings (Table 7.1) given before the start of, and in addition to the regular lectures and tutorial sessions. A studio course combines lecture and problem solving session [Wilson & Jennings, 2000]. These studio courses were developed throughout multiple research rounds, using a developmental research approach [Gravemeijer, 1994]. Relevant preparatory courses in the first semester are Calculus and General Chemistry. The latter gives a general overview of the structure of matter by discussing subjects such as: atomic structure, the periodic table, chemical bonding, and the structure of molecules.

After the introductory meetings, the Quantum Chemistry course continues with twelve lectures and corresponding problem solving sessions (tutorials), of two hours each. In the first half of the course the principles of quantum mechanics (Schrödinger’s equation, wave function, operators, expectation values, etc.) and simple applications (e.g. particle in a box, harmonic oscillator) are discussed. In the second half of the course these principles are applied to study atomic structure and molecular bonding. Text book used is Atkins and de Paula’s Physical Chemistry [Atkins & de Paula, 2006]. During the course we maintained the warming-up and cooling-down questions, as well as the remedial mathematics program, as described in the Introduction.

7.3.2 Sample selection, data collection, and analysis

The research questions require qualitative research in which we can closely follow students’ reasoning during classroom sessions. This limits the number
Table 7.1: Topics covered in the four introductory Studio Course meetings.

<table>
<thead>
<tr>
<th>1. matter waves</th>
<th>2. meaning of matter waves</th>
<th>3. probability</th>
<th>4. a new theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>– double-slit experiment (water, light, electrons)</td>
<td>– build-up of electron interference pattern (interpretation of the wave function)</td>
<td>– electron diffraction at a straight edge</td>
<td>– Schrödinger equation</td>
</tr>
<tr>
<td></td>
<td></td>
<td>– classical probabilities</td>
<td>– operators</td>
</tr>
<tr>
<td></td>
<td></td>
<td>– potential energy and force</td>
<td>– expectation values</td>
</tr>
</tbody>
</table>

of subjects we can study. In total 30 first year chemistry majors enrolled for Quantum Chemistry. From this group six students were selected and asked to volunteer as subjects in this research. This sample (20% of the total) should be representative for the whole group as it included both males and females, as well as weak, average, and high achieving students. The students formed groups (two students each), denoted by group 1, 2, and 3 in the following. Groups 1 and 2 varied a little in composition, as one student quit the course, while some others joined these groups occasionally. Sometimes groups cooperated with neighboring groups. Group 3 proved to be the most stable group: the two students worked together throughout the course. Table 7.2 lists the composition of the observed groups during the four introductory meetings. The groups were audio recorded throughout the course and written work was collected.

Table 7.2: Composition of three observed groups during first four introductory meetings (sessions 1-4).

<table>
<thead>
<tr>
<th>Group</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S1 S2</td>
<td>S1 S2</td>
<td>S1 S2</td>
<td>S1 S2</td>
</tr>
<tr>
<td>2</td>
<td>S3 S4</td>
<td>S3 S7</td>
<td>S3 S7</td>
<td>S3 S7</td>
</tr>
<tr>
<td>3</td>
<td>S5 S6</td>
<td>S5 S6</td>
<td>S5 S6</td>
<td>S5 S6</td>
</tr>
</tbody>
</table>

*a S9 and S13 occasionally join group 1 during meeting 4.*

The audio recordings were transcribed and analyzed using three category systems. To identify the phases of the analogical reasoning (research question 1), and whether the term *wave function* is used as model term (research question 2), the categories as given in the theoretical framework were used. Students’ interpretation of the term *wave function* (research question 3) was analyzed by checking which of the following four properties could be attributed
to students’ statements:

1. Electron interference (as seen in the double-slit experiment) can be explained by hypothesizing that an electron can be described by a wave function.

2. Electrons do not interfere with each other; waves interfere with each other.

3. The wave function determines the probability of finding an electron in a certain region.

4. The wave function $\psi(x)$ describes one electron. A superposition of two such wave functions is itself a wave function. In other words: a superposition of two such (interfering) wave functions is not a description of two electrons.

Each property can be answered with either yes, no, or undetermined. In the latter case, there simply are no statements available that enable us to conclude whether this students’ understanding satisfies the corresponding property.

The analysis was carried out independently by two researchers and compared afterwards to check the objectivity of the analysis. In case of disagreement, discussion would follow until agreement was reached. The written work was used, where possible, to check the interpretation of the transcripts.

7.3.3 Teaching materials used

The three phases of the analogical reasoning are implemented in two assignments, described in the following sections: double-slit interference, and the build-up of the electron interference pattern. The use of the double-slit experiment was inspired by a discussion in Feynman et al. (1965).

Double-slit interference

In this assignment students study double-slit interference experiments for three domains: water, light, and electrons. As discussed, an analogy is defined as the mapping of (mainly) relations from a base to a target domain. Water is the base domain for the target domain light, which is the base domain of the target domain electrons.

The structure-mapping of the double-slit analogy is shown in Figures 7.1 and 7.2 for water and light. The domain of electrons is left out, as it very much resembles the structure of light. Relations are preserved as much as possible between domains. However, relations involving displacement and amplitude can only be mapped from water to light if a light wave is hypothesized. Similarly, analogous relations can only be mapped from light to electrons if an electron wave is hypothesized.

Phase 0 of the analogical reasoning consists of studying the base domain: the double-slit experiment for water. The goal of studying this domain is that
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Figure 7.1: Relation network for water domain. Objects are displayed as text without decoration. Attributes appear in ovals connected to objects with a single arrow. Relations appear in ovals connected to other relations, attributes, or objects.

Figure 7.2: Relation network for light domain. The dashed ovals and arrows indicate what relations are missing in this domain and thus need to be hypothesized.
students start using terms such as wave, displacement, and amplitude in a precise manner. For instance, to enable the mapping, the concept *wave* needs to have a specific meaning to students; something which has a displacement changing periodically in time and space between two extremes around some equilibrium value.\(^1\)

The general set-up of the experiment is explained and a top view and movie of the interfering water waves is given. This movie clearly shows that although the water waves change in time, the resulting pattern has a static aspect to it: places of destructive and constructive interference are fixed. To understand this, students are asked to find a function for the displacement, \(u_p(x, y = L, t)\), of the interfering water waves at a certain distance \(L\), parallel to the double slit (e.g. along the screen in Figure 7.3). They are expected to find:

\[
u_p(x, y = L, t) = 2u_0 \cos \left( \frac{kd}{2L} x \right) \exp \left[ i \left( k \sqrt{x^2 + L^2} - \omega t \right) \right], \quad (7.1)
\]

where \(k\) is the wave number, \(\omega\) the angular frequency of the waves, and \(d\) the slit separation.\(^2\) The complex function results from the fact that the students were instructed to use complex notation in adding the separate waves. Students are asked to identify what part of the function \(u_p\) determines the places of destructive and constructive interference. This should lead to the distinction between displacement (changing in time) and the amplitude (the maximum displacement). The expected conclusion is: where the amplitude is zero the water waves do not oscillate, these are the places of destructive interference. Likewise, where the amplitude is maximal, there is constructive interference. This special relation between the displacement and the amplitude is indicated by the double arrow between the functions \(u\) and \(A\) in the sub-network in the top of Figure 7.1. In other words, the amplitude \(A\) expresses

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\(^1\)This “definition” does not account for the possibility that the amplitude of the wave depends on the position.

\(^2\)This expression follows when adding two circular waves, described by \(u(r, t) = u_0 \exp [i(kr - \omega t)]\), at a distance \(d\) from each other. For simplicity, it is ignored that the amplitude \(u_0\) of these waves will also depend on \(r\).
the static characteristics of the interference pattern. This distinction between
displacement and amplitude will become useful in the domains of light and
electrons.

Phases 1 (note homonymous relations) and 2 (hypothesis of missing terms) of
the analogical reasoning are planned by subsequently studying the target
domains light and electrons interfering in a double-slit experiment. First the
double-slit experiment for light is presented, the first target domain in the
analogical reasoning. Students compare the interference patterns for water
and light and discuss similarities and differences between the two domains.
They are encouraged to map relations from the base domain (water) to the
target domain (light). In the light domain there is no wave observable, in the
sense of a moving pattern \(u_p(x, y = L, t)\), as was the case in the domain of
water. The observed interference pattern for light is static. It is expected that
students map places where the water does not oscillate to dark places in the
case of light. Thus, the dark places can be explained by the assumption that
light can be described by a wave. The two slits are then assumed to emit
circular waves, and the interference pattern is described by the amplitude
of these interfering light waves. This mechanism is described by the sub-
network in the top of Figure 7.2. To map the relations between amplitude,
displacement, and interference pattern from the water domain to the light
domain, a wave should be hypothesized (phase 2 of the analogical reasoning).
To make the correspondence more explicit students are asked to qualitatively
give the relation between the amplitude of the light wave and the observed
light intensity.

Finally, electrons are considered. This is a second target domain in the
analogical reasoning. As with the comparison between water and light, stu-
dents compare the interference pattern for electrons with the other domains
and map similar relations. In the domain of electrons there is also no wave (or
displacement) observable, making it impossible to map the relation between
displacement and interference pattern. Students are thus encouraged to hy-
pothesize a wave in the domain of electrons. Students are expected to use
the term \textit{electron density} to describe the interference pattern. They might at
this point either identify this density with the amplitude of the hypothesized
\textit{electron wave}, or else formulate a relation between the two.

We expect the use of two target domains (light and electrons) will benefit
students in finding the mapping for electrons. First of all, going from water to
light, students are expected to have some difficulty in comparing the patterns,
because the interfering water waves are viewed from above, whereas the inter-
ference pattern for light is a frontal view on a screen opposite the double slit.
Furthermore, students are used to thinking of light as a wave phenomenon.
This might make it less difficult in finding the relation between a time depen-
dent wave and a static interference pattern on a screen. Mapping from light to
electrons introduces another difficulty: the electrons are observed as discrete
events. This sequence tries not to introduce all these difficulties at once.

This completes phases 0-2 of the analogical reasoning. Next, students need
to evaluate their hypothesized term *wave function* for electrons, which is phase 3 of the analogical reasoning, discussed in the following section.

**Electron interference pattern build-up**

The double-slit assignment results in a hypothesized wave describing a collection of discrete events (electrons hitting a screen). This seems like a contradiction. What does the wave describe? What is interfering with what? At this stage students might think electrons are waves interfering with each other. To test such a hypothesis the build-up of the interference pattern by single electron events is studied (Tonomura, Endo, Matsuda, Kawasaki, & Ezawa, 1989). First students convince themselves that at any one instant there is at most one electron in the apparatus. Next they watch a movie of the interference pattern build-up and are asked what aspects of this movie can be described by the wave hypothesized in the double-slit assignment. This is designed to let them evaluate and fine tune the interpretation of the wave function used to describe the electron interference pattern. It is expected that students arrive at the conclusion that electrons do not interfere with each other, and that the wave function describes the probability of finding an electron at a certain location on the screen.

### 7.4 Results

To answer the research questions the transcripts were analyzed. Relevant parts were translated from Dutch and are presented in the following. Students are denoted by $S_n$, where $n$ is some integer, $O$ is the observer. The results are organized to follow the phases of the analogical reasoning.

#### 7.4.1 Phase 0: studying the water domain

After studying the base domain students should at least distinguish amplitude and displacement, and use the term wave in a consistent manner. Groups 1 and 3 make the distinction between amplitude and displacement, although they interchange the terms sometimes. This becomes clear from the following fragment, in which group 3 (students S5 and S6) try to answer the question what the maximum displacement is for each point $x$ along the screen:

$$A(x) = 2u_0 \left| \cos \left( \frac{kdx}{2L} \right) \right|.$$  

3The exact wordings of this question is:

The waves move up and down in time. The **maximum displacement** is called the **amplitude**. What is the amplitude as function of the location $x$? Show that this equals:

$$A(x) = 2u_0 \left| \cos \left( \frac{kdx}{2L} \right) \right|.$$  

**Hint** Show that the amplitude of a complex wave $\hat{u}$ is equal to the amplitude of the real wave and that you can write this as $|\hat{u}| \equiv \sqrt{\hat{u}^* \hat{u}}$, with $\hat{u}^*$ the complex conjugate of $\hat{u}$. (Also refer to the syllabus of Calculus.)
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We said, amplitude is \( u_0 \). They call the maximum displacement the amplitude, what is the amplitude as function of the location? Yes, that is independent of time.

But here is only an \( x \), that is why.

What?

The place \( x \), what is the amplitude as function of the place \( x \)?

The maximum amplitude at this point, that will have a maximum. It will always see a highest point. That is what they want to know. They want to know, for instance this one, it has an amplitude of at most two. It goes up and down. So the maximum is at two and minus two. So the maximum amplitude equals two. They want to know it in that sense.

Yes

So as a function of \( x \), like: on this place two is the maximum, but here it is much smaller.

The following is an example where S6 gets a little sloppy, interchanging the two terms, and where S5 corrects her:

Not the amplitude, the displacement changes. The amplitude does not change.

Yes, that. I usually interchange these words. Yes, the maximum displacement is the amplitude.

Group 2 barely used terms such as amplitude or displacement. They had much more difficulties answering the questions from the assignment. It can not be concluded that they make a distinction between amplitude and displacement.

The students in group 1 (S1 and S2) show that they understand the relation between the displacement of the interfering water waves they have calculated and the interference pattern they observe. They find the expected function \( u_p \), Equation (7.1), and understand that this function must be set equal to zero to find places of destructive interference, S1: “But how can I read off the extinction from the function? Oh, that is a node of the function.” S2 understands what part of the function \( u_p \) describes the interference pattern, S2: “No, cosine describes it, but we have this function and it can only become zero if that is zero, because the exponential function can never be zero. Yes?” Later on S2 says: “As a function of \( x \). The maximum displacement is just \( 2u \) times the cosine. Because that is what determines the maximum displacement.”

7.4.2 Phases 1-3: the analogy with light and electrons

In phases 1-3 of the analogical reasoning students compare water with light and electrons and apply what they have found in case of water in those two target domains. The three phases are discussed separately.
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**Phase 1: Homonymous relations**

None of the three groups compare water with light, and light with electrons, as explicitly as was expected. However, all students recognize that the three domains are very much alike, and agree that they are all cases of interference. An implicit mapping of terms does occur, as can be seen for example in the next fragment in which group 1 compares water and light:

S1 What is meant with pattern? Just the whole thing, or...?
S2 The pattern, this is the pattern.
S1 Oh yes, OK. So this here applies to above. And this here is, oh these here are, the black ones are the amplitude and the white one the extinctions.
S2 No, the other way around.
S1 Why?
S2 Light, if it fades out, it will be black?

Implicitly the interference pattern of light is mapped to that of water, and to the mathematical description. S1 at first makes the mapping: \( \text{amplitude} \mapsto \text{black ones} \) and \( \text{extinctions} \mapsto \text{white ones} \), which might be translated as: \( \text{high amplitude} \mapsto \text{black bands} \) and \( \text{low amplitude} \mapsto \text{white bands} \). S2 notes that it should be the other way round: \( \text{high amplitude} \mapsto \text{white bands} \) and \( \text{low amplitude} \mapsto \text{black bands} \). So in mapping the patterns of water onto that of light, students use the functional description of water.

The following quotes of student S6 (group 3), also show that the mapping is very implicit: “They both have upper and lower parts, they are both waves.”, “Well, just things with interference? The properties correspond to each other?” and: “Well, you can see these bright stripes, that is when it amplifies each other, and dark stripes when it extinguishes each other.” This might be read as the following mapping: \( \text{bright stripes} \mapsto \text{waves constructively interfering} \), and \( \text{dark stripes} \mapsto \text{waves destructively interfering} \). Clearly she already tries to explain why the two patterns are so much alike, before examining how they are alike.

Group 2 did not reach the part of the assignment in which water and light are compared. The comparison with electrons is done only by group 3. Now the mapping is more explicit, as S6 notes: “Well, I see exactly the same as with light, bright bands and dark bands.” This is a literal similarity between the two domains, not a mapping between terms from the two domains.

It can be concluded that phase 1 does occur in the protocols. However, the mapping is not seen to be made as explicitly as expected. Apparently, because of the way the assignment presents the analogy, the similarity between the three domains is already convincing. Students do not appear to see the need

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4Students try to answer the following question:

*How can the animation of the expression for \( \hat{u}_p \) help us to compare the two patterns with each other? Describe similarities and differences between the two patterns.*

The animation is given to the students to help them compare the two domains.
of doing a detailed mapping. The question remains whether this is enough for students to introduce wave as hypothesis in the domains of light and electrons.

**Phase 2: Hypothesis**

In this phase it is expected students introduce model terms (as hypothesis) to be able to map relations between the domains which involve these new terms. In this case, the wave, which is not observable in case of light and electrons is to be introduced. In the domain of light, the students in group 3 most clearly articulate this:

S5  Yes, just wait a moment... [after a long silence] Right, three [reads question aloud:] “Do we observe a movement in case of light: something that changes in time? Do we need such a property for our answer to the previous question?”

S6  Well, you see nothing happening.

S5  No

S6  It just remains a bunch of stripes. But yes, to explain why they are visible in this way, you need to say: it is a wave.

The relation between the amplitude of the hypothesized wave and the light intensity is also established:

S5  Well, I think that the, the higher the amplitude...
S6  The stronger the light.

And next, for electrons it seems a clear shot, as S6 notes: “But are electrons, they are a wave in principle, that pull each other in a sort of wave motion or something?” The students also agree upon the relation between amplitude and number of electrons:

S5  Well, four [reads question aloud:] “What connection might there be between the amplitude and the observed electrons?”
S6  I have answered: the higher the amplitude, the more electrons!
S5  The higher... Yes, I have got the same again.

The students in group 2 also seem to use the term wave as a model term in case of light, because they discuss whether or not the displacement of the wave is observable. They do, however, agree upon introducing a wave, because this can describe the interference pattern. From the discussion in group 1 it does not become clear if they use the term wave as model term.

Phase 2 has been reached by most students, and a model term wave has been introduced. We see that in working on phase 2, students are still mapping observable terms between the domains, and are thus moving back and forth between phases 1 and 2.
Chapter 7. An Analogical Reasoning to let Students Hypothesize the Wave Function

Phase 3: Evaluation

In the final phase of the analogical reasoning, the introduced term is to be evaluated for its usefulness. Of course this has in part been done in the double-slit assignment, as introducing the term enabled the students to describe the interference pattern of electrons. Introducing this term, however, also raised questions about the mechanism behind the interference, as is clear from the following discussion between S5 and S6:

S6 Ah, but then it is like a... If because of that an electron propagates as a wave, and there comes another electron, then it will pull itself in its trajectory. And if they are both waving, and they meet, and they are in anti-wave, then they both stand still, or something?
S5 Then they will amplify each other.
S6 Not if they are an anti-wave.
S5 Yes, then you will not see them.
S6 That is not amplification, that is extinguishing, right? Anti-wave.
S5 They extinguish each other, and that is impossible. But that is funny.
S6 But maybe, would it be here like, that they push each other, away, or something?
S5 If they are anti-wave, they would.
S6 Yes, anti-wave, then they push each other away, and happen to exactly end up on the nearby slit. [laughs]
S5 [laughs] Nice theory.

Later in this discussion, S6 considers the possibility that, instead of “extinguishing” each other, an electron might hit the screen at a bright spot, instead of a dark spot. In any case, this shows that the concept wave needs fine tuning. This is done in the assignment on the build-up of the electron interference pattern, discussed in the remainder of this section.

The first task when looking at the build-up of the interference pattern, was for students to calculate the distance between successive electrons shot at the double-slit. Most students find a correct value, or a value of the right order. Together with an estimate of the length of the apparatus, they can use this distance to determine how many electrons will be in the apparatus at any given moment. S6 (group 3) puts it this way: “Yes, then there is at most one at a time... Then there are zero or one electrons. That is not logical.” S6 feels that this cannot be right; apparently she had expected at least two electrons in the device: “But then I think our apparatus should be 260 kilometers, because then there would be two in the device.” Up to now this student clearly believed that the electrons interfere with each other. This interpretation is incommensurable with the finding that at most one electron is in the device. She therefore tries to adjust the length of the device, in order to be able to save her idea. After the students hear that other students also reach at the conclusion that at most one electron can be in the device, S6 concludes:
"I think it is really a feat that we found out that the device is smaller than 130 kilometers and so there is only zero or one in the machine." It appears that S6 defines *interference* as an interaction between particles. Because of the conclusion so far, she needs to adjust this view:

S5 Yes, you have to say something about that wave based on the movie you have just seen. Say something about the interference pattern.

S6 But there is no interference, because there is only one in the device, right?

S5 Yes

S6 Then I do not understand why a wave [inaudible].

S5 But right, if only one comes flying out, that one stays there.

S6 Yes

S5 Right? If then by chance one... If you wait long enough and there are already a lot. Then on a given moment you do get...

S6 Yes, but that is...

S5 ... interference.

S6 But it is not like they interfere with things already attached to the screen?

S5 Yes, but one by one... They surely cannot go and interfere with each other?

S6 Yes, so that is the problem. [addressed to O:] Aren’t there only one or zero things in the machine?

O Yes

S6 Then how can they interfere? They will not interfere with each other, and I assume that the screen is not some sort of source of attraction.

O No

S6 Yes, I do not understand it.

O No, I do not understand it either. But, what I wanted to do with this assignment, is that you systematically explore... The experiment is actually very elegant, that they were able to do it this way. That with the help of the experiment you ask yourself, OK, how many are there in the machine, what do we see on the screen, do we still call this interference? Then the conclusion can only be, you said it yourself, that they do not interfere with each other.

S6 No, interference is some sort of interaction between particles, or waves?

O Yes, waves, indeed.

S6 Yes, waves.

O So there we have to be careful. Because in the last couple of assignments we have looked into waves that interfere. We have used that to explain the electron interference pattern. But now we have to carefully rethink, what do we mean by interfering?
S6 That is an interaction between two waves? An electron is not two waves, is it?

Actually, to understand the interference pattern, the electron has to be described by two waves.

Next, S5 has a feeling that the wave describes the pattern, not the position where individual electrons will hit, S6 seems not so sure about this:

S6 Well, what aspects can you explain? Well none, because it is not a wave.
S5 Just not much, because they one by one, eh... Because the pattern that eventually emerges is the same as with light and eh...

The students in group 1 also reach the conclusion that the electrons cannot interfere with each other. S1 finds this disturbing and tries to keep up the idea of electrons interacting with matter, by suggesting that the electrons interfere with the surroundings, the air, or the backstop/screen. S2 objects to this idea, but S1 keeps insisting and suggest that the electron is itself composed of particles that might interfere with each other:

S1 [...] It can only interfere with something in the device, right? S2?
S2 Yes?
S1 Yes, well.
S2 No, but, yes, no it just cannot be. It cannot [interfere] with the air, or with the backstop.
S1 Yes, OK, but what more is in the device?
S2 Nothing more. Yes! What would you like to put into that device? You fire off electrons, you accelerate them...
S1 Then you should say: OK an electron consists of other particles that interfere with each other, that is what you see?
S2 Yes, an electron exists of muons, I guess.
S1 [inaudible] other particle, but is that the question? If you say there is nothing more in the machine, nothing can interfere...
S2 That might in itself be possible, yes...
S1 ...then an electron has to interfere with itself. Or with the things of which it is composed. Well, I do not know whether this is the answer to the question. Right.
S2 Yes, an electron consists of, what are these things called again, quarks right? But I do not know if they...
S1 Yes, but if you are talking about quantum chemistry, then it is possible.
S2 [uncertain] Yes, might be.

5The protocol follows after the students have read the following question:
What aspects of the movie can we explain by the wave function we introduced in the assignment on the double slit?
Table 7.3: Students’ interpretation of the wave function, using the four properties as introduced in the Methodology section. For each student, each property is answered with either yes (y), no (n), or undetermined (?). Student S4 was left out in this overview, as this student only participated one session and thus not enough data was available.

<table>
<thead>
<tr>
<th>Property</th>
<th>Group 1</th>
<th>Group 2</th>
<th>Group 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Electron interference explained by hypothesizing wave function.</td>
<td>y y y</td>
<td>y y y</td>
<td>y y y</td>
</tr>
<tr>
<td>2. Electrons do not interfere with each other.</td>
<td>y y ?</td>
<td>? ? y</td>
<td>y y</td>
</tr>
<tr>
<td>3. Wave function determines probability of finding an electron in a certain region.</td>
<td>y y ?</td>
<td>? ? y</td>
<td>y y</td>
</tr>
</tbody>
</table>

Group 2 does find the correct electron separation, but after this there is little time left. They do not draw the conclusion that the electrons cannot interfere with each other.

Most students reach the conclusion that the electrons should be described by waves in order to understand the interference pattern. They also conclude that electrons do not interfere with each other. However, they are left with an uneasy feeling how this is possible. Students are trying to find a mechanism to visualize what happens to the electrons between the double-slit and the screen. In a group discussion, teacher and students carefully review the steps taken to arrive at the wave description for electrons. It is emphasized what we now do and do not understand using this picture. Students are told that we still do not understand why one electron can act like it passes through two slits simultaneously, as two interfering waves. However strange this appears to us, it is not much different from what happens in other fields of physics, such as gravitation. Do we there understand why two masses attract each other? They just do, and with Newton’s law of gravitation, we can understand and predict the behavior of a collection of masses.

7.5 Conclusion and Discussion

Although the double-slit analogy is commonly used in teaching quantum mechanics, research on its use in education is not widespread. This article wants to contribute to the understanding of the double-slit analogy. Based on research on analogical thinking and the usage of analogies in education a teaching sequence was developed that incorporates the double-slit analogy. The goal of the designed teaching sequence was to let students make an analogical rea-
soning and compare the three domains: water, light, and electrons. This was expected to result in students formulating the electron wave function as a hypothesis. Fine tuning of the interpretation of the hypothesized wave was to take place by studying the build-up of the electron interference pattern. The research questions for which we sought an answer are repeated below.

1. Can the phases 0-3 of the analogical reasoning be discerned in the student discussions?
   It was possible to discern the phases 0-3 in the protocols. Phase 0, studying the water domain, took more time than expected. Many of the physics and mathematics involved was considered prior knowledge for the students. It appears that this pre-knowledge did not function on a level required for the assignments. The mapping between the three domains (phase 1) was not done as explicitly as expected. We conclude that from the presented interference patterns, students already saw the analogy. This undermined the motivation to do a detailed, explicit mapping. Because of this, some of the students found this part rather annoying. The idea that the phases of the analogical reasoning take place sequentially, might need to be adjusted. Phase 0 still is necessary to have a thorough understanding of the base domain. Phase 1 then needs to motivate students to move to phase 2 and use the structure of the base domain in the target domain, thus introducing hypothesized terms in the latter. During phase 2 it might be necessary to go back to phase 1 and map other terms to assist in the description of the target domain. Finally, after being confident that an appropriate description has been found, students evaluate the introduced terms (phase 3).

2. Is a model term wave function introduced by the students in the target domains?
   Groups 2 and 3 both use the term wave function as a means to explain the observed interference pattern in case of electrons, group 3 being most explicit. For these groups the term wave function is used as a model term. From the discussion in group 1 it does not become clear whether or not they see wave function as a hypothesis. However, they do see the applicability of such a term in case of electrons.

3. What interpretation do students give this hypothetical wave? How does this interpretation compare to the scientific term wave function?
   Groups 1 and 3 both reach the conclusion that the electrons cannot interfere with each other. They understand that the wave function is able to explain the fact that there is an interference pattern visible, but cannot predict where an individual electron will hit the screen. They do see that the build-up of the pattern appears to be random. However, a probabilistic interpretation of the wave function does not follow. Their conception of the term wave function does not yet correspond to the scientific term, but is at least consistent with it. It does, however, need fine tuning. Groups 1 and 3 are searching for a mechanism that might explain why single electrons still give rise to the interference pattern. This shows that there is still work to do for the teacher in formulating a conclusion to both assignments. Group 2, unfortunately, did
References


