Entanglement and order in many-body systems

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Introduction

1.1 Entanglement

Entanglement is the most mysterious property of quantum mechanics. Although the notion itself was introduced [1] only 9 years after the discovery of the Schroedinger equation, it immediately caught attention of both physicists and outsiders. Indeed, quantum entanglement appeared as a result of the famous Schroedinger’s cat paradox within still ongoing discussion about the interpretation of quantum mechanics. Entanglement stems from the principle of quantum superposition and as such has no analog in classical world. Despite being an indispensable part of quantum theory, for a long time entanglement was largely treated as a philosophical concept, which would appeal to imagination on qualitative basis, such as discussions of EPR paradox [2], non-locality of quantum mechanics, etc.

The mystery of entanglement only increases if one asks the question of why it is absent in the classical world. Clearly the EPR paradox challenges our understanding of the physical reality as being local in the sense of the special theory of relativity. Nevertheless EPR like experiments have been performed and discarded attempts to extend quantum mechanics with hidden variable theories. On the other hand Schroedinger’s thought experiment has not yet been realized, although there is no fundamental principles which would forbid us to do so. Qualitatively it is clear that entangled macroscopic states, such as alive and dead cat, are killed by decoherence. And indeed there are plenty of models which show that coupling to the thermal bath destroys quantum coherence and entanglement faster than any other time-scales in the system. One proposal is therefore to say that interaction with environment selects an outcome of the experiment into one of the ”pointer states” [3]. This, of course does not explain, why one of the states selected by the environment is not a ”half-alive” superposition, which leaves us with many open questions about the origin of the classical world [4]. Fortunately, this thesis does not address these fundamental questions, as it seems that little progress has been achieved in this area despite numerous heated debates resulting in such outlandish proposals as ”quantum Darwinism”, ”Mind-Matter Unification Project” [5], etc. However amusing some of the experiments carried out to support those speculations look like, such as collecting a database of 750,000 human/machine interaction trials [6], many of the suggested solutions of the
quantum-classical problem lie outside the normal development of science [7].

A more pragmatic approach towards entanglement emerged in the field of quantum computation and communication [8]. Here entanglement between qubits serves as a necessary resource for performing various quantum algorithms, quantum cryptology or teleportation. For this purpose entanglement has to be quantified and classified and there are different measures and schemes available in the literature. On the experimental side, the progress in the field is driven by the increasing ability to manipulate and detect single qubits, which are usually represented by spins, photons or Josephson currents. The corner stone problem in the area of quantum computation is the presence of decoherence, which destroys entanglement of qubits and as a result precludes implementation of quantum algorithms.

![Figure 1.1: Possible spatial partitioning of the system into A and B regions, which can be connected, simply connected, having smooth or non-smooth boundaries.](image)

Condensed matter physics describes macroscopic objects and their properties. Although this description is usually based on quantum mechanical principles, until re-
cently there was little interest in quantum entanglement and its applications. The main reason is because the properties of solids, when derived from quantum mechanics, are usually cast in terms of single particle physics. A standard textbook approach begins with the independent electron approximation, Landau’s Fermi liquid theory etc., which emphasizes as starting point the free Fermi or Bose gas and treats interactions via perturbation. Most of the systems with order parameter are described with mean field approximation, where the ground state wave function is explicitly assumed to be in a non-entangled state. Surprisingly such approximations often give a satisfactory description of the system. On the other hand there are many more complicated systems with exotic ground states, such as BCS or RVB states, which do contain non-trivial entanglement. In BCS state electrons constituting a Cooper pair are entangled with each other to form a spin singlet. The presence of non-trivial entanglement is however a byproduct rather than a requirement, as we do not start the analysis of the system demanding that a ground state should contain certain amount of entanglement. A successful model should be able to reproduce experimental observations, such as magnetization or resistance, whose relation to entanglement of the constituents is not obvious, if there is any. The systematic study of entanglement and its connection with physical properties, such as order parameters has started only recently. The driving force behind this was a realization that certain 1d and 2d systems can be conveniently characterized via entanglement measures. For example, the conformal charge of critical 1d systems can be best extracted numerically by scaling of the entanglement entropy. Another example is the density-matrix renormalization group (DMRG) approach, which is designed to preserve entanglement in the ground state wave functions, and as a result offers an extremely efficient description of the physics in 1d.

This chapter presents an overview of the most important results which relate entanglement entropy and physical properties of various systems. This section continues with the definition of the entanglement entropy and its expected generic features. We then give a congested set of results, which are chosen to develop intuition about entanglement entropy in the context of specific solvable systems, although they do not have direct relation to the results of the rest of the thesis. It is important to realize that analytic calculation of entanglement entropy is possible for a very limited number of models, therefore it is worth to pay attention to a few available results. In the section 1.2 we present the notion of ”topological entanglement entropy”, which is a necessary starting point for a discussion which follows in the next Chapters.

Different measures of entanglement should provide a quantitative way to characterize non-local correlations in a macroscopic system. We primarily focus attention on entanglement entropy as a measure of quantum entanglement. Moreover we assume that the macroscopic system is described by a unique pure wave function $\Psi$. Although this assumption is not necessary, we will not consider systems in mixed states. After separating a subsystem $A$ of the full system $A \bigcup B$, with $A \bigcap B = \emptyset$, we define the entanglement between $A$ and $B$. The division of the full system into $A$ and $B$ can be of very different nature. Usually the system is divided into spatial regions as depicted at the Fig. 1.1, however one can introduce other ways of partitioning. For example, $A$ can consist of certain modes of the full Hilbert space. In this chapter we consider partitioning into spatial blocks only.
Now that a system is divided into two parts, we can represent a normalized state as:

$$|\Psi\rangle = \sum_{i,j} M_{i,j} |\Psi^A_i\rangle |\Psi^B_j\rangle,$$

(1.1)

with $|\Psi^A_i\rangle$ and $|\Psi^B_j\rangle$ forming an orthonormal basis in corresponding Hilbert spaces. A rectangular matrix $M$ can be diagonalized via singular-value decomposition as:

$$M = UDV^T,$$

(1.2)

where $U$ is unitary, $D$ is diagonal and $V$ has orthonormal rows. Defining a new basis $|u_i\rangle = U_{i,j} |\Psi^A_j\rangle$ and $|v_i\rangle = V_{i,j} |\Psi^B_j\rangle$ we can write:

$$|\Psi\rangle = \sum_i \lambda_i |u_i\rangle |v_i\rangle,$$

(1.3)

which is called Schmidt decomposition. The weight factors $\lambda_i$ are the diagonal elements of the $D$ matrix and their number is restricted by the size of the smaller of the two Hilbert spaces of either $A$ or $B$ subsystems. In general case $\lambda$ are zero or positive and encode the information about the entanglement between $A$ and $B$. In a specific case when there is no entanglement and the state is separable only one $\lambda$ is equal to 1 and the rest have zero value. In another extreme case when all $\lambda$’s are equal the state is called maximally entangled.

Entanglement entropy can be conveniently defined from the reduced density matrices associated with the state $|\Psi\rangle$. We start with the total density matrix

$$\rho = |\Psi\rangle\langle\Psi|$$

(1.4)

and integrate over $A$ or $B$ degrees of freedom

$$\rho_A = \text{tr}_B \rho \quad \rho_B = \text{tr}_A \rho,$$

(1.5)

where $\text{tr}_B$ denotes the the summation over degrees of freedom in $B$. Using (1.3) we obtain:

$$\rho_A = \sum_i |\lambda_i|^2 |u_i\rangle\langle u_i|,$$

(1.6)

$$\rho_B = \sum_i |\lambda_i|^2 |v_i\rangle\langle v_i|.$$  

(1.7)

Thus in the basis of the Schmidt decomposition the reduced density matrices are diagonal with $|u_i\rangle$, $|v_i\rangle$ being the eigenvectors. It is trivial but important to note that $\rho_A$ and $\rho_B$ are Hermitian and have the same set of nonzero eigenvalues $|\lambda_i|^2$ lying in the range $0 \leq |\lambda_i|^2 \leq 1$.

The von Neumann entanglement entropy $S_A$ for the region $A$ is defined as:

$$S_A = -\text{tr}_A (\rho_A \ln \rho_A) = -\sum_n \lambda_n \ln \lambda_n$$

(1.8)

and the same for the subsystem $B$. It is clear that for a pure ground state $S_A = S_B$, and $S_A = S_B = 0$ if the subsystems $A$ and $B$ are separable, i.e. the ground state wave function factorizes into two pure state wave functions from $A$ and $B$. 


As an example of the above definitions consider a simple system of two spin-$\frac{1}{2}$ particles in the singlet state $|\Psi\rangle = (1/\sqrt{2})(|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle)$. The probability of finding the left spin in a state $|\uparrow\rangle$ ($|\downarrow\rangle$) is $1/2$ ($1/2$). This means that the reduced density is diagonal $\rho_A = \frac{1}{2}1$ and the entanglement entropy of the left spin with the right one is $S = \ln 2$.

In the general case the computation of entanglement entropy for macroscopic systems is nontrivial and shows complex behavior, which reveals intricate properties of underlying field theories. However the most important attribute of entanglement is its inherent non-local nature, and therefore one expects that studying entanglement entropy may provide additional information about the ground states of the systems, unaccessible via local measurements. In principle entanglement entropy may depend not only on the properties of the system but also on the shape of the region $A$, whether $A$ has corners, if it is connected or simply connected as shown at the Fig. 1.1. One obvious question to start with is the dependence of the entanglement entropy on the boundary $L$ between regions $A$ and $B$. Originally this question has been addressed to shed light on the enigmatic black hole entropy [9,10]. It is now agreed [11,12] that a classical Schwarzschild black hole should have a thermodynamic entropy proportional to its surface $S_{BH} = AM^{2}_{pl}/4$ ($A$ is area and $M_{pl}$ is Planck’s mass), whose origin however remains mysterious. Another connected issue is the so called “information loss” paradox, which raises a question of non-unitary evolution of the quantum mechanical system after falling through the event horizon and being emitted as thermal Hawking radiation. One could imagine that degrees of freedom under the event horizon are effectively cut out and hidden from the external observer, who therefore has to operate with the reduced density matrix $\rho_B$. Although this motivation for the computation of the entanglement entropy is clearly speculative, the results nevertheless uncovered scaling law of entanglement entropy in free quantum field theories. It turns out that the entanglement entropy scales with the area of the boundary between subsystems $A$ and $B$, e.g. for a spherical region $A$ with radius $R$ [10]:

$$S_A = \kappa M^2 R^2 .$$

(1.9)

This has been established via a combination of analytical and numerical methods for a free bosonic quantum field theory represented by discretized harmonic oscillators with ultraviolet cut-off $M = 1/a$. This scaling is shown at the Fig 1.2. A somewhat different and purely analytical computation has appeared earlier in the Ref. [9], which treated a bosonic quantum field theory where the region $A$ was chosen as a half-plane of the whole space. Thus it has been suggested that entanglement entropy scales with the surface rather than with the volume of the subsystem unlike classical entropy. This proposition is known as the area law scaling of the entanglement entropy.

A simplistic argument for the area law scaling follows from the identity $S_A = S_B$, which is valid for any partitioning of the system described by a pure state. In case of spatial partitioning into $A$ and $B$ the subsystems share the same boundary, therefore $S_A$ and $S_B$ should be some function of this boundary. A natural assumption is then that this function is linear with the size of the boundary. Although this argument looks persuasive it should be noted that for an arbitrary chosen quantum state, entanglement scales with the volume rather than area [13]. This means that the ground states of the local physical hamiltonians are less entangled than an arbitrarily chosen quantum
state. In other words physical states occupy only a small fraction of all available Hilbert space. This has practical applications for variational methods as one does not have to vary over the whole Hilbert space, but rather over the states which satisfy the area law [14].

Similar results were obtained in other works. In the Ref. [15] a free bosonic field theory was considered and entanglement entropy was shown to diverge according to the area-law, where the emphasis was made on geometric entropy appearing as response to a conical singularity in the underlying geometry. The entanglement entropy is shown to be for \( D > 2 \) (\( D = d + 1 \) is the space-time dimension)

\[
S = \frac{L^{D-2}}{(2\sqrt{\pi})^{D-2}(D/2 - 1)} \epsilon^{2-D},
\]

where \( \epsilon \) is UV cut-off and \( L \) is the size of spatial region in accordance with the area law. Despite these early calculations even in the free case there are not many results for other geometries, and generally the computation of the entanglement entropy requires solving difficult questions about the spectrum of certain integral operators. The leading term in the scaling of entanglement is divergent due to unbounded number of local degrees of freedom and on general grounds one also expects point-like vertex induced logarithmic terms in any dimensions. The computation leading to the scaling law for entanglement entropy does not directly reveal the difference between free and interacting fields and instead is related to ultra-violet behavior of quantum field theories. A thorough analysis of entanglement entropy for free scalar and Dirac fields in various \( d > 2 \) dimensions has been presented in the Ref. [16], where also a few universal features related to non-trivial choice of spatial geometries of \( A \) and \( B \) regions have been noted.
1.1 Entanglement

For $d = 2$, the area law scaling of entanglement has been confirmed in the Ref. [17] where a relativistic $\phi^4$ model with global $O(N)$ symmetry has been studied in the large $N$ limit. The quantity that has been computed is the Renyi entanglement entropy, which is the generalization of von Neumann entanglement entropy $S_n = \ln \text{tr} \rho^n/(1 - n)$. The limit $n = 1$ gives the back the von Neumann entropy. Apart from the area law scaling, Renyi entropies contain corrections $\kappa$ due to finite size

$$ S_n = C_n \frac{L^{d-1}}{a^{d-1}} + \kappa_n $$

and $r$ due to finite correlation length:

$$ S_n = C_n \frac{L^{d-1}}{a^{d-1}} + r_n \frac{L^{d-1}}{a^{d-1}}. $$

The correction due to correlation length appears because the model is studied in the vicinity of the Gaussian and Wilson-Fisher critical points. Both $\kappa$ and $r$ are different at different fixed points. For example, for Wilson-Fisher $r_1 = -\frac{N}{144\pi}$ and Gaussian $r_1 = -\frac{N}{24\pi}$ which is obtained with the $\epsilon$-expansion technique and $D = 4 - \epsilon$. As a function of $n$ $r_n$ and $\kappa_n$ have non-analytic structure. Thus entanglement entropy contains additional corrections to the area law, which contain potentially useful information about the structure of fixed points.

Although the area law scaling of the entanglement entropy has been confirmed for many systems [18] it is not rigorously proven. On the contrary an important counterexample has been presented in the Refs. [19, 20], where it was shown that a model of fermions on a lattice with Fermi surface has an additional logarithmic factor in scaling of entanglement entropy:

$$ S_A = \alpha L^{d-1} \ln L. \quad (1.11) $$

Fermionic systems without finite Fermi surface have been considered numerically in both gapped and critical regimes in the Ref. [21]. In both regimes the sub-area correction has been identified, while the leading term scales with the system size $L$ without a logarithmic factor.

Another interesting topic is the calculation of the entanglement entropy using the AdS/CFT approach. Loosely speaking the AdS/CFT correspondence is a tentative proposition, which arose in the context of string theory, that correlation functions of conformal field theories are equal to the correlators of theories of quantum gravity in anti-de Sitter spaces. Because on the gravity side a theory has one extra dimension, this is also sometimes referred to as an instance of ”holographic principle” or simply ”holography”. It was proposed in the Ref. [22] that a similar correspondence holds for entanglement entropy, where the entanglement between the spatial regions on the CFT side is equal to the area of some minimal surfaces on the AdS side. The proposal is spectacularly verified for CFT$_2$/AdS$_3$ systems, where the familiar expression $1.17$ for the entanglement entropy is equal to the length of geodesics on AdS manifold (see the Ref. [23] for a review).

In this Chapter we are going to be interested in universal features in the scaling of the entanglement entropy. As it has been discussed the leading term in scaling is
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non-universal (except in 1d), therefore we are interested in other sub-leading terms, corrections to the area law which can be universal for some systems. An example of universal coefficients has been given for the fixed points of \( \phi^4 \) theory. Generally, one may expect universality in scaling close to a quantum critical states which are fixed points of the renormalization group and describe continuous phase transitions. Criticality is characterized by scaling of local observables, such as correlation length or susceptibility, and little is known about the behavior of non-local quantities such as entanglement entropy.

It is difficult to analytically compute entanglement entropy for a generic Hamiltonian, and system-specific methods are sometimes employed [24–26]. A common way to calculate entanglement entropy analytically is to use the ”replica trick” [27]. In this approach one looks for an expression for \( \rho^n_A \), i.e. the \( n \)-th power of the reduced density matrix, which is obtained by gluing together \( n \) independent copies of the system (replicas). It is instructive to present the approach in detail as it is a generic method available for computing the entanglement entropy, which also will be used later in the Chapter. We assume that the field theory is Lorentz invariant, then in the Euclidian time formalism the wave function is:

\[
\Psi(\{\phi(x)\}) = \int \phi(x) \mathcal{D}\phi \exp(-S_E(\phi(x))) ,
\]

where the field \( \phi(x) \) are the degrees of freedom of the QFT, the wave function is defined on the field configuration \( \phi \) and \( S(\phi) \) is the action. From this relation the reduced density matrix is expressed as

\[
\rho_A(\phi(x), \phi'(x')) = Z^{-1} \Psi(\phi) \overline{\Psi}(\phi') =
\]

\[
= Z^{-1} \int_{t=-\infty}^{t=\infty} \mathcal{D} \phi(y,t) e^{-S(\phi)} \lim_{\epsilon \to 0} \prod_{x \in A} \delta(\phi(x) - \phi(y,\epsilon)) \delta(\phi'(x') - \phi(y, -\epsilon))
\]

(1.13)

where \( Z \) is the partition function which insures the normalization \( \text{tr} \rho^n_A = 1 \), and delta functions introduce a cut in the space-time manifold along the spatial extent of the region A. The crucial ingredient is to recognize that we can compute any power of \( \text{tr} \rho^n_A \) by gluing together different fields as shown at the figure 1.3 i.e. gluing fields \( \phi \) from different replicas along the cut. We denote the partition function of the system on this Riemann surface as \( Z_n \), then:

\[
\text{tr} \rho^n_A = \frac{Z_n(A)}{Z^n} .
\]

(1.14)

The entanglement entropy can be computed as:

\[
S_A = - \lim_{n \to 1} \frac{\partial}{\partial n} \text{tr} \rho^n_A .
\]

(1.15)

The main challenge is the computation of the partition function \( Z_n \) on a non-trivial manifold and it can be done in special cases, such as systems described by CFT, and will be discussed in the following sections. Most importantly the method allows to
introduce and calculate topological entanglement entropy, which is the focus of the
thesis.

Universal scaling of different local quantities at fixed points is predicted using
various perturbative methods: $1/N$ expansion, $4 - \epsilon$ expansion, $2 + \epsilon$ expansion.
One dimensional systems constitute an exceptional case, where the comprehensive
description of critical points can be given using $1 + 1$ dimensional conformal field
theories (CFT). Coincidentally there is a large class of 1-dimensional models, that are
integrable and can be exactly solved. These two different and powerful approaches
render our understanding of entanglement entropy most complete in 1d in comparison
to higher spatial dimensions.

A detailed description of entanglement in 1d spin-systems near criticality from a
CFT perspective has been developed in [27,28]. The treatment is based on the replica
trick described above, so the result for the power of the reduced density matrix,
assuming that the whole system $A \cup B$ is infinite is:

$$\text{tr} \rho^n_A = c_n \left( \frac{l}{a} \right)^{-c(n-1/n)/6}. \tag{1.16}$$

Here $c$ is conformal charge, $a$ is ultraviolet cut-off, $l$ is the length of the interval in the
subsystem $A$ and $c_n$ some non-universal constant. Applying formula 1.15 one gets:

$$S_A = \frac{c}{3} \ln \frac{l}{a} + c'_1. \tag{1.17}$$

The main feature of this formula is the logarithmic scaling of the entanglement with
subsystem size, where the prefactor $c$ is the universal quantity for a particular field
theory or model at criticality. This expression can be easily generalized for finite
temperatures (or finite size) with the result:

$$S_A = \frac{c}{3} \ln \left( \frac{\beta}{\pi a} \sinh \frac{\pi l}{\beta} \right) + c'_1, \tag{1.18}$$
where $\beta$ is the inverse temperature (or the size of the whole system times imaginary $i$). Based on this approach other interesting results can be obtained. For massive (with $m = 1/\xi$) 1D theories in the scaling limit $a \ll \xi$, when $A$ is a semi-infinite interval:

$$S_A = \frac{c}{6} \ln \frac{\xi}{a},$$

i.e. entanglement saturates to an upper value restricted by the correlation length $\xi = 1/m$ and does not scale with the size $l$. Note the change of a prefactor from $1/3$ to $1/6$. This is because semi-infinite interval has only one boundary with its compliment, while a usual interval of the length $l$ has two. A similar logarithmic scaling is also observed for the 1d chains with random disorder, which is reviewed in [29].

These results are successfully verified by numerical simulations, which also show a good convergence even for small system sizes. Moreover the calculation of entanglement provides a way to extract numerically the central charge $c$. This procedure can be easily implemented within DMRG approach and is technically simpler then the extraction of $c$ via finite-size scaling of the free energy, because the latter requires the knowledge of the speed of sound.

An expression for 1d entanglement entropy beyond CFT is available for massive theories, see Ref. [30] for a review. This result is based on replica trick coupled with a form-factor approach to massive integrable theories. The entanglement entropy is equal to

$$S_A(l) = -\frac{c}{3} \ln(\alpha m) + U_{\text{model}} - \frac{1}{8} \sum_\alpha K_0(2 l m_\alpha) + O(e^{-3l m}).$$

(1.19)

The first term is the usual saturation of the entanglement entropy for massive theories; the second term is some non-universal model-dependent constant, which can be computed in certain cases; the third term is the universal sub-leading term which depends only on the masses of the particles in QFT $m_\alpha > m$.

Another interesting feature of the entanglement entropy in 1d CFT models is the dependence on boundary conditions. Within a CFT framework boundary conditions should be generalized in a conformaly invariant way, resulting in the so-called conformaly invariant boundary conditions (CIBC). As a result the entanglement entropy acquires a universal dependance on CIBC of the form

$$S_A(l) = \frac{c}{6} \ln l/a + \ln g + O(1),$$

(1.20)

where $\ln g$ is the sub-leading correction which encodes the boundary conditions.

A similar correction appears in the case of impurities in 1d models. For example in the $k$-channel Kondo model in a low-$T$ limit $T << T_K$ $g = 2 \cos(\pi/(k + 2))$, which coincides with the thermal entropy of impurity. The impurity entanglement entropy has a surprising connection with the topological entanglement entropy, which will be discussed later. The full information about a CFT (such as conformal dimensions of operators) appears in the scaling of the entanglement entropy if one considers a block $A$ consisting of small disjoint intervals [31].
To sum up, the scaling of entanglement entropy in 1 + 1 dimensions is now relatively well understood as well as its connection with CFT structure at fixed points. The most interesting aspects entanglement entropy reveal themselves in various universal features appearing in the scaling limit. Unfortunately there are only limited number of general analytical results available. Therefore it is worth to consider them in detail. The most general description is given for 1 + 1 dimensional theories with scale-invariant action which can be solved with CFT formalism as presented above. In particular, these theories describe continuous limits of many 1D spin models in the critical phases. Therefore the entanglement entropy can be computed both via CFT and explicit microscopic Hamiltonians.

Universal scaling of entanglement in 2+1 dimensions in the vicinity of critical points is less explored except for a few cases, which we will describe in detail. It is possible to find entanglement entropy for a class of quantum critical points with dynamic critical exponent \( z = 2 \) in 2 + 1 dimensions, whose ground states wave functions are scale invariant. This set of theories is called quantum Lifshitz models [32]. Note that the scale invariance of the wave functions is different from the scale invariance of the actions. Again the scale invariance is the reason for the solvability of the systems. In fact the wave functions of these systems correspond to the configurations of the associated 2D classical systems. This is explored in detail in the context of various quantum dimer models which give microscopic description of quantum Lifshitz points. As quantum dimer models have also topological phases it is interesting to compare entanglement entropy in different phases. We will see that in both phases entanglement entropy scales linearly with boundary length as suggested by the area law but there are also universal sub-linear corrections. For quantum Lifshitz points the precise formula for the entanglement entropy of the simply connected block \( A \) with smooth boundary is:

\[
S_A = \alpha L_A + \gamma_{QCP} + O\left(\frac{1}{L_A}\right),
\tag{1.21}
\]

where \( \gamma_{QCP} \) is universal finite size correction coming from the conformal structure of the wave function. This expression is deduced below.

Most of this thesis will be concerned with the computation of entanglement in idealized quantum Hall systems which constitute a phase of matter in 2 + 1 dimension with topological order. These phases are described by topological field theories, which allows us to compute entanglement entropy with the result:

\[
S_A = \alpha L_A - \gamma_{\text{topo}} + O\left(\frac{1}{L_A}\right),
\tag{1.22}
\]

where the first term scales non-universally with the boundary size and the second term is universal and is dubbed as topological entanglement entropy. Topological field theories are highly idealized representations of realistic physical systems, for example, the correlation length is taken to be zero. Therefore it is worth to check that the same result can be obtained independently for concrete microscopic models with topological phases, such as Kitaev’s toric code model. A detailed presentation of entanglement in topological phases is given in the Sec[1.3]
1. Introduction

Let us present the reasoning behind the Eq. 1.21. Quantum Dimer Models (QDM) at criticality are described by the effective theory of quantum Lifshitz model [32]. A dimer model on a bipartite lattice can be represented using the so-called "height" configurations on a dual lattice. These degrees of freedom in a coarse-grained limit transform into a scalar field $\varphi$ with a compactification radius $r$, which reflects certain constraints on the height configurations. The Hamiltonian of the model is:

$$H = \int d^2x \left[ \frac{1}{2} \Pi^2 + \frac{1}{2} \left( \frac{k}{4\pi} \right)^2 (\nabla^2 \varphi)^2 \right], \quad (1.23)$$

where field $\varphi$ and momentum $\Pi$ obey canonical equal-time commutation relations, $[\varphi(\vec{x}), \Pi(\vec{y})] = i\delta^2(\vec{x} - \vec{y})$, and $k$ is an arbitrary parameter.

The ground state wave function $\Psi_0[\varphi]$ of the Hamiltonian 1.23 is scale invariant and given by

$$\Psi_0[\varphi] \propto e^{-\frac{k}{8\pi} \int d^2x \ (\nabla \varphi(x))^2}. \quad (1.24)$$

It has the form of a local Gibbs weight in classical statistical mechanics. Indeed, it corresponds to the simplest critical classical system in 2D - the free bosonic Gaussian model. The norm of the 2D wave function is the partition function of this classical critical conformally invariant system:

$$\|\Psi_0\|^2 = \int \mathcal{D}\varphi \ e^{-\frac{k}{4\pi} \int d^2x \ (\nabla \varphi(x))^2}. \quad (1.25)$$

Thus the coarse-grained wave function of the QDM at its critical point is mapped to a 2D CFT. In this case the wave function is scale invariant and corresponds to the simplest CFT of the massless free bosons with $c = 1$. With this identification one can match correlation functions in the QDM at the Rokshar-Kivelson (RK) point and in CFT and find the parameters in the Hamiltonian 1.23, $k = r = 1$. The same scheme works for other models such as the 2D quantum Baxter wave functions or can be generalized to states with non-Abelian statistics.

Using the QDM - CFT connection it is instructive to outline the computation of the universal properties of entanglement entropies in these systems following [33]. The geometry is depicted at the Fig. 1.4 where the circumferences of the regions $A$ and $B$ are chosen such that $L \gg \ell \gg a$ (where $a$ is short-distance cut-off). The important part of this construction are boundary conditions that are chosen to be fixed (Dirichlet) as shown at the figure.

The ground state wave function $|\Psi_0\rangle$ of the system is given by CFT action $S$:

$$|\Psi_0\rangle = \frac{1}{\sqrt{Z}} \int \mathcal{D}\phi \ e^{-S(\{\phi\})/2} |\{\phi\}\rangle \quad (1.26)$$

with

$$Z = \int \mathcal{D}\phi \ e^{-S(\{\phi\})}. \quad (1.27)$$

The Hilbert space $|\{\phi\}\rangle$ is labelled by classical configurations $\{\phi\}$ and expectation values in this state reproduce CFT correlators.
The computation of the entanglement entropy is based on a replica trick, which gives the expression for the $n$-th power of the reduced density matrix:

$$\text{tr}\rho^n_A = \frac{Z_n}{Z^n} = \left(\frac{Z_A Z_B}{Z_{A\cup B}}\right)^{n-1},$$  \hspace{1cm} (1.28)

where $Z_n$ is the partition function of $n$ copies (replicas) of the 2D systems coupled with each other through the boundary conditions on $\Gamma$ and $Z$ is a partition function of a single system.

The partition functions on the r.h.s of Eq.(1.28) are given by $Z_A = ||\Psi^A_0||^2$ and $Z_B = ||\Psi^B_0||^2$ with supports on $A$ and $B$ which satisfy Dirichlet boundary conditions on $\Gamma$; $Z_{A\cup B} = ||\Psi_0||^2$ is the partition function of the whole system. The entanglement entropy $S$ is computed as:

$$S_A = -\text{tr}(\rho_A \ln \rho_A) = -\lim_{n \to 1} \frac{\partial}{\partial n} \text{tr}\rho^n_A = -\ln \left(\frac{Z_A Z_B}{Z_{A\cup B}}\right).$$  \hspace{1cm} (1.29)

Thus the computation of the entanglement entropy is reduced to the computation of a ratio of partition functions in a 2D classical statistical mechanical problem, each satisfying specific boundary conditions. This can be rewritten as

$$S_A = F_A + F_B - F_{A\cup B}.$$  \hspace{1cm} (1.30)

where $F_A$, $F_B$ and $F_{A\cup B}$ are the free energies of the CFT associated with the ground state wave function. The boundary conditions in the case of the quantum Lifshitz wave function 1.24 are as follows: $F_A$, $F_B$ have Dirichlet boundary conditions on $\Gamma$, $F_B$ and $F_{A\cup B}$ obey the same boundary conditions on the outer boundary of $B$-region. The equation 1.30 has a general validity, with Dirichlet boundary conditions replaced by some appropriately chosen fixed boundary conditions. We arrived to the conclusion that the computation of the entanglement entropies of conformal wave functions reduces to a problem in boundary CFT.
The scaling of the entanglement entropy with the size of the region is determined by the scaling of the free energies in (1.30). This problem has been intensively studied and for a large bounded region of linear size $L$ and smooth boundary $F$ obeys [35]:

$$F = \alpha L^2 + \beta L - \frac{c}{6} \chi \ln L + O(1),$$

(1.31)

where $\alpha$ and $\beta$ are non-universal constants, $c$ is the central charge, and $\chi$ is the Euler characteristic of the manifold:

$$\chi = 2 - 2h - b.$$  

(1.32)

Here $h$ and $b$ are the number of handles and boundaries of the region. Hence, we see that the universal coefficients in the entanglement entropy are grouped with logarithmically dependent terms:

$$S_{A,\text{log}} = -\frac{c}{6}(\chi_A + \chi_B - \chi_{A \cup B}) \ln L.$$  

(1.33)

It turns out that the $O(1)$ correction has also a universal piece related to the "boundary entropy." It is clear that in the case $A \subseteq B$ the coefficient of the logarithmic term vanishes because there is no change in the Euler characteristic:

$$\chi_A + \chi_B = \chi_{A \cup B} \Rightarrow \Delta S = 0.$$  

(1.34)

However if $A$ and $B$ regions are physically disjoint then $\chi_A + \chi_B - \chi_{A \cup B} \neq 0$ and there is a ln term in the entanglement entropy. Logarithmic term also appears if $A$ and $B$ share a common non-smooth boundary. In this case the coefficient of $\ln L$ terms is determined by the angles at singular points of the boundary [33].

In the case when the coefficient of the logarithmic term vanishes the $O(1)$ correction becomes universal. This finite term is determined by the contributions of the winding modes to the respective partition functions, so the the conformal charge alone is not sufficient and more detailed information about the structure of the CFT is needed. Fortunately, the required results are widely available in the literature [36]. The $O(1)$ term is determined by exact formulas for 2D partition functions and therefore differs for cylinder, torus and disk geometries.

For example, the partition function of a boson with compactification radius $R$ on a cylinder is

$$Z_{DD}(L, \ell) = \mathcal{N} \frac{1}{R} \frac{\vartheta_3 \left( \frac{2\pi i}{R} \right)}{\eta(q^2)}.$$  

(1.35)

Here $L$ is the length of a cylinder, $\ell$ is its circumference, $\tau = i \frac{L}{\ell}$ is the modular parameter and $q = e^{2\pi i \tau}$; $\vartheta_3(\tau)$ is the elliptic theta-function and $\eta(q)$ is Dedekind eta-function. Dirichlet boundary conditions are imposed on both ends. The important feature of the Eq. (1.35) is the factor $1/R$, which is the contribution of the winding modes of the compactified boson on the cylinder. It is straightforward to find an expression for the entanglement entropy using Eq. (1.29). The entanglement entropy depends on the geometry of the cylinders encoded in ratios of theta and eta functions, however, in the limit $L_A \gg \ell$, in which the length of the cylinders are long compared to their circumference, it takes a simple form

$$S_A = \mu \ell + \ln R,$$  

(1.36)
where \( \mu \) is a non-universal constant that depends on the prefactor \( N \) of Eq. (1.35). Thus the universal \( O(1) \) contribution to the entanglement entropy is

\[
\gamma_{QCP, \text{cylinder}} = \ln R
\]

for the cylindrical geometry.

![Diagram of cylinder and torus geometries](image)

**Figure 1.5:** Cylinder and torus geometries. The figures are taken from [34].

The compactification radius is \( R = \sqrt{2kr^2} \) and for the RK quantum dimer model, \( k = r = 1 \) we obtain \( R = \sqrt{2} \). As a result the universal finite term in the entanglement entropy is \( \gamma_{QCP} = \ln \sqrt{2} \), which is different from the value \( \gamma_{\text{topo}} = -\ln 2 \) in the nearby topological phase of the \( Z_2 \) universality class of Kitaev’s toric code.

Similarly we can also consider the case in which the full system \( A \cup B \) is a torus and \( L/\ell \gg 1 \) as shown in Fig. 1.5. The two subsystems \( A \) and \( B \) are two cylinders of the length \( L_A \) and \( L_B \) respectively \( (L = L_A + L_B) \), both with the same circumference \( \ell \). Again the partition function for this case is known and in the limit \( L_A \gg \ell \gg a \) and \( L_B \gg \ell \gg a \) the entanglement entropy is:

\[
S_A = \mu \ell + 2 \ln \left( \frac{R^2}{2} \right).
\]

For the toroidal geometry the universal term is \( \gamma_{QCP} = 2 \ln(R^2/2) = 2 \ln (kr^2) \). In the Eq. (1.38) \( \mu \) is a non-universal regularization-dependent constant. The contribution is determined by the winding modes of the bosonic field but the values of \( \gamma_{QCP} \) in the Eq. (1.38) and the Eq. (1.36) are different. For a disk, shown at the Fig. 1.4, the finite universal term \( S_{\text{disk}} \) also depends on the aspect ratio and the universal finite term is [37]

\[
S_A = \frac{1}{2} \ln \left[ \frac{1}{\pi} \ln \left( \frac{L}{\ell} \right) \right] + \ln R.
\]

Since the universal contribution \( \gamma_{QCP} \) at critical point is different from a similar term \( \gamma_{\text{topo}} \) in the adjacent topological phase we can expect that this correction jumps as we tune across the quantum phase transition.

One can consider more general systems where conformally invariant wave functions correspond to a rational conformal field theory. The norm of the wave function is the
partition function of the RCFT, which are no longer free theories so the correspondence is more complicated [38]. The defining feature of the RCFT is the finite number of primaries, which are closed under the operator product expansion

$$\Phi_a \times \Phi_b = \sum_j N_{ab}^j \Phi_j , \quad (1.40)$$

where integer-valued $N_{ab}^j$ are the fusion coefficients. Each primary field defines a set of conformally invariant boundary conditions labelled by $a$.

Let us consider the computation of the entanglement entropy in cylindrical geometry for these systems. The partition function for a RCFT on a cylinder of length $L$ and circumference $\ell$, with boundary conditions $a$ and $b$ on the left and right ends is expressed in terms of the Virasoro characters $\chi_i$ of the RCFT:

$$Z_{a/b} = \sum_j N_{ab}^j \chi_j \left( e^{-\pi \ell / L} \right) . \quad (1.41)$$

Under the modular transformation $\tau \to -1/\tau$, which exchanges horizontal and vertical coordinates on a cylinder, the characters behave as

$$\chi_i \left( e^{-\pi \ell / L} \right) = S_i^j \chi_j \left( e^{-4\pi L / \ell} \right) , \quad (1.42)$$

where $S_i^j$ is the modular $S$-matrix of the RCFT. The modular $S$-matrix and the fusion coefficients are related by the Verlinde formula [39].

A simple expression for the entanglement entropy can be obtained in the limit of $L \gg \ell$. Under the modular transformation the partition function of Eq.(1.41) transforms as

$$Z_{a/b} = \sum_{i,j} N_{ab}^i S_i^j \chi_j \left( e^{-4\pi L / \ell} \right) . \quad (1.43)$$

The limit $\ell / L \to 0$ is dominated by the the descendants of the identity 1 up to exponential corrections, so

$$Z_{a/b} \to \sum_i N_{ab}^i S_i^0 \chi_0 \left( e^{-4\pi L / \ell} \right) \to e^{\pi \ell c / 6\ell} \sum_i N_{ab}^i S_i^0 . \quad (1.44)$$

Here the definition of the characters is used:

$$\chi_j (e^{-\pi \ell / L}) = e^{\pi \ell c / 24L} \text{tr}_a \left( e^{-\frac{\pi \ell c}{6L}} \hat{L}_0 \right) , \quad (1.45)$$

Thus $\ln Z_{a/b}$ becomes

$$\ln Z_{a/b} = \frac{\pi L c}{6\ell} + \ln g_{ab} . \quad (1.46)$$

The quantity $\ln g_{ab}$ in Eq.(1.46) is also called boundary entropy of a boundary RCFT, where the "ground state degeneracy" $g_{ab}$ is given by

$$g_{ab} = \sum_i N_{ab}^i S_i^0 . \quad (1.47)$$
The Eq.1.46 allows us to find the scaling and constant correction of the entanglement entropy:

$$S_A = - \ln \left( \frac{Z_A^A Z_B^B}{Z_{A \cup B}^D} \right) = \mu \ell - \ln \left( \frac{g^{a_0} g^{b_0}}{g_{a_b}} \right).$$

(1.48)

This result shows that $\gamma_{QCP}$ is determined by the fusion coefficients $N_{ba}^c$ and by the modular matrix $S_i^j$ of the RCFT associated with the norm squared of the many-body wave function at the quantum critical point.

It is instructive to consider a simple case, when the boundary conditions of $Z_A$, $Z_B$ and $Z_{A \cup B}$ are fixed and the boundary states are in the conformal block of the identity $1$ state. In this case the only non-vanishing fusion coefficient is $N_{00}^0 = 1$ and the universal contribution is:

$$\gamma_{QCP} = - \ln S_0^0.$$

(1.49)

Note the similarity of this expression with the topological entanglement entropy. A thorough analysis of these results for the cases of the Gibbs weights of 2D Ising model, quantum loop models and quantum net models is given the Ref. [37].

1.2 Topological phases of matter

Topological phases of matter are gapped liquid-like phases of electrons and spins in two dimensions without long range order [40]. The defining feature of topological phases is the robust ground state degeneracy on a high genus manifolds. The quasiparticles in these theories are similar to vortices (i.e. characterized by some topological numbers), have fractional statistics and fractional charge. The phases cannot be characterized by local order parameters and the observables are akin to Wilson loops in gauge theories. The effective low-energy theories of the topological phases are Chern-Simons gauge theories.

Another important feature of the topological phases is that the quasiparticle wave functions have a non trivial transformation under braiding operations. In other words because the Braid group in 2d can have non trivial representation (unlike in 3 dimensions) the excitations in the topological phases realize irreducible representations of the Braid group. This property is also called fractional statistics and serves as basis for a proposal for topological quantum computation. The simplest case of the fractional statistics is the abelian one and it is believed to be realized in the abelian quantum Hall states. In addition there are more interesting cases of topological phases where the quasiparticles transform under non-Abelian representation of the Braid group. The state in the presence of the non-Abelian quasiparticles is degenerate and there is a finite number of linearly independent states with fixed coordinates of quasiparticles. These topologically protected quasiparticle states define a basis suitable for topological quantum computations [41]. The real physical systems with topological order are gapped and at the energies low compared with value of the gap are described by topological field theories. This gives another tautological definition of the topological order, namely the phase of the system which has topological field theory as a low energy effective action is called topological.

The experimental observation of the topological order is not yet fully achieved. The best know example of a topological quantum liquid is realized in the FQH states of
1. Introduction

a 2D electronic gas at large magnetic fields. The most stable FQH states correspond to Abelian order and they are manifested as plateaus in the Hall conductance $\sigma_{xy} = \nu e^2/h$ where $\nu$ is the filling fraction of the Landau level. The fractional charge in these states has been measured via shot-noise in a tunneling constriction. However the attempts to unambiguously demonstrate the Abelian fractional statistics have not yet been successful.

The best candidate for the non-Abelian FHQ state is the Moore-Read state at the filling fraction $\nu = 5/2$. Recent experimental efforts have been directed at the detection of the quarter charge ($q = e/4$) in this state via shot noise through point contact and transport experiments [42,43], however the results are not conclusive. The best evidence for the realization of the Moore-Read state in nature is still numerical and provided by exact diagonalization on a computer. Experimentally there is also a plateau at $\nu = 12/5$, for which a possible candidate is the Read-Rezayi state. There are also interesting proposals for the realization of non-Abelian states in ultracold rapidly rotating gases, which are however difficult to realize at present [44,45]. A recent progress has been reported in the Ref. [46], where a small number (up to 10) of trapped bosons has been transferred into FQH regime.

We will give a short description of the Abelian fractional quantum Hall states and their effective theory. The Abelian FQH states are generalizations of the Laughlin states whose wave functions for a system of $N$ electrons in $N_\phi$ magnetic fluxes, at filling fraction $\nu = N/N_\phi = 1/m$ (with $m$ an odd integer) are [47]:

$$\Psi_m(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4l_\phi^2}, \quad (1.50)$$

where $z = x + iy$ are the complex coordinates of the plane and $l_\phi = \sqrt{\hbar c/eB}$ is the magnetic length. The effective theory of the Laughlin FQH states is an Abelian Chern-Simons gauge theory and it is instructive to follow the approach described by Wen [48] to see how it appears.

The first thing to notice is that electronic gas is a charged fluid and therefore has a conserved charge current $j_\mu$, $\mu = 0, 1, 2$:

$$\partial_\mu j^\mu = 0. \quad (1.51)$$

In 2D the we can introduce a vector field $a_\mu$ dual to the current defined as

$$j_\mu = \frac{1}{2\pi} \epsilon_{\mu\nu\lambda} \partial^\nu a^\lambda. \quad (1.52)$$

The current $j_\mu$ is invariant under the transformation $a_\mu \rightarrow a_\mu + \partial_\mu \Lambda$ so the vector field $a_\mu$ is gauge invariant. The aim is to find an effective theory in terms of the gauge field $a_\mu$, which is related to the current $j_\mu$. Because the gauge field is fictitious, and the only physical observable is the current, the effective action should be gauge invariant. From the experiment we know the linear response of the current to the changes in the external electromagnetic field. The incompressibility of the quantum Hall fluids leads to a relation between the density and magnetic field:

$$j_0 = \frac{\nu B}{2\pi}.$$
Combined with the quantization of the Hall conductance and the requirement of the action to be odd under time-reversal symmetry (because of the large perpendicular magnetic field) this gives the only possible form of the effective action for $a_\mu$:

$$ S(a) = \frac{m}{4\pi} \int_{\Sigma \times S^1} d^3x \, \epsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda . \tag{1.53} $$

Here $\Sigma$ is the spatial manifold and $S^1$ is the time direction. For a closed manifold $\Sigma$ the value of the $m$ is restricted to be integer if we require the action to be invariant under ”large” gauge transformations. The coupling to the external field is given in a usual way:

$$ L_{\text{int}} = -e j_\mu a^\mu = -\frac{e}{2\pi} \epsilon_{\mu\nu\lambda} A^\mu \partial^\nu a^\lambda , $$

which obviously satisfies the requirement of the ordinary gauge invariance. A further important ingredient is to specify the coupling of the quasiparticles to the vector gauge field $a$. This is done via a minimal coupling prescription $L_{\text{qp}} = j_{\mu}^{\text{qp}} a_\mu$.

The theory described by this Lagrangian indeed predicts the quantization of the Hall conductance $\sigma_{xy} = \frac{1}{m} \frac{e^2}{h}$ and $\sigma_{xx} = 0$. The statistical angle due to braiding of the quasiparticles is $\theta = \pi/m$ and the quasiparticle charge is $\pi/m$. Another important fact is that the theory does not have any internal dynamics, the Hamiltonian is equal to zero, i.e. it is topological. This, however does not meant that the theory is trivial. The purpose of the abelian Chern-Simons theory is to explain the behavior of quantum Hall states under under external perturbations such as changes in electric field (Hall current) or braiding of quasiparticles (topological defects).

On a manifold with a a boundary the theory has chiral edge states described by the chiral boson $U(1)_m$ CFT with the compactification radius $R = \sqrt{m}$ and $c = 1$. This allows to access the topological states experimentally by performing various transport measurements at the edge of the sample.

The appearance of the CFT at the edge of topological phases can be exploited to give a heuristic derivation of the topological entanglement entropy in the Eq. 1.22 presented in the Ref. [49]. First let us rewrite the reduced density matrix as

$$ \rho_A = e^{-\beta H} . $$

This can be regarded as a trivial definition of the effective hamiltonian $H$ and $\beta$. For a $2 + 1$ dimensional system in a topological phase can be made an additional assumption that the resulting ”hamiltonian” is described by $1 + 1$D CFT

$$ H = H_{\text{CFT}} . $$

The motivation of this assumption stems form the work of Witten [50], which showed that the bulk degrees of freedom of Chern-Simons theory correspond to conformal blocks of the associated CFT at the boundary. For quantum Hall systems a similar assumption is empirically justified and can be used as a signature of topological order [51] (see also Chapter 4). Effectively, it is suggested that bulk degrees of freedom are gapped out and do not affect physical observables, therefore the contribution to the reduced density matrix comes only from boundary degrees of freedom, which are
described by CFT. Taking this assumption, we consider the reduced density matrix as a density matrix of some physical system, and, correspondingly, the entanglement entropy will be the entropy of this physical (CFT) system. The entropy itself can be found using standard CFT machinery. First we compute the partition function in the presence of one quasiparticle of type \(a\): 

\[
Z_a = \text{tr}_a e^{-\beta H}.
\]

Here the subscript of the trace means that there is a Wilson loop in \(a\) representation in \(\beta\) direction. The same can be calculated via modular transformation: 

\[
Z_a = \sum_b S_b^a \tilde{Z}_b.
\]

We take a large \(L\)-limit and all the terms in the summation can be neglected with an exponential precision except the identity sector \(b = 1\):

\[
\ln Z_a \approx \ln(S_a^1 \tilde{Z}_1) \approx \frac{\pi}{12} (c + \tilde{c}) L/\beta + \ln S_a^1.
\]

By definition:

\[
S_a^1 = d_a/D.
\]

The (entanglement) entropy (which can be found as usual \(S = -\partial F/\partial T; F\) is the free energy) is:

\[
S_A(\rho_A) = \frac{\partial}{\partial T} T \ln Z = \alpha L - \ln(D/d_a).
\]

This elegant derivation, although based on some physical and non-rigorous assumptions, provides us also a universal sub-linear correction in the presence of bulk quasi-particles.

Another class of systems with topological order are quantum dimer models (QDM), which were introduced to describe frustration in doped Mott insulator [52]. The dimers are valence-bond singlets of adjacent spins on a lattice. QDM’s have a special point on a phase diagram, Rokhsar-Kivelson point, where the ground state wave function has a form of ”resonating valence bond” (RVB). This wave function is an equal weight sum of all possible dimer configuration of the lattice. The useful feature of RK state is the weight of a particular configuration coincides with a Gibbs weight of the configuration on classical dimer model. It follows that equal time correlation functions of QDM can be computed from corresponding classical dimer models. There is a number of different dimer models with more complex degrees of freedom than original RK hamiltonian. Among them some are designed to describe non-Abelian topological phases at their RK points via quantum loops, nets and string condensates [53, 54]. RK points on non-bipartite lattices describe topological phases of \(Z_2\) universality class [55]. A similar model is Kitaev’s ”toric code” which is considered below. For bipartite lattices RK points are generally in critical phase. This phase has the universality of quantum Lifshitz model type with the dynamical exponent \(z = 2\). The ’quantum-classical’ correspondence allows one to compute entanglement via CFT methods, see the preceding Sec.1.1.

Perhaps the simplest exactly tractable model with topological order is Kitaev’s toric code [56]. The model is defined on a square lattice as shown at the Fig.1.6 with spins 1/2 at the links by the Hamiltonian:

\[
H = -\sum_s A_s - \sum_p B_p.
\]
where sum is over all plaquettes (p) and sites (s); plaquette and vertex operators are 4-spin interactions:

\[ B_p = \sigma_x \sigma_x \sigma_x \sigma_x, \quad A_s = \sigma_z \sigma_z \sigma_z \sigma_z. \]

The product is taken among 4 spins of a plaquette or adjacent to a site. As \([B_p, A_s] = 0\) one can see that the ground states should satisfy the condition that the eigenvalues of \(B_p\) and \(A_s\) operators is 1 on all plaquettes and sites. It follows that a ground state \(\Psi\) is the superposition of all possible spin-up loops on a lattice \(c\) or dual lattice \(c'\) as schematically shown at the Fig. 1.7.

\[ \text{Figure 1.6: Kitaev’s Toric code model, the figure taken from [57].} \]

A pair of excited sites \(i, j\) is created by \(\prod_{c(i,j)} \sigma_x\) where \(c(i,j)\) a path of links connecting \(i\) and \(j\). Similarly there is a pair of excited plaquettes produced by \(\prod_{c'(i,j)} \sigma_z\) with \(c'(i,j)\) denoting links crossed by a path on a dual lattices from a plaquette \(i\) to \(j\). The two types of excitations have a semionic statistics with respect to each other and are called electric \(e\) and magnetic \(m\) excitations. They obey fusion rules \(e \times e = 1, \quad m \times m = 1, \quad e \times m = \psi, \quad \psi \times e(m) = m(e), \quad \psi \times \psi = 1\). Thus there are 4 types of excitations with quantum dimension 1 so the total quantum dimension \(D = 4\). The ground state on the torus is four-fold degenerate corresponding to combinations of even or odd number of loops around the two cycles of the torus. The model describes a topological phase, as it has excitations with abelian (semionic) statistics and non-trivial ground state degeneracy.

Let us calculate the entanglement entropy of a simply connected region \(A\) in this model \([58, 59]\). We factorize the wave function into \(\Psi = \sum_q \Psi^A_q \Psi^B_q\) where \(\Psi^A(B)\) is a part of the wave function inside region \(A(B)\). A precise decomposition and multi-index \(q\) can be specified as follows: \(q = q_1, \ldots, q_n\) with \(n\) number of sites at the boundary and \(q_i = 0, 1\). The wave function \(\Psi(X) = 1\) if \(X\) form a closed loop and if the string intersects the \(k - th\) link then \(q_k = 1\). Clearly the sum \(q_1 + \cdots q_n\) is even as every loop intersects the boundary of a region \(A\) even number of times.

The density matrix of the region \(A\) is an equal weight superposition of \(\Psi^A_q\) with
∑ q even. There are 2^{n-1} states with different q’s so the entanglement entropy is
\[ S_A = \ln 2^{n-1} = n \ln 2 - \ln 2. \] (1.57)

In this expression the first term corresponds to the area law and is proportional to the size of the region n and the second term is topological entanglement entropy \( \ln \sqrt{D} = \ln \sqrt{4} \) for this particular model. Intuitively the correction appears because the ground state is a superposition of loops of up-spins and every loop intersects a boundary zero or two times as shown at the Fig. 1.7.

A similar calculation can also be done for more complicated Hamiltonians with topological order. In particular, the \( O(1) \) corrections to the scaling due to finite correlation length has been found in the Ref. [60]. Those, however, appear due to sharp features of the A boundary.

![Figure 1.7: Contribution of the GS loops to the entanglement, the figure taken from [58].](image)

### 1.3 Entanglement Entropy and Chern-Simons theory

The entanglement entropy can be computed using the effective topological Chern-Simons theory of the FQH states. The description of the FQH wave functions by topological theories retain only topological properties of the theory so one can compute the universal piece of the entanglement entropy, and the linear ("area law") non-universal cut-off dependent term in will be omitted. The presented here calculation of the topological entanglement entropy follows Refs. [34, 61] and the result is given in terms of the modular matrix of the Chern-Simons theory and its conformal blocks.

The main ingredient of the computation are the famous results of Witten [50] for the Chern-Simons theory. The action of a non-Abelian Chern-Simons gauge theory is (using wedge notation)
\[ S(A) = \frac{k}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \] (1.58)
where $A_\mu$ is a vector field taking values in the algebra of a compact gauge group $G$. We will be primarily interested in the case $G = SU(2)$. This action looks like some complicated non-linear theory and it is surprising that this theory can be solved. The states on a closed 2D surface $\Sigma$ are realized as a path integral over a 3D volume. The Chern-Simons states on a spatial manifold $\Sigma$ are in one-to-one correspondence with the conformal blocks of a Wess-Zumino-Witten (WZW) CFT. The ground state degeneracy depends on the level $k$ and on the topology of the surface $\Sigma$. The partition function depends on the modular $S$-matrix, e.g. the partition function on $S^3$ with a Wilson loop in representation $\rho_j$ is

$$Z(S^3, \rho_j) = S_j^0.$$  

(1.59)

As usual the $S$-matrix defines the transformation of the degenerate ground states under modular transformation.

The computation of the Chern-Simons path integral can be reduced to its computation on a sphere $S^3$ using the method of surgeries. If a 3D manifold $M$ is the connected sum of two 3D manifolds $M_1$ and $M_2$ which are joined along the $S^2$-sphere, then the partition functions are related as:

$$Z(M)Z(S^3) = Z(M_1)Z(M_2).$$  

(1.60)

It follows that if $M$ is $M_1$ and $M_2$ which are joined along $n$ $S^2$-spheres then the resulting partition function is

$$Z(M) = \frac{Z(M_1)Z(M_2)}{Z(S^3)^n}. $$  

(1.61)

The entanglement entropy is computed (again) using the replica tick, where one computes the $n$-th power of the reduced density matrix. This procedure leads to a 3D manifold shown at the Fig. 1.8 for $n = 3$. In the simplest case the spatial manifold is $\Sigma = S^2$-sphere so the space-time manifold is a 3-sphere $\Sigma \times S^1 \cong S^3$.

**Figure 1.8:** Left: the cut-manifold defining the reduced density matrix $\rho_A$. Right: the manifold used to compute $\text{Tr} \rho_A^3$ with replica method. $\beta \to \infty$ is the inverse temperature. The figures are taken from [34].

We consider the case of $S^2$ with the two regions $A$ and $B$ being two hemispheres (disks). $\text{tr} \rho_A^3$ is obtained by glueing $2n$ such pieces together. When glued together
$S^2$ rotated about the axis which has the topology of $S^3$. For $n > 2$ the $S^2$ is obtained by sequentially gluing $2n$ disks and the normalized trace of $\rho^n_A$ is

$$\frac{\text{tr} \rho^n_{A(S^2,1)}}{(\text{tr} \rho_{A(S^2,1)})} = \frac{Z(S^3)}{(Z(S^3))^n} = \left(\frac{Z(S^3)}{Z(S^3)}\right)^{1-n} = S^{1-n}_{00}$$

(1.62)

In the replica limit $n \to 1$ we get the entanglement entropy

$$S_A^{(S^2,1)} = \ln S_{00} = -\ln D.$$  

(1.63)

This result coincides with the Refs. [49] and is [58] and is called topological entanglement entropy. In this formalism there is no ”area law” term like in the Eqs. 1.22 and 1.56 because the theory is topological so there is no metric to introduce the notion of distance or area. In the case of a sphere $S^2$ and a disconnected region $A$ with $M$ boundaries one can find $S_A^{(S^2,M)} = M \ln S_{00} = -M \ln D$. Thus each boundary adds the same contribution to the topological entropy. It is worth to note that the only quantity we can extract from the entanglement entropy and Renyi entropies is the total quantum dimension $D$. Thus computation in this geometry and without quasi-particles does not allow us to observe other properties of the topological theories. This has also been explicitly verified for lattice models with topological order in the Ref. [62].

It is possible to find a dependence of the entanglement entropy not only on the total quantum dimension (or $S_{00}$) if we consider the system on a more complicated geometries such as toroidal or in the presence of quasi-particles in the block $A$. For example if there is one quasiparticle $\alpha$ in the region $A$ then the entanglement entropy is:

$$S_A = \ln S_0^\alpha = \ln S_{00} + \ln d_\alpha,$$  

(1.64)

which is consistent with the previous derivation 1.56 More complicated expressions can be obtained for the cases of two quasiparticles in the block $A$.

In the real physical systems the correlation length is not zero and therefore entanglement entropy scales with the system size and $\gamma$ appears as $O(1)$ correction 1.22. There are ways to combine the different subsystems so that the non-universal parts, which depend on the boundary length, are cancelled. For example, one can combine the entanglement entropies of the regions shown at the Fig.1.9 to extract the topological entanglement entropy as:

$$S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC} = -2\gamma,$$  

(1.65)

$$S_{A_1} + S_{A_4} - S_{A_2} - S_{A_3} = -2\gamma.$$  

(1.66)

To provide the numerical values of topological entanglement entropies we will need some properties of the modular $S$ matrix and of the conformal blocks for physical systems of interest. For the simplest gauge groups $U(1)_m$ and $SU(2)_k$ the modular $S$ matrix is well known:

$$S_{l'} = \frac{1}{\sqrt{m}} e^{2\pi il'/m}$$

(1.67)
and

\[ S_j^{(k)j'} = \sqrt{\frac{2}{k+2}} \sin \left( \frac{\pi (2j + 1)(2j' + 1)}{k+2} \right) , \]  

(1.68)

where labels run through \( l = 0 \ldots m - 1 \) and \( j, j' = 0, 1/2, \ldots, k/2 \). The definitions of the quantum dimensions \( d_j \) and total quantum dimension \( D \) are:

\[ d_j = \frac{S_j^0}{S_0^0}, \quad D \equiv \sqrt{\sum_i |d_i|^2} = \frac{1}{S_0^0} , \]  

(1.69)

which measure the exponential growth \( n^{d_j} \) of the dimension of the Hilbert space in the presence of \( n \) quasiparticles in the representation \( \rho_j \).

The Laughlin states at filling fraction \( \nu = 1/m \) are described by effective field theory with \( U(1)_m \) Abelian Chern-Simons action. It follows that the Laughlin state has a fundamental anyon (of fractional charge \( q = e/m \)), which generates \( m \) abelian sectors. The quantum dimension \( d_I \) is unity in all sectors so that \( D = \sqrt{m} \) for the \( \nu = 1/m \) Laughlin state. For \( m = 3 \) this gives \( \gamma = \ln \sqrt{3} \simeq 0.55 \). It is also possible to show that the entanglement entropy for any \( U(1)_m \) state is

\[ S_m = -\frac{1}{2} \ln g \]  

where \( g \) is the ground state degeneracy of the state on the torus.

For states with non-abelian quasiparticles, the situation is more interesting because some anyon sectors contribute \( d_i > 1 \). Details for some examples have been provided in Refs. [49, 63]. In particular, for the \( m = 2 \) Moore-Read state, there are six sectors (two each of quasiparticles denoted by \( I, \sigma, \psi \)) which contribute \( d_I = 1, d_\sigma = \sqrt{2}, d_\psi = 1 \), leading to \( D = \sqrt{6} \) and \( \gamma \simeq 1.04 \). The non-abelian nature shows up in the fact that \( \gamma \) is larger than \( \ln \sqrt{6} \), six being the degeneracy of the \( m = 2 \) Moore-Read state on the torus.

A compact general expression for the total quantum dimension for a Read-Rezayi state with order-\( k \) clustering and at filling fraction \( \nu = k/(kM + 2) \) is

\[ D_{RR}[k, M] = \sqrt{\frac{(k+2)(kM+2)}{2\sin(\pi/(k+2))}} . \]  

(1.70)

It includes the Laughlin states \( (k = 1, M = m-2) \) and the Moore-Read states \( (k = 2, M = m-1) \) as special cases. For a general Ardonne-Schoutens (AS) spin-singlet non-abelian FQH state with order \( k \) clustering and filling fraction \( \nu = 2k/(2kM + 3) \), the result is

\[ D_{AS}[k, M] = \frac{(k+3)\sqrt{(2kM+3)}}{16\cos(\pi/(k+3))\sin^3(\pi/(k+3))} , \]  

(1.71)

giving \( \gamma \simeq 1.62 \) for the paired spin-singlet state \( (k = 2, M = 1) \) at \( \nu = 4/7 \).

In general, the \( M \)-dependence of these expressions for total quantum dimensions is linked to the ground state degeneracy in torus geometry. Denoting the latter by \( \#[k, M] \) we have the relation

\[ D[k, M] = D[k, 0] \sqrt{\frac{\#[k, M]}{\#[k, 0]} . \]  

(1.72)
The conformal field theories underlying the states at $M = 0$ are of Wess-Zumino-Witten type ($SU(2)_k$ for the RR states and $SU(3)_k$ for the AS series) and the quantities $D[k,0]$ can be expressed in the modular $S$-matrix for these WZW models.

![Figure 1.9: The choice of the regions to compute entanglement entropy. The figures are taken from [49,58]](image)

1.4 Conclusions

We described the behavior of the entanglement entropy for a wide class of physical systems, including relativistic free quantum field theories, spin chains, a class of 2D quantum critical points, topological phases of matter etc. We demonstrated that for most systems entanglement entropy scales with the area rather than volume of the subsystem with generally non-universal coefficients. The most understood description of the entanglement entropy is given in 1d, where powerful methods are available such as CFT and integrability. In 2d the physically interesting universal coefficients appear as sub-linear terms in scaling. Two important classes of the universal correction are considered in detail corresponding to quantum critical points with scale-invariant wave functions (quantum Lifshitz points) and topological phases of matter. Interestingly both quantum Lifshitz points and topological phase are realized for different range of parameters for certain microscopic hamiltonians.

For topological phases the universal correction to the entanglement entropy is called the topological entanglement entropy, which depends on the total quantum dimension of the quasiparticles. In the presence of quasiparticles or on toroidal manifold entanglement entropy depends on the detail structure of the Chern-Simons gauge theories, which are effective low-energy description of topological phases. These results suggest that it is possible to determine the structure of the topological field theory by means of entanglement entropy measurements. As topological order can not be characterized via local observables, perhaps the entanglement entropy can serve as a tool to detect
topological order. Indeed, the aim of the next chapters is to describe topological order in terms of entanglement entropy for the certain Abelian and non-Abelian quantum Hall states.

It should be emphasized however that entanglement entropy is not an experimentally accessible observable, unlike the usual thermodynamic entropy. The main problem is that by definition it is a highly non-local quantity which depends on high-order correlation functions. For the system of free fermions some highly idealized set up has been proposed which relates entanglement entropy and fluctuations of electrical current through quantum point contact [64]. However, the analysis in the Ref. [65] showed that there is no simple relation between counting statistics of shot noise and the generation of the entanglement for interacting 1D theories of quantum Hall edges.