Entanglement and order in many-body systems
Zozulya, O.S.

Citation for published version (APA):
Zozulya, O. S. (2011). Entanglement and order in many-body systems
3 Entanglement entropy in fractional quantum Hall states

3.1 Introduction

In the previous chapter the entanglement entropy was studied between subsets of orbitals, which is related to a spatial partitioning of a system. This allowed us to extract the topological entanglement entropy with a good precision for Laughlin wave functions. In this chapter, we present a systematic discussion of bipartite entanglement entropies for FQH states, elaborating on the results in the Chap. 2. In addition to the abelian Laughlin (L) states, we consider non-abelian FQH states: the Moore-Read (MR) (or pfaffian) states [73, 74]. We carefully study different γ extraction procedures, which are important due to severe finite size effects. Finally we notice some intriguing scaling properties of entanglement entropies, which, if taken for granted, can also be used to extract the topological entanglement entropy.

In the Sec. 3.3 we consider the entanglement entropy between particles rather than orbitals for various FQH states. We find upper bounds and give an interpretation of entanglement in terms of exclusion statistics. Thus the entanglement between particles gives another handle to study intricate properties of the quantum Hall wave functions. It is worth to note that a few generic results are available for this type of partitioning and the Chap. 5 is devoted to the study of this type of entanglement for more general physical systems. For brevity we will refer to the entanglement between particles as particle entanglement. For both series of states, we derive upper bounds $S_A^{\text{bound}}$ for particle entanglement entropies. An important difference between the $m = 3$ Laughlin state and the $m = 2$ Moore-Read states is that in the latter the leading correlations have a 3-body nature, whereas those in the Laughlin states are due to 2-body effects. This difference is nicely manifested in the leading terms of a $1/N$ expansion of the upper bounds $S_A^{\text{bound}}$, which are given by (see the Sec. 3.3 for

---

This chapter is based on a collaboration with M. Haque, K. Schoutens and E. H. Rezayi.
3.2 Partition choices on the sphere

The entanglement entropy is a bipartite measure of entanglement and depends on the particular partitions being considered. Obviously, a many-particle system can be partitioned in many ways. Rather than asking which partition is the “correct” one, it is more useful to ask what information one can extract from various kinds of partitioning. Accordingly, we have partitioned both the spatial degrees of freedom (as in the chapter 2), and the particles themselves. We find that both schemes are useful, for revealing distinct features of the many-particle state. The kinds of partitioning one is able to study depend on the available degrees of freedom. The calculations are all performed for FQH states in a spherical geometry described in the Sec. 2.3. Since the FQH wave functions on a sphere are obtained in terms of orbital occupancies, one can either partition orbitals or partition particles. Because of the spatial arrangement of the orbitals, partitioning orbitals is in fact equivalent to partitioning spatial regions.

Another important distinction between abelian and non-abelian states is in the value for the topological entropy $\gamma$. Comparing the Laughlin and Moore-Read states at the same filling fraction $\nu = 1/m$ we have

$$\gamma_L = \ln \sqrt{m} , \quad \gamma_{MR} = \ln \sqrt{4m} ,$$

the difference being due to the non-abelian nature of the Moore-Read states (see the Sec. 1.2). Thus $\gamma$ allows to distinguish between different topological states.

In the Section 3.2 we give a general discussion of possible bipartite entanglement measures in itinerant many body systems, carefully separating particle and spatial partitioning schemes. More details on particle partitioning are given in the Chapter 5. In the Section 3.3 we present analytical and numerical results for particle entanglement in the Laughlin and Moore-Read FQH states. In this, the eigenvalue distribution of the reduced density matrix $\rho_{nA}$ plays a central role. In the subsection 3.3.6 we relate the eigenvalue distribution for $\rho_{nA=2}$ to the two-particle correlation function $g_2(r)$. In the Section 3.4 we discuss spatial partitioning, paying particular attention to the numerical procedure used in extracting the topological entropy $\gamma$. In the Section 3.5 we present empirical results on finite-size scaling in quantum Hall wave functions, which can be successfully used to extract $\gamma$. While most results in this chapter are for fermionic FQH states (meaning $m$ odd in the Laughlin sequence and $m$ even for the Moore-Read states), we briefly comment on bosonic states in sections 3.3.3 and 3.5.

3.2 Partition choices on the sphere

The entanglement entropy is a bipartite measure of entanglement and depends on the particular partitions being considered. Obviously, a many-particle system can be partitioned in many ways. Rather than asking which partition is the “correct” one, it is more useful to ask what information one can extract from various kinds of partitioning. Accordingly, we have partitioned both the spatial degrees of freedom (as in the chapter 2), and the particles themselves. We find that both schemes are useful, for revealing distinct features of the many-particle state. The kinds of partitioning one is able to study depend on the available degrees of freedom. The calculations are all performed for FQH states in a spherical geometry described in the Sec. 2.3. Since the FQH wave functions on a sphere are obtained in terms of orbital occupancies, one can either partition orbitals or partition particles. Because of the spatial arrangement of the orbitals, partitioning orbitals is in fact equivalent to partitioning spatial regions.
as was shown in the Sec. 2.3. The difference between spatial and particle partitioning is not usually stressed because the most common systems studied (in the context of entanglement entropies in many-particle states) are spin models, for which there is no such distinction. In the cases of itinerant particles where there is a difference, the common scheme is spatial partitioning. Thus the the established results for entanglement scaling discussed in the Chapter 1 pertain to the blocks in space. This permits to use the powerful machinery of field theory (in particular CFT [27,28] in 1D), where the notion of an individual particle is absent. The generic methods of this kind are absent for particle entanglement and only a few particular cases have been explored so far.

3.3 Entanglement for particle partitioning

In this section we provide close upper bounds to the entropy of entanglement between \( n_A \) particles of the state and the remaining \( n_B = N - n_A \) particles. We also discuss the \( n_A \)-particle reduced density matrices \( \rho_{n_A} \) that arise in this context. As an example consider the wave function (2.4). There are 2 particles in 4 orbitals. After tracing one of them, we are left with \( n_A = 1 \) particle in 4 orbitals. This makes the basis states for 1-particle reduced density matrix as: [1000], [0100], [0010] and [0001]. For symmetry reasons alone all the states are equally possible so the reduced density matrix is diagonal with equal eigenvalues and the 1-particle entropy is simply the logarithm of the size of the Hilbert space: \( S_1 = \ln 4 \). This can be generalized for any total number of particles so that the 1-particle entanglement entropy for a given number of particles \( N \) and a total flux \( N_\phi \) is:

\[
S_{n_A=1} = \ln(N_\phi + 1).
\]

In the rest of the section we extend this simple result for any \( n_A \).

3.3.1 Multiplet structure and particle entropy bounds

For FQH states on a sphere, the \( n_A \)-particle reduced density matrices \( \rho_{n_A} \) commute with the total angular momentum operators \( L^2_{n_A} \) and \( L^z_{n_A} \) of the selected \( n_A \) particles. This implies that the eigenvalues of \( \rho_{n_A} \) are organized in a multiplet structure of the corresponding \( SU(2) \) algebra: an eigenvalue for total angular momentum \( L_{n_A} \) will be \( (2L_{n_A} + 1) \)-fold degenerate.

For \( n_A = 2 \) fermions, each having angular momentum \( L \), the 2-particle states have total angular momenta \( L_2 = 2L - 1, 2L - 3, \ldots, 1(0) \), for \( L \) integer (half-integer), giving a total number of \( (2L + 1)(2L)/2 \) states. A naive upper bound to the entanglement entropy is thus

\[
S_{n_A=2} \leq \ln \left(\frac{(2L + 1)(2L)}{2}\right).
\] (3.1)

Inspecting the explicit structure of the fermionic Laughlin states with \( m = 3, 5, \ldots, \), one finds that the eigenvalues corresponding to 2-particle states with \( L_2 = 2L - 1, 2L - 3, \ldots, 2L - (m - 2) \) all vanish. The reason is that the correlations in the Laughlin states are such that particles cannot come too close together. For example,
if a first fermion occupies the $l = 0$ orbital, localized near the north pole, the Laughlin wavefunction has zero amplitude for finding a second fermion in orbitals $l = 1$, $l = 2$, \ldots, $l = m - 1$. The highest possible value of the angular momentum of the two fermions combined is thus $L_2 = L + (L - m)$. The remaining number of non-zero eigenvalues is $(2L + (2 - m))(2L + (1 - m))/2$, leading to an improved bound on the entropy $S_{n_A = 2}$

$$S_{n_A = 2} \leq \ln [(2L + (2 - m))(2L + (1 - m))/2]$$

(3.2)

with $2L = m(N - 1)$ as before. For $n_A > 2$, the multiplet structures are more complicated and we need to resort to a different method for finding a non-trivial upper bound to the particle entropy. In the next subsection we give a general derivation for both the Laughlin and the Moore-Read series of fermionic FQH states.

### 3.3.2 Upper bounds for fermionic states

For $N$ fermionic particles, $n_A$ particles in the $A$ block, and the total number of orbitals given by $N_\phi + 1 = 2L + 1$, fermionic statistics lead to an obvious upper limit $S^F_A$ to the entropy $S_A$

$$S_A \leq S_A^F = \ln \left( \frac{N_\phi + 1}{n_A} \right).$$

(3.3)

In the FQH states the correlations are such that the particles avoid each other and the entropy is further reduced. To obtain a handle on this, one may reason as follows. The model FQH states in the Laughlin and Moore-Read series can be characterized as zero-energy eigenstates of a Hamiltonian penalizing pairs and/or triplets of particles coming to the same position. After tracing out the coordinates for the $B$ set, the dependence on those in the $A$ set is such that one still has a zero-energy eigenstate. However, the number of orbitals available to the $A$ particles is larger than what is needed to make the model FQH state in the $A$ sector, and one instead has a certain number of quasi-holes on top of the $A$ set model state. The total ground state degeneracy for this situation has been studied in the literature: see Ref. [74] for the Laughlin and Moore-Read states and Refs. [75, 76] for the Read-Rezayi and Ardonne-Schoutens series of non-abelian FQH states.

For the Laughlin states the details are as follows. The $N$-particle Laughlin state is realized on a total of $N_\phi + 1$ Landau orbitals, corresponding to $N_\phi = m(N - 1)$ flux quanta. The Laughlin state for $n_A$ particles would need $N_\phi^A = m(n_A - 1)$ flux quanta; we thus have an excess flux of $\Delta N_\phi = N_\phi - N_\phi^A = m(N - n_A)$. With the Laughlin gauge argument this corresponds to the presence of $n_{qh} = \Delta N_\phi$ quasi-holes over the groundstate. According to Ref. [74] each of the quasi-holes has a number of $n_A + 1$ effective orbitals to choose from, with bosonic counting rules (meaning that two or more quasi-holes can be in the same effective orbital). This gives a number of quasi-hole states equal to

$$\binom{(n_A + 1) + n_{qh} - 1}{n_{qh}},$$
leading to the following upper bound to the entropy $S_A$

$$S_A^{\text{bound}} = \ln \left( \frac{N_\phi + 1 - (m-1)(n_A - 1)}{n_A} \right). \quad (3.4)$$

We remark that this expression has a clear interpretation in terms of exclusion statistics: the counting factor in Eq. (3.4) gives the number of ways $n_A$ particles can be placed in $N_\phi + 1$ orbitals, in such a way that a particle placed in a given orbital $l$ excludes particles from orbitals $l'$ with $|l - l'| < m$.

In a $1/N$ expansion we find (assuming $n_A \ll N$)

$$S_A^F - S_A^{\text{bound}} = \frac{1}{N} \frac{m-1}{m} n_A (n_A - 1)$$

$$+ \frac{1}{N^2} \frac{m-1}{2m^2} n_A (n_A - 1) \left[ 2m + (n_A - 1)(m + n_A - 4) \right]$$

$$+ \mathcal{O}(1/N^3) \quad (3.5)$$

The particle entropy reaches a maximum for $n_A = N/2$. For this case our Eq. (3.4) gives, in the limit of large $N$,

$$S_{n_A=N/2} \leq N \left[ (m+1) \ln(m+1) - m \ln(m) \right]/2. \quad (3.6)$$

This bound is sharper than a bound presented in Ref. [77], which gives a larger coefficient for the linear-in-$N$ behavior.

For the fermionic Moore-Read states at $\nu = 1/m$, with $m = 2, 4, \ldots$, we can reason in a similar way, with now $N_\phi = m(N - 1) - 1$. As for the Laughlin states we have an excess flux of $\Delta N_\phi = N_\phi - N_\phi^A = m(N - n_A)$ but now the number of quasi-holes is twice this number due to the fact that the fundamental quasi-holes correspond to half a flux quantum. Thus, $n_{\text{qh}} = 2\Delta N_\phi$. We now take from Ref. [74] the following result for the total quasi-hole degeneracy

$$\sum_{F \equiv n_A \mod 2} \binom{n_{\text{qh}}/2}{F} \binom{(n_A - F)/2 + n_{\text{qh}}}{n_{\text{qh}}} \quad (3.7)$$

This gives us an upper bound $S_A^{\text{bound}}$ as before.

Putting $m = 2$, one easily checks that $S_A^{\text{bound}}$ coincides with $S_A^F$ for $n_A = 2$. In a $1/N$ expansion, the leading deviation from $S_A^F$ is a 3-body term at order $1/N^2$,

$$S_A^F - S_A^{\text{bound}} = \frac{1}{N^2} \frac{3}{4} n_A (n_A - 1)(n_A - 2) + \ldots \quad (3.8)$$

This result nicely illustrates the fact that the leading correlations in the $m = 2$ Moore-Read state have a 3-body character: the wave-function vanishes if at least three particles come to the same position.

For $m \neq 2$ the leading correlations do have a 2-body character, as for the Laughlin states,

$$S_A^F - S_A^{\text{bound}} = \frac{1}{N} \frac{m-2}{m} n_A (n_A - 1) + \ldots \quad (3.9)$$
Inspecting the particle entanglement at $n_A = N/2$ and for $N$ large, our bound implies that for the $m = 2$ Moore-Read state

$$S_{n_A = N/2} \leq 1.044N.$$  \hspace{1cm} (3.10)

This bound is reduced from the Fermi bound $S_F^A \sim N(4 \ln 4 - 3 \ln 3)/2 \sim 1.125N$, but it is larger than the bound for the $m = 2$ (bosonic) Laughlin state, which has asymptotic form $N(3 \ln 3 - 2 \ln 2)/2 \sim .955N$. This indicates that, at equal filling $\nu = 1/2$, the particles in a Moore-Read state are more entangled than those in a Laughlin state.

The quasi-hole counting rules for the order-$k$ clustered spin-polarized (Read-Rezayi) and spin-singlet (Ardonne-Schoutens) states are all known in the literature. [75] They can be used to generalize the upper bounds on particle entanglement entropy given in this subsection to these more intricate non-abelian FQH states.

### 3.3.3 Bosonic quantum Hall states

We briefly comment on the case of bosonic FQH states. The realization that a rapidly rotating Bose gas may eventually enter a regime of bosonic quantum Hall states motivates the theoretical study of the effects of bosonic statistics.

We consider bosonic Laughlin states at filling fraction $\nu = \frac{1}{m}$ with $m = 2, 4, \ldots$. The naive upper bound to the the entropy associated to placing $n_A$ bosons in $N_\phi + 1$ orbitals is

$$S_A^B = \ln \left( \frac{N_\phi + n_A}{n_A} \right)$$  \hspace{1cm} (3.11)

The expression for $S_A^{\text{bound}}$ remains unchanged, giving the following leading correction in a $1/N$ expansion

$$S_A^B - S_A^{\text{bound}} = \frac{1}{N} n_A(n_A - 1) + \ldots$$  \hspace{1cm} (3.12)

For a bosonic Moore-Read state, with filling fraction $\nu = 1/m$ with $m = 1, 3, \ldots$, the leading $1/N$ correction becomes

$$S_A^B - S_A^{\text{bound}} = \frac{1}{N} \frac{m-1}{m} n_A(n_A - 1) + \ldots$$  \hspace{1cm} (3.13)

In the case $m = 1$ the leading correlations have 3-body character, leading to the vanishing of the leading $1/N$ correction.

### 3.3.4 Numerical results

In deriving the upper bound $S_A^{\text{bound}}$ we relied on the fact that a certain number of eigenvalues of the reduced density matrix vanish. The bounds would be exact if all non-zero eigenvalues were equal, but since they are not the bounds overestimate the actual values for the entropies.

Fig. 3.1 plots the eigenvalues for the $n_A = 2$-particle reduced density matrix for $N = 9$ particles on a sphere in the $m = 3$ Laughlin state, for which the single particle angular momentum is $L = 12$. The horizontal axis represents the degeneracy
2\(L_2 + 1\) of the eigenvalues, in descending order. The eigenvalue at \(L_2 = 2L - 1 = 23\), with degeneracy 47, vanishes; the non-zero eigenvalues show some scatter around an asymptotic value. Due to this scatter the entropy \(S = 5.509\) is somewhat lower than the upper bound \(S_{\text{bound}}^A = 5.533\).

An important difference between the \(m = 3\) Laughlin and the \(m = 2\) Moore-Read states is the absence of vanishing eigenvalues for the 2-particle reduced density matrix. The eigenvalue distribution shown in Fig. 3.2 illustrates this point.

In the \(m = 2\) Moore-Read state, there are vanishing eigenvalues in the reduced density matrix of \(n_A \geq 3\) particles. The number of nonzero eigenvalues predicted by Eq. (3.7) agrees with numerical results. For example, for \(n_A = 3\) and \(N = 10\) particles there are 770 nonvanishing eigenvalues, in agreement with Eq. (3.7).

In Figs. 3.3, 3.4 we compare numerically computed particle entanglement entropies with the bounds derived above.

### 3.3.5 Corrections to \(S_{\text{bound}}^A\) due to eigenvalue spread

It is interesting to consider in some detail the deviation between the bounds \(S_{\text{bound}}^A\) and the actual entropies computed numerically. As mentioned above, this deviation arises from the fact that the non-zero eigenvalues of the reduced density matrices are not all equal.

To estimate the effect on \(S_{n_A}[N]\) of the spread in the non-zero eigenvalues, we do a rough modeling of the eigenvalue distribution (Fig. 3.1) of \(\rho_{n_A=2}\) for the \(m = 3\) Laughlin state. For this case the number \(D\) of non-zero eigenvalues is \(D = (3N - 4)(3N - 5)/2\). If these nonzero eigenvalues were all equal (to \(1/D\)), the entanglement
3.3 Entanglement for particle partitioning

**Figure 3.2:** Eigenvalues for the 2-particle reduced density matrix, plotted against their multiplicities, for $N = 12$ particles in the $m = 2$ Moore-Read state.

**Figure 3.3:** Entanglement entropy for $n_A = 2$ and $n_A = 3$ particles for the $m = 3$ Laughlin state. Dots are numerical exact values, the dotted line represents $S_A^{F}$ and the solid curve is the bound $S_A^{bound}$. 
Figure 3.4: Entanglement entropy for \( n_A = 2 \) and \( n_A = 3 \) particles for the \( m = 2 \) Moore-Read state. Dots are numerical exact values, the dotted line represents \( S_A^F \) and the solid curve is the bound \( S_A^{\text{bound}} \).

entropy would have the maximum value \( S_D = \ln D \), which is the predicted upper bound \( \text{(3.4)} \). We now take into account the deviations from \( 1/D \), guided by Fig. \( \text{3.1} \) with the following toy distribution: we take \( D_0 \) out of \( D \) of the eigenvalues to be equal to \( \alpha/D \), with \( \alpha > 1 \), while the rest of the eigenvalues are at value \( \beta/D \), \( \beta < 1 \), such that the sum of eigenvalues is unity. Assuming \( D_0/D \ll 1 \) leads to

\[
S \approx \ln D - (\alpha \ln \alpha - \alpha + 1) \frac{D_0}{D} .
\]  
(3.14)

Guided by the eigenvalue distributions in Fig. \( \text{3.1} \) we assume that \( D_0/D \) is of order \( 1/N \); for concreteness we put \( D_0 \) equal to the multiplicity of the largest eigenvalue, which is \( 6N - 7 \). Taking \( \alpha \) between 1.2 and 1.5 (as observed for the largest available Laughlin wavefunctions) gives a \( 1/N \) correction in the entropy with coefficient in the range 0.03 – 0.14.

Fitting the difference \( S_A^F - S_{n_A=2} \) to a form \( a/N + b/N^2 \) gives a coefficient \( a \approx 1.38 \). The vanishing eigenvalues account for \( a = 4/3 \), see Eq. \( \text{(3.5)} \), and we see that the remaining difference \( b \alpha \approx 0.05 \) is consistent with the \( 1/N \) correction due to the spread in the non-zero eigenvalues.

We made similar estimates for the \( m = 2 \) Moore-Read state with up to \( N = 12 \) particles, where the eigenvalues are all non-zero and \( S_A^{\text{bound}} \) agrees with \( S_A^F \). In this case the deviation between data and bound show a \( 1/N \) dependence with a coefficient of about 0.14.

These considerations are of some general interest, as they make the point that a \( 1/N \) expansion of particle entanglement entropies are indicative of correlations in a many-body state. In the concrete case studied here, the sizeable value of the leading \( 1/N \) correction in the Laughlin state indicates strong 2-body correlations, while the small value for the Moore-Read state indicates the absence of such correlations.

### 3.3.6 Reduced density matrices and correlation functions

Since the \( n_A \)-particle reduced density matrices \( \rho_{n_A} \) are obtained by integrating out all but \( n_A \) of the particles, one expects these matrices to be related to the \( n_A \)-particle
correlation functions. This is indeed the case as will be shown in the Chapter 5. In this subsection we study the relation for the case \( n_A = 2 \). In particular, the eigenvalue distributions of \( \rho_{n_A=2} \) in Figs. 3.1, 3.2 although discrete, are reminiscent of the well-known two-particle correlation functions \( g_2(r) \) for Laughlin and Moore-Read states. We will show that the eigenvalue distributions are in fact very closely related to the correlation functions – the eigenvalue distribution function is a kind of discretized version of \( g_2(r) \).

The two-particle correlation function \( g_2(r) \) is conventionally defined as

\[
g_2(r) = \frac{N(N-1)}{n^2} \int d^2r_3 \cdots d^2r_n \Psi^*(0,r,r_3,\ldots,r_n)\Psi(0,r,r_3,\ldots,r_n) \langle \Psi | \Psi \rangle,
\]

where \( N \) is the number of particles and \( n \) is a density, which is chosen such that \( g_2(r) \rightarrow 1 \) as \( r \to \infty \).

We express the 2-particle reduced density matrix \( \rho_2 \) on a sphere in a basis of polar spherical coordinates. Because of the rotational symmetry it should be a function of the angular distance \( \theta \) between the two particles. In the Appendix A we show that \( \rho_2(\theta) \) can be written in the form

\[
\rho_2(\theta) = \sum_l \lambda_l \frac{2l+1}{4\pi} R_l(\theta).
\]

In this expression, \( \lambda_l \) is the eigenvalue with multiplicity \( 2l+1 \), corresponding to the total angular momentum of the two particles equal to \( l \). The functions \( R_l(\theta) \) are explicitly given in Eq. (A.12).

Since the distance \( r \) between two particles is simply equal to \( r = R \theta \), with \( R \) the radius of the sphere, the 2-particle correlation function \( g_2(r) \) is directly proportional to \( \rho_2(\theta) \). Through Eq. (3.16) it is expressed as a transform from \( l \) space to \( \theta \) space, with basis functions \( R_l(\theta) \). In Fig. 3.5, we show some curves for \( \rho_2(\theta) \); they agree with known results.

As illustrated in the inset to Fig. 3.5, the basis functions \( R_l(\theta) \) have a peak structure, with the position of the peak depending on the total angular momentum \( l \). Large values of \( l \) correspond to small angular distances and vice versa. This is easy to understand from the following classical picture. When two particles with angular momenta \( L \) have total angular momentum \( 2L \), the corresponding vectors \( \vec{L}_1 \) and \( \vec{L}_2 \) should point at the same direction. The angular momentum vector points at the position of the particle on a sphere, therefore two particle should be close to each other. On the other hand, if total angular momentum is zero then the angular momentum vectors should point in opposite directions. This means that particles are placed at the opposite sides of the sphere.

The fact that the \( R_l(\theta) \) are localized functions, peaked at \( \theta \) values monotonically decreasing with \( l \), indicates that \( g_2(r) \) curve is simply a continuous form of the \( \lambda_l \) versus descending \((2l+1)\) curves of Figs. 3.1 and 3.2. The similarity between the discrete \( \lambda_l \) and the continuous \( g_2(r) \) curves is not accidental.

At small distances \( R_l(\theta) \propto (\theta^2)^{2L-l} \). For a Laughlin state the lowest value of \( 2L-l \) is \( m \), thus \( p_2(\theta) \propto \theta^{2m} \). This behavior is a direct consequence of the vanishing of eigenvalues with the largest multiplicities. For the \( m = 2 \) Moore-Read state the
lowest value of $2L - l$ is 1 because there are no vanishing eigenvalues. Therefore at small distances $\rho_2(\theta) \propto \theta^2$.

Of course, our observations on the 2-particle correlations agree with known results. The main point has been to stress the intimate relation with the eigenvalue distribution of the 2-particle reduced density matrices.

### 3.4 Spatial entanglement and topological entropy

We now turn to dividing the Landau level orbitals into two blocks and calculating the entropy of entanglement between them. In the Chapter 2 we used this scheme to extract the topological entropy of the Laughlin state. Here we mainly focus on the fermionic ($m = 2$) Moore-Read state. We give details on some issues with taking the thermodynamic limit necessary for extracting $\gamma$ from numerical data (3.4.1), and then present our numerical results (3.4.2). We also briefly comment on the spectral structure of the reduced density matrices (3.4.3).

#### 3.4.1 $\gamma$ from sphere calculations: extrapolations

In the Chapter 2 we described the procedure of the extraction of the topological entanglement entropy from the scaling law $S_A \xrightarrow{L \to \infty} \alpha L - \gamma$ [49,58], which can briefly summarized as follows. We choose the block $A$ as the first $l_A$ orbitals, extending...
spatially from one pole out to some latitude, corresponds to a disk-shaped block only in the thermodynamic limit. The boundary of the disk is proportional to $\sqrt{l_A}$ while the boundary of a block $A$ on the sphere is proportional to $\sqrt{l_A(N\phi + 1 - l_A)}$; these are equivalent only in the $N \to \infty$ limit. One way to numerically access the thermodynamic limit is to take the entanglement entropy of $l_A$ orbitals with the rest, for accessible wavefunctions of various sizes $N$, and then take the $N \to \infty$ limit. The points $S_{l_A}(N \to \infty)$ versus $\sqrt{l_A}$ obtained in this way should then follow a linear curve at large $l_A$, whose vertical intercept gives the topological entropy. Results following this procedure were provided for the $\nu = 1/3$ Laughlin state in the previous chapter.

Here we will focus on the Moore-Read state. The extrapolation of $S_{l_A}(N)$ values to the thermodynamic limit is a tricky issue. We therefore discuss the extrapolation in some detail here, providing some general results.

We are interested in the function $S_{l_A}(x)$, where $x = 1/N$. We have access to $S_{l_A}(x_i)$ at several integer values of $N$, and would like to estimate $S_{l_A}(0)$. For each dataset that we have access to (each $l_A$; both Laughlin and Moore-Read), we note the following: the $S_{l_A}(x_i)$ versus $x_i$ values form a monotonic curve and this curve gets flatter (slope magnitude decreases) with decreasing $x$. Two examples can be seen in the inset to Fig. 3.6. In other words, the first and second derivatives of the $S_{l_A}(x)$ function have the same sign and neither derivative changes sign.

Motivated by the above observations, we provide the following result. Assuming only that the signs of the first two derivatives of the $S_{l_A}(x)$ function are the same and that the signs remain unchanged until $x = 0$, we have:

1. The value $S_0 = S_{l_A}(x_0)$ corresponding to the smallest value $x_0$ of the available $x_i$ is a strict lower (upper) bound for $S_{l_A}(0)$ if the $S_{l_A}'(x)$ is negative (positive).

2. The intercept found by connecting the $S_{l_A}(x)$ corresponding to the smallest two $x_i$ values $(x_0, x_1)$, namely

$$S_1 = S_{l_A}(x_0)\left(1 - \frac{x_0}{x_1 - x_0}\right) + S_{l_A}(x_1)\left(\frac{x_0}{x_1 - x_0}\right),$$

is a strict upper (lower) bound if $S_{l_A}'(x)$ is negative (positive) and $S_{l_A}''(x)$ is positive (negative).

The limits $S_0$ and $S_1$ thus obtained give us conservative bounds for the required entanglement entropies in the thermodynamic limit, $S_{l_A}(N \to \infty)$. To obtain a sharper extrapolation, one can use various polynomial extrapolations and take the average, as done in the previous chapter. Here, we improve the extrapolation by using the extrapolation algorithm of Bulirsh and Stoer (BST algorithm), based on rational polynomial fraction approximations. [78, 79]

The BST procedure involves successive transformations of the original dataset, leading to transformed datasets successively smaller by one element. If the free parameter ($\omega$) of the algorithm is chosen properly, the successive sets will be more and more convergent and will eventually converge to the $x = 0$ value. The parameter $\omega$ is chosen to optimize convergence; how precisely this is done is a non-trivial issue, especially for large datasets.


<table>
<thead>
<tr>
<th>(\omega = 2.0)</th>
<th>3.46601</th>
<th>5.21183</th>
<th>4.69023</th>
<th>4.77647</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.80408</td>
<td>4.92355</td>
<td>4.74756</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.01821</td>
<td>4.84476</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.16743</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\omega = 3.0002)</th>
<th>3.46601</th>
<th>4.56058</th>
<th>4.51976</th>
<th>4.55722</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.80408</td>
<td>4.53643</td>
<td>4.59865</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.01821</td>
<td>4.56761</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.16743</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\omega = 2.5926)</th>
<th>3.46601</th>
<th>4.74638</th>
<th>4.58027</th>
<th>4.65108</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.80408</td>
<td>4.65109</td>
<td>4.65108</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.01821</td>
<td>4.65108</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.16743</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Extrapolation using the BST algorithm, using three different \(\omega\) values for the same initial dataset (first column), corresponding to \(S_{l_A=11}(N)\) values for \(N = 12, 14, 16, 18\). In the last case, \(\omega\) has been tuned to ensure that the sets obtained after first and second iterations converge to the same value, i.e., the lowest elements of the second and third columns are the same.

In the present case where the initial dataset consists of bipartite entanglement entropies, it is particularly desirable to be able to use a small number of initial \(S_{l_A}(N)\) values, corresponding to the largest available \(N\). This is because the symmetry \(S_{l_A} = S_{N_\phi + 1 - l_A}\) makes the \(S_{l_A}(N)\) data meaningless for \(N \to \infty\) extrapolation when \(l_A\) gets close to \(N_\phi/2 = \frac{1}{2}(mN - S)\). In other words, the \(S_{l_A}(N)\) values used should not be too close to the peak at the midpoint of the respective \(S_{l_A}(N)\) versus \(l_A\) curve (e.g., Figure 3.8). This restricts us to the largest few \(N\) values.

In Table 3.1, we demonstrate some possible choices of \(\omega\) starting from four \(S_{l_A}(N)\) values. The top example uses \(\omega = 2\), which is a common choice used for extrapolation-based numerical integration algorithms. In the second example, \(\omega\) is tuned to give the “most converged” set after the first iteration, by choosing \(\omega\) to minimize the standard deviation of the second column. We find that this procedure does not always lead to great stability for the subsequent iteration results.

In the last example, we have tuned \(\omega\) to give the same last value for the sequences obtained after the first and the second iterations. This also guarantees the third (final) iteration to give the same value (Table 3.1), which indicates that the convergence is very good. This procedure has the added advantage that actually only the last three points of the initial dataset are used to determine the values tuned to be equal. We thus get a nicely converged estimate based only on three initial points, which is highly desirable as explained above. This is therefore our method of choice for tuning the free parameter \(\omega\) in applying the BST algorithm for our problem.

As a side note, we remark that if the largest available entanglement entropies are not known exactly (e.g., if they are calculated from approximate wavefunctions, or
using the density matrix renormalization group (DMRG) technique), it might be more prudent to use a prescription for choosing $\omega$ that actually uses more than the minimal (three) numbers that we have used here. In the cases reported in the Chapters 2 and 3, we have entanglement entropies calculated from numerically exact wavefunctions, so we have no precision issues to worry about when choosing the number of $S_{l_A}(N)$ values to use for the $N \to \infty$ extrapolation.

Once the extrapolation value is (uniquely) determined by the algorithm outlined above, we need an estimate of the uncertainty. We can use the bounds $(S_0, S_1)$ derived earlier to obtain a conservative error estimate:

$$\min(|S_1 - S_{BST}|, |S_{BST} - S_0|),$$

where $S_{BST}$ is the extrapolated value obtained by the BST algorithm.

### 3.4.2 Numerical results

**Moore-Read state.** Fig. 3.6 shows results of numerical calculations for the $\nu = 1/2$ Moore-Read state. We used exact wavefunctions up to $N = 18$ particles. These wavefunctions were obtained by diagonalizing $\hat{L}^2$ in an $L_z = 0$ Hilbert space spanned by the “squeezed states”. After numerically obtaining the entanglement entropies $S_{l_A}(N)$ from these wavefunctions, we obtain estimates and uncertainties for the $N \to \infty$ extrapolations by the procedure outlined in the previous subsection. The resulting data are plotted in Fig. 3.6.

The linear $S_{l_A}$ versus $\sqrt{l_A}$ behavior is expected only for large $l_A$; however our large-$l_A$ points have the greatest uncertainty. For estimating the topological entropy, we therefore make linear fits after discarding 0 to 5 of the smallest-$l_A$ points and/or 0 to 2 of the largest-$l_A$ points. This results in estimates of $\gamma$ (magnitude of the vertical intercept) scattered between 0.85 and 1.35. The error propagated into our $\gamma$ estimate from our extrapolation uncertainties is $\sim 0.3$, larger than that obtained from this scatter. With all this we arrive at the result $\gamma \simeq 1.1 \pm 0.3$, quite consistent with the expected value of $\gamma \approx 1.04$.

**Laughlin state.** We used the well-defined procedure of the previous subsection to revisit our previous estimate of the topological entropy for the $\nu = 1/3$ Laughlin state. To get the extrapolated $S_{l_A}$, we now use the BST estimates rather than doing several polynomial fits. Dropping 0 to 4 of the smallest-$l_A$ points and 0 to 2 of the largest-$l_A$ points leads to $\gamma \simeq 0.51 \pm 0.14$, consistent with the previously reported estimate $(0.60 \pm 0.15)$ and with the expected value $\gamma \approx 0.55$. The error estimate in the Chapter 2 only took into account this variation, due to dropping various number of points. There is also some error propagated from the extrapolation uncertainty. Using the conservative uncertainty estimate proposed in the previous subsection gives us a more conservative and more rigorous error estimate, $\gamma \simeq 0.51 \pm 0.25$.

### 3.4.3 Eigenvalue distribution for reduced density matrix

In Fig. 3.7 we show the largest eigenvalues of reduced density matrices obtained by orbital or spatial partitioning. The eigenvalues are ordered according to decreasing magnitude and plotted on a log scale; the resulting curves are roughly linear,
3. Entanglement entropy in fractional quantum Hall states

Figure 3.6: Entanglement entropies in Moore-Read state wavefunctions, extrapolated to the thermodynamic limit. Dashed line is a fit to $-\gamma + c_1 \sqrt{l_A}$, with some points dropped. Inset plots $S_{l_A}$ against $1/N$ for various fixed $l_A$. 

\[ S_{l_A}(N) \]

\[ \text{Sq. root of # orbitals in block, } l_A^{1/2} \]

\[ 0 1 2 3 \]

\[ 0 0.05 0.1 0.15 \]

\[ \frac{1}{N} \]

\[ l_A = 7 \]

\[ l_A = 4 \]
3.5 Scaling of entanglement entropies.

An interesting observation was briefly made in the previous chapter at the Fig. 2.2 where we have plotted the maxima of entanglement entropies for different system sizes against the square root of their positions. The plot (the subplot of the Fig. 2.2) is surprisingly linear and more intriguingly the value of an intercept with the vertical axis is close the expected $\gamma$. With this in mind we plot the entanglement entropies for different system sizes at the Fig. 3.8 in another way: the coordinates of the horizontal axis are fractions of the total sphere $x = l_A/(N_{\phi} + 1)$ and on the vertical axis the values of the entanglement entropy are divided by $\sqrt{N - 1}$. The arcs of the Fig. 2.2 collapse into one “universal” curve as shown at the Fig. 3.8. This can be explained as follows. The length of the boundary between $l_A$ and $l_A + 1$ orbitals is given by the

Figure 3.7: Density matrix eigenvalues in decreasing order for orbital partitioning with $l_A$ orbitals in a block for $N = 9$, $m = 3$ Laughlin state.

suggesting a roughly exponential decay of the eigenvalue distribution function. The distribution of eigenvalues of 1D systems in a scaling regime has been studied in detail in the Ref. [80], however there are no known general results for 2D systems.

A more interesting perspective on the eigenvalues distribution was presented in the Ref. [51]. It was shown that a carefully chosen subsets of the eigenvalues plotted against the $L_z$ angular momentum closely resemble a spectrum of a CFT describing quantum Hall state. This will be discussed in the Chapter 4 in the context of the quantum topological phase transitions.
This can be dramatically improved if we assume that the next correction term in the treatment of entanglement entropy as a function of the continuous parameter $x$ to 0 for every $N$ late the values between the available discrete $x$ glement entropies for the same $N$ for

Moreover the constant $c_0$ is explained. Moreover the constant $c_0$ should be identified with $\gamma$.

We made extrapolations based on a such identification for $m = 3$ Laughlin state for $N = 14, 13, 12, 11^2$. First we fix the $x$, let say $x = 0.5$. Then we find entanglement entropies for the same $x$ and different $N$ and fit it to the Eq. [3.17]. To treat entanglement entropy as a function of the continuous parameter $x$ we interpolate the values between the available discrete $x$ points. This step is necessary because for every $N$ there are entanglement entropies for different sets of the sphere fractions $x = l_A/(N\phi + 1)$. The results are presented in the Table [3.2]. As we can see for $x$ close to 0.5 (corresponding to the central regions of the arcs) the fit gives $\gamma$ about 0.36. This can be dramatically improved if we assume that the next correction term in the scaling is of the form $c_1/\sqrt{N-1}$ . The extracted $\gamma$ becomes about 0.52 (expected is 0.55) and the residual errors of the fit decrease dramatically. A similar analysis was done for $m = 2$ Moore-Read state and the extracted value of $\gamma$ was 0.95 (expected is 1.08). However in this case the $1/\sqrt{N}$ correction terms do not improve the quality of the fit.

Surprisingly, the scaling of the entanglement allows us to extract the topological entanglement entropy with a precision not inferior to the ”standard” procedure in-

<table>
<thead>
<tr>
<th>$x$</th>
<th>$c_0$</th>
<th>MSE</th>
<th>$c_0$</th>
<th>MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>-0.413815</td>
<td>$6.27312 \cdot 10^{-11}$</td>
<td>-0.443606</td>
<td>$1.64896 \cdot 10^{-9}$</td>
</tr>
<tr>
<td>0.32</td>
<td>-0.426276</td>
<td>$1.99506 \cdot 10^{-11}$</td>
<td>-0.424276</td>
<td>$1.72626 \cdot 10^{-11}$</td>
</tr>
<tr>
<td>0.34</td>
<td>-0.439133</td>
<td>$1.32974 \cdot 10^{-11}$</td>
<td>-0.407769</td>
<td>$1.79969 \cdot 10^{-9}$</td>
</tr>
<tr>
<td>0.36</td>
<td>-0.452154</td>
<td>$4.80787 \cdot 10^{-11}$</td>
<td>-0.394121</td>
<td>$6.16267 \cdot 10^{-9}$</td>
</tr>
<tr>
<td>0.38</td>
<td>-0.469674</td>
<td>$1.12964 \cdot 10^{-10}$</td>
<td>-0.382875</td>
<td>$1.37889 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.482865</td>
<td>$7.55007 \cdot 10^{-11}$</td>
<td>-0.373864</td>
<td>$2.16938 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>0.42</td>
<td>-0.494728</td>
<td>$2.89335 \cdot 10^{-11}$</td>
<td>-0.36697</td>
<td>$2.9765 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>0.44</td>
<td>-0.507752</td>
<td>$3.90589 \cdot 10^{-12}$</td>
<td>-0.361785</td>
<td>$3.88367 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>0.46</td>
<td>-0.514155</td>
<td>$3.93659 \cdot 10^{-11}$</td>
<td>-0.358249</td>
<td>$4.43236 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>0.48</td>
<td>-0.516911</td>
<td>$6.15833 \cdot 10^{-12}$</td>
<td>-0.35621</td>
<td>$4.70742 \cdot 10^{-8}$</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.519773</td>
<td>$1.11894 \cdot 10^{-11}$</td>
<td>-0.355508</td>
<td>$4.91877 \cdot 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 3.2: Extraction of the sublinear correction $c_0$ assuming scaling relations. First column: fraction of the sphere $x$ in a block $A$. Second and third: $c_0$ and mean squared error (MSE) of the fit with a $1/\sqrt{N-1}$ correction. Fourth and fifth: $c_0$ and MSE of the fit without the correction.

Eq. [2.3] $C_{l_A}(N) = 2\pi \sqrt{2N\phi \sqrt{x(1-x)}}$ ,

where $x = l_A/(N\phi + 1)$. For Laughlin state $N\phi = m(N-1)$. If we directly put this expression into the area law defining $\gamma$ we get:

$$S_x(N) = c_1 \sqrt{N-1} f(x) - c_0 + O(1/\sqrt{N}) . \quad (3.17)$$

The $\sqrt{x(1-x)}$ is substituted by a more general function $f(x)$ to account for possible finite size effects. If the constant $c_1$ (related to the ultraviolet cutoff) does not depend on number of particles $N$ (or depends weakly) then the scaling presented at the Fig. [3.8] is explained. Moreover the constant $c_0$ should be identified with $\gamma$.

The data was kindly provided by Nicolas Regnault.
Figure 3.8: (above) Entanglement entropy for as a function of the number of orbitals in a block A for different number of particles $N$. (below) Rescaled entanglement entropies as a function of the fraction of the sphere in a block A $x$ with $x = l_A/(N\phi + 1)$. 
3. Entanglement entropy in fractional quantum Hall states

3.6 Conclusion

We have extended the results of the previous chapter in several ways. In addition to the abelian Laughlin states we have considered the simplest example of the non-Abelian state - a Moore-Read wave function. We have introduced a more rigorous procedure for the extrapolation of the entanglement entropies from the finite sized data to the thermodynamic limit based on a BST algorithm. Using this procedure, we calculated the topological entropy for the Moore-Read state, and found our numerical result to be consistent with the expected value $\ln \sqrt{8}$.

We have presented another 'phenomenological' method based on the scaling of the entanglement entropies with the system size. It permits to extract the topological entropy with a good accuracy for both Laughlin and Moore-Read states.

Our results on Laughlin and Moore-Read states indicate that the computation of topological entanglement entropies can be used for diagnosing topological order in FQH states that are only available for a limited number of particles. Clearly, the practical use of this method depends on the accuracy by which a value for $\gamma$ can be extracted. In the subsequent chapter we will use this approach to investigate topological order in systems that correspond to physical Hamiltonians with Coulomb interactions, rather than in ideal Moore-Read or Laughlin wave functions.

For entanglement between subsets of particles, we have demonstrated the effects of particle-particle correlations in the deviation of the entanglement entropies from an upper bound $S_A^{\text{bound}}$ set by fermionic or bosonic statistics only. We presented a close upper bounds for particle entanglement entropies for both bosonic and fermionic states and related them to the underlying exclusion statistics of the quasiparticles. We showed that the distribution of the eigenvalue spectrum of the two-particle density matrix can be directly related to a two-particle correlation function $g_2(r)$. A more detailed study of particle entanglement for other systems is present in the Chapter 5.