Entanglement and order in many-body systems
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4 Entanglement signatures of quantum Hall phase transitions

4.1 Introduction

In the previous chapters we have developed a method to detect topological order in fractional quantum Hall (FQH) systems by extracting the topological entanglement entropy from numerically generated wave functions. However, the method was applied to variational Laughlin and Moore-Read wave functions, which are ground states of model Hamiltonians with artificial interactions, rather than realistic wave functions corresponding to Coulomb interactions. Laughlin and Moore-Read wave functions have inbuilt topological order so the main question was whether it is possible to extract it for finite size systems we can obtain numerically. After the efficacy of the methods was established in the Chapters 2 and 3 it is natural to ask if we could detect topological order or its signatures in systems with realistic Hamiltonians. In this chapter we explore to what extent the entanglement calculations can be helpful in exploring topological order in numerically generated FQH systems.

Usually topologically ordered ground states are characterized by fractionalized excitations, degeneracies on higher-genus surfaces, and an energy gap in the excitation spectrum above the ground state [48]. For example, on a genus-\(g\) surface, the Laughlin state at filling \(\nu = 1/m\) has \(m^g\) ground states. A notable property is that there is no local order parameter characterizing topological states, therefore one has to develop unconventional methods to observe topological order in numerical simulations. This is in contrast to the usual symmetry-breaking states within Landau-Ginzburg paradigm, where a local order parameter exists and can be computed for a given wave function. Therefore the studies of topological order should rely on different techniques. Although a number of spin models theoretically possess topological order, the only experimental realizations of topological order are FQH states of two-dimensional electrons in a magnetic field. The numerical methods used to study FQH states usually include computations of overlaps with variational wave functions, low energy spectrum in toroidal geometries, edge excitations in disk geometries. These methods are affected by finite size effects and except for Laughlin states the nature of many FQH states corresponding to different plateaus (e.g. 5/2, 12/5) is not yet firmly established.

\(^1\)This chapter is based on a collaboration with M. Haque N. Regnault
both theoretically and experimentally. Thus it is interesting to use new methods of extraction of topological entanglement entropy developed in the previous chapters to explore topological order in another way.

A quantity characterizing a phase should also provide signatures of quantum phase transitions leading into or away from that phase. This idea is already fruitful in 1D physics, where the (asymptotic) block entanglement entropy can distinguish between gapless and gapped phases. This allows one to pinpoint the phase transition between gapped and gapless phase, in particular in DMRG calculations where the block entanglement entropy is readily available.

As in the previous chapters considering entanglement properties, we will focus on bipartite entanglement between two parts \((A \text{ and } B)\) of the system. This is characterized by the reduced density matrix \(\rho_A = \text{tr}_B \rho\) of subsystem \(A\), obtained by tracing out all the \(B\) degrees of freedom. While it is instructive to study the upper part of the eigenvalue spectrum of \(\rho_A\), the so-called entanglement spectrum, it is also often convenient to extract a single number from this spectrum. For the latter purpose we will use the entanglement entropy, \(S_A = -\text{tr}[\rho_A \ln \rho_A]\). To study topological order in FQH states, Refs. [81–83] exploited the new concept of topological entanglement entropy [49, 58], which appears in the entanglement entropy \(S_A\). On the other hand, Ref. [51] studies the spectrum of \(\rho_A\) and extracts information relevant to the topological order from one end of the spectrum.

For the case of quantum phase transitions involving topologically ordered states, entanglement studies have been exploited for the simpler ‘toy’ case of Kitaev models with order-destroying additional terms [84, 85]. We will examine the utility of entanglement calculations for studying more ‘realistic’ topological phase transitions involving FQH states. With FQH states being the only experimental examples of topological order, this is an important step toward developing entanglement-based tools that are useful for experimentally relevant situations.

We study phase transitions between FQH and non-FQH states driven by a change in the interaction potential. Specifically, using the Haldane pseudopotential description of interactions projected to specific Landau levels [67], we vary the first pseudopotential \(V_1\) away from its Coulomb potential value. The transition we mainly focus on involves the best-known FQH state, for fermions at filling \(\nu = 1/3\). With a Coulomb interaction between the fermions, the ground state is known to be an FQH state topologically equivalent to the Laughlin state [47]. If the first pseudopotential is reduced enough, the Laughlin state ceases to be a good description of the ground state. There is thus a phase transition as a function of \(V_1\) [86]. We will present entanglement calculations which probe this phase transition. We also present results for filling fraction \(\nu = 5/2\), where the possibility of a more intricate quantum Hall state (the Moore-Read state [73]), provides a more challenging situation.

In Section 4.2, we summarize necessary background material on the topological entanglement entropy \(\gamma\) and on FQH wavefunctions and transitions, and discuss the tools necessary for treating phase transitions. In the subsequent sections, we use three different aspects of entanglement to probe these phase transitions. First, we consider the entanglement entropy of a block with the rest of the system, and track phase transitions through the behavior of this quantity as a function of block size and system size. Since our calculations are based on finite-size wave functions, there are
limiting procedures discussed in the Chapters 2, 3 which can be performed in different orders. Results obtained using different limit orders are discussed in Sections 4.3, 4.4 and 4.5.

Second, we consider the top part of the reduced density matrix spectrum. This analysis is based on the identification of features of the entanglement spectrum in terms of topological order and related edge physics [51]. We show in section 4.6 how the entanglement spectrum is affected when the system is driven through a quantum phase transition.

Finally, in section 4.7 we use the concept of majorization, which is based on comparing complete reduced density matrix spectra for two wavefunctions. Majorization relations between reduced density matrices obtained from condensed matter wavefunctions has been the subject of intriguing recent studies [24, 80, 87]. While the full implications are not yet clear, this work adds to the growing understanding of majorization in condensed-matter systems.
4.2 Entanglement and FQH states

4.2.1 Block entanglement entropy and topological phase transitions

We base our discussion on a result for 2D topologically ordered systems \[49,58\] which relates the block entanglement entropy \(S_A\) to the topological order. The dependence of the block entanglement entropy \(S_A\) on the length \(L\) of its boundary is asymptotically linear, in according with the “area law” and has a topological sub-leading term:

\[
S_A(L) \xrightarrow{L \to \infty} \alpha L - \gamma.
\]

called topological entanglement entropy.

The concept of topological entropy is an important new development, because the usual definition of topological order is difficult to use in practice for theoretical or numerical studies of such order. The entanglement-based characterization provides a new route for identifying topological order and, by extension, topological phase transitions. We will employ considerations of both the leading linearity and the sub-leading invariant term to characterize topological phase transitions.

Let us consider, very generally, quantum phase transitions between a topologically ordered phase and another phase. For the block entanglement entropy, let us imagine that we have determined the asymptotic relationship

\[
S_L \xrightarrow{L \to \infty} \alpha L - s_0
\]

where \(L\) is the boundary of the block. Note that this is not always possible; in some 2D phases the leading term might not be purely linear \[19,20,88,89\]. Note also that the above behavior does not necessarily imply topological order; Ref. \[90\] gives an example of a non-topological state following such a relationship with nonzero \(s_0\).

If we are in a topologically ordered phase, the negative intercept \(s_0\) will by definition be equal to the topological entropy, \(\gamma = \ln D\). Figure 4.1 shows some possibilities of what can happen to \(s_0\) as the 2D system is driven across a quantum phase transition away from the topologically ordered state, by varying a parameter \(X\) across the critical value \(X_c\). In the parameter range \(X < X_c\), where the system is in the topologically ordered phase, \(s_0\) is fixed at a positive plateau (\(s_0 = \gamma\)).

Case A shows a transition into another topologically ordered state with a different quantum dimension; \(s_0\) jumps to another constant value \(\gamma'\). The other figures show transitions to non-topological phases. Case B shows a transition to a gapped state which is not topologically ordered – the intercept \(s_0\) drops to zero. (Ref. \[84\] treats an example.) Case C shows a transition into a non-topological phase, in which \(s_0\) is nonzero but not constant. Finally, Case D shows a transition into a state where the leading term in the asymptotic behavior of \(S_A(L)\) is not linear, so that \(s_0\) is undefined.

4.2.2 Exact diagonalization

In the previous chapters we used variational Laughlin and Moore-Read wave functions to study entanglement. In this Chapter our analysis is based on numerical exact diagonalization of Coulomb-like Hamiltonians projected onto appropriate Landau levels
4.2 Entanglement and FQH states

(lowest Landau level for $\nu = 1/3$ and the second Landau level for $\nu = 5/2$). The universal Hamiltonian

$$H = \sum_i \left| -i\hbar \nabla_i - \frac{e}{c} \vec{A}_i \right|^2 + \sum_{i>j} \frac{e^2}{|\vec{r}_i - \vec{r}_j|}$$

is projected to the $n$-th Landau level in a spherical geometry. After the projection the kinetic term disappears and the interaction term is:

$$H = \sum_{i<j} V_m^{(n)} P_{ij}(m),$$

where operator $P_{ij}(m)$ projects to a state with the relative angular momentum $m$ and interactions are parametrized by pseudopotentials [86]:

$$V_m^{(n)} = \int_0^\infty \left[ L_n(k^2/2) \right]^2 L_m(k^2)e^{-k^2} V(k)dk.$$

Here $L_n(x)$ are Laguerre polynomials and $V(k)$ is a Fourier transform of the interaction potential. The index $(n)$ will be omitted for now on as it is equal to 0 (1) for Laughlin (Moore-Read) state. We employ the spherical geometry described in the Section 2.3. We recall the formula for the boundary between partitions $A$ and $B$

$$C_{l_A}(N_\phi) = \frac{4\pi R \sqrt{l_A(N_\phi + 1 - l_A)}}{N_\phi + 1},$$

where $l_A$ is the number of orbitals in a $A$-block. Tentatively one expects that in the area law 4.1 for large enough $l_A$ and $N_\phi$, $L$ should be substituted by $C_{l_A}(N_\phi)$.

It is important to note that in finite-size spherical geometries, FQH states do not appear exactly at the filling fraction $\nu$. Instead, the number of orbitals $(N_\phi + 1)$ is related to the number of particles $N$ by:

$$N_\phi = N\nu^{-1} - S,$$

where the shift $S$ is an integer determined by the FQH state [91]. Although $S$ is insignificant in the thermodynamic limit, it can create an “aliasing” problem in numerical studies: different FQH states can compete at the same finite values of $N$ and $N_\phi$. On a technical side the problem of aliasing does not allow us to apply methods of extraction of topological entanglement entropy for Read-Rezayi (filling $13/5$) states, because these states compete with Haldane hierarchy states in different ways for different number of particles $N$. E.g., the ground state is a Read-Rezayi-like state for $N = 12$ particles but it becomes a Laughlin-like for $N = 15$ particles for the same range of interaction parameters. Such behavior prevents us from applying methods developed for extraction of topological entanglement entropy.

For this work, we have calculated $\nu = 1/3$ wave functions up to 14 fermions and $\nu = 5/2$ wave functions up to 20 fermions. To give an idea of the scale of these calculations, we note that for the $N = 20$ wave functions at $\nu = 5/2$ the Hilbert space dimension is $193498854 \ (\simeq 10^8)$, while for the $N = 14$ wave functions at $\nu = 1/3$, it
Overlaps of ground state wave functions at interactions relevant to fillings $1/3$ and $5/2$, respectively with Laughlin and Moore-Read wave functions. A large-$N$ and a small-$N$ case are shown for each fraction.

We used the Lanczos algorithm to compute the system ground state. Calculations were performed on a PC cluster with 24 cores (AMD Opteron 265) and 48 Gbytes of memory. A Lanczos iteration typically requires up to 17h of cpu time for the largest Hilbert spaces. In our study of phase transitions, calculations are particularly time-intensive when the system is in or near a gapless phase, in which case the Lanczos procedure requires a larger number of iterations to converge.

### 4.2.3 Phase transitions

For FQH systems, the natural (and standard in the literature) interaction parameter to vary is one of the Haldane pseudopotentials $V_m$ obtained by decomposing the interaction potential into channels specified by the relative angular momentum $m$ of the interacting particles [67, 86]. We use interaction potentials whose $m = 1$ pseudopotential $V_1$ is changed, while the other $V_m$ are fixed at the Coulomb value for that Landau level. We calculate and present wavefunction properties as a function of $\delta V_1 \propto (V_1 - V_{1\text{coul}})$. Tuning of $V_1$ loosely represents variable aspects of FQH experiments, such as the thickness of the quantum well where the 2D electron gas resides.

For the $\nu = 1/3$ case, the exact Laughlin state is obtained for $\delta V_1 \to +\infty$ (short-range interaction), and continues to be a good description of the state for the Coulomb potential, $\delta V_1 = 0$. However, for negative $\delta V_1 \propto (V_1 - V_{1\text{coul}})$, there is a quantum phase transition at some $\delta V_1 = \delta V_{1c} < 0$ into a non-FQH incompressible state [86]. The overlap plots of Fig. 4.2 (left panel) shows the transition to be somewhere between $\delta V_1 = -0.065$ and $\delta V_1 = -0.11$. The non-FQH state for $\delta V_1 < \delta V_{1c}$ presumably has charge-density-wave or Wigner-crystalline order [86]; the details are not important for our purposes.
For $\nu = 5/2$, the overlaps (figure 4.2) suggest that the Moore-Read state is stable in some window of slightly positive $\delta V_1$ (around $\delta V_1 \sim 0.03$), and that there are transitions on either side of this phase to non-FQH phases [92, 93]. The non-FQH phases are possibly a striped charge-density-wave phase on the left of the Moore-Read region and a composite fermi sea on the right [93].

### 4.2.4 Extrapolation for $s_0$

The definition of the topological entanglement entropy $\gamma$, or the quantity $s_0$ in equation 4.1, involves two extrapolations: (1) to the thermodynamic limit, $N \to \infty$, and (2) to the asymptotic limit of the size of the $A$ block, i.e. $l_A \gg 1$. This double scaling limit can be approached via different possible extrapolation paths in the $(N,l_A)$ space. To describe the different extrapolation methods, we first rewrite the relation 4.1 for finite $N$ and $l_A$:

$$S_{l_A}(N) = s_1 C_{l_A}(N) - s_0(N),$$

where $N$ is substituted for $N\phi$ because of the constant filling. We have ignored $l_A$-dependence of $s_0$ and both $N$- and $l_A$-dependence of $s_1$. Our experience with numerical wave functions indicate that these dependences are generally weak, at least for model FQH wave functions. Note that, even for topologically ordered model FQH states, the block entanglements $S_{l_A}$ are not linear in $\sqrt{l_A}$ for finite $N$ (c.f. Fig. 2.2). The reason is that $\sqrt{l_A}$ is proportional to the square root of the area of the $A$ region, which is not proportional to the circumference $C_{l_A}(N)$ in the curved geometry of the sphere.

In Sec. 4.3, we consider the method used in Chapters 2, 3, namely, performing the $N \to \infty$ extrapolation first for each $l_A$, and then using the resulting $S_{l_A}(\infty)$ versus $C_{l_A}(\infty) \sim \sqrt{l_A}$ to extract $s_0(\infty)$ or $\gamma$.

In Sec. 4.4 we consider the extrapolation procedure in reverse order, namely, first extracting $s_0(N)$ from finite-size $S_{l_A}$ versus $C_{l_A}$ dependences, and then taking the $N \to \infty$ limit.

In Sec. 4.5 we take the block $A$ to be half the system (which is a natural choice in a spherical geometry), $l_A = l^*_A = \frac{1}{2}(N_\phi + 1)$. Examining the $S_{l^*_A}(N)$ versus $C_{l^*_A}(N)$ may be regarded as taking the two limits simultaneously and was discussed in the Section 3.5.

### 4.2.5 The entanglement spectrum

Clearly, the complete spectrum of the reduced density matrix $\rho_A$ of a subsystem $A$ contains more information than any one number (such as the entropy of entanglement $S_A$) extracted from this spectrum. Extraction of additional information from the complete spectrum has been reported in several other condensed-matter contexts.

Ref. [51] has empirically shown that, for FQH wavefunctions on a sphere, the spectrum of the reduced density matrix of one hemisphere can be related to the conformal field theory (CFT) describing the 1D edge of the FQH state. We exploit this notion in Sec. 4.6 to show how quantum phase transitions appear in the so-called entanglement spectrum.
Another way of exploiting the complete spectrum of $\rho_A$ is via *majorization* comparisons, which we use in Sec. 4.7.

### 4.3 Extrapolation for each block size

We attempt to employ the extrapolation method of the Chapter 2, first extrapolating $N \to \infty$ for fixed values of $l_A$. From the $S_{l_A}(N)$ versus $1/N$ plots in figure 4.3 (top 4 panels), we note that the extrapolation starts to lose meaning as one reduces $V_1$ beyond the presumed transition. This is made quantitative by estimating goodness-of-fit of the $S_{l_A}(N)$ versus $1/N$ data to simple functions. The plotted $\chi^2$ estimates are obtained from trying $S_{l_A}(N) = c_0 + c_1/N^2$; any other reasonable function gives similar results. This $\chi^2$ versus $\delta V_1$ curve can be regarded as one entanglement signature of the phase transition. The jump in $\chi^2$ gets sharper for smaller $l_A$. In particular, for $l_A = 1$ it is distinctly localized at $\delta V_1 \sim -0.1$. This appears to provide a sharp estimate for the location of the transition. One tempting interpretation is that, since the system is large compared to a single orbital, $S_{l_A=1}(N)$ for moderate $N$ is already a good indicator of the thermodynamic limit; hence the sharp jump.

One could still estimate $S_{l_A}(N \to \infty)$ extrapolates, disregarding the “noise” in the $S_{l_A}(N)$ versus $1/N$ datapoints. The resulting $S_{l_A}(\infty)$ points, plotted against $\sqrt{l_A}$, gives a line whose intercept is by definition $-s_0$. The estimates of $s_0$, thus obtained, move away from the expected value of $\gamma = \ln \sqrt{3} \sim 0.55$ around the transition point. However, it is important to note that in the region $\delta V_1 \lesssim -0.06$, the estimates have relatively little meaning. First, there is the increasing scatter in the $S_{l_A}(N)$ versus $1/N$ data, which makes the $S_{l_A}(\infty)$ values unreliable. Second, we find that the $S_{l_A}(\infty)$ versus $\sqrt{l_A}$ curves are more and more curved as one increases $-\delta V_1$ further away from the Laughlin state. Given the uncertainties, we do not attempt to estimate error bars for $s_0$. The point here is to note how $s_0$ loses meaning, in the large-$(-\delta V_1)$ region for which we have used shaded symbols in figure 4.3. Referring back to figure 4.1, the situation we have should be regarded more like case D and not like case C. Note that neither the overlap not the $s_0$ calculations in Fig. 4.3 pinpoint the transition point $\delta V_{1c}$ very sharply.

### 4.4 Dependence on circumference and area law

We now consider extrapolations in reverse order, i.e., first find a negative intercept $s_0(N)$ for each $N$ by looking at the large-$l_A$ behavior at that $N$, and only afterwards consider the $N$-dependence. For topologically ordered model FQH states, the block entanglements $S_{l_A}(N)$ have good linear dependences on the circumference $C_{l_A}(N) \propto \sqrt{l_A(N_\phi + 1 - l_A)}$, for any fixed $N$.

Here we are of course interested in realistic wave functions, i.e., the ground states of Coulomb-like potentials. In figure 4.4, $S_{l_A}$ versus $C_{l_A}$ plots are shown for several $\delta V_1$, for $\nu = 1/3$ states of 12 particles. It is interesting to note that the plots remain smooth as $-\delta V_1$ is increased past the transition; however they acquire curvature. As the dependence deviates from the linearity of model wave functions, extracting
Figure 4.3: Top four: $S_{l_A}(N)$ versus $1/N$ for several $\delta V_1$ values, displaying how the smooth behavior of these curves disappear. Bottom left: quality of fit ($\chi^2$) to a simple function, $c_0 + c_1/N^2$. Bottom right: Attempt to extract $s_0$. The quantity is less and less meaningful for more negative $\delta V_1$ (discussion in text). The topological value $\frac{1}{2} \ln 3 \approx 0.55$ is shown by dashed horizontal line.
Figure 4.4: Top panels: entanglement entropy plotted against circumference of $A$ block; $
u = 1/3$, $N = 12$. The linearity is progressively destroyed as one reduces the pseudopotential $V_1$. Lower panel: $\chi^2$ for linear fits to the $S_{l_A}$ versus circumference data.
4.5 Entanglement between equal hemispheres

We now consider entanglement between equal hemisphere-shaped $A$ and $B$ blocks, i.e., we use $l_A = l_A^* = (N_\phi + 1)/2$. According to Eq. (4.2), $S_{l_A}^*(N)$ should depend linearly on the circumference $C_{l_A}^*(N) \equiv 2\pi R \propto \sqrt{N_\phi}$. For the exact Laughlin states this was explored in detail in the Section 3.5. We showed that the dependence is indeed linear and the negative intercept $s_0$ is very close to the expected value of $\gamma$.

Here, we explore the fate of this linearity as a function of $\delta V_1$.

The top row in figure 4.5 plots $S_{l_A}^*$ against $\sqrt{N_\phi}$ for several $V_1$ values at $\nu = 1/3$. The points acquire scatter as $-\delta V_1$ is increased, until it becomes meaningless to extract the intercept; the situation is similar to Section 4.3 (figure 4.3). This process is indicated by the $\chi^2$ curve (center panel) signifying the goodness of the linear fit.

The $\nu = 5/2$ case (figure 4.5 bottom panel) similarly shows that the scatter is low (so that a linear $S_{l_A}^*$ versus $\sqrt{N_\phi}$ fit is meaningful) only in the window of $\delta V_1$ where the Moore-Read state describes the physics of the ground state.

A technical note: for even $N_\phi$, the number of orbitals is odd and it is impossible to divide the sphere precisely into halves. In this case we keep $l_A^* = N_\phi/2$ orbitals in block $A$, which nearly divides the sphere into halves, and use values $S_{l_A^* = N_\phi/2}$ for the entropy and $C_{l_A^* = N_\phi/2}(N)$ for the circumference.

4.6 Entanglement spectrum and entanglement gap

Another way to look at topological order via entanglement was presented in the Ref. [51]. The many-body wave function is written in Schmidt decomposition as:

$$|\psi\rangle = \sum_i e^{\frac{1}{2}\xi_i} |i_A\rangle |i_B\rangle$$

and we introduce the "entanglement spectrum" $\xi$ with $\lambda_i = \exp(-\xi_i)$, where $\lambda_i$ are eigenvalues of the reduced density matrix $\rho_A$ of one hemisphere. The partitioning of the sphere conserves rotational symmetry in blocks $A$ and $B$, so the eigenvalues can be classified by the number of fermions $N_A$ in the $A$ block and also by the total "angular momentum" $L_A^z$ of the $A$ block. Thus $\psi_A^i$ and $\psi_B^i$ have defined electron number and $L_z$ number and the entanglement spectrum splits into different sectors labelled by $L_A^z$ and $N_A$. It was argued [51] that the low-lying spectrum $\xi_i$ of the reduced density matrix for fixed $N_A$, plotted as a function of $L_A^z$, should display a structure reflecting the conformal field theory (CFT) describing the edge physics. In figure 4.6 this “CFT spectrum” is marked with an ellipse. For interactions at which the FQH
Figure 4.5: Top row: entanglement entropies for hemisphere partitioning, plotted against sphere radius; \( \nu = 1/3 \). Center row shows quality of linear fit as function of pseudopotential \( \delta V_1 \) for \( \nu = 1/3 \). Bottom row shows \( \chi^2 \) for \( \nu = 5/2 \).
state provides a good description of the physics, the CFT spectrum is well-separated by a gap from a higher “non-CFT” part of the spectrum.

From a CFT point of view the Laughlin state corresponds to a chiral bosonic mode at the edge, so counting the number of modes with decreasing $L_A$ goes as $1,1/2,1,3/2,5/2,7/2,11/2,\ldots$ [94]. Another way of counting consists in assigning a specific ”root configuration” to each mode in a space of occupation numbers [51]. Qualitatively one may view the tracing out of $B$ degrees of freedom as a ”natural” boundary, as opposed to the boundaries introduced by external potentials when working in disk geometries. Because the $A$ system size is a half of the original, entanglement spectrum is numerically restricted, compared to edge modes extracted from numerics on a disk. The number of particles in typical computations in a disk geometry is $12-14$ for Laughlin states [95], but half of the sphere for entanglements spectrum includes only $6-7$ particles, so the counting of edge modes deviates from the predictions of the thermodynamic limit ($1,1/2,1,3/2,5/2,7/2,11/2,\ldots$) at some $L_A$. At the Fig.4.6 we see that at $\Delta L_A = 6$ the number of modes deviates from the expected value in thermodynamic limit and more crucially the edge spectrum merges with the bulk. This empirically supports the Moore-Read state as good description of $5/2$ plateau.

As in Ref. [51], we denote the gap between the lowest two $\xi_i$, at the $L_A$ value where the highest-$L_A$ member of the CFT spectrum occurs, as $\delta_0$. In figure 4.6 this is the gap between the lowest two states at $L_A = 54$ (marked by arrow). We study what happens to the spectrum as we tune the interaction away from the FQH state across a quantum phase transition. We quantify the change of the spectrum in terms of the quantity $\delta_0$, defined above. (The quantities $\delta_1,2$ defined in Ref. [51], the gaps at other $L_A$ values, are expected to have similar dependence on $\delta V_1$.)

In figure 4.6 (lower panel), we plot $\delta_0$ as a function of the pseudopotential $\delta V_1$ for the $\nu = 1/3$ case. This clearly shows a dramatic decrease of the ”entanglement gap” around the region of the phase transition. The two levels in question are also individually plotted with open dots; there is a level crossing around $\delta V_1 \sim -0.1$. We note that for values of $\delta V_1 < -0.1$ the CFT-like structure of the entanglement spectrum is lost so it is no longer meaningful to think of $\delta_0$ as the gap between CFT and non-CFT energy levels.

A similar picture is observed for Moore-Read wave functions and is presented in figure 4.8. In this case there are three different possible entanglement spectra corresponding to different choices of the $A$ block and number of particles $N_A$. We present one of them with $l_A = 15 N_A = 8$. The expected counting of modes is $1,1/2,1,3/2,5/2,7/2,11/2,\ldots$ [96]. We see a clear change of the entanglement spectrum as the interaction parameter is tuned across the phase transition.

4.7 Majorization

The concept of majorization involves comparison of two complete spectra. In the context of condensed-matter applications, it generally involves the comparison of two reduced density matrix spectra corresponding to the ground states of two Hamiltonians with slightly different parameters [24,80,87].

To define majorization, we consider two sets of $n$ real elements $\{\lambda_i\}$ and $\{\mu_i\}$ sorted
Figure 4.6: Top panels: entanglement spectrum, $\nu = 1/3$, $N = 12$, block $A$ containing $l_A = 17$ orbitals and $N_A = 6$ fermions. Main plot: ground state for $\delta V_1 = 0.04$. Ellipse indicates the most prominent "conformal" part of the spectrum. Arrow indicates the "entanglement gap" $\delta_0$ between CFT and non-CFT parts of the spectrum. Inset shows exact Laughlin state, which has no higher-lying non-CFT part. Lower panel: Empty dots show two lowest levels at $L_z^{(A)} = 54$, plotted against $\delta V_1$. Filled squares show "entanglement gap", the difference of the two lowest levels.
4.7 Majorization

in decreasing order and satisfying $\sum_i \lambda_i = \sum_i \mu_i = 1$. One says that the set $\{\lambda_i\}$ majorizes the set $\{\mu_i\}$ if

$$\forall k \in \{1, \ldots, n\} : \sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i.$$  

This relationship is often expressed as $\lambda \succ \mu$. It can be shown that if $\lambda \succ \mu$ then $S(\lambda) \leq S(\mu)$, where $S(x) = -\sum_i x_i \ln x_i$ is the von Neumann entropy.

In 1D quantum systems, the spectrum of the reduced density matrix for a spatial block has been argued [24, 80, 87] to become more majorized as one moves along a renormalization group (RG) flow, away from an RG fixed point. As of now, there are no established general results for 2D (or higher dimensional) quantum states.

Figure 4.7: $\delta V_1 = -0.01$ and $\delta V_1 = 0.00$

Figure 4.8: $\delta V_1 = 0.03$ and $\delta V_1 = 0.04$. Entanglement spectrum for $\nu = 5/2, N = 18$ for different values of $\delta V_1$; block $A$ contains $l_A = 15$ orbitals and $N_A = 8$ particles.
Figure 4.9: Left panels: Entanglement entropy and majorization plotted against $\delta V_1$ for $N = 12, \nu = 1/3$. Right panels: same for $N = 18, \nu = 5/2$. Majorization at some $\delta V_1$ is taken to be 1 if the corresponding $\rho_A$ spectrum majorizes a spectrum at an adjacent value of $\delta V_1$; otherwise it is set to 0.
In figure [4.9], we examine majorization of reduced density matrices between ground states at different values of the pseudopotential $V_1$. We show results for the $\nu = 1/3$ system with $N = 12$ particles, and the $\nu = 5/2$ system with $N = 18$ particles. For the $\nu = 1/3$ system, we find that the eigenspectrum is continuously majorized as $\delta V_1$ decreases down to the value $\delta V_1 = -0.1$, i.e., down to the phase transition region. In this region, the eigenspectrum of $\rho_A$ for each $\delta V_1$ majorizes the $\rho_A$ eigenspectrum at a smaller (more negative) $\delta V_1$. For $\delta V_1 \leq -0.1$, the $\rho_A$ spectra are not majorized. This result is robust for different sizes ($l_A$) of the $A$ block. A similar picture emerges for the $\nu = 5/2$ system; majorization occurs in a region roughly corresponding to where the ground state has the structure of the Moore-Read state. However, the effect is more fragile, e.g., the extent of $\delta V_1$ values where the majorization is found, depends on the partition size ($l_A$) used.

One has to be careful in verifying the majorization due to numerical roundoff errors introduced during the computation. If a typical numerical error is $\epsilon$ then we say that $\lambda > \mu$ if $\forall k \ T_k \equiv \sum_{i=1}^{k} (\lambda_i - \mu_i) > -\epsilon$.

Note for $\nu = 1/3$ filling that $S_{l_A}$ has a kink near the transition point. No such feature is seen for the $\nu = 5/2$ case, neither in $S_{l_A}(V_1)$ not in its derivative.

To summarize, we have demonstrated for $\nu = 1/3$ that the region where majorization occurs coincides dramatically with the region where the system is in an FQH state; we no longer find majorization away from this phase. The situation is similar but less clear for $\nu = 5/2$. A full understanding is lacking at the moment, but several intriguing speculations present themselves. Most obviously, it is tempting to think of majorization being an indicator of quantum phase transitions. Second, since the reduced density matrix of a block of the sphere contains information about the physics of the quantum Hall edge [51], it is possible that our majorization results can be interpreted in terms of the evolution of the edge as a function of $\delta V_1$.

## 4.8 Conclusions

In this Chapter we have presented a numerical study showing how quantum phase transitions involving fractional quantum Hall states are manifested in entanglement measures and related quantities. We used three extrapolation methods developed in the Chapters [2, 3] to examine the double-scaling limit where the block entanglement intercept ($s_0$ in equation [4.1]) is defined. We showed that the breakdown of these extrapolation procedures signals the quantum phase transitions away from the topologically ordered incompressible FQH states. In addition, the entanglement spectrum was used in more detail in two different ways to characterize the quantum phase transitions, exploiting concepts of CFT spectrum and majorization.

We have explored topological phase transitions on spherical geometries only, so it would be instructive to look for analogous signatures on a toroidal geometry. Entanglement scaling behaviors have been analyzed in the Ref. [97] for FQH states on the torus and the results are in agreement with our findings. [2]

Our results on majorization invite a thorough investigation of this concept, in [2]. In this regard it is worth to note that the results of entanglement computations for FQH states in the toroidal geometry presented in the Ref. [98] are wrong.
general for two-dimensional systems and in particular both for 2D phase transitions and for topologically ordered states. It would appear that the knowledge necessary for putting our findings in context does not yet exist. The study of majorization, and the upper part of the reduced density matrix spectrum, also raises the question of other signatures of topological phase transitions one might yet extract from the full $\rho_A$ spectrum.

Finally, for topologically ordered states, it is promising to explore our three extrapolation methods, for extracting the topological quantity $\gamma$. In the previous chapters with FQH states on spherical geometries the focus has been on the first extrapolation procedure (section 4.3). If the topological entanglement entropy is to be used to identify intricate FQH states and their CFT’s, improved methods of calculating $\gamma$ are vital. Alternate extrapolation methods is one direction that needs to be explored in this regard.