Entanglement and order in many-body systems
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It is intuitively clear that reduced density matrices are proportional to correlation functions. The precise relation is:

\[ \rho_n(x_n, y_n) = \int dz_{n+1} \cdots dz_N \Psi^* (x_1, \cdots, x_n, z_{n+1}, \cdots, z_N) \times \Psi (y_1, \cdots, y_n, z_{n+1}, \cdots, z_N) \]

\[ = \frac{(N-n)!}{N!} \langle \Psi | \phi^\dagger (x_1) \cdots \phi^\dagger (x_n) \phi (y_n) \cdots \phi (y_1) | \Psi \rangle . \]

Proving it requires straightforward but lengthy bookkeeping of numerous indices. Rather than doing so we illustrate the identity on an example of a two-particle reduced density matrix:

\[ \langle \Psi | \phi^\dagger (x_1) \phi^\dagger (x_2) \phi (y_2) \phi (y_1) | \Psi \rangle = N(N-1) \rho (x_1, x_2, y_1, y_2) . \]

Let us first introduce the usual notations. We denote by \( \phi_i (x) \) single particle basis states with quantum number \( i \) and by \( \phi^\dagger (x) \equiv \sum_i a_i^\dagger \phi_i (x) \) a particle creation operator at the position \( x \). A many-body state is written as:

\[ \Psi(x_1, x_2, \cdots, x_N) = \sum_{i_1 < i_2 < \cdots < i_N} f(i_1, \cdots, i_N) \Phi_{i_1, \cdots, i_N} (x_1, x_2, \cdots, x_N) , \]

where

\[ \Phi_{i_1, \cdots, i_N} (x_1, x_2, \cdots, x_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_{i_1} (x_1) & \cdots & \phi_{i_N} (x_1) \\ \vdots & \ddots & \vdots \\ \phi_{i_1} (x_N) & \cdots & \phi_{i_N} (x_N) \end{vmatrix} \]

is a Slater determinant. In the second quantized notations the same state is

\[ \Psi(x_1, x_2, \cdots, x_N) = \sum_{i_1 < i_2 < \cdots < i_N} f(i_1, \cdots, i_N) a_{i_1}^\dagger \cdots a_{i_N}^\dagger | vac \rangle . \]
First, consider the reduced density matrix for a single Slater determinant wave function \( \Phi_{i_1, \ldots, i_N}(x_1, x_2, \cdots, x_N) \). By definition:

\[
\rho_2(x_1, x_2; y_1, y_2) = \int \Phi^*_{i_1, \ldots, i_N}(x_1, x_2, z_3, \cdots, z_N) \Phi_{i_1, \ldots, i_N}(y_1, y_2, z_3, \cdots, z_N) dz_3 \cdots dz_N
\]

\[
= \frac{1}{N(N-1)} \sum_{p_{i_1} \cdots p_{i_N}} \phi_{p_{i_1}}^*(x_1) \phi_{p_{i_2}}^*(x_2) (\phi_{p_{i_1}}(y_1) \phi_{p_{i_2}}(y_2) - \phi_{p_{i_2}}(y_1) \phi_{p_{i_1}}(y_2))
\]

where \( \{p_{i_1} \cdots p_{i_N}\} \) is a permutation of \( \{i_1 \cdots i_N\} \). In the second quantized language the same computation goes as:

\[
\langle \Psi | \phi^\dagger(x_1) \phi^\dagger(x_2) \phi(y_2) \phi(y_1) | \Psi \rangle = \sum_{i,j,k,l} \langle vac | a_{i_N} \cdots a_{i_1} | a_i^\dagger a_j^\dagger a_k a_l | a_{i_{N-1}}^\dagger \cdots a_{i_1}^\dagger | vac \rangle
\]

\[
\times \phi_i^*(x_1) \phi_j^*(x_2) \phi_k(y_2) \phi_l(y_1)
\]

\[
= \sum_{i,j} \phi_i^*(x_1) \phi_j^*(x_2) (\phi_i(y_1) \phi_j(y_2) - \phi_j(y_1) \phi_i(y_2))
\]

\[
= N(N-1) \rho_2(x_1, x_2; y_1, y_2)
\]

which proves the case.

For a generic wave function a non-zero contribution to a two-particle reduced density matrix comes from a convolution of two Slater wave functions such that their quantum numbers \( i_1 < i_2 < \cdots < i_N \) and \( j_1 < j_2 < \cdots < j_N \) have at least \( N-2 \) the same elements. Let us consider the case of \( N-2 \) equal elements and denote the different quantum numbers as \( i_k < i_l, j_m < j_n \). Then

\[
\rho_2(x_1, x_2; y_1, y_2) = \frac{1}{N(N-1)} \sum_{i_k \cdots i_l, j_m \cdots j_n} \phi_{p_{i_k}}^*(x_1) \phi_{p_{i_l}}^*(x_2) (\phi_{p_{i_m}}(y_1) \phi_{p_{i_n}}(y_2))
\]

\[
\times \text{sign}(i_k, i_l, i_1 < \cdots < \hat{i}_k < \cdots < \hat{i}_l < \cdots i_N)
\]

\[
\times \text{sign}(j_m, j_n, j_1 < \cdots < \hat{j}_m < \cdots < \hat{j}_n < \cdots j_N)
\]

\[
= \frac{1}{N(N-1)} \sum_{i,j,k,l} \langle vac | a_{i_N} \cdots a_{i_1} | a_i^\dagger a_j^\dagger a_k a_l | a_{j_{N-1}}^\dagger \cdots a_{j_1}^\dagger | vac \rangle
\]

\[
\times \phi_i^*(x_1) \phi_j^*(x_2) \phi_k(y_2) \phi_l(y_1)
\]

\[
= \frac{1}{N(N-1)} \langle \Psi_i | \phi^\dagger(x_1) \phi^\dagger(x_2) \phi(y_2) \phi(y_1) | \Psi_j \rangle
\]

The case with \( N-1 \) equal quantum numbers is similar. One can also verify that the above relation is valid for bosonic particles.