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On superconformal characters and partition functions in three dimensions

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Possible short and semishort positive energy, unitary representations of the Osp(2N|4) superconformal group in three dimensions are discussed. Corresponding character formulas are obtained, consistent with character formulas for the SO(3,2) conformal group, revealing long multiplet decomposition at unitarity bounds in a simple way. Limits, corresponding to reduction to various Osp(2N|4) subalgebras, are taken in the characters that isolate contributions from fewer states, at a given unitarity threshold, leading to considerably simpler formula. Via these limits, applied to partition functions, closed formula for the generating functions for numbers of BPS operators in the free field limit of superconformal \( U_n \times U_n \) \( \mathcal{N}=6 \) Chern–Simons theory and its supergravity dual are obtained in the large \( n \) limit. Partial counting of semishort operators is performed and consistency between operator counting for the free field and supergravity limits with long multiplet decomposition rules is explicitly demonstrated. Partition functions counting certain protected scalar primary semishort operators, and their superconformal descendants, are proposed and computed for large \( n \). Certain chiral ring partition functions are discussed from a combinatorial perspective. © 2010 American Institute of Physics. [doi:10.1063/1.3211091]

I. INTRODUCTION

With the resurgence of interest recently in three dimensional superconformal field theory, due largely to the discovery by Bagger and Lambert\(^1\) of a new superconformal \( \mathcal{N}=8 \) Chern–Simons theory and, more recently, by Aharony \textit{et al.}\(^2\) of a superconformal \( \mathcal{N}=6 \) Chern–Simons theory, with \( U(n) \times U(n) \) gauge symmetry, much attention has been devoted to uncovering dualities,\(^3\)\(^4\) investigating integrability,\(^5\)\(^6\) spectra,\(^6\)\(^7\) etc. One issue that has hitherto perhaps not been explored in very much detail is the representation theory, underlying these theories, for positive energy, unitary representations of Osp(\(2N|4\)). This paper is an attempt to address in some detail this issue and focuses on the case of even \( N \).\(^9\)

For \( \mathcal{N}=4 \) superconformal symmetry in four dimensions such a detailed study of the representation theory\(^11\) proved fruitful in many ways, e.g., investigating the operator product expansion, in terms of conformal partial wave expansions of four point functions, see Ref. 12, for example, in partially determining the operator spectrum of \( \mathcal{N}=4 \) super Yang Mills. In this context, superconformal characters provide for an alternative and arguably more straightforward and powerful way of organizing the sometimes detailed multiplet structure that arises. They easily lead to the rules for long multiplet decomposition into short/semishort multiplets at unitarity bounds, formulas for decompositions in terms of subalgebra representations, and indicate how partition functions may be computed for decoupled sectors.

The outline of this paper is as follows. In Sec. II the superconformal algebra for Osp(2N|4), i.e., for \( \mathcal{N}=2N \), is given in detail. Mainly for the purposes of discussing shortening conditions,
constructing Verma module characters and finding subalgebras, it proves easiest to use the orthonormal basis for the SO(2N) R-symmetry algebra. Section II gives a brief account of the possible shortening conditions for semيشort representations, denoted by \( (N, A, n) \), \( n = 1, \ldots, N - 1 \), and \( (N, A, \pm) \), and short or BPS representations, denoted by \( (N, B, n) \) and \( (N, B, \pm) \), as well as for the conserved current multiplet, denoted \( (N, \text{cons}) \). Here, highest weight states in the \( (N, A, n) \), respectively, \( (N, A, \pm) \), \( (N, B, n) \), \( (N, B, \pm) \), representations are annihilated by \( n/4N \), respectively 1/4, \( n/2N \), 1/2 of the supercharges. [Hence, the \( (N, B, \pm) \) representations are referred to here as half BPS. For \( N = 4 \), it is well known that there are two types of half BPS representation, (see Refs. 13 and 14 for a discussion.)

In Sec. IV the characters for positive energy, unitary representations of the \( \text{Osp}(2N|4) \) superconformal group are obtained through use of Verma module characters and the Weyl symmetrizer for the maximal compact subgroup, \( U(1) \otimes SU(2) \otimes SO(2N) \). BPS and conserved conserved current multiplet characters are decomposed in terms of \( SO(3,2) \otimes SO(2N) \) characters. The decomposition rules for long multiplets at the unitarity bounds are found. In particular, they imply that in any \( N = 2N \) superconformal field theory in three dimensions, operators in all \( (N, B, n) \), \( n > 1 \), as well as in \( (N, B, \pm) \) and certain \( (N, B, 1) \), BPS multiplets must have protected conformal dimensions.

In Sec. V, various limits are taken in the characters that lead to nonvanishing formulas only for subsets of the characters corresponding to different short/semi-short multiplets. These are shown, from Appendix A, to match with various \( \text{Osp}(2N|4) \) subalgebra characters and, in terms of partition functions, isolate different sectors of operators. For the subset being \( \{ (N, B, m), m \geq n, (N, B, +), (N, B, -) \} \) the subalgebra is \( U(1) \otimes SO(2N - 2m) \) while for the subsets \( \{ (N, B, \pm) \} \) there are two \( U(1) \) subalgebras. Similarly, for the subset being \( \{ (N, A, m), m \geq n, (N, A, +), (N, A, -), (N, B, l), 1 \leq l \leq N, (N, B, +), (N, B, -), (N, \text{cons}), \} \), where \( (N, B, l), l \leq m \), multiplets have particular \( SO(2N) \) R-symmetry eigenvalues, the subalgebra is \( U(1) \otimes \text{Osp}(2N - 2m) \). Finally, for the subsets \( \{ (N, A, \pm), (N, B, l), 1 \leq l \leq N, (N, B, +), (N, B, -), (N, \text{cons}), \} \), where \( (N, B, l), l \leq m \), \( (N, B, \pm) \) multiplets have particular \( SO(2N) \) R-symmetry eigenvalues, there are two \( U(1) \otimes SU(1,1) \) subalgebras. The limits are shown to be consistent with long multiplet decomposition rules and also with the index of Ref. 6.

In Sec. VI, the general free field multiparticle partition function for \( \mathcal{N} = 6 \) superconformal Chern–Simons theory is written in detail in terms of superconformal characters and, for large \( n \), computed using symmetric polynomial techniques. Also, the corresponding partition function is given in detail for the supergravity limit.

In Sec. VII, operator counting for short and certain semishort multiplets is investigated using the limits taken in Sec. V, and other symmetric polynomial techniques, described in Appendix B, and shown to be consistent with expectations from long multiplet decomposition rules.

In the conclusion, partition functions counting certain protected semishort operators are discussed. Some related comments are made about \( \mathcal{N} = 4 \) super Yang Mills, explained in more detail in Appendix C, where a certain class of chiral ring partition functions are computed using the Polya enumeration theorem.

II. THE SUPERCONFORMAL ALGEBRA IN THREE DIMENSIONS

In \( d \) dimensions, the standard nonzero commutators of the conformal group \( SO(d, 2) \) are given by, for \( \eta_{ab} = \text{diag.}(-1, 1, \ldots, 1) \), \( a, b = 0, 1, \ldots, d - 1 \),

\[
\begin{align*}
[M_{ab}, P_{c}] &= i(\eta_{ac} P_b - \eta_{bc} P_a), \\
[M_{ab}, K_c] &= i(\eta_{ac} K_b - \eta_{bc} K_a), \\
[M_{ab}, M_{cd}] &= i(\eta_{ac} M_{bd} - \eta_{bc} M_{ad} - \eta_{ad} M_{bc} + \eta_{bd} M_{ac}), \\
[D, P_a] &= P_a, \\
[D, K_a] &= -K_a, \\
[K_a, P_b] &= -2iM_{ab} + 2\eta_{ab} D,
\end{align*}
\] (2.1)

where the generators of translations are \( P_a \), those of special conformal transformations are \( K_a \), those of \( SO(d - 1, 1) \) are \( M_{ab} = -M_{ba} \), while that of scale transformations is \( D \).
For later application it is convenient to write the $\text{Sp}(4,\mathbb{R})/\mathbb{Z}_2 \cong \text{SO}(3,2)$ algebra for three dimensions in the spinor basis so that for \(^{16}\)

\[
P_{\alpha\beta}=(\gamma^\rho)_{\alpha\beta}P_{\rho}, \quad K^{\alpha\beta}= (\gamma^\rho)_{\alpha\beta}K_{\rho}, \quad M^{\alpha\beta}_a = \frac{i}{2}(\gamma^\rho)_{\alpha\beta}M_{\rho a},
\]  

the algebra (2.1) becomes

\[
[M^{\alpha\beta}_a,P_{\alpha\beta}]=\delta^{\alpha}_{\gamma}P_{\alpha\delta} + \delta^{\beta}_{\delta}P_{\alpha\gamma} - \delta^{\alpha}_{\delta}P_{\beta\gamma},
\]

\[
[M^{\alpha\beta}_a,K^{\gamma\delta}]= -\delta^{\alpha}_{\gamma}K^{\beta\delta} - \delta^{\beta}_{\delta}K^{\alpha\gamma} + \delta^{\alpha}_{\delta}K^{\beta\gamma},
\]

\[
[M^{\alpha\beta}_a,M^{\gamma\delta}_a]= -\delta^{\alpha}_{\gamma}M^{\beta\delta}_a + \delta^{\beta}_{\delta}M^{\alpha\gamma}_a + \delta^{\alpha\gamma}_{\delta}(M^{\beta\gamma}_a - \frac{1}{4}\delta^{\beta}_{\gamma}M^{\alpha\delta}_a),
\]

\[
[D,M^{\alpha\beta}_a]=\frac{1}{2}Q_{\alpha\beta}, \quad [D,S^{\alpha}_r]= -\frac{1}{2}S^{\alpha}_r,
\]

\[
[R_{\alpha\beta},Q_{\gamma\delta}]=i(\delta^{\alpha}_{\gamma}Q^{\beta\delta}_a + \delta^{\beta}_{\delta}Q^{\alpha\gamma}_a), \quad [P_{\alpha\beta},S^{\gamma}_r]= -i(\delta^{\alpha}_{\gamma}Q^{\beta\gamma}_a + \delta^{\beta}_{\gamma}Q^{\alpha\gamma}_a),
\]

\[
[Q_{\alpha\beta},Q_{\gamma\delta}]=2i(M^{\alpha\beta}_a \delta^{\gamma}_{\delta} - M^{\gamma\delta}_a \delta^{\alpha}_{\beta}), \quad [R_{\alpha\beta},Q_{\gamma\delta}]=i(\delta^{\alpha}_{\gamma}R^{\beta\delta}_a - \delta^{\beta}_{\delta}R^{\alpha\gamma}_a), \quad [R_{\alpha\beta},S^{\gamma}_r]=i(\delta^{\alpha}_{\gamma}S^{\beta}_a - \delta^{\beta}_{\gamma}S^{\alpha}_a),
\]  

along with

\[
[Q_{\alpha\beta},S^{\gamma}_r]=2i(M^{\alpha\beta}_a \delta^{\gamma}_{\delta} - M^{\gamma\delta}_a \delta^{\alpha}_{\beta}), \quad [R_{\alpha\beta},R_{\gamma\delta}]=i(\delta^{\alpha}_{\gamma}R^{\beta\delta}_a - \delta^{\beta}_{\delta}R^{\alpha\gamma}_a - \delta^{\alpha}_{\delta}R^{\beta\gamma}_a + \delta^{\beta}_{\gamma}R^{\alpha\delta}_a),
\]  

where the $\text{SO}(n)$ $R$-symmetry generators $R_{\alpha\beta}=-R_{\beta\alpha}=R_{\alpha\beta}^\dagger$ satisfy

\[
[R_{\alpha\beta},R_{\gamma\delta}]=i(\delta^{\alpha}_{\gamma}R^{\beta\delta}_a - \delta^{\beta}_{\delta}R^{\alpha\gamma}_a - \delta^{\alpha}_{\delta}R^{\beta\gamma}_a + \delta^{\beta}_{\gamma}R^{\alpha\delta}_a),
\]  

In order to discuss highest weight states and shortening conditions, (2.6) is now rewritten in terms of generators in the orthonormal basis of $\text{SO}(2N)$, which has rank $N$. In this basis, the Cartan subalgebra $H_a$, and raising/lowering operators $E^{\pm}_{mn}$, $m < n$, are given by, for $m,n=1,\ldots,N$,

\[
H_a=R_{2n-1\ 2n}, \quad E^{\pm}_{mn}=R_{2m-1\ 2n-1} \pm iR_{2m\ 2n-1} \pm iR_{2m-1\ 2n} \mp R_{2m\ 2n},
\]

\[
E^{\pm}_{mn}=E^{\pm \dagger}_{mn}, \quad E^{-}_{mn}=-E^{\pm}_{mn}, \quad E^{\pm}_{mn}=-E^{\mp}_{mn},
\]  

so that the nonzero commutators from (2.6) are, for $l=1,\ldots,N$,

\[
[H_l,E^{\pm}_{mn}]=(\delta^{\pm}_{\mu} \mp \delta^{\pm}_{\nu})E^{\pm}_{mn}, \quad [H_l,E^{-}_{mn}]=(\delta^{\pm}_{\mu} \pm \delta^{\pm}_{\nu})E^{-}_{mn},
\]

\[
[E^{\pm}_{mn},E^{-}_{mn}]=4(H_m \pm H_n),
\]
\[ [E^\pm_{lm}, E^\pm_{ln}] = 2iE^\pm_{mn}, \quad [E^\pm_{lm}, E^\mp_{ln}] = 2iE^\mp_{mn}, \]

where \( m \neq n \) in the last line.

In this basis a convenient choice for the supercharges and their superconformal extensions is given by

\[ Q_{\alpha a} = \frac{1}{\sqrt{2}}(Q_{2n-1} \alpha + iQ_{2n} \alpha), \quad \bar{Q}_{\alpha a} = \frac{1}{\sqrt{2}}(Q_{2n-1} \alpha - iQ_{2n} \alpha), \]

\[ S^\alpha_n = \frac{1}{\sqrt{2}}(S_{2n-1} \alpha + iS_{2n} \alpha), \quad \bar{S}^\alpha_n = \frac{1}{\sqrt{2}}(S_{2n-1} \alpha - iS_{2n} \alpha). \]

(2.9)

Trivially, from (2.4),

\[ [K^{\alpha\beta}, Q_{\gamma a}] = i(\delta^{\alpha}_{\gamma}Q_{\beta a} + \delta^{\beta}_{\gamma}Q_{\alpha a}), \quad [P_{\alpha\beta}, S^\gamma_n] = -i(\delta^{\gamma}_{\alpha}Q_{\beta a} + \delta^{\beta}_{\gamma}Q_{\alpha a}), \]

\[ [M^\alpha \beta, Q_{\gamma a}] = \delta^\beta_{\gamma}Q_{\alpha a} - \frac{1}{2} \delta^\alpha_{\gamma}Q_{\beta a}, \quad [M^\alpha \beta, S^\gamma_n] = -\delta^\gamma_{\alpha}S^\beta_n + \frac{1}{2} \delta^\beta_{\gamma}S^\alpha_n, \]

\[ [D, Q_{\alpha a}] = \frac{1}{2} Q_{\alpha a}, \quad [D, S^\alpha_n] = -\frac{1}{2} S^\alpha_n, \]

(2.10)

along with identical equations for \( Q_{\alpha a}, \bar{S}^\alpha_n \rightarrow \bar{Q}_{\alpha a}, \bar{S}^\alpha_n \). The nonzero anticommutators among the supercharges and their superconformal extensions are

\[ \{Q_{\alpha a}, \bar{Q}_{\beta a}\} = 2\delta_{\alpha\beta}P_{\alpha\beta}, \quad \{S^\alpha_n, \bar{S}^\alpha_n\} = 2\delta_{\alpha\beta}K^{\alpha\beta}, \]

\[ \{Q_{\alpha a}, \bar{S}^\beta_m\} = 2i(M^\alpha \beta + \delta^\alpha_{\beta}H - \delta^\alpha_{\beta}H_m), \]

\[ \{\bar{Q}_{\alpha a}, S^\beta_m\} = 2i(M^\alpha \beta + \delta^\alpha_{\beta}D + \delta^\alpha_{\beta}H_m), \]

\[ \{Q_{\alpha a}, S^\beta_n\} = \delta^\alpha_{\beta}E_{mn}, \quad \{\bar{Q}_{\alpha a}, \bar{S}^\beta_n\} = \delta^\alpha_{\beta}E_{mn}, \]

\[ \{Q_{\alpha a}, \bar{S}^\beta_n\} = \delta^\alpha_{\beta}E_{mn}, \quad \{\bar{Q}_{\alpha a}, S^\beta_n\} = \delta^\alpha_{\beta}E_{mn}, \]

(2.11)

where \( m \neq n \) in the last two lines. Finally, under the action of the generators (2.7) the nonzero commutators are

\[ [H_m, Q_{\alpha a}] = \delta_{mn}Q_{\alpha a}, \quad [H_m, S^\alpha_n] = \delta_{mn}S^\alpha_n, \]

\[ [H_m, \bar{Q}_{\alpha a}] = -\delta_{mn}\bar{Q}_{\alpha a}, \quad [H_m, \bar{S}^\alpha_n] = -\delta_{mn}\bar{S}^\alpha_n, \]

\[ [E^+_{lm}, Q_{\alpha a}] = -[E^+_{ml}, Q_{\alpha a}] = 2i\delta_m Q_{\alpha a}, \quad [E^+_{lm}, S^\alpha_n] = -[E^+_{ml}, S^\alpha_n] = 2i\delta_m S^\alpha_n, \]

\[ [E^-_{lm}, Q_{\alpha a}] = 2i(\delta_l Q_{\alpha a} - \delta_m Q_{\alpha a}), \quad [E^-_{lm}, S^\alpha_n] = 2i(\delta_l S^\alpha_n - \delta_m S^\alpha_n), \]

\[ [E^-_{lm}, \bar{Q}_{\alpha a}] = -[E^-_{ml}, Q_{\alpha a}] = 2i\delta_m \bar{Q}_{\alpha a}, \quad [E^-_{lm}, \bar{S}^\alpha_n] = -[E^-_{ml}, \bar{S}^\alpha_n] = 2i\delta_m \bar{S}^\alpha_n, \]

\[ [E^+_{lm}, \bar{Q}_{\alpha a}] = 2i(\delta_l \bar{Q}_{\alpha a} - \delta_m \bar{Q}_{\alpha a}), \quad [E^+_{lm}, \bar{S}^\alpha_n] = 2i(\delta_l \bar{S}^\alpha_n - \delta_m \bar{S}^\alpha_n). \]

(2.12)
Suppressing spinor indices, the action of the linearly independent set of lowering operators on $Q_n$ may be expressed by the following diagram:

\[ Q_1 \overset{E_{12}^-}{\rightarrow} Q_2 \overset{E_{23}^-}{\rightarrow} Q_3 \rightarrow \cdots \overset{E_{N-1,N}^-}{\rightarrow} Q_{N-1} \overset{E_{N-1,N}^+}{\rightarrow} Q_N \]

(2.13)

Of course, an identical diagram applies to $\bar{Q}_n$, $\bar{S}_n$. (This diagram is helpful later for discussing the shortening conditions and the computation of superconformal characters.)

The spin generators in terms of SU(2) generators may be expressed as follows:

\[ [M_\alpha^\beta] = \begin{pmatrix} J_3 & J_+ \\ J_- & -J_3 \end{pmatrix}, \quad [J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm, \]

(2.14)

where $Q_{n1}$, $Q_{n2}$, $(\bar{Q}_{n1}, \bar{Q}_{n2})$, $(S_n^2, -S_n^1)$, $(\bar{S}_n^2, -\bar{S}_n^1)$ transform as usual spin $\frac{1}{2}$ doublets, each with $J_3$ eigenvalues $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

### III. SHORTENING CONDITIONS FOR UNITARY MULTIPLES

For physical applications, we require states with positive real conformal dimensions, non-negative half integer spin eigenvalues, and in finite dimensional irreducible representations of the $R$-symmetry group $SO(2N)$. Hence it is sufficient to consider superconformal representations defined by highest weight states $|\Delta, j, r\rangle^{h.w.}$ with

\[ (K^{\alpha\beta}, S_n^\alpha, \bar{S}_n^\alpha, J_+, E_{mn}^\pm)|\Delta, j, r\rangle^{h.w.} = 0, \quad 1 \leq m < n \leq N, \]

\[ (D, J_3, H_m)|\Delta, j, r\rangle^{h.w.} = (\Delta, j, r_m)|\Delta, j, r\rangle^{h.w.}, \]

\[ r = (r_1, \ldots, r_N), \]

(3.1)

where $\Delta$ is the conformal dimension, $j \in \frac{1}{2}N$ is the spin, and with $SO(2N)$ Dynkin labels expressed in terms of $r_j \in \frac{1}{2}Z$ required to satisfy

\[ [r_1 - r_2, \ldots, r_{N-2} - r_{N-1}, r_N, r_{N-1} - r_N] \in \mathbb{N}^N, \]

(3.2)

so that, in particular, $r_i \geq r_{i+1} \geq \cdots \geq r_{N-1} \geq |r_N|$. For a representation space with basis $V_{\Delta, j, r}$, the states are given by

\[ V_{\Delta, j, r} = \left\{ \prod_{\alpha, \beta, \gamma, \delta=1,2} (Q_{n\alpha})^{\kappa_{n\alpha}}(\bar{Q}_{m\beta})^{\bar{K}_{m\beta}}(P_\gamma)\delta_{\gamma, \delta}(J_\delta)k(E_{rs}^\pm)^{K_{rs}}|\Delta, j, r\rangle^{h.w.} \right\} \]

(3.3)

for $\kappa_{n\alpha}, \bar{K}_{m\beta} \in \{0,1\}$ and $k_{\Delta, j, r}, K, K_{rs} \in \mathbb{N}$.

Unitarity requires that

\[ \Delta \geq \begin{cases} r_1 + j + 1 & \text{for } j > 0 \\ r_1 & \text{for } j = 0. \end{cases} \]

(3.4)

The superconformal multiplets may be truncated by various shortening conditions which are considered now. For the BPS shortening condition,

\[ Q_{n\alpha}|\Delta, j, r\rangle^{h.w.} = 0, \quad \alpha = 1, 2, \]

(3.5)

then this leads to the following equations, using (2.11):
leads to a conservation condition on the highest weight state, using
\[(E_{\alpha}^{m,n}, J_\pm D \pm J_3 - H_n) |\Delta, j, r\rangle^{h.w.} = 0, \quad 1 \leq m \leq N, \quad m \neq n, \]  
so that, using (2.8), (2.14), and (3.1),
\[\Delta = r_n, \quad r_1 = r_2 = \cdots = r_n, \quad j = 0. \]  
(3.7)
Clearly, (3.5) with (3.6) implies that \(Q_{\alpha} |\Delta, j, r\rangle^{h.w.} = 0, \ m < n.\)
Imposing the constraint,
\[\widetilde{Q}_{N\alpha} |\Delta, 0, r\rangle^{h.w.} = 0, \quad \alpha = 1, 2, \]  
(3.8)
leads to, using (2.11),
\[(E_{\alpha}^{m,n}, J_\pm D \pm J_3 + H_N) |\Delta, j, r\rangle^{h.w.} = 0, \quad 1 \leq m < N,\]  
so that, using (2.8), (2.14), and (3.1),
\[\Delta = -r_N, \quad r_1 = r_2 = \cdots = -r_N, \quad j = 0. \]  
(3.10)
We may also consider the semishort multiplet condition,
\[\left(\frac{1}{2j} Q_{\alpha} J_- \right) |\Delta, j, r\rangle^{h.w.} = 0, \]  
(3.11)
whereby applying \(S_1^2 \) and using (2.11), we obtain
\[\left( E_{\alpha}^{m,n}, D - J_3 - H_n - \frac{1}{2j} J_\pm J_- \right) |\Delta, j, r\rangle^{h.w.} = 0, \quad 1 \leq m \leq N, \quad m \neq n, \]  
so that, using (2.8), (2.14), and (3.1),
\[\Delta = r_n + j + 1, \quad r_1 = r_2 = \cdots = r_n. \]  
(3.13)
We may also consider
\[\left( \frac{1}{2j} Q_{N\alpha} J_- \right) |\Delta, j, r\rangle^{h.w.} = 0, \]  
(3.14)
whereby applying \(S_1^2 \) and using (2.11) then
\[\left( E_{\alpha}^{m,n}, D + J_3 + H_N - \frac{1}{2j} J_\pm J_- \right) |\Delta, j, r\rangle^{h.w.} = 0, \quad 1 \leq m < N, \]  
so that, using (2.8), (2.14), and (3.1),
\[\Delta = -r_N + j + 1, \quad r_1 = r_2 = \cdots = -r_N. \]  
(3.16)
Imposing both,
\[\left( \frac{1}{2j} Q_{\alpha} J_- \right) |\Delta, j, r\rangle^{h.w.} = 0, \quad \left( \frac{1}{2j} Q_{N\alpha} J_- \right) |\Delta, j, r\rangle^{h.w.} = 0 \]  
(3.17)
leads to a conservation condition on the highest weight state, using (2.11),
\[((2j - 1)(2jP_{22} - 2P_{12}J_-) + P_{11}J_-^2) |\Delta, j, r\rangle^{h.w.} = 0, \]  
(3.18)
and also to, using (3.13), for \(n = N\), and (3.16),
\[\Delta = j + 1, \quad r = 0. \]  
(3.19)
It is easy to see that imposing BPS or semishortening conditions, other than the above, using other $Q_{i\alpha}$, $n \neq N$, leads to violations of (3.4), implying that corresponding multiplets are nonunitary. These are not considered here.

For the truncated supermultiplets $M$, the Verma modules $V_{(\Delta,j)} \rightarrow V_{(\Delta,j)}^M$ are generated by a subset of the generators in (3.3) so that it is sufficient to set some $\kappa_{i\alpha}$, $\overline{\kappa}_{m\beta}$, $\overline{k}_{\gamma\delta}$ to zero.

Using the information encapsulated in (2.13), then the condition (3.5), for BPS multiplets, entails omitting $Q_{j\alpha}$, $j=1,\ldots,N$ from (3.3), so that $\kappa_{j\alpha}=0$. Similarly, for (3.8), then $\kappa_{j\alpha}$, $\overline{\kappa}_{N\alpha}=0$, $j=1,\ldots,N-1$, while for the semishortening conditions, (3.11), $\kappa_{j\alpha}=0$, $j=1,\ldots,n$, and (3.14), $\overline{\kappa}_{j\alpha}=0$, $j=1,\ldots,N-1$. Corresponding to (3.17) with (3.18), then $\kappa_{j\alpha}$, $\overline{\kappa}_{N\alpha}=0$, $j=1,\ldots,N$, $k_{22}=0$ for the multiplet of conserved currents. This information along with notation is summarised in Table I.

### IV. SUPERCONFORMAL CHARACTERS FOR SO(2N) R-SYMMETRY

A procedure for computing conformal characters for higher than two dimensions and $N=4$ superconformal characters for four dimensions has been explained in detail elsewhere.\cite{20,21} The procedure is also closely related to that in Ref. 11 for constructing supermultiplets by employing the Racad–Speiser algorithm. We proceed by analogy with Refs. 20 and 21.

Introducing variables $s, x, y=(y_1,\ldots,y_N)$, we may write the character corresponding to the restricted Verma module, (3.3), for $V_{(\Delta,j)} \rightarrow V_{(\Delta,j)}^M$, as a formal trace,

$$C_{M}^{(\Delta,j)}(s,x,y) = \tilde{\text{Tr}}_{M_{(\Delta,j)}}(s^{2D}x^{2j}y_1^{H_1}\cdots y_N^{H_N}) = s^{2\Delta}C_{2j}(x)C_{r}^{(N)}(y) \times \sum_{k,\beta=0}^{r} (s^{2j}x^{k_1}y_1^{k_{12}}y_n^{k_{22}})k_{12}\beta_{22},$$

where the sum over $k_{\gamma\delta}$ gives the contributions of $P_{\gamma\delta}$ and that over $\kappa_{i\alpha}$, $\overline{\kappa}_{m\beta}$ gives those of $Q_{i\alpha}$, $\overline{Q}_{m\beta}$, and where

$$C_{j}(x) = \frac{x^{j+1}}{x-x^{-1}},$$

$$C_{r}^{(N)}(y) = \prod_{j=1}^{N} y_j^{r_j}y^j/\Delta(y + y^{-1}), \quad \Delta(y) = \prod_{1 \leq i < j \leq N} (y_i - y_j),$$

are the Verma module characters for SU(2) and SO(2N), giving contributions from the $J_{\pm}, E_{rs}^{-z}$ generators, and the highest weight state, in (3.3).

### TABLE I.OSP(2N|4) unitary, positive energy irrep.

<table>
<thead>
<tr>
<th>Type</th>
<th>$\Delta$</th>
<th>$r$</th>
<th>Omitted</th>
<th>Denoted</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long</td>
<td>$r_1 + j + 1$</td>
<td>$r_1 \cdots r_N$</td>
<td>None</td>
<td>$(N, A, 0)$</td>
</tr>
<tr>
<td>Semishort</td>
<td>$r_1 + j + 1$</td>
<td>$r_1 \cdots r_N$</td>
<td>${Q_{2j}}_{r_1}$</td>
<td>$(N, A, n)$ for $n &lt; N$</td>
</tr>
<tr>
<td>Semishort</td>
<td>$r_1 + j + 1$</td>
<td>$r_1 \cdots r_N$</td>
<td>${Q_{2j}}_{r_1}^r$</td>
<td>$(N, A, +)$</td>
</tr>
<tr>
<td>Semishort</td>
<td>$r_1 + j + 1$</td>
<td>$r_1 \cdots r_N$</td>
<td>${Q_{2j}, Q_{2j}}_{r_1}$</td>
<td>$(N, A, -)$</td>
</tr>
<tr>
<td>BPS</td>
<td>$r_1$</td>
<td>$r_1 \cdots r_N$</td>
<td>${Q_{2j}}_{r_1}$</td>
<td>$(N, B, n)$ for $n &lt; N$</td>
</tr>
<tr>
<td>$\frac{1}{2}$ BPS</td>
<td>$r_1$</td>
<td>$r_1 \cdots r_N$</td>
<td>${Q_{2j}}_{r_1}$</td>
<td>$(N, B, +)$</td>
</tr>
<tr>
<td>$\frac{1}{2}$ BPS</td>
<td>$r_1$</td>
<td>$r_1 \cdots r_N$</td>
<td>${Q_{2j}, Q_{2j}}_{r_1}$</td>
<td>$(N, B, -)$</td>
</tr>
<tr>
<td>Cons. current</td>
<td>$j + 1$</td>
<td>$r_1 = 0$</td>
<td>${Q_{2j}, Q_{2j}, P_{2j}}_{r_1}$</td>
<td>$(N, cons.)$</td>
</tr>
</tbody>
</table>
Once the correct generators are omitted from (3.3), so that various $\kappa_{\alpha\beta}$, $\bar{\kappa}_{\rho\sigma}$, $k_{\gamma\delta}$ are zero in (4.1), the prescription for finding the characters of corresponding unitary irreducible representations $R_{(\Delta; j, r)}^M$ is simply given by \(^2\)

$$
\chi_{(\Delta; j, r)}^M(s, x, y) = \text{Tr}_{R_{(\Delta; j, r)}^M} \left( s^{2D} x^{2j} y_1^{H_1} \cdots y_N^{H_N} \right) = \mathcal{M}^{(N)}(s, x, y),
$$

(4.3)

where $\mathcal{M}^{(N)} = \mathcal{M}^{S_2 \otimes S_\nu (S_\nu)^{N-1}}$ is the Weyl symmetrizer for the maximal compact subgroup of the superconformal group, $U(1) \times SU(2) \times SO(2N)$. Here $S_2$ and $S_\nu (S_\nu)^{N-1}$ are the Weyl symmetry groups for $SU(2)$ and $SO(2N)$ and, for some functions $\mathcal{F}(x)$, $\mathcal{F}(y) = f(y_1, \ldots, y_N)$, the action of the relevant Weyl symmetrizers is given by

$$
\mathcal{M}^{S_2}(x) = f(x) + f(x^{-1}),
$$

$$
\mathcal{M}^{S_\nu (S_\nu)^{N-1}} f(y) = \sum_{\sigma \in S_N} \sum_{\varepsilon = \pm 1} f(y^{\sigma(1)}_{\varepsilon 1}, \ldots, y^{\sigma(N)}_{\varepsilon N}).
$$

(4.4)

It is important to realize that the resulting characters may be expanded in terms of $SU(2) \times SO(2N)$ characters using

$$
\chi_f(x) = \mathcal{M}^{S_2} C_f(x) = \frac{x^{j+1} - x^{-j-1}}{x - x^{-1}},
$$

$$
\chi_{(N)}^{(N)}(y) = \mathcal{M}^{S_\nu (S_\nu)^{N-1}} C_{(N)}^{(N)}(y) = \frac{\det[y_{i, j}^{N+N-j} + y_{-j, i}^{N+N-j}] + \det[y_{i, j}^{N+N-j} - y_{-j, i}^{N+N-j}]}{2\Delta(y + y^{-1})},
$$

(4.5)

the usual Weyl character formulas for $SU(2)$ and $SO(2N)$ finite dimensional, irreducible representations.

Defining

$$
P(s, x) = \frac{1}{(1 - x^2)(1 - x^2+S_\nu^2)(1 - x^2-S_\nu^2)},
$$

$$
\mathcal{Q}_n(y, x) = \prod_{j=1}^N (1 + y_j x), \quad \bar{\mathcal{Q}}_n(y, x) = \prod_{j=1}^N (1 + y_j^{-1} x),
$$

(4.6)

then this prescription leads to the following character formulas for the unitary irreducible representations, using (4.1) with (4.3) and with the notation of Table I:

$$
\chi_{(\Delta; j_1, \ldots, j_r; r_1, \ldots, r_n; \ldots, r_N)}^{(N, \ldots, r_n; \ldots; r_N)}(s, x, y) = \mathcal{M}^{(N)}(s, x, y) C_{(\Delta; j_1, \ldots, j_r; r_1, \ldots, r_n; \ldots, r_N)}^{(N, \ldots, r_n; \ldots; r_N)}(s, x, y)
$$

$$
= s^{2\Delta} P(s, x) \mathcal{M}^{(N)}(s, x, y) R_{(\Delta; j_1, \ldots, j_r; r_1, \ldots, r_n; \ldots, r_N)}^{(N, \ldots, r_n; \ldots; r_N)}(s, x, y) \prod_{\varepsilon = \pm 1} \mathcal{Q}_n(s^{-1} x, x^\varepsilon),
$$

n < N,

$$
\chi_{(\Delta; j_1, \ldots, j_r; r_1, \ldots, r_n; \ldots, r_N)}^{(N, \ldots, r_n; \ldots; r_N)}(s, x, y) = \mathcal{M}^{(N)}(s, x, y) C_{(\Delta; j_1, \ldots, j_r; r_1, \ldots, r_n; \ldots, r_N)}^{(N, \ldots, r_n; \ldots; r_N)}(s, x, y)
$$

$$
= s^{2\Delta} P(s, x) \mathcal{M}^{(N)}(s, x, y) R_{(\Delta; j_1, \ldots, j_r; r_1, \ldots, r_n; \ldots, r_N)}^{(N, \ldots, r_n; \ldots; r_N)}(s, x, y) \prod_{\varepsilon = \pm 1} \mathcal{Q}_N(s^{-1} x, x^\varepsilon),
$$

(4.7)

where
\[ R^{(N,i,n)}(s,x,y) = \begin{cases} Q_0(s,y,x)Q_n(s,y,x^{-1}) & \text{for } i = A \\ \prod_{e = \pm 1} Q_n(s,y,x^e) & \text{for } i = B, \end{cases} \]

\[ R^{(N,i,z)}(s,x,y) = \begin{cases} Q_0(s,y,x)(1 + sy_N^{-1}x)(1 + sy_N^{-1}x^{-1}) & \text{for } i = A \\ \prod_{e = \pm 1} (1 + sy_N^{-1}x^e) & \text{for } i = B, \end{cases} \] (4.8)

and appropriate \( \Delta, j \) are as given in Table I. (The conserved current multiplet is discussed separately below.)

Using invariance of \( \prod_{e = \pm 1} Q_0(s,y,x^e) \) \( \tilde{Q}_N(s^{-1}y,x^e) \) under \( \mathcal{W}^{(N)} \) then the long multiplet character is given by

\[ \chi^{(N,\text{long})}_{(\Delta,j;\bar{j})}(s,x,y) = \chi^{(N,A,0)}_{(\Delta,j;\bar{j})}(s,x,y) = s^{2N}P(s,x)\chi_2(x)\chi_{2N}^{(N)}(y) \prod_{e = \pm 1} Q_0(s,y,x^e)\tilde{Q}_N(s^{-1}y,x^e). \] (4.9)

This may be expanded in terms of \( SU(2) \times SO(2N) \) characters using the identities

\[ P(s,x) = \frac{1}{1 - s^2} \sum_{n=0}^{\infty} s^{2n} \chi_{2n}(x), \] (4.10)

along with, for later use,

\[ \prod_{j=1}^{N} (1 + ty_j)(1 + ty_j^{-1}) = \sum_{n=0}^{N-1} (t^n + t^{N-1-n}) \chi^{(N)}_{(1^{n},0^{N-n})}(y) + \chi^{(N)}_{(1^{N})}(y) + t^{N} \chi^{(N)}_{(1^{N-1},-1)}(y). \] (4.11)

### A. Simplification of BPS multiplet characters

Half BPS characters may be simplified by first writing, easily obtained from (4.7),

\[ \chi^{(N,B,\pm)}_{(r;0,r,\ldots,r,z)}(s,x,y) = s^{2r}P(s,x) \sum_{a_1,\ldots,a_N = 0}^{2} s^{a_1 + \cdots + a_N} \chi_{j_{a_1}}(x) \cdots \chi_{j_{a_N}}(x) \chi^{(N)}_{(r-a_1,a_2,\ldots,z+a_N)}(y), \] (4.12)

where, \( j_{a} = \frac{1}{2}(1 - (-1)^a) \) so that

\[ j_0 = j_2 = 0, \quad j_1 = 1. \] (4.13)

Some further manipulation shows that (4.12) may be simplified further to

\[ \chi^{(N,B,\pm)}_{(r;0,r,\ldots,r,z)}(s,x,y) = s^{2r}P(s,x) \sum_{0 \leq a_1 \leq \cdots \leq a_N \leq 2} s^{a_1 + \cdots + a_N} \chi_{j_{a_1}}(x) \cdots \chi_{j_{a_N}}(x) \chi^{(N)}_{(r-a_1,a_2,\ldots,z+a_N)}(y), \] (4.14)

where

\[ j_{a_1,\ldots,a_N} = \frac{1}{2}(N - (-1)^{a_1} - \cdots - (-1)^{a_N}). \] (4.15)

Another consistency check is for \( N = 4 \), or \( SO(8) \) \( R \)-symmetry, whereby this formula leads directly to
\[ \chi^{(4,R,-)}_{(r_0, r, r, r, r)}(s, x, y) = s^{2r_0} P(s, x) \chi^{(4)}_{(r, r, r, r, r)}(y) + s^2 \chi^{(4)}_{(r, r, r, r, r)}(y) + s^2 \chi^{(4)}_{(r, r, r, r, r)}(y) + s^2 \chi^{(4)}_{(r, r, r, r, r)}(y) + s^2 \chi^{(4)}_{(r, r, r, r, r)}(y) + s^2 \chi^{(4)}_{(r, r, r, r, r)}(y) + s^2 \chi^{(4)}_{(r, r, r, r, r)}(y) + s^2 \chi^{(4)}_{(r, r, r, r, r)}(y) + s^2 \chi^{(4)}_{(r, r, r, r, r)}(y) + s^2 \chi^{(4)}_{(r, r, r, r, r)}(y) + s^2 \chi^{(4)}_{(r, r, r, r, r)}(y) \]

which corresponds exactly to the graviton spectrum derived in Ref. 23.

Further simplifications to the half BPS character (4.14) occur for \( r < 2 \). For \( r = 1 \) then,

\[ \chi^{(N,R,\pm)}_{(1; 0, 1, 0, 1, 0)}(s, x, y) = \mathcal{A}_{(1,0)}(s, x) \chi^{(N)}_{(1; 0, 1, 0, 1, 0)}(y) + \mathcal{A}_{(3,2,1,2)}(s, x) \chi^{(N)}_{(1, 0, 1, 0, 1, 0)}(y) + \mathcal{A}_{(2,0)}(s, x) \chi^{(N)}_{(1, 0, 1, 0, 1, 0)}(y) + \sum_{n=0}^{N-2} D_{1/2(N-1)}(s, x) \chi^{(N)}_{(1, 0, 1, 0, 1, 0)}(y), \]

where

\[ \mathcal{A}_{(a,j)}(s, x) = s^{2a} \chi^{(a)}_{(1, 0, 1, 0, 1, 0)}(y), \quad D_{j}(s, x) = s^{2j-2} \chi^{(a)}_{(1, 0, 1, 0, 1, 0)}(y) \]

are the characters for unitary irreducible representations of the conformal group in three dimensions, SO(3,2), see also Ref. 20—by which \( \mathcal{A}_{(a,j)} \) corresponds to an unconstrained spin \( j \) field, conformal dimension \( \Delta = j = 1 \), while \( D_{j} \) corresponds to a conserved current with spin \( j \), conformal dimension \( j+1 \), including all their conformal descendants (or derivatives acting on fields).

Similarly, for \( r = \frac{1}{2} \),

\[ \chi^{(N,R,\pm)}_{(1/2, 0, 1/2, 1/2, 1/2)}(s, x, y) = D_{\text{Rac}}(s, x) \chi^{(N)}_{(1/2, 1/2, 1/2, 1/2)}(y) + D_{\text{Di}}(s, x) \chi^{(N)}_{(1/2, 1/2, 1/2, 1/2)}(y) \]

where

\[ D_{\text{Rac}}(s, x) = \frac{s + s^3}{(1 - s^2 x^2)(1 - s^2 y^2)}, \quad D_{\text{Di}}(s, x) = \frac{s^2 (x + x^{-1})}{(1 - s^2 x^2)(1 - s^2 y^2)} \]

are characters for the free field representations of SO(3,2), the so called “Di,” respectively, “Rac,” singleton representations, corresponding to a free spin \( \frac{1}{2} \), respectively, scalar field with conformal dimension 1, respectively, \( \frac{1}{2} \), and all its descendants.

Finally, from (4.14) it may be shown that

\[ \chi^{(N,R,\pm)}_{(0, 0, 0, 0, 0)}(s, x, y) = 1, \]

the character for the identity representation.

For other BPS characters, from (4.7), we may write, similarly to (4.12) with (4.13),

\[ \chi^{(N,R,n)}_{(r_1; 0, r_1, r_2, r_3, r_4)}(s, x, y) = s^{2r_1} P(s, x) \sum_{a_1, \ldots, a_N} \prod_{a_{i+1} = a_i + 1}^{a_N} \chi^{(N)}_{(r_1; 0, a_{i+1}, a_{i+1}, \ldots, a_N)}(y), \]

and, similarly to (4.14) with (4.15), using that \( r_1 = r_2 = \ldots = r_n \).
\[ \chi_{(N,B,n)}^{(N,B,n)}(s,x,y) = s^{2r_1} \mathcal{P}(s,x) \times \sum_{a_{N} = 0}^{2} \sum_{a_{N+1} = 0}^{2} \cdots \sum_{a_1 = 0}^{2} s^{a_1 + \cdots + a_N + a_{N+1} + \cdots + a_N} \chi_{(r_1, a_1, \ldots, a_N)}(x) \times \prod_{i=a_1+1}^{N} \chi_{j_{a_i}}(x) \times \chi_{(r_{N-1}, a_{N-1}, \ldots, a_1)}^{(N)}(y), \]

where,

\[ j_{a_i} = \frac{1}{2} (n - (-1)^{a_i} - \cdots - (-1)^{a_N}). \]

Without further restrictions on \( r_i \), or \( s, x, y \), there appear to be no further simplifications to (4.22).

### B. The conserved current multiplet character

The conserved current multiplet (which has been excluded so far from the above discussion) corresponding to the semishortening condition (3.17), has character

\[ \chi_{(j+1;0,\ldots,0)}^{(N,\text{cons.})}(s,x,y) = s^{2j+2} \mathcal{P}(s,x) \mathcal{M}^{(N)}(C_2(x)C_0^{(N)}(y)(1 - s^2x^2)Q_0(sy,x)Q_N(s^{-1}y,x)\tilde{Q}_{N-1}(s^{-1}y,x^{-1})), \]

which ensures that the contributions from the supercharges \( Q_{n2}, n = 1, \ldots, N, \tilde{Q}_{N2} \) along with \( P_{N2} \) are omitted from the Verma module character (4.1).

To simplify, it is useful to observe that \( \mathcal{M}^{(N)}(sy,x)\tilde{Q}_N(s^{-1}y,x) \) is invariant under action by \( \mathcal{M}^{(N)}(\mathcal{S}_2)^{N-1} \) and, further, that \( \mathcal{M}^{(N)}(\mathcal{S}_2)^{N-1}C_0(y)\tilde{Q}_{N-1}(y,t) = 1 \) so that

\[ \chi_{(j+1;0,\ldots,0)}^{(N,\text{cons.})}(s,x,y) = s^{2j+2} \mathcal{P}(s,x) \mathcal{M}^{(N)}(C_2(x)(1 - s^2x^2)Q_0(sy,x)\tilde{Q}_N(s^{-1}y,x)) = \sum_{n=0}^{N-1} (D_{j+1/2}^{(1/2)})^{(N)}(s,x) \]

\[ + D_{j-N-1/2}^{(1/2)}(s,x) \chi_{(1/2)}^{(N)}(y) + D_{j+1/2}^{(1/2)}(s,x) \chi_{(1/2)}^{(N)}(y) + \chi_{(1/2)}^{(N)}(y), \]

using (4.11), to expand in \( x \), and subsequently the expression for the conserved current character in (4.17).

### C. Long multiplet decompositions

Using linearity of \( \mathcal{M}^{(N)} \) we may easily obtain, for \( j \geq \frac{1}{2}, r_1 > r_2 \),

\[ \chi_{(r_1+j+1;j_1r_1r_2,\ldots,r_N)}^{(N,\text{long})}(s,x,y) = \mathcal{M}^{(N)}(1 + syx^{-1})C_{(r_1+j+1;j_1r_1r_2,\ldots,r_N)}^{(N,\text{A.1})}(s,x,y) = \chi_{(r_1+j+1;j_1r_1r_2,\ldots,r_N)}^{(N,\text{A.1})}(s,x,y) \]

\[ + \chi_{(r_1+j+3/2;j_1r_1r_2,\ldots,r_N)}^{(N,\text{A.1})}(s,x,y), \]

which expresses the reducibility of a long multiplet with \( \Delta = r_1 + j + 1 \) into a sum of semishort multiplets.

For semishort multiplet characters it may be shown that, for \( r_1 = r_2 = \cdots = r_n > r_{n+1}, r > 0 \),

\[ \chi_{(r_1+j+1;j_1r_1r_2,\ldots,r_N)}^{(N,\text{A.1})}(s,x,y) = \chi_{(r_1+j+1;j_1r_1r_2,\ldots,r_N)}^{(N,\text{A.1})}(s,x,y), \]

\[ \chi_{(r+j+1;j_1r_1r_2,\ldots,r_N)}^{(N,\text{A.1})}(s,x,y) = \chi_{(r+j+1;j_1r_1r_2,\ldots,r_N)}^{(N,\text{A.1})}(s,x,y). \]

This employs
\[ \chi^{(N,A,1)}_{(r_1+j+1;1,r_1\ldots,1;\ldots r_{n+1};\ldots r_N)}(s,x,y) = \mathcal{M}^{(N)}_1 \prod_{i=2}^{n+1} (1 + s y_i x^{-1}) C_{(r_1+j+1;1,r_1\ldots,1;\ldots r_{n+1};\ldots r_N)}(s,x,y), \]

along with the identity \( \chi^{(N)}_{(r)}(y) = (-1)^w \chi^{(N)}_r(y), w \in \mathcal{S}_N \times (S_2)^{N-1} \), where \( r^n = w(r + \rho) - \rho \), with \( w \in \mathcal{S}_N \times (S_2)^{N-1} \), the Weyl group, and \( \rho = (N-1, N-2, \ldots, 0) \), being the Weyl vector. In particular, \( \chi^{(N)}_{(r_1,\ldots,1;\ldots r_{n+1};\ldots r_N)}(y) = 0 \) which implies \( \chi^{(N,A,n)}_{(\Delta;1;\ldots,1;\ldots r_{n+1};\ldots r_N)}(s,x,y) = 0 \). The \((N,A,\pm)\) cases may be found similarly.

Similarly, for the conserved current multiplet character,
\[ \chi^{(N,A,1)}_{(j+1;0,\ldots,0)}(s,x,y) = \chi^{(N,\text{cons.})}_{(j+1;0,\ldots,0)}(s,x,y). \]  

(4.28)

In order to avoid ambiguity, it is assumed in what follows that \( r_n > r_{n+1} \) for \((N,A,n)\) semishort multiplet characters, unless otherwise stated.

From (4.26) with (4.27) and (4.28), then we have, for \( j \geq \frac{1}{2}, \ r_1 > r_{n+1}, \ n < N \),
\[ \chi^{(N,long)}_{(r_1+j+1;1;\ldots,1;\ldots r_{n+1};\ldots r_N)}(s,x,y) = \chi^{(N,A,n)}_{(r_1+j+1;1;\ldots,1;\ldots r_{n+1};\ldots r_N)}(s,x,y) + \chi^{(N,A,1)}_{(r_1+j+1;1;\ldots,1;\ldots r_{n+1};\ldots r_N)}(s,x,y), \]  

(4.29)

along with, for \( r > 0 \),
\[ \chi^{(N,long)}_{(r;\ldots,1;\ldots,r_{n+1};\ldots r_N)}(s,x,y) = \chi^{(N,A,\pm)}_{(r;\ldots,1;\ldots,r_{n+1};\ldots r_N)}(s,x,y) + \chi^{(N,A,1)}_{(r;\ldots,1;\ldots,r_{n+1};\ldots r_N)}(s,x,y), \]  

(4.30)

and
\[ \chi^{(N,long)}_{(j+1;0,\ldots,0)}(s,x,y) = \chi^{(N,\text{cons.})}_{(j+1;0,\ldots,0)}(s,x,y) + \chi^{(N,A,1)}_{(j+1;0,\ldots,0)}(s,x,y). \]  

(4.31)

It is also easily seen that
\[ \chi^{(N,A,1)}_{(r_1+1;1;\ldots,1;\ldots r_{n+1};\ldots r_N)}(s,x,y) = \mathcal{M}^{(N)}_1 (1 + s y_1 x^{-1}) C_{(r_1+1;1;\ldots,1;\ldots r_{n+1};\ldots r_N)}(s,x,y) = \chi^{(N,B,1)}_{(r_1+1;0;1;\ldots,1;\ldots r_{n+1};\ldots r_N)}(s,x,y), \]  

(4.32)

using \( \mathcal{M}^{(N)}_1 C_{-1}(x) = 0 \), \( \mathcal{M}^{(N)}_1 C_{-2}(x) = \mathcal{M}^{(N)}_1 C_0(x) = 1 \), so that, for \( r_1 > r_2 \),
\[ \chi^{(N,long)}_{(r_1+1;0;1;\ldots,r_{n+1};\ldots r_N)}(s,x,y) = \mathcal{M}^{(N)}_1 (1 + s y_1 x^{-1}) C_{(r_1+1;0;1;\ldots,r_{n+1};\ldots r_N)}(s,x,y) = \chi^{(N,A,1)}_{(r_1+1;0;1;\ldots,r_{n+1};\ldots r_N)}(s,x,y) \]
\[ + \chi^{(N,B,1)}_{(r_1+1;0;1;\ldots,r_{n+1};\ldots r_N)}(s,x,y), \]  

(4.33)

which expresses the reducibility of a long multiplet with \( \Delta = r_1 + 1 \) into a sum of a \((N,B,1)\) BPS and a semishort multiplet.

Thus, from (4.33) with (4.27) and (4.28), we have, for \( r_1 = r_n > r_{n+1} \),
\[ \chi^{(N,long)}_{(r_1+1;0;1;\ldots,r_{n+1};\ldots r_N)}(s,x,y) = \chi^{(N,A,n)}_{(r_1+1;0;1;\ldots,r_{n+1};\ldots r_N)}(s,x,y) + \chi^{(N,B,1)}_{(r_1+1;0;1;\ldots,r_{n+1};\ldots r_N)}(s,x,y), \]  

(4.34)

along with, for \( r > 0 \),
\[ \chi^{(N,long)}_{(r;\ldots,1;\ldots,r_{n+1};\ldots r_N)}(s,x,y) = \chi^{(N,A,\pm)}_{(r;\ldots,1;\ldots,r_{n+1};\ldots r_N)}(s,x,y) + \chi^{(N,B,1)}_{(r;\ldots,1;\ldots,r_{n+1};\ldots r_N)}(s,x,y), \]  

(4.35)

and
\[ \chi^{(N,long)}_{(j+1;0,\ldots,0)}(s,x,y) = \chi^{(N,\text{cons.})}_{(j+1;0,\ldots,0)}(s,x,y) + \chi^{(N,B,1)}_{(j+1;0,\ldots,0)}(s,x,y). \]  

(4.36)

(4.26), (4.29)–(4.31), and (4.33)–(4.36) appear to exhaust all possibilities for long multiplet decompositions (and are also consistent with limits, as discussed in Sec. V) and have important consequences for any superconformal field theory in three dimensions. In particular, they imply
that all \((N,B,\pm)\) and \((N,B,n), N>n>1\), short multiplet operators as well as certain \((N,B,1)\) short multiplet operators in \(R_{r_1,r_2,\ldots,r_N}\), \(r_1 \geq r_2\), SO\((2N)\) \(R\)-symmetry representations must remain protected against gaining anomalous dimensions. The decomposition formulas (4.26) and (4.33) have essentially appeared in Ref. 10, where also comments about the protectedness of certain operators in three dimensional superconformal field theories were made. [Note that all the decomposition formulas here are consequences of the basic formula (4.26). This is not surprising as a similar thing happens for decomposition formulas of long multiplets for \(\mathcal{N}=4\) superconformal symmetry in four dimensions.20]

V. REDUCTION TO SUBALGEBRA CHARACTERS

Here are described certain limits that can be taken in the previous BPS and semishort multiplet characters that isolate contributions from fewer states in each multiplet and hence lead to significant simplifications. These limits are equivalent to reductions in the characters to those for various subgroups of the superconformal group, as explained in more detail in Appendix A.

A. BPS limits

1. The \(\mathbf{U}(1) \otimes \text{SO}(2N-2m)\) sector

By considering,

\[
\chi_{(\Delta;\tau)}^M(\delta u^{1/2} x, (\delta^2 u)^{1/m} \bar{y}, \bar{y}) = \text{Tr}_{\mathcal{R}_{(\Delta;\tau)}^M} \left( \delta^2 \lambda_{\mu} u^T e^{2\lambda/\mu} y_1^H \ldots y_m^H y_{m+1}^H \ldots y_{N-m}^H \right),
\]

\[
\bar{y} = (\delta^2 u)^{-1/m}(y_1, \ldots, y_m), \quad \prod_{i=1}^m y_i = 1, \quad \bar{y} = (y_{m+1}, \ldots, y_N),
\]

\[
\mathcal{H}_m = D - \frac{1}{m} \sum_{i=1}^m H_m, \quad \mathcal{I}_m = D + \frac{1}{m} \sum_{i=1}^m H_m,
\]

in the limit \(\delta \to 0\), it is clear that only those states in \(\mathcal{R}_{(\Delta;\tau)}^M\) for which \(\mathcal{H}_m\) has zero eigenvalue contribute. In particular, this applies to the highest weight state in the \((N,B,m)\) BPS multiplet.

It can be shown that

\[
\lim_{\delta \to 0} \chi_{(\Delta;\tau)}^M(\delta u^{1/2} x, (\delta^2 u)^{1/m} \bar{y}, \bar{y}) = \chi_{(\Delta;\tau)}^{\mathbf{U}(1) \otimes \text{SO}(2N-2m)}(u, \bar{y}) = \mu^R \mathcal{X}_{\mathcal{I}_m, N-m}^{(N-m)}(\bar{y}),
\]

in terms of \(\mathbf{U}(1)_{\mathcal{I}_m} \otimes \text{SO}(2N-2m)\) characters, for appropriate \(R, \bar{r}\), see Appendix A.

An identity which is useful for simplifying the limits considered here is the following, for \(r = (r_1, r_2, \ldots, r_N)\):

\[
\chi_{\bar{r}}^{(N)}(\delta^{-2m}u_\bar{r}, \bar{y}) \sim \delta^{-2m(r_1+\ldots+r_m)} s_{\bar{r}}(\bar{u}) \chi_{\bar{r}}^{(N-m)}(\bar{y}),
\]

\[
\bar{r} = (r_1, \ldots, r_m), \quad \bar{r} = (r_{m+1}, \ldots, r_N),
\]

giving the leading behavior as \(\delta \to 0\), where \(s_{\bar{r}}(\bar{u})\) is a Schur polynomial,

\[
s_{\bar{r}}(\bar{u}) = \det[\bar{u}_i^{r_j + m-j}/\Delta(\bar{u})], \quad s_{(r_1,\ldots,r_N)}(\bar{u}) = \prod_{i=1}^m \bar{u}_i^{r_i}.
\]

The identity (5.3) may be obtained by noting that, for small \(\delta\),

\[
C_{r_1}^{(N)}(\delta^{-2m}u_\bar{r}, \bar{y}) \sim \tilde{C}_{\bar{r}}^{(m)}(\delta^{-2m}u) C_{\bar{r}}^{(N-m)}(\bar{y}), \quad \tilde{C}_{\bar{r}}^{(m)}(\bar{u}) = \prod_{i=1}^m \bar{u}_i^{r_j + m-j}/\Delta(\bar{u}),
\]
\[
M^2 \mathcal{S}^{2(N)}_{(S_2)^{N-1}} C_{f}^{N}(\sigma^{-2m} \hat{u}, \hat{y}) \sim M^2 \mathcal{S}^{2(N-m)}_{(S_2)^{N-m-1}} C_{f}^{N}(\sigma^{-2m} \hat{u}) C_{f}^{N}(\hat{y}).
\]

Here, \(C_{f}^{N}(\hat{u})\) is equivalent to the \(U(m)\) Verma module character while \(M^2 \mathcal{S}_{f}\) is the Weyl symmetrizer for \(U(m)\), acting on \(\hat{u}\), so that for any \(f(\hat{u}) = f(\hat{u}_1, \ldots, \hat{u}_m)\), \(M^2 \mathcal{S}_{f}(\hat{u}) = \sum_{\sigma \in S_m} f(\sigma(\hat{u}_1), \ldots, \sigma(\hat{u}_m))\). Hence, \(M^2 \mathcal{S}_{f}^{2(N)}(\hat{u}) = \mathcal{S}_{f}(\hat{u})\), the \(U(m)\) Weyl character. Here also, \(M^2 \mathcal{S}_{f}^{2(N)}_{(S_2)^{N-m}} C_{f}^{N}(\hat{y}) = \mathcal{X}_{f}^{N}(\hat{y})\).

Using (5.3), we may obtain from (4.7) that the limit taken in (5.1) gives the following \(U(1) \otimes SO(2N-2m)\) characters, for \(n \geq m\), see Appendix A:

\[
\begin{align*}
\chi_{(r_1, \bar{z})}^{U(1) \otimes SO(2N-2m)}(\hat{u}, \hat{y}) &= \lim_{\delta \rightarrow 0} \chi_{(r_1, 0, r_2)}^{(N, B, m)}(\delta \hat{u})^{1/2}, \hat{x}, (\delta^{-2} \hat{u})^{1/2}, \hat{y}, \hat{y}) = u^{2r_1} \chi_{f}^{(N-m)}(\hat{y}), \\
\chi_{(r_N, \ldots, \bar{r}_N)}^{U(1) \otimes SO(2N-2m)}(\hat{u}, \hat{y}) &= \lim_{\delta \rightarrow 0} \chi_{(r_1, 0, \ldots, 0, r_2)}^{(N, B, m)}(\delta \hat{u})^{1/2}, \hat{x}, (\delta^{-2} \hat{u})^{1/2}, \hat{y}, \hat{y}) = u^{2r_1} \chi_{(r_1, \ldots, \bar{r}_1, 0, \ldots, 0)}^{(N-m)}(\hat{y}).
\end{align*}
\]

For \(m = 1\), we have in addition to (5.5) that

\[
\chi_{(r_1, \bar{z})}^{U(1) \otimes SO(2N-2)}(\hat{u}, \hat{y}) = \lim_{\delta \rightarrow 0} \chi_{(r_1, 0, r_2)}^{(N, B, m)}(\delta \hat{u})^{1/2}, \hat{x}, \delta^{-2} \hat{u}, \hat{y}, \hat{y})
\]

while other long multiplet characters, for \(\Delta \geq r_1 + j + 1\), \(j \neq 0\), along with semishort multiplet characters vanish, consistent with (4.33). Since, in the limit taken in (5.2), long multiplet characters vanish for \(m > 1\), this provides further evidence, apart from long multiplet decomposition formulas listed in Sec. IV, that \((N, B, m), m > 1\) BPS operators must remain protected in any three-dimensional superconformal field theory.

\section*{2. The \(U(1)\) sectors}

Similarly, by considering

\[
\chi_{(r_1, \bar{z})}^{U(1) \otimes SO(2N-2)}(\hat{u}) = \lim_{\delta \rightarrow 0} \chi_{(r_1, 0, r_2)}^{(N, B, m)}(\delta \hat{u})^{1/2}, \hat{x}, (\delta^{-2} \hat{u})^{1/2}, \hat{y}, \hat{y}) = u^{2r_1} \chi_{f}^{(N-m)}(\hat{y}).
\]

separately in the limit \(\delta \rightarrow 0\), isolating states for which \(\mathcal{H}_{\pm}\) has zero eigenvalues, only the corresponding \(1/2\) BPS character is nonzero, giving

\[
\chi_{(r_1)}^{U(1)}(\hat{u}) = \lim_{\delta \rightarrow 0} \chi_{(r_1, 0, \ldots, 0)}^{(N, B, m)}(\delta \hat{u})^{1/2}, \hat{x}, (\delta^{-2} \hat{u})^{1/2}, \hat{y}, \hat{y}) = u^{2r_1}.
\]

\section*{B. Semishort limits}

\subsection*{1. The \(U(1) \otimes \text{Osp}(2N-2m|2)\) sector}

Considering, for \(\hat{y}, \hat{y}\) as in (5.1),

\[
\chi_{(r_1, \bar{z})}^{U(1) \otimes \text{Osp}(2N-2m|2)}(\hat{u}) = \lim_{\delta \rightarrow 0} \chi_{(r_1, 0, \ldots, 0, \bar{z})}^{(N, B, m)}(\delta \hat{u})^{1/2}, \hat{x}, (\delta^{-2} \hat{u})^{1/2}, \hat{y}, \hat{y})
\]

where

\[
\begin{align*}
\mathcal{H}_{\pm} &= D - \frac{1}{N}(H_1 + \cdots + H_{N-1} \pm H_N), \\
\mathcal{I}_{\pm} &= D + \frac{1}{N}(H_1 + \cdots + H_{N-1} \pm H_N),
\end{align*}
\]

separately in the limit \(\delta \rightarrow 0\), isolating states for which \(\mathcal{H}_{\pm}\) has zero eigenvalues, only the corresponding \(\frac{1}{2}\) BPS character is nonzero, giving

\[
\chi_{(r_1, \bar{z})}^{U(1)}(\hat{u}) = \lim_{\delta \rightarrow 0} \chi_{(r_1, 0, \ldots, 0, \bar{z})}^{(N, B, m)}(\delta \hat{u})^{1/2}, \hat{x}, (\delta^{-2} \hat{u})^{1/2}, \hat{y}, \hat{y}) = u^{2r_1}.
\]
\[ J_m = D - J_3 - \frac{1}{m} \sum_{m=1}^{\infty} H_m, \quad K_m = \frac{1}{m} \sum_{m=1}^{\infty} H_m, \quad D = D + J_3, \]  
(5.9)
in the limit \( \delta \to 0 \), again, it is clear that only those states in \( R_{(\alpha,j,\bar{\alpha})}^M \) for which the eigenvalue of \( J_m \) is zero contribute.

It can be shown that
\[
\lim_{\delta \to 0} \chi_{(\Delta,j,\bar{\Delta})}^M (\delta^{-1/2}, \delta^{-1-1/m}, (\delta^{-2} u)^{1/m}, \bar{y}, \bar{y}) = \chi(r; \Delta) \chi(\bar{r}; \bar{\Delta}) (u, \bar{y}, \bar{y}) = u^{R} \chi_{(\Delta, \bar{\Delta})}^R (\bar{y}, \bar{y}),
\]
(5.10)
in terms of \( U(1) \mathcal{R}_m \otimes \text{Osp}(2N-2m|2) \) characters, for appropriate \( i, R, \bar{\Delta} \), see Appendix A, for notation.

Similar to above, using (4.7) with (5.3), and also
\[
\chi_{2, j} (\delta^{-1} s^{1/2}) \sim \delta^{-2} j^2, \quad (5.11)
giving the leading behavior as \( \delta \to 0 \), then we have, for semishort and conserved multiplet cases, for \( n \gg m \) and with \( \bar{r} \) as in (5.3),
\[
\chi_{(r_1, r_2, \ldots, r_j, j+1)}^{(U(1) \otimes \text{Osp}(2N-2m|2), \text{long})} (u, \bar{y}, \bar{y}) = \lim_{\delta \to 0} \chi_{(r_1, r_2, \ldots, r_j, j+1)}^{(N, A, n)} (\delta s^{1/2}, \delta^{-1} s^{1/2}, (\delta^{-2} u)^{1/m}, \bar{y}, \bar{y})
\]
\[
= u^{r_1+2j+2+m+1} \chi_{(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_j)}^{(N-m)} (\bar{y}) \prod_{e=1}^{N-m} (1 + \bar{y}_i^e \bar{s}),
\]
(5.12)
For \( (N, B, n) \) BPS multiplet characters there are a number of cases to consider. For \( n < m \) we have, for \( r_n = \cdots = r_1 = 1 \) and \( r_{m+1} = \cdots \geq r_m = r_1 - 1 \geq \cdots \geq r_{n+1} \geq |\mathcal{R}_n| \),
\[
\chi_{(r_1, r_2, \ldots, r_j, j+1, m-n, 2)}^{(U(1) \otimes \text{Osp}(2N-2m|2), \text{long})} (u, \bar{y}, \bar{y}) = \lim_{\delta \to 0} \chi_{(r_1, 0, \ldots, 0)}^{(N, B, n)} (\delta s^{1/2}, \delta^{-1} s^{1/2}, (\delta^{-2} u)^{1/m}, \bar{y}, \bar{y})
\]
\[
= u^{r_1+2j+m-n} \chi_{(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_j, 0, \ldots, 0)}^{(N-m)} (\bar{y}) \prod_{e=1}^{N-m} (1 + \bar{y}_i^e \bar{s}),
\]
(5.13)
while for \( n = m \), so that \( r_1 = \cdots = r_n \geq r_{n+1} \geq \cdots \geq |\mathcal{R}_n| \),
\[
\chi_{(r_1, r_2, \ldots, r_j, j+1, m-n, 2)}^{(U(1) \otimes \text{Osp}(2N-2m|2), \text{long})} (u, \bar{y}, \bar{y}) = \lim_{\delta \to 0} \chi_{(r_1, 0, \ldots, 0)}^{(N, B, m)} (\delta s^{1/2}, \delta^{-1} s^{1/2}, (\delta^{-2} u)^{1/m}, \bar{y}, \bar{y})
\]
\[
= u^{r_1+2j+1} \chi_{(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_j, 0, \ldots, 0)}^{(N-m)} (\bar{y}) \prod_{e=1}^{N-m} (1 + \bar{y}_i^e \bar{s}).
\]
(5.14)
For \( n > m \) we have, using results from Appendix A,
\( \chi_{(r_1, r_1 \cdots)}^{(U(1) \otimes \text{Osp}(2N-2m)[2], r_1 \cdots)}(u, \tilde{s}, \tilde{y}) = \lim_{\delta \to 0} \chi_{(r_1, r_1 \cdots)}^{(N, B, \mp)}(\delta \tilde{s}^{1/2}, \delta^{-1/2} \tilde{s}^{-1/2}, (\delta^{-2} u)^{1/m} \tilde{s}, \tilde{y}) \)

\( = u^r \tilde{s}^{1/2} \tilde{s}^{1/2} \sum_{\alpha = 0}^{N-m} \sum_{\alpha = 0}^{N-m} \delta^{-1/2} \tilde{s}^{-1/2} \left( C_F(N-m) \prod_{i=1}^{N-m+1} (1 + \tilde{y}_i \tilde{s}) \prod_{i=1}^{N-m} (1 + \tilde{y}_i^{-1} \tilde{s}) \right) \),

so that

\( \chi_{(r_1, r_1 \cdots)}^{(U(1) \otimes \text{Osp}(2N-2m)[2], r_1 \cdots)}(u, \tilde{s}, \tilde{y}) = \lim_{\delta \to 0} \chi_{(r_1, r_1 \cdots)}^{(N, B, \mp)}(\delta \tilde{s}^{1/2}, \delta^{-1/2} \tilde{s}^{-1/2}, (\delta^{-2} u)^{1/m} \tilde{s}, \tilde{y}) \)

\( = u^r \tilde{s}^{1/2} \tilde{s}^{1/2} \sum_{\alpha = 0}^{N-m} \sum_{\alpha = 0}^{N-m} \delta^{-1/2} \tilde{s}^{-1/2} \left( C_F(N-m) \prod_{i=1}^{N-m+1} (1 + \tilde{y}_i \tilde{s}) \prod_{i=1}^{N-m} (1 + \tilde{y}_i^{-1} \tilde{s}) \right) \),

Finally, for the \( \frac{1}{2} \) BPS cases,

\( \chi_{(r_1, r_1 \cdots, r, \pm)}^{(U(1) \otimes \text{Osp}(2N-2m)[2], r_1 \cdots)}(u, \tilde{s}, \tilde{y}) = \lim_{\delta \to 0} \chi_{(r_1, r_1 \cdots, r, \pm)}^{(N, B, \mp)}(\delta \tilde{s}^{1/2}, \delta^{-1/2} \tilde{s}^{-1/2}, (\delta^{-2} u)^{1/m} \tilde{s}, \tilde{y}) \)

\( = u^r \tilde{s}^{1/2} \tilde{s}^{1/2} \sum_{\alpha = 0}^{N-m} \sum_{\alpha = 0}^{N-m} \delta^{-1/2} \tilde{s}^{-1/2} \left( C_F(N-m) \prod_{i=1}^{N-m+1} (1 + \tilde{y}_i \tilde{s}) \prod_{i=1}^{N-m} (1 + \tilde{y}_i^{-1} \tilde{s}) \right) \),

In particular, this implies

\( \chi_{(r_1, r_1 \cdots, r, \pm)}^{(U(1) \otimes \text{Osp}(2N-2m)[2], r_1 \cdots)}(u, \tilde{s}, \tilde{y}) = \lim_{\delta \to 0} \chi_{(r_1, r_1 \cdots, r, \pm)}^{(N, B, \mp)}(\delta \tilde{s}^{1/2}, \delta^{-1/2} \tilde{s}^{-1/2}, (\delta^{-2} u)^{1/m} \tilde{s}, \tilde{y}) \)

\( = u^r \tilde{s}^{1/2} \tilde{s}^{1/2} \sum_{\alpha = 0}^{N-m} \sum_{\alpha = 0}^{N-m} \delta^{-1/2} \tilde{s}^{-1/2} \left( C_F(N-m) \prod_{i=1}^{N-m+1} (1 + \tilde{y}_i \tilde{s}) \prod_{i=1}^{N-m} (1 + \tilde{y}_i^{-1} \tilde{s}) \right) \),

agreeing with (4.18), in this limit, using (5.3) with (5.4) for \( r = \frac{1}{2} \).

For \( m > 1 \), we have for long multiplet characters,

\( \chi_{(r_1, r_1 \cdots, r, \pm)}^{(U(1) \otimes \text{Osp}(2N-2m)[2], r_1 \cdots)}(u, \tilde{s}, \tilde{y}) = \lim_{\delta \to 0} \chi_{(r_1, r_1 \cdots, r, \pm)}^{(N, B, \mp)}(\delta \tilde{s}^{1/2}, \delta^{-1/2} \tilde{s}^{-1/2}, (\delta^{-2} u)^{1/m} \tilde{s}, \tilde{y}) \),

which is in accord with (4.26) with (4.27). For \( m = 1 \), we have that

\( (1 + u) \chi_{(r_1, r_1 \cdots, r, \pm)}^{(U(1) \otimes \text{Osp}(2N-2m)[2], r_1 \cdots)}(u, \tilde{s}, \tilde{y}) = \lim_{\delta \to 0} \chi_{(r_1, r_1 \cdots, r, \pm)}^{(N, B, \mp)}(\delta \tilde{s}^{1/2}, \delta^{-1/2} \tilde{s}^{-1/2}, (\delta^{-2} u)^{1/m} \tilde{s}, \tilde{y}) \),

compatible with (4.26).

### 2. The U(1) ⊗ SU(1, 1) sectors

Similarly, by considering,

\( \chi_{(r_1, r_1 \cdots)}^{(\Delta, j, r_1 \cdots)}(\delta \tilde{s}^{1/2}, \delta^{-1/2} \tilde{s}^{-1/2}, (\delta^{-2} u)^{1/m} \tilde{s}, \tilde{y}) \)

\( = \text{Tr}_{R_M^{(\Delta, j)}}(\delta \tilde{s}^{1/2}, \delta^{-1/2} \tilde{s}^{-1/2}, (\delta^{-2} u)^{1/m} \tilde{s}, \tilde{y}) \)

\( = \delta \tilde{s}^{1/2} \tilde{s}^{1/2} \sum_{\alpha = 0}^{N-m} \sum_{\alpha = 0}^{N-m} \delta^{-1/2} \tilde{s}^{-1/2} \left( C_F(N-m) \prod_{i=1}^{N-m+1} (1 + \tilde{y}_i \tilde{s}) \prod_{i=1}^{N-m} (1 + \tilde{y}_i^{-1} \tilde{s}) \right) \),

separately in the limit \( \delta \to 0 \), we have for the corresponding \((N, A, \pm)\) multiplet character,
corresponding limit in $F^{-1}$ in the $U/H_{20849}$ for the $/H_{20849}$ along with $/H_{20849}$ all other characters, apart from the long multiplet one corresponding to $/H_{20849}$ vanish.

In particular, from (5.20), this limit ensures that $Osp(2N|4)$ long multiplet characters vanish, and hence do not contribute in the decomposition of partition functions, in the same limit, in terms of $Osp(2N|4)$ characters. Thus, partition functions evaluated in this limit receive contributions from protected operators only. It should be noted, however, that the magnitude of the numbers obtained for counting operators by expansion in terms of characters, in this limit, provide only a lower bound on the numbers of actual protected operators due to the $(-1)^F$ sign in the index, for fermion number $F$.

\[ \chi^{U(1)\otimes Osp(1,1)}_{(r+1;2r+1;N)}(u,\bar{s}) \]

\[ = \lim_{\delta \to 0} \chi^{(N,\pm)}_{(r+1;2r+1;\ldots,2r\pm 1,\ldots,2,1/2)}(\delta^{3/2}, \delta^{-1/2}\bar{s}^{1/2}, (\delta^2 u)^{1/N} \bar{y}_1, \ldots, (\delta^2 u)^{1/N} \bar{y}_{N-1}, (\delta^2 u)^{\pm 1/N} \bar{y}_N^{\pm 1}) \]

\[ = u^{r+1} \frac{\bar{s}^{2r+2}+N+1}{1-\bar{s}^2}, \quad \text{(5.22)} \]

and, for the conserved current character, in either limit in (5.21),

\[ \chi^{U(1)\otimes Osp(1,1)}_{(1;2+N+n)}(u,\bar{s}) = \lim_{\delta \to 0} \chi^{(N,\text{cons.})}_{(1;2+N+n)}(\delta^{3/2}, \delta^{-1/2}\bar{s}^{1/2}, (\delta^2 u)^{1/N} \bar{y}_1, \ldots, (\delta^2 u)^{1/N} \bar{y}_{N-1}, (\delta^2 u)^{\pm 1/N} \bar{y}_N^{\pm 1}) \]

\[ = u^{1/2} \frac{\bar{s}^{2r+N-n}}{1-\bar{s}^2}. \quad \text{(5.23)} \]

For the $(N,B,\pm)$ half BPS cases and in the corresponding limit in (5.21), we have

\[ \chi^{U(1)\otimes Osp(1,1)}_{(r\pm,1;N+n)}(u,\bar{s}) = \lim_{\delta \to 0} \chi^{(N,B,\pm)}_{(r\pm,1;N+n)}(\delta^{3/2}, \delta^{-1/2}\bar{s}^{1/2}, (\delta^2 u)^{1/N} \bar{y}_1, \ldots, (\delta^2 u)^{1/N} \bar{y}_{N-1}, (\delta^2 u)^{\pm 1/N} \bar{y}_N^{\pm 1}) \]

\[ = u^{1/2} \frac{\bar{s}^{r+1}}{1-\bar{s}^2}. \quad \text{(5.25)} \]

along with

\[ \chi^{U(1)\otimes Osp(1,1)}_{(1;2,3/2)}(u,\bar{s}) \]

\[ = \lim_{\delta \to 0} \chi^{(N,B,\pm)}_{(1;2,0,1/2,\ldots,2,1/2)}(\delta^{3/2}, \delta^{-1/2}\bar{s}^{1/2}, (\delta^2 u)^{1/N} \bar{y}_1, \ldots, (\delta^2 u)^{1/N} \bar{y}_{N-1}, (\delta^2 u)^{\pm 1/N} \bar{y}_N^{\pm 1}) \]

\[ = u^{1/2} \frac{\bar{s}^{3/2}}{1-\bar{s}^2}. \quad \text{(5.26)} \]

All other characters, apart from the long multiplet one corresponding to (4.35), in the limit (5.21) vanish.

C. The superconformal index

The (single particle) superconformal indices\(^{1,26}\) may be computed by taking the limit $u \to -1$ in the $U(1) \otimes Osp(2N-2,2)$ characters above, i.e., for $m=1$ in the semishort limits above.\(^{27}\)

In particular, from (5.20), this limit ensures that $Osp(2N|4)$ long multiplet characters vanish, and hence do not contribute in the decomposition of partition functions, in the same limit, in terms of $Osp(2N|4)$ characters. Thus, partition functions evaluated in this limit receive contributions from protected operators only. It should be noted, however, that the magnitude of the numbers obtained for counting operators by expansion in terms of characters, in this limit, provide only a lower bound on the numbers of actual protected operators due to the $(-1)^F$ sign in the index, for fermion number $F$. 
VI. PARTITION FUNCTIONS FOR $\mathcal{N}=6$ SUPERCONFORMAL CHERN–SIMONS THEORY

In Ref. 2, a class of $\mathcal{N}=6$ superconformal Chern–Simons theories, with gauge group $U(n) \times U(n)$, was proposed. For levels $\pm k$ in the Chern–Simons terms, these theories admit a dual description in terms of $M$ theory compactified on $AdS_4 \times S^7/S_4$.

Here, the free field partition function of the theory, where the effective ’t Hooft coupling $n/k \to 0$, so that $k \to \infty$, in the large $n/k$ limit, is computed using appropriate superconformal characters.

The supergravity partition function, obtained using the duality proposed in Ref. 2 in the $n/k \to \infty$ limit, for $n \gg k \rightarrow \infty$, is also computed using appropriate superconformal characters.

A. Free field theory

For $\mathcal{N}=6 U(n) \times U(n)$ superconformal Chern–Simons theory, the dynamical field content consists of free scalars $\phi_i, \overline{\phi}_i, i=1, \ldots, 4$, and free spin half fermions $\psi_{ia}, \overline{\psi}_{ia}, a=1, 2$, where $\phi_i, \overline{\phi}_i$ belong to the fundamental representation of $SO(6) = SU(4)$, with $\psi_{ia}, \overline{\psi}_{ia}$ belonging to the antifundamental, while $\phi_i, \psi_{ia}$ transform in the $(n, \overline{n})$ bifundamental representation of $U(n) \times U(n)$, with $\overline{\phi}_i, \overline{\psi}_{ia}$ transforming in the $(\overline{n}, n)$ representation. Under the action of $SU(6)$ lowering operators,

$$\phi_1 \rightarrow \phi_2 \rightarrow \phi_3 \rightarrow \phi_4, \quad \psi_{1a} \rightarrow \psi_{2a} \rightarrow \psi_{3a} \rightarrow \psi_{4a},$$

$$\overline{\phi}_1 \rightarrow \overline{\phi}_2 \rightarrow \overline{\phi}_3 \rightarrow \overline{\phi}_4, \quad \overline{\psi}_{1a} \rightarrow \overline{\psi}_{2a} \rightarrow \overline{\psi}_{3a} \rightarrow \overline{\psi}_{4a},$$

where $\psi_{1}\rightarrow \psi_{2j}, \overline{\psi}_{1j} \rightarrow \overline{\psi}_{2j}$ under the action of the SU(2) lowering operator. The detailed fields are given below.

Here the SU(6) orthonormal basis labels $(r, q, p)$ are related to SU(4) Dynkin labels $[a, b, c]$ by

$$(r, q, p) \rightarrow [q + p, r - q, q - p],$$

so that $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \rightarrow [1, 0, 0], (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) \rightarrow [0, 0, 1]$.

Notice that the derivatives give rise to a factor $P(s, x)$ as in (4.6) multiplying the character for each conformal primary field in Table II, however, the condition that the fields be free gives rise to an effective factor $(1-s^2)$ for the scalar cases and $(1-s^2)$ for the fermion cases so that the conformal group character contributions are given, respectively, by the “Rac” and “Di” characters given in (4.19). [See Ref. 20 for details of the derivation of SO(d, 2) conformal group characters.]

As may be evident from (4.18), $\phi$ and $\psi$, respectively, $\overline{\phi}$ and $\overline{\psi}$, belong to the half BPS multiplet $(3, B, +)$, respectively, $(3, B, -)$. [The scalar $\phi_1$, respectively, $\overline{\phi}_1$, is the primary field of the $(3, B, +)$, respectively, $(3, B, -)$, multiplet.]

For free field theory, the single particle partition function is given by

$$Y_{\text{free}}(s, x, y, u, v) = \text{Tr}(s^2 d^2 x^2 y^2 H_1 H_2 H_3 u^1 \ldots u^n t^1 M_1 \ldots M_n),$$

where the trace is over states corresponding to $\phi_1, \overline{\phi}_1$, and all their superconformal descendants, of the form (3.3), and $s, x, y, u, v$ are “fugacities” with $L_1, \ldots, L_n, M_1, \ldots, M_n$ being usual U(n) Cartan subalgebra generators.

Thus we may write, in terms of characters,

$$Y_{\text{free}}(s, x, y, u, v) = f_+(s, x, y)p_1(u)p_1(v^{-1}) + f_-(s, x, y)p_1(u^{-1})p_1(v),$$

defining, for subsequent use,
where the signs on tors, is given by the usual integral over the gauge group, namely, so that $p_j(u)p_j(v^{-1}), p_j(u^{-1})p_j(v)$ correspond to the characters for the $(n,\bar{n})$, respectively, $(\bar{n},n)$ representations of $U(n)\times U(n)$, and where

$$f_\pm(s,x,y) = \lambda^{(3,B,\pm)}_{(s/2,0,1/2,1/2,\pm,1/2)}(s,x,y) = (y_1y_2y_3)^{s/2} \left( \sum_{j=1}^{3} y_j^j + (y_1y_2y_3)^{j+1} \right) D_{Rac}(s,x)$$

$$+ (y_1y_2y_3)^{s/2} \left( \sum_{j=1}^{3} y_j^{-j} + (y_1y_2y_3)^{-j-1} \right) D_{D}(s,x),$$

being half BPS characters (4.18) for SU(6) R-symmetry.

The multiparticle partition function, receiving contributions from only gauge invariant operators, is given by the usual integral over the gauge group, namely,

$$Z_{\text{tree}}^{(n)}(s,x,y) = \int_{U(n)} d\mu(u) \int_{U(n)} d\mu(v) \exp \left( \sum_{j=1}^{\infty} \frac{1}{2} Y_j^{(s',(-1)^{i+1})x^i y^j u^j v^j} \right),$$

where the signs on $x$ take account of particle statistics.

The integral may be evaluated in the large $n$ limit by using a method described in Ref. 28 (see also Ref. 29 for other applications). We may first expand

$$Z_{\text{tree}}^{(n)}(s,x,y) = \sum_{\lambda,\rho} \frac{1}{z^{\lambda} z^{\rho}} f_{\lambda,\rho}(s,x,y)f_{\lambda,\rho}(s,x,y) \int_{U(n)} d\mu(u) \int_{U(n)} d\mu(v) p_j(u)p_j(v) p_j(v^{-1}),$$

in terms of partitions,
\[ \lambda = (\lambda_1, \ldots, \lambda_j, \ldots), \quad \sum_{j=1} j \lambda_j = |\lambda| \in \mathbb{N}, \]
\[ \rho = (\rho_1, \ldots, \rho_j, \ldots), \quad \sum_{j=1} j \rho_j = |\rho| \in \mathbb{N}, \quad (6.9) \]

where for \( \sigma = (\sigma_1, \ldots, \sigma_j, \ldots), \)
\[ z_{\sigma} = \prod_{j=1} \sigma_j \eta_j^{\sigma_j}, \quad f_{s,\sigma}(x,y) = \prod_{j=1} f_{s,j}((-1)^{r+1}x^j,y^j)^{\sigma_j}, \quad p_{s,\sigma}(x) = \prod_{j=1} p_j(x)^{\sigma_j}. \quad (6.10) \]

Using the orthogonality relation for power symmetric polynomials,
\[ \int_{U(n)} d\mu(u)p_{\lambda}(u)p_{\rho}(u^{-1}) = z_{\lambda}\delta_{\lambda\rho}, \quad |\lambda|,|\rho| \leq n, \quad (6.11) \]

then we trivially obtain in the large \( n \) limit,
\[ Z^{(n)}_{\text{tree}}(s,x,y) \sim Z_{\text{tree}}(s,x,y) = \sum_{\lambda} f_{s,\lambda}(s,x,y) f_{-\lambda}(s,x,y) = \prod_{j=1} \prod_{h,j>0} \left( f_{s,j}((-1)^{r+1}x^j,y^j)f_{-s,j}((-1)^{r+1}x^j,y^j) \right) \]
\[ \frac{1}{1 - f_{s,j}((-1)^{r+1}x^j,y^j)f_{-s,j}((-1)^{r+1}x^j,y^j)}. \quad (6.12) \]

This result was also derived in Ref. 6 by using saddle point methods, in the context of showing superconformal index matching.

**B. Supergravity limit**

In the strong coupling limit, large \( n/k, n > k \to \infty \), as explained in Ref. 2, using results of Ref. 30, the single particle states (gravitons) effectively belong to scalar superconformal representations with conformal dimensions \( r \) and in the \( R_{(r,r)} \) SU(6) representations, for \( r \in \mathbb{N}, r > 0 \), so that, in terms of notation here, they belong to \( (3,B,2) \) BPS multiplets.

Taking account of superconformal descendants, we may thus write for the single particle partition function,
\[ Y_{\text{supra}}(s,x,y) = \sum_{r=1}^{\infty} X_{(r;0,r,0)}^{(3,B,2)}(s,x,y), \quad (6.13) \]

where using (4.7), for \( \mathfrak{M}^{S_1 \times (S_2)^2} \) acting on \( y = (y_1, y_2, y_3) \),
\[ X_{(r;0,r,0)}^{(3,B,2)}(s,x,y) = s^{2r}P(s,x)\mathfrak{M}^{S_1 \times (S_2)^2} \left( C_{(r,0)}^{(3)}(y) \prod_{x=1}^{3} (1 + sy_x x^e) \prod_{i=1}^{3} (1 + sy_i^{-1} x^e) \right). \quad (6.14) \]

Thus, using the definition of the SU(6) Verma module character (4.2), for \( N=3 \), and summing over \( r \) in (6.13) with (6.14), we may write
\[ Y_{\text{supra}}(s,x,y) = s^2 P(s,x)\mathfrak{M}^{S_1 \times (S_2)^2} \left( \frac{y_1^2 y_2^2 \prod_{x=1}^{3} (1 + sy_x x^e) \prod_{i=1}^{3} (1 + sy_i^{-1} x^e)}{(1 - s^2 y_1 y_2) \prod_{1 \leq j < k \leq 3} (y_j^{-1} - y_k^{-1})} \right). \quad (6.15) \]

Simplifying the \( S_3 \) part of the action of \( \mathfrak{M}^{S_1 \times (S_2)^2} \) in the latter we end up with the more succinct formula,
where

\[ f(s, x, y_1, y_2, y_3) = P(s, x) \prod_{i=1}^{3} (1 + s^3 y_i x^3) \prod_{i=1}^{3} \left( 1 + s y_i x^2 \right). \]  

(6.17)

We then have the multiparticle (free graviton gas) partition function given by the usual plethystic exponential, also taking into account particle statistics,

\[ Z_{\text{sugra}}(s, x, y) = \exp \left( \sum_{j=1}^{\infty} \frac{1}{j} Y_{\text{sugra}}(s^j, y^j, x^j, y^j, x^j) \right). \]  

(6.18)

VII. COUNTING MULTITRACE GAUGE INvariant OPERATORS

The limits in Osp(2N|4) characters discussed in Sec. V, giving reductions to subalgebra characters, are equivalent to decoupling limits that isolate sectors of operators in partition functions. By taking such limits, for \( N=3 \), in (6.12) and (6.18), and decomposing in terms of characters in the same limits, we are able to count corresponding multitrace gauge invariant operators in the free field, supergravity limits. For free field theory, the counting numbers obtained provide an upper bound on the numbers of protected operators in the interacting theory, while for the supergravity limit they are expected to count actually protected ones.

For \( f_\pm(s, x, y) \rightarrow f_\pm^G(u, \bar{y}), \) \( Y_{\text{sugra}}(s, x, y) \rightarrow Y_{\text{sugra}}^G(u, \bar{y}), \) in the BPS limits, where \( G \) is the corresponding subgroup, and \( f_\pm(s, x, y) \rightarrow f_\pm^H(u, \bar{s}, \bar{y}), \) \( Y_{\text{sugra}}(s, x, y) \rightarrow Y_{\text{sugra}}^H(u, \bar{s}, \bar{y}), \) in the semishort limits, where \( H \) is the corresponding subgroup, the limits generically give

\[ f_\pm^G(u, \bar{y}) = \chi_{1(2;0)}^{3(B;\pm)}(u, \bar{y}), \quad Y_{\text{sugra}}^G(u, \bar{y}) = \sum_{r=1}^{\infty} \chi_{(0;0;0;r;0)}^{3(B;2)}(u, \bar{y}), \]

\[ f_\pm^H(u, \bar{s}, \bar{y}) = \chi_{1(2;0;1,2,1/2)}^{3(B;\pm)}(u, \bar{s}, \bar{y}), \quad Y_{\text{sugra}}^H(u, \bar{s}, \bar{y}) = \sum_{r=1}^{\infty} \chi_{(0;0;0;r;0)}^{3(B;2)}(u, \bar{s}, \bar{y}), \]  

(7.1)

where, for the superconformal characters,

\[ \chi_{(\Delta;j)}^{M}(s, x, y) \rightarrow \chi_{(\Delta;j)}^{M}(u, \bar{y}), \quad \chi_{(\Delta;j)}^{M}(s, x, y) \rightarrow \chi_{(\Delta;j)}^{M}(u, \bar{s}, \bar{y}), \]  

(7.2)

in the same limits. An important point for the corresponding multiparticle partition functions is properly taking into account particle statistics which differs in both cases. For \( Z_{\text{free}}(s, x, y) \rightarrow Z_{\text{free}}^G(u, \bar{y}), \) \( Z_{\text{sugra}}(s, x, y) \rightarrow Z_{\text{sugra}}^G(u, \bar{y}), \) in the BPS limits and \( Z_{\text{free}}(s, x, y) \rightarrow Z_{\text{free}}^H(u, \bar{s}, \bar{y}), \) \( Z_{\text{sugra}}(s, x, y) \rightarrow Z_{\text{sugra}}^H(u, \bar{s}, \bar{y}), \) in the semishort limits, consistency requires\(^1\)

\[ Z_{\text{free}}^G(u, \bar{y}) = \frac{1}{\prod_{j=1}^{\infty} 1 - f_\pm^G(u^j, \bar{y}^j) f_\pm^G(u^j, \bar{y}^j)}, \]

\[ Z_{\text{sugra}}^G(u, \bar{y}) = \exp \left( \sum_{j=1}^{\infty} \frac{1}{j} y_{\text{sugra}}^G(u^j, \bar{y}^j) \right). \]
Thus, expanding over the characters in 

\[ Z_H^{\text{free}}(u, \bar{s}, \bar{y}) = \prod_{j=1}^{\infty} \frac{1}{1 - \alpha_j^{\text{SU}(2)}(u' \alpha_j, \bar{s}'/\alpha_j, \bar{y}')} f_H(u' \alpha_j, \bar{s}'/\alpha_j, \bar{y}') \alpha_j^{(-1)^{j+1}}, \]

\[ Z_H^{\text{sugra}}(u, \bar{s}, \bar{y}) = \exp \left( \sum_{j=1}^{\infty} \frac{1}{j} Y_H^{\text{sugra}}(u' \alpha_j, \bar{s}'/\alpha_j, \bar{y}') \right) \alpha_j^{(-1)^{j+1}} \right), \quad (7.3) \]

A. BPS cases

1. The \( U(1) \) sectors

Corresponding to (5.8) we consider the \((3, B, +)\) limit, whereby

\[ \chi^{(3, B, +)}_{(r:s:t)}(u, \bar{s}) = \chi^{U(1)}_{(r)}(u) = u^{2r}, \]

so that \( f^+_U(u) = u, f^-_U(u) = 0 \). The set of fields that contribute from Table II is

\[ (\phi_i)_i \]

where \( i, \bar{i} = 1, \ldots, n \) are \( U(n) \times U(n, \bar{n}) \) bifundamental indices. It is also easily seen that \( Y^{(1)}_{\text{sugra}}(u) = 0 \) in the limit (5.8). Thus, from (7.3),

\[ Z^{U(1)}_{\text{free}}(u) = Z^{U(1)}_{\text{sugra}}(u) = 1, \]

so that, clearly, there are no \((3, B, +)\) BPS gauge invariant operators in the free field or supergravity spectrum, apart from the identity operator. Of course this case is trivial since there is no way of forming \( SU(n) \times SU(n) \) gauge invariants involving solely \( \phi_{i_1 \bar{i}} \).

The same result applies to \((3, B, -)\) BPS operators.

2. The \((U(1) \times U(1))\) sector

Corresponding to (5.5) for \( N=3, m=2 \), we have that

\[ \chi^{(3, B, 2)}_{(r:s:t)}(u, \bar{s}) = \chi^{U(1) \times U(1)}_{(r:d)}(u, y) = u^{2y}, \]

\[ \chi^{(3, B, \pm)}_{(r:0:0, \pm:0)}(u, y) = \chi^{U(1) \times U(1)}_{(r:0:0)}(u, y) = u^{2y}, \quad (7.7) \]

so that \( f^+_U(u, y) = u^{\pm 1/2} \).

In this case the set of fundamental fields that contribute from Table II is

\[ (\phi_i)_i, \quad (\bar{\phi}_{\bar{i}})_{\bar{j}} \]

where \( j, \bar{j} = 1, \ldots, n \) are \( U(n) \times U(n, \bar{n}) \) bifundamental indices.

We also have that

\[ Y^{U(1) \times U(1)}_{\text{sugra}}(u, y) = \sum_{r=1}^{\infty} u^{2r} = \frac{u^2}{1 - u^2}, \]

Thus, from (7.3),

\[ Z^{U(1) \times U(1)}_{\text{free}}(u, y) = Z^{U(1) \times U(1)}_{\text{sugra}}(u, y) = \prod_{j=1}^{\infty} \frac{1}{1 - u^{2j}}. \quad (7.10) \]

Thus, expanding over the characters in (7.7),
where

\[ N_r^{(3,B,2)} = p(r) = 1, 2, 3, 5, \ldots , \quad r = 1, 2, 3, 4, \ldots , \]

(7.12)

where \( p(r) \) is the usual partition number for \( r \). The agreement (7.10) is expected as \( (3,B,2) \) operators are protected due to long multiplet decomposition rules discussed in Sec. IV.

This counting is simple to see in terms of (7.8) as the gauge invariants may be written as, for \( n_k \in \mathbb{N} \),

\[
\text{Tr}(Z^n) \cdots \text{Tr}(Z^r), \quad Z^j = (\phi_1)^i_j(\bar{\phi}_1)^{\bar{i}}_{\bar{j}}, \quad n_1 \leq \cdots \leq n_r \leq r, \quad \sum_{k=1}^r n_k = r, \]

(7.13)
of which, for a given \( r \), there are \( p(r) \).

### 3. The U(1) \( \otimes \) SO(4) sector

Due to (4.33), we expect disagreement between counting of \( (3,B,1) \) operators in the free field and supergravity limits and thus we split the discussion here. Corresponding to (5.5) for \( N=3, m=1 \), we have that

\[
\chi^{(3,B,1)}_{(r,0,r,0,p)}(u,u_+,u_-) = U^{(1)\otimes SO(4)}_{(r,0,0,p)}(u,u_+,u_-) = u^{2r} \chi_{q=-p}(u_+)^{q} \chi_{q=p}(u_-),
\]

\[
\chi^{(3,B,2)}_{(r,0,r,0,p)}(u,u_+,u_-) = U^{(1)\otimes SO(4)}_{(r,0,r,0,p)}(u,u_+,u_-) = u^{2r} \chi_{r+p}(u_+) \chi_{r-p}(u_-),
\]

\[
\chi^{(3,B,3)}_{(r,0,r,2,0)}(u,u_+,u_-) = U^{(1)\otimes SO(4)}_{(r,0,r,2,0)}(u,u_+,u_-) = u^{2r} \chi_{2r}(u_+),
\]

(7.14)
in terms of SU(2) characters in (4.5), using \( SO(4) = SU(2) \otimes SU(2) \), so that, for SO(4) characters,

\[
\tilde{\chi}^{(2)}_{(q,p)}(\bar{y}_1,\bar{y}_2) = \chi_{q+p}(u_+) \chi_{q-p}(u_-), \quad \bar{y}_1 = u_+, \quad \bar{y}_2 = u_/u_-.
\]

(7.15)

We thus have

\[
\tilde{\chi}^{(1)\otimes SO(4)}_{(u)} = u \chi_1(u_+), \quad \text{where} \quad \chi_1(u_+) = u_+ + u_-^{-1}.
\]

In this case the fundamental fields that contribute from Table II are

\[
(\phi_1)^i_j, \quad (\phi_2)^i_{\bar{j}}, \quad (\bar{\phi}_1)^{\bar{i}}_{\bar{j}}, \quad (\bar{\phi}_2)^{\bar{i}}_{\bar{j}}.
\]

(7.16)

Then from (7.3),

\[
Z^{U(1)\otimes SO(4)}_{\text{free}}(u,u_+,u_-) = \prod_{j=1}^{\infty} \frac{1}{1 - u^{2j} \chi_1(u_+)^j \chi_1(u_-)^j}.
\]

(7.17)

The numbers of multitrace \( (3,B,1) \) BPS operators are then determined by expansions over the characters (7.14), using (7.12),

\[
Z^{U(1)\otimes SO(4)}_{\text{free}}(u,u_+,u_-) = 1 + \sum_{r=1}^{\infty} p(r) \chi^{(3,B,2)}_{(r,0,r,0,r)}(u,u_+,u_-) + \sum_{r,q,p \geq 0} N^{(3,B,1)}_{\text{free},(r,0инд,0,p)} \chi^{(3,B,1)}_{(r,0,0,p)}(u,u_+,u_-),
\]

(7.18)

for which formulas are obtained in Appendix B. We may determine for the first few cases, for \( r = 0, 1, 2, \ldots \).
Again we may expand \( N_{\text{free},(r+1,r,\pm 1)}^{(3,B,1)} \) so that, from (7.19),

\[
N_{\text{free},(r+1,r,\pm 1)}^{(3,B,1)} = \sum_{j=0}^{r} p(j) - p(r + 1) = 0, 1, 2, 5, 8, \ldots
\]  

and

\[
N_{\text{free},(r+2,r,0)}^{(3,B,1)} = 2 \sum_{j=1}^{r} \sum_{k=0}^{j} p(k) - \sum_{j=0}^{r+1} p(j) + p(r + 2) = 2, 5, 12, 23, 44, \ldots
\]  

along with

\[
N_{\text{free},(r+4,r+2,\pm 2)}^{(3,B,1)} = 2 \sum_{j=1}^{r} \sum_{k=0}^{\lfloor j/2 \rfloor} p(k) \left( \frac{1}{2} (1 - (-1)^j) + 2k \right) + \sum_{j=0}^{r+2} p(j) - p(r + 3) = 2, 4, 10, 17, 32, \ldots
\]  

Note that generally, from the results in Appendix B, \( N_{\text{free},(r,q,p)}^{(3,B,1)} \) is a potentially nonzero integer only for \( N_{\text{free},(r,s-t,s-t)}^{(3,B,1)} \), \( r \in \mathbb{N}, s, t=0, \ldots, \lfloor r/2 \rfloor \), \( s \neq 0 \).

[From (6.2), \( (r,r-r-s-t,s-t) \rightarrow [r-2s,s+t,r-2t] \), in terms of SU(4) Dynkin labels.]

For the supergravity limit we may determine

\[
Z_{\text{sugra}}^{U(1)\otimes SO(4)}(u,u_+,u_-) = \prod_{j=1}^{\infty} \frac{1}{\prod_{k=0}^{\lfloor j/2 \rfloor} (1 - u^2 u_+^{2k-j} u_-^{2j-f})}.
\]  

Again we may expand

\[
Z_{\text{sugra}}^{U(1)\otimes SO(4)}(u,u_+,u_-) = 1 + \sum_{r=1}^{\infty} p(r) \chi_{(r,0,0,0)}^{(3,B,2)}(u,u_+,u_-) + \sum_{r,q,p=0}^{\infty} \chi_{\text{sugra},(r,q,p)}^{(3,B,1)} \chi_{(r,r,q,p)}^{(3,B,1)}(u,u_+,u_-)
\]  

to find for the first few cases, using results from Appendix B, for \( r=0, 1, 2, \ldots \),

\[
N_{\text{sugra},(r+1,r,\pm 1)}^{(3,B,1)} = N_{\text{free},(r+1,r,\pm 1)}^{(3,B,1)},
\]  

while

\[
N_{\text{sugra},(r+2,r,0)}^{(3,B,1)} = \sum_{j=0}^{r} \sum_{k=0}^{j} p(k) - \sum_{j=0}^{r+1} p(j) + p(r + 2) = 1, 2, 5, 9, 18, \ldots
\]  

along with

\[
N_{\text{sugra},(r+4,r+2,\pm 2)}^{(3,B,1)} = \sum_{j=1}^{r} \sum_{k=0}^{\lfloor j/2 \rfloor} p(k) \left( \frac{1}{2} (1 - (-1)^j) + 2k \right) + \sum_{j=0}^{r+2} p(j) - p(r + 3) = 2, 4, 10, 17, 32, \ldots
\]  

As emphasized, the matching (7.25) is expected from (4.33) along with the free field restrictions implied for general \( r, q, p \) in \( N_{\text{free},(r,q,p)}^{(3,B,1)} \) mentioned above.
TABLE III. BPS primary operators with conformal dimensions $\Delta$ belonging to SO(6) representations $\mathcal{R}_{(r,q,p)}$, as obtained from expansion of partition functions. (For the free field case, the extra operators appear in the rightmost column.)

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>BPS primary operators</th>
<th>Remaining operators</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathcal{R}_{(1,1,0)}$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$2\mathcal{R}<em>{(2,2,0)},\mathcal{R}</em>{(2,0,0)}$</td>
<td>$\mathcal{R}_{(2,0,0)}$</td>
</tr>
<tr>
<td>3</td>
<td>$3\mathcal{R}<em>{(3,3,0)},\mathcal{R}</em>{(3,2,0)},\mathcal{R}_{(3,1,0)}$</td>
<td>$\mathcal{R}<em>{(3,2,0)},7\mathcal{R}</em>{(1,2,0)}$</td>
</tr>
<tr>
<td>4</td>
<td>$5\mathcal{R}<em>{(4,4,0)},2\mathcal{R}</em>{(4,3,1)},2\mathcal{R}<em>{(4,2,2)},5\mathcal{R}</em>{(4,2,0)}$</td>
<td>$3\mathcal{R}<em>{(4,1,1)},6\mathcal{R}</em>{(4,0,0)}$</td>
</tr>
</tbody>
</table>

The first few numbers of operators may be easily obtained by performing series expansions to low orders in $u$ and using the orthogonality relation for SU(2) characters in Appendix B and are listed in Table III.

Note that more generally, as may be easily argued from results in Appendix B, $N_{\text{sugra},(r,q,p)}^{(3,B,1)}$ is potentially nonvanishing only for $N_{\text{sugra},(r,s,t,s,t)}^{(3,B,1)}$, $r \in \mathbb{N}$, $s,t=0,\ldots,[r/2]$, $st \neq 0$, consistent with the free field theory result.

B. Semishort cases

1. The $U(1) \otimes SU(1,1)$ sectors

In these sectors, where the relevant limits in characters are given by (5.21), for $N=3$, fermion contributions become important in multiparticle partition functions.

The surviving characters, for the $(N,A,+)$ limit, are given by

$$\chi^{(3,B,1)}_{(1/2,0,1/2,-1/2)}(u,\bar{s}) = \frac{u^{1/2}\bar{s}^{3/2}}{1-\bar{s}^2}, \quad \chi^{(3,B,1)}_{(r,0,r,r)}(u,\bar{s}) = \frac{u^r\bar{s}^{r+2}}{1-\bar{s}^2}, \quad \chi^{(3,\text{cons.})}_{(j+1/2,0,0,0)}(u,\bar{s}) = \frac{u^{2j+4}}{1-\bar{s}^2}, \quad \chi^{(3,A,+)\otimes SU(1)}_{(r+j+1/2,0,0,0)}(u,\bar{s}) = \frac{u^{r+j+2}\bar{s}^{2j+4}}{1-\bar{s}^2}. \quad (7.28)$$

Thus, we have that $f^+_{U(1)\otimes SU(1,1)}(u,\bar{s}) = (u\bar{s})^{1/2}/(1-\bar{s}^2)$, $f^-_{U(1)\otimes SU(1,1)}(u,\bar{s}) = u^{1/2}\bar{s}^{3/2}/(1-\bar{s}^2)$.

In this case the fields that contribute from Table II are

$$(\phi_1)^i_1, \quad (\bar{\psi}_{11})^j_1, \quad \partial_{11}. \quad (7.29)$$

Then from (7.3),

$$Z^U_{\text{free},SU(1,1)}(u,\bar{s}) = \prod_{j=1}^{\infty} \frac{1}{1 + (-1)^j u^{2j}(1-\bar{s}^2)^{j+2}}. \quad (7.30)$$

Expanding in $u$ we find
Using the foregoing we may find relatively simple formulas for at least the first few cases, write
\[
Z_{\text{free}}^{U(1) \otimes SU(1,1)}(u, \bar{s}) = 1 + \frac{u s^2}{(1 - s)^2} + \frac{4 u^2 s^6}{(1 - s)^2(1 - s^4)^2} + O(u^3, s^8) = 1 + \chi^{(3,R,2)}_{(1;0,1,1,0)}(u, \bar{s}) + \sum_{j=0}^{\infty} \lambda^{(3,A,+)}_{(j+1;0,0,0)}(u, \bar{s}) + \sum_{r=1}^{\infty} \sum_{j=2r-1}^{\infty} N^{(3,A,+)}_{\text{free},(r,j)} \chi^{(3,A,+)}_{(r+j+1;1,1,0)}(u, \bar{s}).
\]
\(7.31\)

It appears nontrivial to determine \(N^{(3,A,+)}_{\text{free},(r,j)}\), generally, however, we may easily determine,
\[
N^{(3,A,+)}_{\text{free},(1,0)} = 2 \left[ \frac{1}{2} j + 1 \right] \left[ \frac{1}{2} j + 2 \right] = 4, 12, 12, \ldots, \quad j = 1, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots.
\]
\(7.32\)

In this limit, the supergravity single particle partition function reduces to
\[
Y_{\text{sugra}}^{U(1) \otimes SU(1,1)}(u, \bar{s}) = \chi^{(3,R,2)}_{(1;0,1,1,0)}(u, \bar{s}) = \frac{u s^2}{1 - s^2},
\]
so that, from (7.3),
\[
Z_{\text{sugra}}^{U(1) \otimes SU(1,1)}(u, \bar{s}) = \prod_{j=1}^{\infty} (1 + u s^2 j),
\]
which is a reflection of \(Y_{\text{sugra}}^{U(1) \otimes SU(1,1)}(u, \bar{s})\) receiving purely fermionic operator contributions [it is odd under \((u, \bar{s}) \to -(u, \bar{s})]\).

Using the identities,
\[
\prod_{j=1}^{\infty} (1 + z q^j) = \sum_{n=0}^{\infty} \frac{g^{(1/2)n+1} z^n}{\prod_{j=1}^{n} (1 - q^j)} = \prod_{j=2}^{\infty} \frac{1}{1 - q^j} = \sum_{m=0}^{\infty} P_r(m) q^m,
\]
where \(P_r(m)\) is the number of partitions of \(m\) into no more than \(n\) parts, each part \(\geq 2\), we may write
\[
Z_{\text{sugra}}^{U(1) \otimes SU(1,1)}(u, \bar{s}) = 1 + \chi^{(3,R,2)}_{(1;0,1,1,0)}(u, \bar{s}) + \sum_{r=1}^{\infty} \sum_{j=1/2r^2+r-1}^{\infty} N^{(3,A,+)}_{\text{sugra},(r,j)} \chi^{(3,A,+)}_{(r+j+1;1,1,0)}(u, \bar{s}),
\]
\(7.36\)

where
\[
N^{(3,A,+)}_{\text{sugra},(r,j)} = P_{r+1} \left( j + 1 - \frac{1}{2} r^2 - r \right).
\]
\(7.37\)

Using the foregoing we may find relatively simple formulas for at least the first few cases,
\[
N^{(3,A,+)}_{\text{sugra},(1,j)} = \begin{cases} 
1, & j - \frac{1}{2} = 0 \text{ mod } 2, \\
, & j = \frac{1}{2}, \frac{3}{2}, \ldots, \\
0, & j - \frac{3}{2} = 0 \text{ mod } 2,
\end{cases}
\]
Assuming that the partition functions in the supergravity limit receive contributions from only protected operators, then a number of observations from (7.31), (7.32), (7.36), and (7.38) are evident.

First, there are no \( \chi_{r(0,0,r,r-1,1)}^{(3, B, 1)}(u, \bar{s}) \) contributions and one contribution from \( \chi_{r(0,0,r,r-1,1)}^{(3, B, 2)}(u, \bar{s}) \), for \( r=1 \), both consistent with previous analysis for BPS cases. Second, the conserved current contributions in (7.31) disappear from the spectrum in the strong coupling limit, as happens for \( N=4 \) super Yang Mills. Third, there is one scalar \((3, A, +)\) semishort primary operator in the SU(6) representation \( R_{(2,2,2)} \), and its conformal descendants, contributing in the free field theory limit, due to \( N_{\text{free},(2,0)}=1 \) in (7.31), and no such contribution in the supergravity limit, from (7.36). In the interacting theory, these operators, from the long multiplet decomposition formulas (4.35), must then pair up with a \((3, B, 1)\) BPS primary operator in the SU(6) representation \( R_{(2,0,0)} \), and its descendants. This is consistent with Table III as there is precisely one such remaining \((3, B, 1)\) primary operator. (Similarly, the scalar conserved current, and its descendants, contributing to (7.31) pairs with the single \((3, B, 1)\) BPS primary operator in the SU(6) representation \( R_{(2,0,0)} \), and its descendants, listed in Table III.)

Note that the analysis for \((3, A, -)\) semishort operators is very similar and gives the same result for counting of BPS and conserved current multiplets along with

\[
N^{(3, A, -)}_{\text{free},(r,j)} = N^{(3, A, +)}_{\text{free},(r,j)}, \\
N^{(3, A, -)}_{\text{supra},(r,j)} = N^{(3, A, +)}_{\text{supra},(r,j)}.
\]

In particular, again there is one scalar \((3, A, -)\) primary semishort operator contributing in the free field theory limit, absent from the supergravity spectrum, that pairs up with a \((3, B, 1)\) BPS primary operator in the \( R_{(4,2,2)} \) SU(6) representation, consistent with Table III.

**The \( U(1) \otimes \text{Osp}(2|2) \) sector**

In the limit (5.9), for \( N=3, m=2 \), the nonvanishing characters are, defining \( Q(\bar{s}, y) = (1 + \bar{s}y)^{-1}/(1 - \bar{s}^2) \),

\[
\chi_{r(0,0,r,r,1)}^{(3, B, +)}(u, \bar{s}, y) = \frac{u^{r+1}y^r}{1 - \bar{s}^2}, \quad \chi_{r(0,0,0,0,0)}^{(3, B, 2)}(u, \bar{s}, y) = (u, \bar{s})^{r}y^rQ(\bar{s}, y),
\]

\[
\chi_{(3, B, 1)}^{(3, B, 1)}(u, \bar{s}, y) = u^{r+1}y^rQ(\bar{s}, y), \quad \chi_{(3, \text{cons.})}^{(3, \text{cons.})}(u, \bar{s}, y) = u^{r+1}y^rQ(\bar{s}, y),
\]

\[
\chi_{(3, A, +)}^{(3, A, +)}(u, \bar{s}, y) = u^{r+1}y^rQ(\bar{s}, y),
\]

\[
\chi_{(3, A, 2)}^{(3, A, 2)}(u, \bar{s}, y) = u^{r+1}y^rQ(\bar{s}, y).
\]

Thus \( \chi^{U(1) \otimes \text{Osp}(2|2)}_{(r, r, r, r, 0)}(u, \bar{s}, y) = (u \bar{s})^{-1/2}(y^{1/2} + \bar{s}y^{-1/2})/(1 - \bar{s}^2) \).

The fields that contribute from Table II in this case are

\[
(\phi_i)^j, \quad (\tilde{\phi}_i)^j, \quad (\tilde{\phi}_i)^j, \quad (\tilde{\phi}_i)^j, \quad (\tilde{\phi}_i)^j, \quad (\tilde{\phi}_i)^j.
\]

From (7.3) we have
Using the previous results for counting of BPS operators, we may determine

\[ W^{(3,B,2)}_{\text{free}}(u, \bar{s}, y) = \sum_{r=1}^{\infty} p(r) \chi^{(3,B,2)}_{(r,0;r,0,0)}(u, \bar{s}, y) = \left( \prod_{k=1}^{\infty} \frac{1}{1 - (us^k)} - 1 \right) Q(\bar{s}, y) \]  

(7.43)

for contributions from \((3,B,2)\) multiplets, using (7.12), and

\[ W^{(3,B,1)}_{\text{free}}(u, \bar{s}, y) = \sum_{r=1}^{\infty} \left( N^{(3,B,1)}_{\text{free},(r,r-1)} \chi^{(3,B,1)}_{(r,0;r,0,0)}(u, \bar{s}, y) + N^{(3,B,1)}_{\text{free},(r-1,r)} \chi^{(3,B,1)}_{(r-1;0;r,0,0)}(u, \bar{s}, y) \right) = \bar{s}(y + y^{-1}) \times \left( 1 - \frac{2us - 1}{us} \prod_{k=1}^{\infty} \frac{1}{1 - (us^k)} - 1 \right) \]  

(7.44)

for contributions of \((3,B,1)\) multiplets, using the form of the corresponding characters in (7.40), along with (7.19). Similarly, using the free field results for the \(U(1) \otimes SU(1,1)\) sector in (7.31), we may determine

\[ W^{(3,\text{cons.})}_{\text{free}}(u, \bar{s}, y) = \sum_{j=0}^{\infty} \chi^{(3,\text{cons.})}_{(j+1;j,0,0,0)}(u, \bar{s}, y) = \frac{us^j}{1 - \bar{s}^2} Q(\bar{s}, y) \]  

(7.45)

for conserved current multiplet contributions and

\[ W^{(3,A,\pm)}_{\text{free}}(u, \bar{s}, y) = \sum_{r=1}^{\infty} \sum_{j=0}^{r-1} N^{(3,A,\pm)}_{\text{free},(j+r;j,r,\pm)} \chi^{(3,A,\pm)}_{(j+r;j,r,\pm)}(u, \bar{s}, y) = y^{r-1}(1 + \bar{s}y)(1 + \bar{s}y^{-1}) \times \left( \prod_{k=1}^{\infty} \frac{1}{1 - (\bar{s}y^2)^k} \right) \frac{1}{1 - ((1 - \bar{s}^2)^2)^{\frac{1}{2}}} = 1 \]  

(7.46)

for \((3,A,\pm)\) semishort multiplets, evident by using (7.31) and (7.39), and the form of the corresponding characters in (7.40). We may then determine numbers \(N^{(3,A,2)}_{\text{free},(r,j,q)}\) of \((3,A,2)\) semishort operators from

\[ Z_{\text{free}}^{(1) \otimes \text{osp}(2^2)}(u, \bar{s}, y) = 1 + W^{(3,B,2)}_{\text{free}}(u, \bar{s}, y) + W^{(3,B,1)}_{\text{free}}(u, \bar{s}, y) + W^{(3,\text{cons.})}_{\text{free}}(u, \bar{s}, y) + W^{(3,A,+)_{\text{free}}}(u, \bar{s}, y) + W^{(3,A,-)}_{\text{free}}(u, \bar{s}, y) + \sum_{r>1,j>0}^{r} N^{(3,A,2)}_{\text{free},(r,j,q)} \chi^{(3,A,2)}_{(r+1;j+1;0,r,0,0)}(u, \bar{s}, y). \]  

(7.47)

Finding general formulas for \(N^{(3,A,2)}_{\text{free},(r,j,q)}\) is nontrivial, however, series expansion of (7.47), using \textsc{Mathematica}, suggests the following results for particular cases, for \(r=0,1,2,\ldots:\)

\[ N^{(3,A,2)}_{\text{free},(r,j,0)} = \sum_{j=0}^{r} \sum_{k=0}^{r-j} p(k) + \sum_{j=0}^{r} p(j) = 4, 10, 21, 40, 71, \ldots, \]  

(7.48)

along with

\[ N^{(3,A,2)}_{\text{free},(r+3,0,\pm)} = \sum_{j=0}^{r+1} \sum_{k=0}^{[j/2]} p\left( \frac{1}{2} (1 - (-1)^j) + 2k \right) - \sum_{j=0}^{r+3} p(j) + p(r+4) = 2, 5, 10, 19, 33, \ldots. \]  

(7.49)
For the U(1) ⊗ Osp(2|2) sector, the supergravity single particle partition function reduces to
\[ Z_{\text{sugra}}^{U(1) ⊗ Osp(2|2)}(u, ū, y) = \sum_{r=1}^{\infty} \prod_{j,k=1}^{\infty} \frac{(1 + u^{j+2k-1}y)(1 + u^{j+2k-1}y^{-1})}{(1 - u^{j+2k})(1 - u^{j+2k})}, \]  
so that, from (7.3),
\[ Z_{\text{sugra}}^{U(1) ⊗ Osp(2|2)}(u, ū, y) = \prod_{j,k=1}^{\infty} \frac{(1 + u^{j+2k-1}y)(1 + u^{j+2k-1}y^{-1})}{(1 - u^{j+2k})(1 - u^{j+2k})}. \]  
This time, we may determine
\[ W_{\text{sugra}}^{U(1) ⊗ Osp(2|2)}(u, ū, y) = W_{\text{free}}^{U(1) ⊗ Osp(2|2)}(u, ū, y), \]  
due to the relevant sector of BPS operators remaining protected and the conserved current multiplet operators disappearing from the spectrum, as shown previously. Similarly, using the supergravity limit results for the U(1) ⊗ SU(1, 1) sector in (7.36), we may determine
\[ W_{\text{sugra}}^{U(1) ⊗ Osp(2|2)}(u, ū, y) = W_{\text{free}}^{U(1) ⊗ Osp(2|2)}(u, ū, y), \]  
and the form of the corresponding characters in (7.40). We may then determine numbers \( N_{\text{sugra},(r,j,q)}^{(3,A,2)} \) of (3, A, 2) semishort operators from
\[ Z_{\text{sugra}}^{U(1) ⊗ Osp(2|2)}(u, ū, y) = 1 + W_{\text{sugra}}^{U(1) ⊗ Osp(2|2)}(u, ū, y) + W_{\text{sugra}}^{U(1) ⊗ Osp(2|2)}(u, ū, y) + W_{\text{sugra}}^{U(1) ⊗ Osp(2|2)}(u, ū, y) + \sum_{r=1, j=0}^{\infty} N_{\text{sugra},(r,j,q)}^{(3,A,2)}(u, ū, y). \]  
Again, general formulas for \( N_{\text{sugra},(r,j,q)}^{(3,A,2)} \) seem nontrivial to obtain, however, using MATHEMATICA suggests, for particular cases, for \( r = 0, 1, 2, \ldots, \)
\[ N_{\text{sugra},(1,j,0)}^{(3,A,2)} = 1, \quad j = 0, 1, 2, \ldots, \]  
and
\[ N_{\text{sugra},(r+1,0,0)}^{(3,A,2)} = \sum_{j=0}^{r} \sum_{k=0}^{j} p(k) = 1, 3, 7, 14, 26, \ldots, \]  
along with
\[ N_{\text{sugra},(r+3,0,±2)}^{(3,A,2)} = \sum_{j=0}^{r+3} \sum_{k=0}^{[j/2]} p(k) \left( \frac{1}{2} (1 - (-1)^j) + 2k \right) - \sum_{j=0}^{r+3} p(j) + p(r + 4) = 0, 0, 1, 2, 5, \ldots. \]  
A consistency check is provided by (4.34) which implies that the unprotected scalar (3, A, 2) semishort primary operators in SU(6) representation \( R_{(r,r,0)} \), respectively, \( R_{(r,r,±2)} \), and their con-
formal descendants, counted above, should pair with unprotected \((3, B, 1)\) BPS primary operators in the \(SU(6)\) representation \(R_{(r+2, r, 0)}\), respectively, \(R_{(r+2, r, \pm 2)}\) and their descendants, counted previously. Using (7.20), (7.26), (7.48), and (7.56), along with (7.21), (7.27), (7.48), and (7.57), we find

\[
N^{(3, B, 1)}_{\text{free}, (r+2, r, 0) - N^{(3, B, 1)}_{\text{sugra}, (r+2, r, 0)} = N^{(3, A, 2)}_{\text{free}, (r, 0, 0) - N^{(3, A, 2)}_{\text{sugra}, (r, 0, 0)} = \sum_{j=0}^{r-1} \sum_{k=0}^{j} p(k),}
\]

\[
N^{(3, B, 1)}_{\text{free}, (r+2, r, \pm 2) - N^{(3, B, 1)}_{\text{sugra}, (r+2, r, \pm 2)} = N^{(3, A, 2)}_{\text{free}, (r, 0, \pm 2) - N^{(3, A, 2)}_{\text{sugra}, (r, 0, \pm 2)} = \sum_{j=0}^{r-2} \sum_{k=0}^{j} p \left( \frac{1}{2} (1 - (-1)^j) + 2k \right),}
\]

expressing perfect agreement with this expectation.

3. The \(U(1) \otimes Osp(4|2)\) sector

This sector is nontrivial to analyze in similar terms as above due to necessary and nontrivial expansions over \(SO(4)\), characters, as done for the \(U(1) \otimes SO(4)\) sector above, so here are simply given formulas for (7.1). Using (5.15), for \(N=3, m=1\), with (7.15), we have

\[
\chi^{(3, B, 2)}_{(r, r, 0)}(u, \bar{s}, u_+, u_-) = \chi^{(U(1) \otimes Osp(4|2))}_{(r, r, 0)}(u, \bar{s}, u_+, u_-) = \frac{(u \bar{s})}{1 - s^2} \sum_{e, q=1} (u_x)^q(u_x)^q(1 + \bar{s}u_x u_-^q)(1 + \bar{s}u_x u_-^q)(1 + \bar{s}u_x u_-^q)
\]

\[
= \frac{(u \bar{s})}{1 - s^2} \sum_{e, q=1} (u_x)^q(u_x)^q(1 + \bar{s}u_x u_-^q)(1 + \bar{s}u_x u_-^q)(1 + \bar{s}u_x u_-^q)
\]

\[
= \frac{(u \bar{s})}{1 - s^2} \sum_{e, q=1} (1 + u_x u_-^q)(1 + \bar{s}(u_x u_-)^q)(1 + \bar{s}(u_x u_-)^q)(1 + \bar{s}(u_x u_-)^q)
\]

\[
= \frac{(u \bar{s})}{1 - s^2} \sum_{e, q=1} (1 + u_x u_-^q)(1 + \bar{s}(u_x u_-)^q)(1 + \bar{s}(u_x u_-)^q)(1 + \bar{s}(u_x u_-)^q)
\]

(7.59) so that, with (5.18),

\[
Y^{(U(1) \otimes Osp(4|2))}_{\text{sugra}}(u, \bar{s}, u_+, u_-) = \frac{1}{1 - s^2} \sum_{e, q=1} (1 + u_x u_-^q)(1 + \bar{s}(u_x u_-)^q)(1 + \bar{s}(u_x u_-)^q)(1 + \bar{s}(u_x u_-)^q) - 1.
\]

(7.60) is consistent with (6.16) in the limit (5.9) and the formula in the second line can also be shown to be symmetric under exchange of \(u_+, u_-\), as is necessary.

In terms of the fields in Table II, the contributions are from

\[
(\phi_1)_i^j, \quad (\phi_2)_i^j, \quad (\psi_1)_i^j, \quad (\psi_2)_i^j,
\]

\[
(\bar{\phi}_1)_j^i, \quad (\bar{\phi}_2)_j^i, \quad (\bar{\psi}_1)_j^i, \quad (\bar{\psi}_2)_j^i, \quad \partial_{11}.
\]

VIII. CONCLUSION

While much progress has been made recently in terms of determining the spectra of superconformal field theories, there are many open questions, for instance, for the new superconformal Chern–Simons theories or \(N=4\) super Yang Mills.

Focusing on the former, while the spectra of the \(N=6\) superconformal Chern–Simons theory at zero (effective) ’t Hooft coupling and in the large \(n, k\) limits has been partially addressed here by use of character methods, it may be interesting to investigate operator counting for finite \(n, k\).
For large $n$, $k$, the results here provide extra confirmation of expectations that the primary operators dual to Kaluza Klein modes, and multitraces of these operators, in $[r,0,r]$ SU(4) representations, conformal dimension $r$, are protected, as argued in another way in Ref. 2. Also, the counting here implies that these are the only gauge invariant multitrace primary operators in the $(3,B,2)$ superconformal representation.

For the $(3,B,1)$ representations, the only gauge invariant primary operators belong to $[r-2s,s+t,r-2t]$, $s,t=0,\ldots,[r/2]$, $st \neq 0$, SU(4) representations, for which generating functions for counting are given in Appendix B. Furthermore, in accord with long multiplet decomposition rules, counting of $(s,t)=(1,0)$ and $(0,1)$ cases shows matching between the free field and supergravity limits, providing a consistency check of the character procedure used here and further evidence, perhaps, in favor of the duality proposed in Ref. 2.

Turning to semishort cases, conserved current multiplet operators disappear from the spectrum in the supergravity limit while the primary operators for $(3,A,\pm)$ semishort operators, belonging to $[2r,0,0]$, $[0,0,2r]$ SU(4) representations, have spins $j \geq \frac{1}{2}r-1$ in the free field limit, while for the supergravity limit, $j \geq \frac{1}{2}r^2 + r - 1$, a reflection of the very simple partition function obtained in (7.51).

While only partial counting is obtained here for semishort operators, the numbers of primary operators obtained are consistent with the following formula, implied by (4.34)–(4.36),

$$N^M_{\text{prot}(r,q)} = N^M_{\text{free}(r,q)} - N^M_{\text{free}(r+2,q)} + N^M_{\text{free}(r+2,q)}$$  \hspace{1cm} (8.1)

where $N^M_{\text{prot}(r,q)}$, $N^M_{\text{free}(r,q)}$ denote numbers of corresponding protected/free scalar semishort primary operators, in appropriate $\mathcal{R}_{(r,q)}$ SU(6) representations [so that for $\mathcal{M}=(3,\text{cons.})$, then $r,q=0$, for $\mathcal{M}=(3,A,\pm)$ then $r=\pm q \neq 0$ and for $\mathcal{M}=(3,A,2)$ then $r>|q|>0$, while $N^M_{\text{free}(r+2,q)}$, $N^M_{\text{free}(r+2,q)}$ denote numbers of protected/free primary operators in $(3,B,1)$ BPS multiplets, in $\mathcal{R}_{(r+2,q)}$ SU(6) representations.

Note also that, for free field theory in the large $n$ limit, $r,q$ are further restricted to $r=2$, $q = \pm 2$ for the $\mathcal{M}=(3,A,\pm)$ cases and $r \in \mathbb{N}$, $q=0$, $\pm 2$, $r>|q|$, for the $\mathcal{M}=(3,A,2)$ cases.

(8.1) implies that it is possible to compute the partition function corresponding to such protected operators using only free field theory and the knowledge of which $(3,B,1)$ BPS operators remain protected.

In particular, this applies to the $\mathcal{N}=6$ Chern–Simons theory, having finite $n$ and large levels $k$, in which case allowable $r,q$ in (8.1), deriving from free field theory, may change. The generating functions,

$$F_q(z) = \sum_{r>0} N^M_{\text{free}(r,q)} z^r,$$

$$G_q(z) = \sum_{r>0} (N^M_{\text{free}(r+2,q)} - N^M_{\text{prot}(r+2,q)}) z^r,$$  \hspace{1cm} (8.2)

may then be determined using the free field partition functions $Z^{n,\text{free}}_{U(1)\otimes \text{Osp}(2|2)}(u,\vec{x},y)$, $Z^{n,\text{free}}_{U(1)\otimes \text{SO}(4)}(u,\vec{u}_+,\vec{u}_-)$, denoting (6.7) evaluated in the relevant $H \subset \text{Osp}(6|4)$ sector, and $Z^{\text{prot}}_{\text{Osp}(1)\otimes \text{SO}(4)}(u,\vec{u}_+,\vec{u}_-)$, the partition function for protected $(3,B,1)$ operators evaluated in the $U(1)\otimes \text{SO}(4)$ sector. It is evident from the form of the characters in (7.40) that

$$F_q(z) = \frac{1}{(2\pi i)^2} \int \int \frac{1-x^2}{x^q y^{q}(x+y)(1+xy)} (Z^{n,\text{free}}_{U(1)\otimes \text{Osp}(2|2)}(z,x,y) - 1) dx dy$$

in terms of a double contour integral, where the contribution from the identity operator is subtracted. (Depending on free field constraints for finite $n$ it may also be necessary to subtract contributions from other short multiplet representations as well.) $G_q(z)$ may, in principle, be computed in terms of a double contour integral similar to that in Appendix B, where $f(u,x,y)$
polynomials for the symmetric permutation group. 

more generally, perhaps using known extensions of the Polya enumeration theorem, taking into

is equivalent to the chiral ring partition function, for finite

N

with other

Hq

chiral ring partition functions, useful in the context of asymptotics, see, for example, Ref.37.

tions regarding operator spectra that perhaps may be more easily answered for

N

counting for protected operators.

Recently, the latter issue received some attention in Ref.32, where, among other things, the

problem of counting chiral ring operators has been addressed from the perspec-

In Appendix C, a more combinatorial approach is described using a natural basis for multi-

trace operators involving commuting matrices, the simplest sort of chiral ring that is relevant for

N=4 superconformal Chern–Simons theories due to the latter having fewer decoupled sectors, as demonstrated here. One concerns finite N counting for protected operators. (Addressing this question may help answer the difficult black hole entropy/microstate counting problem.)

Recently, the latter issue received some attention in Ref. 32, where, among other things, the problem of counting certain gauge invariant operators, consisting of products of single trace chiral ring operators acted on by derivatives, was considered. (These operators should remain protected and certainly enumerating them is interesting and worthwhile.)

The simpler problem of counting chiral ring operators has been addressed from the perspective of computing Hilbert series for the ring of symmetric polynomials in Ref. 33 (equivalently, the latter issue has been investigated from a perhaps more formal perspective in Ref. 34), whereby the nontrivial part of the computation is in taking account of syzygies (or trace identities in terms of matrices). The same difficulty applies to counting operators of the sort, referred to above, considered in Ref. 32 and perhaps could be circumvented by a judicious choice of basis for the operators.

In Appendix C, a more combinatorial approach is described using a natural basis for multi-

trace operators involving commuting matrices, the simplest sort of chiral ring that is relevant for

N=4 super-Yang Mills. The technique involves employing the Polya enumeration theorem, applied to counting graph colorings where the relevant graphs, in this case, have a particular wreath product group automorphism symmetry. (This is similar to the approach used for counting single trace operators, in the large N limit, for N=4 SYM,35,36 where the relevant graphs there were necklaces with cyclic group automorphism symmetry.) The result obtained by this approach (equivalent to the Hilbert series mentioned above) is naturally expressed in terms of cycle index polynomials for the symmetric permutation group. (For alternative polynomial expansions of chiral ring partition functions, useful in the context of asymptotics, see, for example, Ref. 37.)

It may be worth trying to find other combinatorial methods of counting chiral ring operators more generally, perhaps using known extensions of the Polya enumeration theorem, taking into

\[
Z_q(s, x, y, z) = \sum_{r=0}^{\infty} N_{prot, r, q}^{(3A, 1)}(s, x, y, z) = 2w^3(H_q(s^2x^2z^3)C_{1,0,0,0,0}^{(3A, 1)})
\]

\[
H_q(z) = F_q(z) - G_q(z)
\]

for the partition function restricted to relevant protected scalar primary operators and their super-

conformal descendants.

Such partition functions should be consistent with the large n result where, using that there are no relevant scalar (3, cons.), (3,A, ±) primary operators in the supergravity spectrum and that

N=4 SYM,35,36 where the relevant graphs there were

Turning to N=4 super-Yang Mills, with gauge U(N), there are still many outstanding ques-

For alternative polynomial expansions of

\[
= Z^{(n), U(1) \otimes SO(4)}_{\text{free}}(u^{1/2}, x, y) - Z^{U(1) \otimes SO(4)}_{\text{prot}}(u^{1/2}, x, y) \text{ in (B1). (Presumably the latter partition function is equivalent to the chiral ring partition function, for finite } n.)
\]

These generating functions may then be used to write, using also (4.7), (4.27), and (4.28) for

N=3,

\[
Z_q(s, x, y, z) = \sum_{r=0}^{\infty} N_{prot, r, q}^{(3A, 1)}(s, x, y, z) = 2w^3(H_q(s^2x^2z^3)C_{1,0,0,0,0}^{(3A, 1)})
\]

\[
\times (s, x, y, z)
\]

\[
H_q(z) = F_q(z) - G_q(z)
\]

with other \(H_q(z)\) vanishing.

Turning to N=4 super-Yang Mills, with gauge U(N), there are still many outstanding questions regarding operator spectra that perhaps may be more easily answered for N=6 superconfor-

\[
H_0(z) = \sum_{r=1}^{\infty} N_{\text{supra}, r, 0, 0}^{(3A, 2)}(s, x, y, z) = \frac{z}{1-z^2}\prod_{k=1}^{\infty} \frac{1}{1-z^k},
\]

\[
H_{\pm 2}(z) = \sum_{r=3}^{\infty} N_{\text{supra}, r, 0, \pm 2}^{(3A, 2)}(s, x, y, z) = \left( \frac{z^2}{(1-z)(1-z^2)} - \frac{1}{1-z} + \frac{1}{1-z^2} - \frac{1}{z} \right) \prod_{k=1}^{\infty} \frac{1}{1-z^k} = \frac{1}{z},
\]

\[
(8.4)
\]
account symmetry in colors,\textsuperscript{38} for example. In any case, Hilbert series appear to have very interesting connections with counting graph colorings that perhaps may lead to even other extensions of the Polya enumeration theorem.

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APPENDIX A: Osp(2\(N\)|4) SUBALGEBRAS

Corresponding to the shortening conditions (3.5), (3.8), (3.11), and (3.14), with notation as in Table I, there are subalgebras which are now discussed. The characters for these subalgebras lead to the limits considered in Sec. V.

Corresponding to (3.5) for (\(N,B,m\)) short multiplets, we have

\[
\text{Osp}(2\!\!\!N|4) \cong (\text{SU}(2|2m) \otimes \text{SO}(2N-2m)) \ltimes U(1)_{T_m}.
\]

(A1)

The generators of SU(2|2m) are \(M_{\alpha}^{\beta} , Q_{\tilde{\alpha} \tilde{\beta}}, \bar{S}_n^{\beta} , \bar{m}, \bar{n}=1, \ldots , m\), of Sec. II, along with \(\mathcal{H}_n\) as in (5.1), and \(T_{\bar{n}\bar{m}}\), generators of SU(m) \(\subset \text{SO}(2m)\), with, in terms of the generators in (2.7),

\[
T_{\bar{n}\bar{m}} = H_{\bar{n}} - \frac{1}{m} \sum_{\bar{m}=1}^{m} H_{\bar{m}}, \quad T_{\bar{n}\bar{m}} = \begin{cases} 
\frac{i}{2} E_{\bar{m}\bar{n}}^{\bar{m}\bar{n}}, & \bar{m} < \bar{n} \\
- \frac{i}{2} E_{\bar{m}\bar{n}}^{\bar{n}\bar{m}}, & \bar{n} < \bar{m}
\end{cases},
\]

(A2)

so that \(\sum_{\bar{m}=1}^{m} T_{\bar{n}\bar{m}}=0\) and, for \(\bar{k}, \bar{l}=1, \ldots , m\),

\[
[T_{\bar{n}\bar{m}}, T_{\bar{k}\bar{l}}] = \delta_{\bar{m}\bar{l}} T_{\bar{n}\bar{k}} - \delta_{\bar{m}\bar{k}} T_{\bar{n}\bar{l}}.
\]

(A3)

SU(2|2m) has usual algebra with, in particular,

\[
\{Q_{\tilde{\alpha} \tilde{\alpha}}, \bar{S}_n^{\beta}\} = 2i(M_{\alpha}^{\beta} \delta_{\tilde{\alpha} \tilde{\beta}} - \delta_{\tilde{\alpha} \tilde{\beta}} \bar{T}_{\bar{n}\bar{m}} + \delta_{\tilde{\alpha} \tilde{\beta}} \bar{\mathcal{H}}_m)
\]

(A4)

and

\[
[\mathcal{H}_m, Q_{\tilde{\alpha} \tilde{\alpha}}] = \left( \frac{1}{2} - \frac{1}{m} \right) Q_{\tilde{\alpha} \tilde{\alpha}}, \quad [\mathcal{H}_m, \bar{S}_n^{\beta}] = - \left( \frac{1}{2} - \frac{1}{m} \right) \bar{S}_n^{\beta}.
\]

(A5)

For \(m=2\), \(\mathcal{H}_m\) is evidently then a central extension.

The SO(2N-2m) subgroup in (A1) is generated by \(R_{\bar{r}\bar{s}}, \bar{r}, \bar{s}, \bar{u}=2m+1, \ldots , 2N\),

\[
[R_{\bar{r}\bar{s}}, R_{\bar{u}\bar{v}}] = i(\delta_{\bar{s}\bar{v}} R_{\bar{u}\bar{r}} - \delta_{\bar{r}\bar{u}} R_{\bar{v}\bar{s}} - \delta_{\bar{r}\bar{s}} R_{\bar{u}\bar{v}} + \delta_{\bar{u}\bar{v}} R_{\bar{r}\bar{s}}),
\]

(A6)

while \(I_m\) in (5.1) is an external automorphism with

\[
[I_m, Q_{\tilde{\alpha} \tilde{\alpha}}] = \left( \frac{1}{2} + \frac{1}{m} \right) Q_{\tilde{\alpha} \tilde{\alpha}}, \quad [I_m, \bar{S}_n^{\beta}] = - \left( \frac{1}{2} + \frac{1}{m} \right) \bar{S}_n^{\beta}.
\]

(A7)

Corresponding to (3.5) for \(n=N\) for (\(N,B,\pm\)) multiplets and, separately, (3.8) for (\(N,B,\mp\)) multiplets, we have

\[
\text{Osp}(2\!\!\!N|4) \cong \text{SU}(2|N) \ltimes U(1)_{I_\pm},
\]

(A8)

The generators of SU(2|N) are \(M_{\alpha}^{\beta} , Q_{\tilde{\alpha} \tilde{\alpha}}, \bar{S}_n^{\beta} , \bar{m}, \bar{n}=1, \ldots , N-1, +\) for the (\(N,B,\pm\)) case and \(\bar{m}, \bar{n}=1, \ldots , N-1,\) for the (\(N,B,\mp\)) case, where we define
\[ Q_{\pm \alpha} = Q_{N\alpha}, \quad Q_{-\alpha} = \bar{Q}_{N\alpha}, \quad \bar{S}_n^\alpha = \bar{S}_N^\alpha, \quad \bar{S}_\pm^\alpha = S_N^\alpha, \]

along with \( \mathcal{H}_\pm \) as in (5.7), and \( T_{\pm \hat{m}} \), generators of \( \text{SU}(N) \) given by, for \( \hat{m} < N, \)

\[ T_{\pm \hat{m}} = H_\hat{m} = \frac{1}{N} (H_1 + \cdots + H_{N-1} \pm H_N), \]

\[ T_{\pm \hat{n}} = \pm H_N - \frac{1}{N} (H_1 + \cdots + H_{N-1} \pm H_N), \]

\[ T_{\pm \hat{m}} = \pm H_N - \frac{1}{N} (H_1 + \cdots + H_{N-1} \pm H_N), \]

\[ T_{\pm \hat{n}} = \frac{i}{2} E_{N\hat{n}}, \quad T_{\pm \hat{m}} = \begin{cases} \frac{i}{2} E_{\hat{m}N}, & \hat{m} < \hat{n} < N \\ \frac{i}{2} E_{\hat{n}N}, & \hat{n} < \hat{m} < N, \end{cases} \] (A10)

satisfying the same commutation relation (A3) for \( \hat{m}, \hat{n}, \hat{k}, \hat{i}, \hat{l} = 1, \ldots, N-1, \pm \).

The \( Q_{\pm \alpha}, \bar{S}_n^\beta \) generators satisfy (A4) for \( \mathcal{H}_m \) replaced by \( \mathcal{H}_\pm \), as appropriate, and \( \hat{m}, \hat{n} = 1, \ldots, N-1, \pm \), and with

\[ [\mathcal{H}_\pm, Q_{\pm \alpha}] = \left( \frac{1}{2} - \frac{1}{N} \right) Q_{\pm \alpha}, \quad [\mathcal{H}_\pm, \bar{S}_n^\beta] = -\left( \frac{1}{2} - \frac{1}{N} \right) \bar{S}_n^\beta, \] (A11)

so that for \( R \)-symmetry group \( \text{SO}(4) \), i.e., \( N=2, \mathcal{H}_\pm \) is a central extension.

\( \mathcal{I}_\pm \) in (5.7) acts and as external automorphism with

\[ [\mathcal{I}_\pm, Q_{\pm \alpha}] = \left( \frac{1}{2} + \frac{1}{N} \right) Q_{\pm \alpha}, \quad [\mathcal{I}_\pm, \bar{S}_n^\beta] = -\left( \frac{1}{2} + \frac{1}{N} \right) \bar{S}_n^\beta. \] (A12)

The expression (5.2) may be understood as follows. By taking the limit \( \delta \to 0 \), only \( (N,B,n), \)
\( n \gg m, \) and \( (N,B,\pm) \) BPS multiplets have states, including the highest weight state, with zero \( \mathcal{H}_m \)

\[ \text{eigenvalues. Characters for the} \ (\text{SU}(2|m) \otimes \text{SO}(2N-2m)) \otimes \text{U}(1)_{x_m} \text{ subalgebra, when restricted to these representations, reduce to} \ U(1)_{x_m} \otimes \text{SO}(2N-2m) \text{ characters, as these representations are} \]

\[ \text{trivial representations of the} \ (\text{SU}(2|m) \text{ subalgebra, evident from Sec. III.} \]

\[ \text{The limit in (5.8) may be understood similarly to other BPS cases whereby the characters} \]

\[ \text{reduce in the half BPS cases to} \ U(1)_{\mathcal{I}_\pm} \text{ characters.} \]

\[ \text{Corresponding to (3.11) for} \ (N,A,m) \text{ semishort multiplets, we have} \]

\[ \text{Osp}(2N|4) \supset (\text{SU}(1|m) \otimes \text{Osp}(2N-2m|2)) \otimes \text{U}(1)_{x_m}. \] (A13)

\[ \text{The generators of} \ \text{SU}(1|m) \text{ are} \ Q_{\pm \alpha}, \bar{S}_n^2, \ T_{\pm \hat{m}}, \hat{m}, \hat{n}=1, \ldots, m, \) along with \( \mathcal{J}_m \) as in (5.9), where \( \text{SU}(1|m) \) has usual algebra with, in particular,

\[ \{Q_{\pm \alpha}, \bar{S}_n^2\} = \pm T_{\pm \hat{m}} + \delta_{\pm \hat{m}} \mathcal{J}_m, \] (A14)

with

\[ [\mathcal{J}_m, Q_{\pm \alpha}] = \left( 1 - \frac{1}{m} \right) Q_{\pm \alpha}, \quad [\mathcal{J}_m, \bar{S}_n^2] = -\left( 1 - \frac{1}{m} \right) \bar{S}_n^2. \] (A15)

For \( m=1, \mathcal{J}_m \) is then a central extension.

\[ \text{The generators of} \ \text{Osp}(2N-2m|2) \) are \( \text{Sp}(2,R) = \text{SU}(1,1) \) generators \( P_{11}, K^{11} \) of Sec. II,

\[ \text{along with,} \ D=D+J_3 \text{ in (5.9) satisfying} \]

\[ \frac{(5.2)}{5.8} \]

\[ \text{as in (5.7), and} \ T_{\pm \hat{m}} \text{, generators of} \ \text{SU}(N) \text{ given by, for} \ \hat{m} < N, \]

\[ \text{along with} \ \mathcal{H}_\pm \text{ as in (5.7), and} \ T_{\pm \hat{m}} \text{, generators of} \ \text{SU}(N) \text{ given by, for} \ \hat{m} < N, \]

\[ \text{satisfying the same commutation relation (A3) for} \ \hat{m}, \hat{n}, \hat{k}, \hat{i}, \hat{l} = 1, \ldots, N-1, \pm \).
\[ [\tilde{D}, P_{11}] = 2P_{11}, \quad [\tilde{D}, K^{11}] = -2K^{11}, \quad [K^{11}, P_{11}] = 4\tilde{D}, \quad (A16) \]

along with \( Q_{\bar{I}}, S^1_{\bar{I}}, \bar{r}, \bar{s} = 2m + 1, \ldots, 2N \), of Sec. II, and SO(2N−2m) generators \( R_{\bar{I}} \) as before, satisfying

\[ [\tilde{D}, Q_{\bar{I}}] = Q_{\bar{I}}, \quad [\tilde{D}, S^1_{\bar{I}}] = -S^1_{\bar{I}}, \]

\[ [K^{11}, Q_{\bar{I}}] = 2iS^1_{\bar{I}}, \quad [P_{11}, S^1_{\bar{I}}] = -2iQ_{\bar{I}}, \quad (A17) \]

and

\[ \{Q_{\bar{I}}, Q_{\bar{J}}\} = 2\delta_{\bar{I}\bar{J}}P_{11}, \quad \{S^1_{\bar{I}}, S^1_{\bar{J}}\} = 2\delta_{\bar{I}\bar{J}}K^{11}, \quad \{Q_{\bar{I}}, S^1_{\bar{J}}\} = 2i\delta_{\bar{I}\bar{J}}\tilde{D} + 2R_{\bar{J}}. \quad (A18) \]

In (A13), \( K^m \) as in (5.9) is an outer automorphism with

\[ [K^m, Q_{\bar{n}\bar{m}}] = \frac{1}{m}Q_{\bar{n}\bar{m}}, \quad [K^m, \bar{S}^2_{\bar{n}}] = -\frac{1}{m}\bar{S}^2_{\bar{n}}, \quad [K^m, Q_{\bar{I}}] = [K^m, S^1_{\bar{I}}] = 0. \quad (A19) \]

Corresponding to (3.11) for \( n=N \) for \( (N,A,+ \) multiplets and, separately, (3.14) for \((N,A,−) \) semishort multiplets, we have

\[ \text{Osp}(2N|4) \supset (\text{SU}(1|N) \otimes \text{SU}(1,1)) \ltimes U(1)_{\mathbb{K}_R}. \quad (A20) \]

The generators of SU(1|N) are \( Q_{\bar{n}\bar{m}}, \bar{S}^1_{\bar{n}}, T_{\bar{m}\bar{n}} \) for \( \bar{m}, \bar{n} = 1, \ldots, N-1,+, \) for the \( (N,A,+ \) case, and \( \bar{m}, \bar{n} = 1, \ldots, N-1,− \) for the \( (N,A,−) \) case, with the definitions (A9) for \( \alpha = 2 \), with \( T_{\bar{m}\bar{n}} \) as in (A10) and \( J_\pm \) as in (5.21).

The supercharges satisfy (A14) with \( J_m \) replaced by \( J_\pm \), as appropriate, and \( \bar{m}, \bar{n} = 1, \ldots, N - 1, \pm \), with

\[ [J_\pm, Q_{\bar{n}\bar{m}}] = \left( 1 - \frac{1}{N} \right) Q_{\bar{n}\bar{m}}, \quad [J_\pm, \bar{S}^2_{\bar{n}}] = -\left( 1 - \frac{1}{N} \right) \bar{S}^2_{\bar{n}}. \quad (A21) \]

The supercharges satisfy (A14) with \( J_m \) replaced by \( J_\pm \), as appropriate, and \( \bar{m}, \bar{n} = 1, \ldots, N - 1, \pm \), with

\[ [J_\pm, Q_{\bar{n}\bar{m}}] = \frac{1}{N} Q_{\bar{n}\bar{m}}, \quad [J_\pm, \bar{S}^2_{\bar{n}}] = -\frac{1}{N} \bar{S}^2_{\bar{n}}. \quad (A22) \]

(5.10) can be understood similarly to BPS limits. Again by taking the limit \( \delta \to 0 \), only those states in relevant semishort multiplets and various BPS multiplets with zero \( J_m \) eigenvalue contribute to corresponding characters. Note, however, that for semishort multiplets, the states do not include highest weight states, but some superconformal descendants. Such characters, when restricted to the SU(1|m) \( \otimes \) Osp(2N−2m|2) superalgebra, reduce to U(1) \( \otimes \) Osp(2N−2m|2) representations as these representations are trivial representations of the SU(1|m) superalgebra.

To see the equivalence in terms of Osp(2N−2m|2) characters, we may, as in Sec. II, make a change of basis for \( Q_{\bar{I}}, S^1_{\bar{I}}, R_{\bar{I}} \) to the orthonormal basis, as in (2.8) and (2.9), to \( \{Q_{\bar{m}\bar{1}}, \bar{S}^1_{\bar{m}}, T_{\bar{m}\bar{n}}, \bar{H}_{\bar{m}}, E^+_{\bar{m}\bar{n}}, E^-_{\bar{m}\bar{n}}, \bar{m}, \bar{n} = m+1, \ldots, N \} \) in an obvious way. Denoting highest weight states by \( |\Delta; \bar{r}^{h.w.}\rangle \), where

\[ (K^{11}, S^1_{\bar{I}}) |\Delta; \bar{r}^{h.w.}\rangle = 0, \quad (\tilde{D}, H_{\bar{m}}) |\Delta; \bar{r}^{h.w.}\rangle = (\Delta, r_{\bar{m}}) |\Delta; \bar{r}^{h.w.}\rangle, \quad (A23) \]

unitarity requires \( \Delta \geq r_{m+1} \). Compatible shortening conditions are given by, for \( \bar{n} = m+1, \ldots, N \),
and using usual the orthogonality relation for SU(2) characters in (4.5),

\[- \frac{1}{4\pi i} \oint \frac{dz}{z} (z - z^{-1})^2 \chi_j(z) \chi_j(z) = \delta_{ii}, \]

which may be equivalently expressed by, due to \(\chi_j(z) = \chi_j(z^{-1})\),

\[- \frac{1}{2\pi i} \oint \chi_j(z) \chi_j(z) (z - z^{-1}) dz = \delta_{ii}, \]

we have that

\[ N_{rst} = \frac{1}{(2\pi i)^3} \oint \oint f(u,x,y) u^{r-1} \chi_s(x) \chi_t(y) (x - x^{-1})(y - y^{-1}) du dx dy, \]

where each contour is the relevant unit circle. Using \(\chi_j(z) = -\chi_{-j}(z)\), we have

\[ Q_{ii}|\tilde{\Delta};\varphi\rangle^{h.w.} = 0 \Rightarrow \tilde{\Delta} = r_{m+1} = \cdots = r_n, \]

\[ Q_{ij}|\tilde{\Delta};\varphi\rangle^{h.w.} = 0 \Rightarrow \tilde{\Delta} = r_{m+1} = \cdots = r_{N-i} = -r_N. \]
\[ N_{r,s} = -N_{r-s-2s} = -N_{r-s-2r} = N_{r-s-2r}. \] (B5)

It is convenient below to construct generating functions

\[ F_{rs}(z) = \sum_{n=0}^{\infty} N_{n-r-2n}z^n, \] (B6)

so that using (B4), summing over \( r \), and performing the subsequent contour integral over \( u \) for \( |z| < |xy| \),

\[ F_{rs}(z) = \frac{1}{(2\pi i)^2} \oint \oint (x^{-2s}y^{-2r}f(z,xy,x,y) - x^{-1-2s}y^{-1+2r}f(z/x/y,x,y) - x^{-1+2s}y^{-1-2r}f(z/y/x,x,y)
+ x^{-1+2s}y^{-1+2r}f(z/(xy),x,y))dx dy. \] (B7)

Note that for application to the main text, in terms of the notation used below,

\[ N_{\text{free},(r,s-t,t-t)} = N_{F, r-2s, r-2t}, \quad N_{\text{sugra},(r,s-t,t-t)} = N_{S, r-2s, r-2t}. \] (B8)

1. The free field case

For the free field case, due to (7.17) with (7.18), we consider

\[ f_F(u, x, y) = \prod_{j=1} \frac{1}{1 - u^j \chi_1(x^j) \chi_1(y^j)} = \sum_{r,s,t \geq 0} N_{F, ru, t} \chi_1(x) \chi_1(y), \] (B9)

which may be expanded as

\[ f_F(u, x, y) = \sum_{j} u^{[j]} p_x(x, x^{-1}) p_y(y, y^{-1}), \] (B10)

in terms of power symmetric polynomials \( p_x(z), (6.10) \) with (6.5). Using an expansion formula for power symmetric polynomials in terms of Schur polynomials, we may expand the latter in terms of \( \text{SU}(2) \) characters via

\[ p_x(z, z^{-1}) = \sum_{m,n \geq 0} \omega_{\lambda}^{(m,n)} \chi_m(z), \] (B11)

where \( \omega_{\lambda}^{(m,n)} \) are symmetric group characters. Introducing the notation, \( \chi_{[\lambda]} = \prod_{j=1} \chi_{\rho}^{(j, \lambda)} \) then it is easily determined from (B11) that (other formulas for these characters may be found in Ref. 40 but the following is the most useful for purposes here, and is perhaps simpler)

\[ \omega_{\lambda}^{([\lambda], 0)} = 1, \quad \omega_{\lambda}^{([\lambda]+2n, n)} = \sum_{\rho} \omega_{\lambda}^{(\rho) \rho} \chi_{[\lambda]}^{(\rho, \rho)}, \quad n \in \{0, \ldots, [\lambda]/2\}, \] (B12)

with \( \omega_{\lambda}^{(\rho, \rho)} \) being otherwise zero. Thus, using (B3) and (B4) for (B10) with (B11),

\[ N_{F, r-2s, r-2t} = \sum_{\lambda \vdash n} \omega_{\lambda}^{(r-2s, s)} \omega_{\lambda}^{(r-2t, t)}, \quad s, t = 0, \ldots, \lfloor r/2 \rfloor. \] (B13)

Thus, \( N_{F, r-2s, r-2t} \) is a potentially nonzero, non-negative integer only for \( r, s = 0, \ldots, \lfloor r/2 \rfloor \).

A useful identity, which may be easily generalized, is the following, namely, for
\[ n_{ij} = \sum_{|\lambda|=r} \binom{\lambda_j}{i}, \quad (\text{B14}) \]

then \(^{41}\)

\[ g_{ij}(z) = \sum_{r=0}^{\infty} n_{ij} z^r = \prod_{k=1}^{\infty} \frac{1}{1-z^k}. \quad (\text{B15}) \]

This may be seen in a simple way by first introducing, \( h(x,z) = [1/(1-xz)] \prod_{k=1}^{\infty} 1/(1-z^k) \), so that, in a series expansion, \( h(x,z) = \sum_{k,j} \lambda_k \lambda_j | \lambda \rangle \). We then have \( \lim_{x \to 1} (1/x^r) \{1/(1-xz)\} h(x,z) = g_{ij}(z) \) using \( (1/x^r) (d^r/dx^r) [1/(1-xz)] = z^r/(1-xz)^{r+1} \).

The simplest case is for \( s=t=0 \), whereby for (B6) we find, using (B12) along with (B15),

\[ F_{F,00}(z) = g_{00}(z) = \prod_{k=1}^{\infty} \frac{1}{1-z^k}, \quad (\text{B16}) \]

and thus have, giving (7.12),

\[ N_{F,rr} = n_{r00} = p(r). \quad (\text{B17}) \]

Similarly, for (B6) using (B12) along with (B15),

\[ F_{F,10}(z) = F_{F,01}(z) = g_{11}(z) - g_{00}(z) = \frac{2z - 1}{1-z} \prod_{k=1}^{\infty} \frac{1}{1-z^k}, \quad (\text{B18}) \]

and so, agreeing with (7.19),

\[ N_{F,r-2} \rightarrow r = N_{F,r-2} = n_{r11} - p(r) = \sum_{j=0}^{r-1} p(j) - p(r). \quad (\text{B19}) \]

Similarly, for (B6) using (B12) along with (B15),

\[ F_{F,11}(z) = 2g_{21}(z) - g_{11}(z) + g_{00}(z) = \frac{4z^2 - 3z + 1}{(1-z)^2} \prod_{k=1}^{\infty} \frac{1}{1-z^k}, \quad (\text{B20}) \]

and so, giving (7.20),

\[ N_{F,r-2} \rightarrow 2r = 2n_{r21} - n_{r11} + p(r). \quad (\text{B21}) \]

Notice that by a very similar argument, it may be shown that from (B12) with (B13), for \( t = 0 \),

\[ F_{F,t0}(z) = F_{F,0t}(z) = \sum_{\rho_1, \ldots, \rho_t = 0} (z^\rho \delta_{\rho_1 \cdots \rho_t \lambda_1 S_\rho} - z^{\rho+1} \delta_{\rho_1 \cdots \rho_t, \lambda_1 S_\rho+1}) \prod_{j=1}^{\infty} \frac{1}{1-z^j} \prod_{k=1}^{\infty} \frac{1}{1-z^k}, \quad (\text{B22}) \]

enabling determination of \( N_{F,r-2} r = 2r \), for \( st = 0 \).

2. The supergravity case

In this case we consider, due to (7.23) with (7.24),
\[ f_S(u,x,y) = \prod_{j=1}^{\infty} \prod_{k,l=0}^{\infty} \frac{1}{1 - u^j z^{2k-j} y^{2l-j}} = \sum_{r,s,t=0} N_{S,rs} u^r \chi_s(x) \chi_t(y), \]  

(B23)

and we use (B7) and (B6) to determine generating functions for low \( s,t \).

For application to (B6), we have the leading terms

\[
\begin{align*}
 f_S(zxy,x,y) &= \prod_{k=1}^{\infty} \frac{1}{1 - z^k} + \cdots , \\
 f_S(zx/y,x,y) &= \prod_{k=1}^{\infty} \frac{1}{(1 - z^k)(1 - z^k y^2)} + \cdots , \\
 f_S(zy/x,x,y) &= \prod_{k=1}^{\infty} \frac{1}{(1 - z^k)(1 - z^k x^2)} + \cdots , \\
 f_S(z/(xy),xy) &= \prod_{k=1}^{\infty} \frac{1}{(1 - z^k)(1 - z^k x^2)(1 - z^k y^2)(1 - z^k x^2 y^2)} + \cdots ,
\end{align*}
\]

(B24)

where \( \cdots \) denotes terms that in a series expansion in \( x,y \) involve powers \( x^{2a}, y^{2b}, a,b \in \mathbb{Z}, a,b \neq 0,-1 \), that do not contribute below.

We have, from (B24),

\[
\begin{align*}
 \frac{1}{(2\pi i)^2} \oint \oint x y f_S(zxy,x,y) dx dy &= 0, \\
 \frac{1}{(2\pi i)^2} \oint \oint x f_S(zx/y,x,y) dx dy &= 0, \\
 \frac{1}{(2\pi i)^2} \oint \oint y f_S(zy/x,x,y) dx dy &= 0, \\
 \frac{1}{(2\pi i)^2} \oint \oint \frac{1}{x} f_S(z/(xy),xy) dx dy &= \prod_{k=1}^{\infty} \frac{1}{1 - z^k},
\end{align*}
\]

(B25)

so that for (B6) we have, from (B7),

\[ F_{S,00}(z) = g_{00}(z). \]

(B26)

agreeing with (B16). Hence, \( N_{S,rr} = N_{F,rr} \) in (B17), leading to the first equation in (7.25).

Similarly, using (B24) and

\[
\prod_{k=1}^{\infty} \frac{1}{1 - t_k^2} = 1 + \frac{zt}{1 - z} + O(t^2, z^2),
\]

(B27)

we have

\[
\frac{1}{(2\pi i)^2} \oint \oint \frac{x}{y} f_S(zxy,x,y) dx dy = 0,
\]
\[
\frac{1}{(2\pi)^2} \int \int y f_S(z, x, y, dx, dy) = 0,
\]

\[
\frac{1}{(2\pi)^2} \int \int \frac{1}{x} f_S(z, x, y, dx, dy) = \prod_{k=1}^{\infty} \frac{1}{1 - z^k},
\]

\[
\frac{1}{(2\pi)^2} \int \int \frac{y}{x} f_S(z, x, y, dx, dy) = \frac{z}{1 - z} \prod_{k=1}^{\infty} \frac{1}{1 - z^k},
\]

so that for (B6), from (B7),

\[
F_{S,01}(z) = g_{11}(z) - 6_{00}(z),
\]

agreeing with (B20), so that \(N_{S,rr-2} = N_{E,rr-2}\) in (B21). It is easy to show that \(F_{S,10}(z) = F_{S,01}(z)\) so that \(N_{S,rr-2} = N_{S,rr-2}\). Thus we have (7.25).

Similarly, using (B24) with (B27),

\[
\frac{1}{(2\pi)^2} \int \int \frac{1}{x} f_S(z, x, y, dx, dy) = \prod_{k=1}^{\infty} \frac{1}{1 - z^k},
\]

\[
\frac{1}{(2\pi)^2} \int \int \frac{y}{x} f_S(z, x, y, dx, dy) = \frac{z}{1 - z} \prod_{k=1}^{\infty} \frac{1}{1 - z^k},
\]

\[
\frac{1}{(2\pi)^2} \int \int \frac{y}{x} f_S(z, x, y, dx, dy) = \frac{z}{1 - z} \prod_{k=1}^{\infty} \frac{1}{1 - z^k},
\]

\[
\frac{1}{(2\pi)^2} \int \int \frac{y}{x} f_S(z, x, y, dx, dy) = \left( \frac{z}{1 - z} + \frac{z^2}{(1 - z)^2} \right) \prod_{k=1}^{\infty} \frac{1}{1 - z^k},
\]

so that for (B6), from (B7),

\[
F_{S,11}(z) = g_{21}(z) - 6_{11}(z) + g_{00}(z) = \frac{3z^2 - 3z + 1}{(1 - z)^2} \prod_{k=1}^{\infty} \frac{1}{1 - z^k},
\]

giving, from (B15),

\[
N_{S,rr-2r-2} = n_{r21} - n_{r11} + p(r),
\]

leading to (7.26).

**APPENDIX C: COUNTING MULTITRACES OF COMMUTING MATRICES**

In Ref. 34 it was shown that a basis for multiple traces of \(J\) commuting \(N \times N\) matrices \(X_k, k=1, \ldots, J\), is provided by

\[
\text{Tr} U_1 \text{Tr} U_2 \cdots \text{Tr} U_N, \quad U_i = \prod_{j=1}^{n} Y_{ij}, \quad Y_{ij} \in \{X_k, k=0, \ldots, J\}
\]

for \(n \to \infty\), where \(X_0 = \text{I}\), the identity matrix, which, of course, also commutes with \(X_k, k =1, \ldots, J\), so that initially we treat it on an equal footing with the other matrices. (C1) is linearly independent up to the action of the automorphism symmetry group described below.
Each $U_i$ has an automorphism symmetry $S_n$ since $U_i = \Pi_{j=1}^n Y_{ij} = (U_i)^\sigma = \Pi_{j=1}^n Y_{i\sigma(j)}$ for every $\sigma \in S_n$ (since $X_i$ commute). Hence each $\text{Tr } U_i$ corresponds to a graph with $S_n$ symmetry $K_n$, where $Y_{ij}$ to corresponds to the $j$th vertex. Here, $K_n$ is taken to be the complete graph on $n$ vertices.\footnote{The action of $(S_n)^N \times S_N$ on $C_n$ is given by \( \text{Tr}(U_{(a)}^{(1)})^{n_1} \cdots \text{Tr}(U_{(a)}^{(N)})^{n_N} \cdot \text{Tr}(U_{(b)}^{(1)})^{n_1} \cdots \text{Tr}(U_{(b)}^{(N)})^{n_N} \)} Particular $\text{Tr } U_i$ corresponds to a automorphism of the vertices of $K_n$, in colors $c_k$, $k = 0, \ldots, J$, with the exact value of $Y_{ij} \in \{X_k, k = 0, \ldots, J\}$ corresponding to the color of the $j$th vertex.

$\text{Tr } U_1 \cdot \text{Tr } U_2 \cdots \text{Tr } U_N$ corresponds to $N$ copies of $K_n$, denoted $K_n^N$. This graph has a corresponding automorphism group, in graph theory known as the wreath product group, in this case given by the semidirect product $(S_n)^N \times S_N$. For this group, for $\sigma = (\sigma_1, \ldots, \sigma_n) \in (S_n)^N$, $\tau \in S_N$, and defining $\sigma' = (\sigma_1^{-1}, \ldots, \sigma_n^{-1})$ for $\sigma = (\sigma_1, \ldots, \sigma_n) \in (S_n)^N$, group multiplication is defined by $(\sigma, \tau) = (\sigma_\tau, \tau')$ so that, with $1_{(S_n)^N \times S_N} = (1_{S_n}, 1_{S_n})$, then the inverse $(\sigma, \tau)^{-1} = ((\sigma_\tau)^{-1}, \tau^{-1})$.

The action of $(S_n)^N \times S_N$ on (C1) is given by $(\text{Tr} U_{(a)}^{(1)})^{n_1} \cdots \text{Tr}(U_{(a)}^{(N)})^{n_N} \cdot \text{Tr}(U_{(b)}^{(1)})^{n_1} \cdots \text{Tr}(U_{(b)}^{(N)})^{n_N}$, with elements of the basis related by such a group transformation being identical. To achieve a linearly independent basis, it is necessary to divide out by this automorphism symmetry.

Generically, depending on the graph $G$, with $n$ vertices, having automorphism group symmetry $\Gamma$, the generating function for the number $N_{n_0, \ldots, n_j}$ of colorings of the vertices of $G$, in $n_k$ colors $c_k$, $k = 0, \ldots, J$, with $\sum k n_k = n$, may be written as

$$C_\Gamma(x_0, \ldots, x_J) = \sum_{n_0, \ldots, n_J = 0}^{n_0 + \cdots + n_J = n} N_{n_0, \ldots, n_J} x_0^{n_0} \cdots x_J^{n_J}. \tag{C2}$$

We may use the Polya enumeration theorem to determine the generating function for the number of colorings of $K_n^N$, equivalent to the number of independent multiple trace operators, (C1), subject to the automorphism symmetry $(S_n)^N \times S_N$, in terms of cycle indices for the symmetric permutation group.

The cycle index for the symmetric permutation group $S_n$ is given by, with the notation of Sec. VI,

$$Z_{S_n}(s_1, \ldots, s_n) = \frac{1}{n!} \prod_{j=1}^n \frac{e^{s_j x_j}}{j!} = \frac{1}{2\pi i} \oint_{|z| = 1} \frac{dz}{z^{n+1}} \exp \left( \sum_{j=1}^n \frac{1}{j} z^j s_j \right), \tag{C3}$$

and the generating function for the number of colorings of $K_n$ is, by the Polya enumeration theorem,

$$C_{S_n}(x_0, x_1, \ldots, x_J) = Z_{S_n}(s_1, \ldots, s_n), \quad s_j = p_j(x_0, \ldots, x_J),$$

$$= \frac{1}{2\pi i} \oint_{|z| = 1} \frac{dz}{z^{n+1}} \frac{1}{(1 - z x_0) \cdots (1 - z x_J)}. \tag{C4}$$

where $p_j(x) = \sum_{k=0}^J x^k$, a power symmetric polynomial. Similarly, for $K_n^N$, with automorphism group $(S_n)^N \times S_N$, the generating function for the number of colorings is given by, using the Polya enumeration theorem applied to wreath product groups,\footnote{The action of $(S_n)^N \times S_N$ on the generating function is given by $(\text{Tr} U_{(a)}^{(1)})^{n_1} \cdots \text{Tr}(U_{(a)}^{(N)})^{n_N} \cdot \text{Tr}(U_{(b)}^{(1)})^{n_1} \cdots \text{Tr}(U_{(b)}^{(N)})^{n_N}$, with elements of the basis related by such a group transformation being identical. To achieve a linearly independent basis, it is necessary to divide out by this automorphism symmetry.}

$$C_{(S_n)^N \times S_N}(x_0, x_1, \ldots, x_J) = Z_{S_n}(s_1, \ldots, s_n), \tag{C5}$$

where

$$\bar{s}_j = Z_{S_n}(s_1, \ldots, s_n) = C_{S_n'}(x_0', \ldots, x_J'), \quad s_{ji} = p_j(x_0', x_1', \ldots, x_J'). \tag{C6}$$

In order that contributions to $\lim_{n \to \infty} C_{S_n}(x_0, x_1, \ldots, x_J)$ from finite numbers of $X_k$, $k = 1, \ldots, J$ in $\text{Tr } U_i$ not vanish then we must also take $x_0 \to 1$. Following from (C4),
\[
\lim_{n \to \infty} C_{s_n}(1, x_1, \ldots, x_j) = \lim_{s \to -1} \left( 1 - s \right) \sum_{n=0}^{\infty} s^n C_{s_n}(1, x_1, \ldots, x_j) = \frac{1}{(1 - x_1) \cdots (1 - x_j)}. \tag{C7}
\]

Thus the generating function for the number of distinct basis elements of the form (C1) as \( n \to \infty \) is

\[
\lim_{n \to \infty} C_{s_n}(\vec{s}^N_1, \ldots, \vec{s}^N_N) = Z_{s_n}(\vec{s}_1, \ldots, \vec{s}_N), \quad \vec{s}_i = \frac{1}{(1 - x_i^1) \cdots (1 - x_i^j)},
\]

which matches exactly with the chiral ring partition function obtained using the plethystic approach. 33

9. Note that there is some overlap in the discussion here with some related issues explored in Ref. 10 particularly with regard to long multiplet decomposition formulas. \( \bar{N} \) would require some modification of the analysis here particularly with regard to possible short/semishort multiplets. While straightforward, an extension to odd \( \bar{N} \) is avoided here to ensure notation is not overly cumbersome.
15. I thank Troels Harmark for pointing out to me that different sectors, along with the SU(2) x SU(2) case, were also briefly considered in Ref. 5. For \( N = 6 \) superconformal Chern Simons the \( N = 6 \) superconformal Chern Simons th.
16. The conventions used here for the gamma matrices are that \( \gamma^a = 1 \), \( \gamma^a = i \), \( \gamma^a = \sigma_3 \), in terms of Pauli matrices \( \sigma_1, \sigma_2, \sigma_3 \), so that \( \gamma^a \) are real, symmetric matrices. We take \( \gamma^a = e^{-\gamma} \gamma^a e^{\gamma} \), where \( e^{\gamma} = 1 \), \( e^{-\gamma} = 1 \), \( e^{-\gamma} = 1 \), \( e^{-\gamma} = 1 \), \( e^{-\gamma} = 1 \), \( e^{-\gamma} = 1 \). We thus have \( \gamma^a \gamma^a = 2 \gamma^a \gamma^a = 2 \gamma^a \gamma^a = 2 \gamma^a \gamma^a \) and the completeness relation \( \gamma^a \gamma^b = \delta^a_b \). 17. \( E_{sh}^{i+} E_{sh}^{-} \) correspond to the positive/negative roots \( e_{n+} = e_{n-} = e_n, m < n \), where \( e_{n+}, e_{n-} \) are usual \( R^N \) orthonormal vectors. A linearly independent basis for raising/lowering operators is \( \{ E_{sh}^{i+}, E_{sh}^{-}\} \) for \( n = 1, \ldots, N - 1 \), corresponding to the positive/negative simple roots.
19. In the basis (2.3)–(2.6), all the generators of \( \text{Osp}(2N|4) \) are Hermitian, apart from \( D \) and \( M^\rho_{\beta} \) which are anti-Hermitian.
20. In order to impose the physically necessary unitarity conditions arising from a scalar product defined by the two point function for the conformal fields, it is sufficient to perform a similarity transformation, see Ref. 10, for example, where, in particular, \( D \) and \( M_{\beta}^\rho \) become Hermitian so that \( M^\rho_{\beta} \) generates \( \text{SO}(3) \), rather than \( \text{SO}(2,1) \), and \( D \) has real eigenvalues which are required to be positive except for the trivial representation. Of course, such a similarity transformation does not affect the shortening conditions derived here.
23. As discussed in Refs. 15 and 16, the action of the Weyl symmetrizer on the Verma module character corresponds to quotienting out null states in the Verma module, (3.3) for \( V_{(s,\lambda,\rho)} \to \lambda^M_{(s,\lambda,\rho)} \), to obtain the reducible module, (3.3) for \( V_{(s,\lambda,\rho)} \to \lambda^M_{(s,\lambda,\rho)} \).
It is easily checked that for \( N=3, m=1 \), then (5.18) for \((u,x,y_2,y_3)\) → \((-1,-x,y_1,y_2)\) matches with the index for fundamental fields computed in Sec. II of Ref. 6.

No signs are necessary for the BPS limits as the \( x \) dependence drops out. For the traces in (5.9) and (5.21), as the \( J_m, J_r \) eigenvalues are zero for states contributing to subalgebra characters, in the limit as \( \delta \to 0 \), then \( \mu \to iu, \bar{s} \to \bar{s}/\delta^2 \) introduces a factor of \( \alpha^{-25} \) in the trace. For \( \alpha=-1 \) this is equivalent to a sign change in the original variable \( x \) before the \( \delta \to 0 \) limit is taken.

This may be seen in a simple way by first introducing, \( h(x,z)=\prod_{\text{odd } j} \frac{1}{1-1-z^j} \), so that, in a series expansion, \( h(x,z)=\sum_{j} (1/j!) z^j x^j \). We then have \( \lim_{\delta \to 0} (1/j!) ((d^j/d\bar{z}^i) h(x,z)=g_{ij}(z) \) using \( (1/j!) ((d^j/d\bar{z}^i) [1/(1-x^j)^{\delta^2}]=z^i/(1-x^j)^{\delta^2} \).