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Long Multiplication by Instruction Sequences with Backward Jump Instructions

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Abstract. For each function on bit strings, its restriction to bit strings of any given length can be computed by a finite instruction sequence that contains only instructions to set and get the content of Boolean registers, forward jump instructions, and a termination instruction. Backward jump instructions are not necessary for this, but instruction sequences can be significantly shorter with them. We take the function on bit strings that models the multiplication of natural numbers on their representation in the binary number system to demonstrate this by means of a concrete example. The example is reason to discuss points concerning the halting problem and the concept of an algorithm.

Keywords: single-pass instruction sequence, backward jump instruction, bit string, long multiplication, algorithm, halting problem.


1 Introduction

In [5], an approach to non-uniform complexity is presented which is based on the simple idea that, for each function on bit strings, its restriction to bit strings of any given length can be computed by an instruction sequence that contains only instructions to set and get the content of Boolean registers, forward jump instructions, and a termination instruction. It is among other things shown that a function on bit strings whose result is a bit string of length 1 belongs to P/poly iff it can be computed by polynomial-length instruction sequences of this kind. In [4], instruction sequences are considered which contain backward jump instructions in addition to the above-mentioned instructions. It is among other things shown that a function on bit strings whose result is a bit string of length 1 belongs to PSPACE/poly iff it can be computed by polynomial-length instruction sequences of this latter kind.

It is known that NP ⊆ PSPACE/poly (see e.g. [14]). Under the assumption that the reasonable complexity theoretic conjecture that NP ⊈ P/poly (see e.g. [17]) is right, it then follows that there exists a function on bit strings that can be computed by polynomial-length instruction sequences with backward jump instructions and cannot be computed by polynomial-length instruction sequences without backward jump instructions. With this it remains among other things
unanswered whether there exists a function on bit strings that can be computed by linear-length instruction sequences with backward jump instructions while it is commonly assumed that the function concerned cannot be computed by linear-length instruction sequences without backward jump instructions. In this paper, we answer this question in the affirmative by means of a concrete example from binary arithmetic.

In [7], a description is given of instruction sequences of the kind used in [5] that compute the hash function SHA-256 according to the algorithm whose pseudo-code description serves as the definition of SHA-256 in the Secure Hash Standard [26]. In [6], a description is given of instruction sequences of the kind used in [5] that compute the function on bit strings that models the multiplication of natural numbers on their binary representation according to the Karatsuba multiplication algorithm [15,16] and a description is given of instruction sequences of this kind that compute this function according to the standard multiplication algorithm, which is known as the long multiplication algorithm. Thus, mathematically precise alternatives are provided to the natural language and pseudo-code descriptions of these algorithms found in the literature on them.

In [6], the descriptions are further used to determine lower and upper estimates for the length of the representation in the binary number system of natural numbers at which the Karatsuba multiplication algorithm becomes more efficient than the long multiplication algorithm. As expected, the function on bit strings that models the multiplication of natural numbers on their representation in the binary number system can be computed according to the long multiplication algorithm by quadratic-length instruction sequences without backward jump instructions. Although it would not make the algorithm more efficient, it would be of practical value if this could be reduced to linear-length instruction sequences with backward jump instructions. At first sight, this seems impossible without the addition of an indirect addressing mechanism for Boolean registers. However, using a minor variant of the long multiplication algorithm, we will show in this paper that such a reduction is possible without this addition.

It is customary that computing practitioners phrase their explanations of issues concerning programs from an empirical perspective such as the perspective that a program is in essence an instruction sequence. An attempt to approach the semantics of programming languages from this perspective is made in [2]. The groundwork for the approach is an algebraic theory of single-pass instruction sequences, called program algebra, and an algebraic theory of mathematical objects that represent the behaviours produced by instruction sequences under execution, called basic thread algebra.

As a continuation of this work on an approach to programming language semantics, (a) the notion of an instruction sequence was subjected to systematic and precise analysis using the groundwork laid earlier and (b) selected issues relating to well-known subjects from the theory of computation and the area of computer architecture were rigorously investigated thinking in terms of instruc-

\footnote{In [2], basic thread algebra is introduced under the name basic polarized process algebra.}
tion sequences (see e.g. [4]). As in the work referred to above, the work presented in this paper is carried out in the setting of program algebra. Different from usual in the work referred to above, but as in the work presented in [6,7], the accent is this time on a practical issues such as efficiency of algorithms and compactness of instruction sequences.

This paper is organized as follows. First, we survey program algebra and the particular fragment and instantiation of it that is used in this paper (Section 2). Next, we describe how we deal with \( n \)-bit words by means of Boolean registers (Section 3) and how we compute the basic operations on \( n \)-bit words that are used in the multiplication algorithms (Section 4). Then, we show that the function that models the multiplication of natural numbers on their representation in the binary number system can be computed according to a minor variant of the long multiplication algorithm by quadratic-length instruction sequences without backward jump instructions and by linear-length instruction sequences with backward jump instructions (Section 5). After that, we discuss two points, concerning the halting problem and the concept of an algorithm, which were raised by the preceding material (Sections 6 and 7). Finally, we make some concluding remarks (Section 8).

The preliminaries to the work presented in this paper are the same as the preliminaries to the work presented in [6,7], which are in turn a selection from the preliminaries to the work presented in [5]. For this reason, there is some text overlap with those papers. The preliminaries concern program algebra. We only give a brief summary of program algebra. A comprehensive introduction, including examples, can among other things be found in [4].

2 Program Algebra

In this section, we present a brief outline of PGA (ProGram Algebra) and the particular fragment and instantiation of it that is used in the remainder of this paper. A mathematically precise treatment can be found in [5].

The starting-point of PGA is the simple and appealing perception of a sequential program as a single-pass instruction sequence, i.e. a finite or infinite sequence of instructions of which each instruction is executed at most once and can be dropped after it has been executed or jumped over.

It is assumed that a fixed but arbitrary set \( \mathcal{A} \) of basic instructions has been given. The intuition is that the execution of a basic instruction may modify a state and produces a reply at its completion. The possible replies are 0 and 1. The actual reply is generally state-dependent. Therefore, successive executions of the same basic instruction may produce different replies. The set \( \mathcal{A} \) is the basis for the set of instructions that may occur in the instruction sequences considered in PGA. The elements of the latter set are called primitive instructions. There are five kinds of primitive instructions, which are listed below:

- for each \( a \in \mathcal{A} \), a plain basic instruction \( a \);
- for each \( a \in \mathcal{A} \), a positive test instruction \( +a \);
- for each \( a \in \mathcal{A} \), a negative test instruction \( -a \);
– for each \( l \in \mathbb{N} \), a forward jump instruction \( \#l \);
– a termination instruction \( ! \).

We write \( \mathcal{I} \) for the set of all primitive instructions.

On execution of an instruction sequence, these primitive instructions have the following effects:

– the effect of a positive test instruction \(+a\) is that basic instruction \( a\) is executed and execution proceeds with the next primitive instruction if 1 is produced and otherwise the next primitive instruction is skipped and execution proceeds with the primitive instruction following the skipped one — if there is no primitive instruction to proceed with, inaction occurs;
– the effect of a negative test instruction \(-a\) is the same as the effect of \(+a\), but with the role of the value produced reversed;
– the effect of a plain basic instruction \( a\) is the same as the effect of \(+a\), but execution always proceeds as if 1 is produced;
– the effect of a forward jump instruction \( \#l\) is that execution proceeds with the \( l\)th next primitive instruction of the instruction sequence concerned — if \( l\) equals 0 or there is no primitive instruction to proceed with, inaction occurs;
– the effect of the termination instruction \( !\) is that execution terminates.

To build terms, PGA has a constant for each primitive instruction and two operators. These operators are: the binary concatenation operator \( ;\) and the unary repetition operator \( \omega\). We use the notation \( \overset{n}{\underset{i=0}{\Pi}} P_i\), where \( P_0, \ldots, P_n \) are PGA terms, for the PGA term \( P_0 ; \ldots ; P_n \). We also use the notation \( P^n\). For each PGA term \( P \) and \( n > 0 \), \( P^n\) is the PGA term defined by induction on \( n\) as follows: \( P^1 = P\) and \( P^{n+1} = P ; P^n\).

The instruction sequences that concern us in the remainder of this paper are the finite ones, i.e. the ones that can be denoted by closed PGA terms in which the repetition operator does not occur. Moreover, the basic instructions that concern us are instructions to set and get the content of Boolean registers. More precisely, we take the set

\[
\{ \text{in}:i.\text{get} \mid i \in \mathbb{N}^+ \} \cup \{ \text{out}:i.\text{set}:b \mid i \in \mathbb{N}^+ \land b \in \{0, 1\} \} \\
\cup \{ \text{aux}:i.\text{get} \mid i \in \mathbb{N}^+ \} \cup \{ \text{aux}:i.\text{set}:b \mid i \in \mathbb{N}^+ \land b \in \{0, 1\} \}
\]

as the set \( \mathfrak{A} \) of basic instructions.

Each basic instruction consists of two parts separated by a dot. The part on the left-hand side of the dot plays the role of the name of a Boolean register and the part on the right-hand side of the dot plays the role of a command to be carried out on the named Boolean register. For each \( i \in \mathbb{N}^+\):

– \( \text{in}:i\) serves as the name of the Boolean register that is used as \( i\)th input register in instruction sequences;
– \( \text{out}:i\) serves as the name of the Boolean register that is used as \( i\)th output register in instruction sequences;
– aux:i serves as the name of the Boolean register that is used as ith auxiliary register in instruction sequences.

On execution of a basic instruction, the commands have the following effects:

– the effect of get is that nothing changes and the reply is the content of the named Boolean register;
– the effect of set:0 is that the content of the named Boolean register becomes 0 and the reply is 0;
– the effect of set:1 is that the content of the named Boolean register becomes 1 and the reply is 1.

We are also interested in the extension of PGA with, for each \( l \in \mathbb{N} \), a backward jump instruction \( \#l \) as additional primitive instruction. On execution of an instruction sequence, the effect of a backward jump instruction \( \#l \) is that execution proceeds with the \( l \)th previous primitive instruction of the instruction sequence concerned — if \( l \) equals 0 or there is no primitive instruction to proceed with, inaction occurs. We write PGA_{bj} for PGA with these additional primitive instructions.

Regarding the behaviours produced by finite instruction sequences with backward jump instructions under execution, we refer to the treatment of C, which is a variant of PGA, in [8]. The fragment of PGA_{bj} without the repetition operator coincides with the fragment of C without backward instructions other than backward jump instructions.

Let \( n, m \in \mathbb{N} \), let \( f : \{0, 1\}^n \rightarrow \{0, 1\}^m \), and let \( X \) be a finite instruction sequence that can be denoted by a closed PGA or PGA_{bj} term in the case that \( \mathfrak{A} \) is taken as specified above. Then \( X \) computes \( f \) if there exists a \( k \in \mathbb{N} \) such that, for all \( b_1, \ldots, b_n \in \{0, 1\} \), if \( X \) is executed in an environment with \( n \) input registers, \( m \) output registers, and \( k \) auxiliary registers, the content of the input registers with names in:1, \ldots, in:n are \( b_1, \ldots, b_n \) when execution starts, and the content of the output registers with names out:1, \ldots, out:m are \( b'_1, \ldots, b'_m \) when execution terminates, then \( f(b_1, \ldots, b_n) = b'_1, \ldots, b'_m \).

Let \( f : \{0, 1\}^* \rightarrow \{0, 1\}^* \) be such that for all \( \beta, \beta' \in \{0, 1\}^* \), \( \text{len}(\beta) = \text{len}(\beta') \) implies \( \text{len}(f(\beta)) = \text{len}(f(\beta')) \), and let \( F \subseteq \{ g : \mathbb{N} \rightarrow \mathbb{N} \} \). Then \( f \) can be computed by \( F \)-length instruction sequences if there exists a \( g \in F \) such that, for all \( n \in \mathbb{N} \), there exists a finite instruction sequence \( X \) that can be denoted by a closed PGA or PGA_{bj} term such that \( X \) computes the restriction of \( f \) to \( \{0, 1\}^n \) and \( \text{len}(X) \leq g(n) \). We write polynomial-length instead of \( F \)-length if \( F \) is the set of all polynomial functions \( g : \mathbb{N} \rightarrow \mathbb{N} \). The phrases quadratic-length and linear-length are used similarly.

3 Dealing with \( n \)-Bit Words

This section is concerned with dealing with bit strings of length \( n \) by means of Boolean registers. It contains definitions which facilitate the description of instruction sequences that compute the function on bit strings that models the
multiplication of natural numbers on their representation in the binary number system according to the long multiplication algorithm or a minor variant thereof. In the sequel, bit strings of length \( n \) will mostly be called \( n \)-bit words. The prefix “\( n \)-bit” is left out if \( n \) is irrelevant or clear from the context.

Let \( \kappa : i \) (\( \kappa \in \{ \text{in, out, aux} \}, i \in \mathbb{N}^+ \)) be the name of a Boolean register. Then \( \kappa \) and \( i \) are called the kind and number of the Boolean register. Successive Boolean registers are Boolean registers of the same kind with successive numbers. Words are stored by means of Boolean registers such that the successive bits of a stored word are the content of successive Boolean registers.

Henceforth, the name of a Boolean register will mostly be used to refer to the Boolean register in which the least significant bit of a word is stored. Let \( \kappa : i \) and \( \kappa' : i' \) be the names of Boolean registers and let \( n \in \mathbb{N}^+ \). Then we say that \( \kappa : i \) and \( \kappa' : i' \) lead to partially coinciding \( n \)-bit words if \( k = k' \) and \( |i - i'| < n \).

The words that represent the two natural numbers whose product is to be computed are stored in advance of the whole computation in input registers, starting with the input register with number 1. It is convenient to have available, for each \( n > 0 \), the names \( I^{(n)}_1 \) and \( I^{(n)}_2 \) for the input registers in which the least significant bit of these words are stored. The word that represents the product is stored before the end of the whole computation in output registers, starting with the output register with number 1. It is convenient to have available, for each \( n > 0 \), the name \( O^{(n)} \) for the output register in which the least significant bit of this word is stored.

A number of words that represent intermediate values computed are temporarily stored during the whole computation in auxiliary registers, starting with the auxiliary register with number 1. It is convenient to have available, for each \( n > 0 \), names \( T^{(n)}_1 \), \( T^{(n)}_2 \), \ldots for auxiliary registers in which the least significant bit of these words are stored. Moreover, it is convenient to have available the name \( c \) for the auxiliary register that contains the carry bit that is repeatedly stored when computing the function on bit strings that models the addition of natural numbers on their representation in the binary number system.

Therefore, we define for each \( n > 0 \) and \( i > 0 \):

\[
I^{(n)}_1 \triangleq \text{in:1},
I^{(n)}_2 \triangleq \text{in:k} \quad \text{where } k = n + 1,
O^{(n)} \triangleq \text{out:1},
T^{(n)}_i \triangleq \text{aux:k} \quad \text{where } k = 2 \cdot n \cdot (i - 1) + 2,
c \triangleq \text{aux:1}.
\]

For each \( n > 0, I^{(n)}_1, I^{(n)}_2, O^{(n)}, T^{(n)}_1, T^{(n)}_2, T^{(n)}_3, \) and \( T^{(n)}_4 \) are the names that will be used in Section 5 to define instruction sequences that compute the function on bit strings of length \( n \) that models the multiplication of two natural numbers less than \( 2^n \) on their representation in the binary number system. Moreover, we will write \( I^{(n)}_i[j] \) (\( 0 \leq j < n \)) for \( \text{in:k} \) where \( k = (i - 1) \cdot n + j + 1 \) and \( T^{(n)}_i[j] \) (\( 0 \leq j < 2 \cdot n \)) for \( \text{aux:k} \) where \( k = 2 \cdot (i - 1) \cdot n + j + 2 \).
4 Computing Operations on $n$-Bit Words

This section is concerned with computing operations on bit strings of length $n$. It contains definitions which facilitate the description of instruction sequences that compute the function on bit strings that models the multiplication of natural numbers on their representation in the binary number system according to the long multiplication algorithm or a minor variant thereof.

Henceforth, we will write $\beta\beta'$, where $\beta$ and $\beta'$ are bit strings, for the concatenation of $\beta$ and $\beta'$. In other words, we will use juxtaposition for concatenation. Moreover, we will use the bit string notation $b^n$. For $n > 0$, the bit string $b^n$, where $b \in \{0, 1\}$, is defined by induction on $n$ as follows: $b^1 = b$ and $b^{n+1} = b^n b$.

The basic operations on words that are relevant to the different multiplication algorithms are test on nonzero, decrement by one, shift left over $\mathbb{N}$, and addition on $n$-bit words ($n > 0$). For these operations, we define parameterized instruction sequences computing them in case the parameters are properly instantiated (see below):

$\text{ISNZ}_n(s;k) \triangleq$
- $\gamma_{i=0}^{n-1} (+s;k+i . \text{get} ; #2 ; #3 ; #1)$,

$\text{DEC}_n(s;k;d) \triangleq$
- $\gamma_{i=0}^{n-1} (-s;k+i . \text{get} ; #3 ; d:d+i . \text{set} : 0 ; #5 ; d:d+i . \text{set} : 1 ; #1 ; #1 ; #1)$,

$\text{SHL}_n^m(s;k;d:l) \triangleq$
- $\gamma_{i=0}^{n-1-m} (+s;k+n-1-m-i . \text{get} ; #2 ; +d:d+n-1-i . \text{set} : 0 ; d:d+n-1-i . \text{set} : 1)$,
- $\gamma_{i=0}^{m-1} (d:d+m-1-i . \text{set} : 0)$,

$\text{SHR}_n^m(s;k;d:l) \triangleq$
- $\gamma_{i=0}^{n-1-m} (+s;k+m+i . \text{get} ; #2 ; +d:d+i . \text{set} : 0 ; d:d+i . \text{set} : 1)$,
- $\gamma_{i=0}^{m-1} (d:d+n-m+i . \text{set} : 0)$,

$\text{ADD}_n(s_1;k_1,s_2;k_2;d:l) \triangleq$
- $c . \text{set} : 0$;
- $\gamma_{i=0}^{n-1} (+s_1;k_1+i . \text{get} ; #4 ; +s_2;k_2+i . \text{get} ; #7 ; #9 ; +s_2;k_2+i . \text{get} ; #10$;
  +c . \text{get} ; #10 ; #16 ; +c . \text{get} ; #7 ; #13 ; +c . \text{get} ; #11 ; #9 ; +c . \text{get} ; #4$;
  $d:d+i . \text{set} : 0 ; c . \text{set} : 1 ; #6 ; d:d+i . \text{set} : 1 ; c . \text{set} : 1 ; #3$;
  $+d:d+i . \text{set} : 0 ; d:d+i . \text{set} : 1)$,

where $s, s_1, s_2$ range over $\{\text{in, aux}\}$, $d$ ranges over $\{\text{aux, out}\}$, and $k, k_1, k_2, l$ range over $\mathbb{N}^+$. For each of these parameterized instruction sequences except the first one, all but the last parameter correspond to the operands of the operation concerned and the last parameter corresponds to the result of the operation concerned. The intended operations are computed provided that the instantiation of the last parameter and the instantiation of none of the other parameters
lead to partially coinciding n-bit words. In this paper, this condition will always be satisfied. No result is stored on execution of ISNZ\(_n\). Instead, the first primitive instruction following ISNZ\(_n\) is skipped if the test on nonzero fails.

Transferring n-bit words (n > 0) is also relevant to multiplication algorithms. For this, we define parameterized instruction sequences as well. By one the successive bits in a constant n-bit word become the content of n successive Boolean registers and by the other the successive bits in a n-bit word that are the content of n successive Boolean registers become the content of n other successive Boolean registers:

\[
\begin{align*}
SET_n(b_0 \ldots b_{n-1}, d:l) & \triangleq \sum_{i=0}^{n-1} (d:l+i.set:b_i), \\
MOV_n(s:k, d:l) & \triangleq \sum_{i=0}^{n-1} (+s:k+i.get; \#2 ; +d:l+i.set:0 ; d:l+i.set:1),
\end{align*}
\]

where \(b_0, \ldots, b_{n-1}\) range over \(\{0, 1\}\), \(s\) ranges over \(\{\text{in, aux}\}\), \(d\) ranges over \(\{\text{aux, out}\}\), and \(k, l\) range over \(\mathbb{N}^+\). In the case of MOV\(_n\), the intended transfer is performed provided that the instantiation of the last parameter and the instantiation of the first parameter do not lead to partially coinciding n-bit words. In this paper, this condition will always be satisfied.

For convenience’s sake, we define a special case of the parameterized instruction sequences for transferring n-bit words (0 < m < n):

\[
ZPAD^m_n(d:l) \triangleq SET_{n-m}(0^{n-m}, d:l+m),
\]

where \(d\) ranges over \(\{\text{aux, out}\}\) and \(l\) range over \(\mathbb{N}^+\). ZPAD\(_n^m\) is meant for turning a stored m-bit word into a stored n-bit word by zero padding.

The calculation of the lengths of the parameterized instruction sequences defined above is a matter of simple additions and multiplications. The lengths of these instruction sequences are as follows:

\[
\begin{align*}
\text{len}(ISNZ_n(s:k)) & = 3 \cdot n + 1, \\
\text{len}(DEC_n(s:k, d:l)) & = 5 \cdot n + 3, \\
\text{len}(SHL_n^m(s:k, d:l)) & = 4 \cdot n - 3 \cdot m, \\
\text{len}(SHR_n^m(s:k, d:l)) & = 4 \cdot n - 3 \cdot m, \\
\text{len}(ADD_n(s_1:k_1, s_2:k_2, d:l)) & = 26 \cdot n + 1, \\
\text{len}(SET_n(b_0 \ldots b_{n-1}, d:l)) & = n, \\
\text{len}(MOV_n(s:k, d:l)) & = 4 \cdot n, \\
\text{len}(ZPAD_n^m(d:l)) & = n - m.
\end{align*}
\]

Note that the instruction sequences defined in this section do compute the intended operations in case of fully coinciding n-bit words. Slightly shorter instruction sequences are defined for addition on n-bit words and transfer of a stored n-bit word in [7], but those instruction sequences do not compute the intended operations in case of fully coinciding n-bit words.
5 Long Multiplication and Backward Jump Instructions

This section shows that the function on bit strings that models the multiplication of natural numbers on their representation in the binary number system can be computed according to a minor variant of the long multiplication algorithm by quadratic-length instruction sequences without backward jump instructions and by linear-length instruction sequences with backward jump instructions.

We begin with defining instruction sequences without backward jump instructions that compute this function according to the long multiplication algorithm. The additions are done on the fly and the shifts are restricted to one position by shifting the result of all preceding shifts.

We uniformly define instruction sequences $LMUL_n$ $(n > 0)$ by

$$MOV_n(I_1^{(n)}, T_1^{(n)}); ZPAD_n(T_1^{(n)}) ; SET_2^n(0^{2n}, T_2^{(n)});$$

$$\sum_{i=0}^{n-1} (-I_2^{(n)}[i].get; \#l_i ; ADD_n+i+1(T_1^{(n)}, T_2^{(n)}, T_2^{(n)}); SHL_n+i+1(T_1^{(n)}, T_2^{(n)}));$$

$$MOV_2^n(T_2^{(n)}, O^{(n)}); ! ,$$

where

$$l_i = \text{len}(ADD_n+i+1(T_1^{(n)}, T_2^{(n)}, T_2^{(n)})) + 1 = 26 \cdot n + 26 \cdot i + 28 \ (0 \leq i \leq n - 1).$$

Using the property that $\sum_{i=0}^{k} i = (k \cdot (k+1))/2$, we obtain by simple calculations that

$$\text{len}(LMUL_n) = 45 \cdot n^2 + 30 \cdot n + 1 .$$

This means that the function on bit strings that models the multiplication of natural numbers on their representation in the binary number system can be computed by quadratic-length instruction sequences without backward jump instructions if it is computed according to the long multiplication algorithm.

For each bit of the representation of the multiplier, $LMUL_n$ contains a different instruction sequence. This seems to exclude the use of backward jump instructions to obtain linear-length instruction sequences, unless an indirect addressing mechanism for Boolean registers is added. However, there exists a minor variant of the long multiplication algorithm that makes it possible to have the same instruction sequence for each bit of the representation of the multiplier. From the least significant bit of the representation of the multiplier onwards, the algorithm concerned shifts the representation of the multiplier one position to the right after it has dealt with a bit. In this way, the next bit remains the least significant one throughout.

We proceed with defining instruction sequences without backward jump instructions that compute the function on bit strings that models the multiplication of natural numbers on their representation in the binary number system according to this minor variant of the long multiplication algorithm.
We uniformly define instruction sequences $LMUL_n' (n > 0)$ by

\[
MOV_n(I_1^{(n)}, T_1^{(n)}) \; ; \; ZPADD_{2n}(I_1^{(n)}) \; ; \; MOV_n(I_2^{(n)}, T_2^{(n)}) \; ; \; SET_{2n}(0^{2n}, T_3^{(n)}) \; ; \\
(\neg T_2^{(n)})[0].get \; ; \#l \; ; ADD_{2n}(T_1^{(n)}, T_3^{(n)}) \; ; \\
SHL_{2n}(T_1^{(n)}, T_1^{(n)}) \; ; \; SHR_{n}(T_2^{(n)}, T_2^{(n)}) \; ; \; \\
MOV_{2n}(T_3^{(n)}, O^{(n)}) \; ; ! \; .
\]

where

\[
l = \text{len}(ADD_{2n}(T_1^{(n)}, T_3^{(n)}, T_3^{(n)})) + 1 = 52 \cdot n + 2 .
\]

We obtain by simple calculations that

\[
\text{len}(LMUL_n') = 64 \cdot n^2 + 16 \cdot n + 1 .
\]

This means that the function on bit strings that models the multiplication of natural numbers on their representation in the binary number system can still be computed by quadratic-length instruction sequences without backward jump instructions if it is computed according to the minor variant of the long multiplication algorithm. Moreover, we have that $\text{len}(LMUL_n') > \text{len}(LMUL_n)$ for all $n > 0$.

For each bit of the representation of the multiplier, $LMUL_n'$ contains the same instruction sequence. That is, it contains $n$ duplicates of the same instruction sequence. This duplication can be eliminated by implementing a for loop by means of a backward jump instruction.

We proceed with defining instruction sequences with backward jump instructions that compute the function on bit strings that models the multiplication of natural numbers on their representation in the binary number system according to the minor variant of the long multiplication algorithm. In the definition to come, we write $\underline{n}$ for the shortest representation of the natural number $n$ in the binary number system.

We uniformly define instruction sequences $LMUL_n'' (n > 0)$ by

\[
MOV_n(T_1^{(n)}, T_1^{(n)}) \; ; \; ZPADD_{2n}(T_1^{(n)}) \; ; \; MOV_n(T_2^{(n)}, T_2^{(n)}) \; ; \; SET_{2n}(0^{2n}, T_3^{(n)}) \; ; \\
\neg T_2^{(n)}[0].get \; ; \#l_1 \; ; ADD_{2n}(T_1^{(n)}, T_3^{(n)}, T_3^{(n)}) \; ; \\
SHL_{2n}(T_1^{(n)}, T_1^{(n)}) \; ; \; SHR_{n}(T_2^{(n)}, T_2^{(n)}) \; ; \\
DEC_{\log_2(n)+1}(T_4^{(n)}) \; ; \; ISNZ_{\log_2(n)+1}(T_4^{(n)}) ; \#l_2 \; ; \\
MOV_{2n}(T_3^{(n)}, O^{(n)}) \; ; ! \; .
\]

where

\[
l_1 = \text{len}(ADD_{2n}(T_1^{(n)}, T_3^{(n)}, T_3^{(n)})) + 1 = 52 \cdot n + 2 ,
\]

\[
l_2 = \text{len}(-T_2^{(n)}[0].get) + \ldots + \text{ISNZ}_{\log_2(n)+1}(T_4^{(n)}) = 64 \cdot n + 9 \cdot \lfloor \log_2(n) \rfloor + 11 .
\]
We obtain by simple calculations that

$$\text{len}(\text{LMUL}'^n) = 83 \cdot n + 9 \cdot \lfloor \log_2(n) \rfloor + 12.$$  

This means that the function on bit strings that models the multiplication of natural numbers on their representation in the binary number system can be computed by linear-length instruction sequences with backward jump instructions if it is computed according to the minor variant of the long multiplication algorithm. Moreover, we have that $\text{len}(\text{LMUL}'^n) < \text{len}(\text{LMUL}^n)$ for all $n > 1$.

6 Long Multiplication and the Halting Problem

In this section, a point concerning the halting problem is discussed which was raised by the material in Section 5 but for which space could not be found there.

Turing’s result regarding the undecidability of the halting problem (see e.g. [23]) is a result about Turing machines. In [3], we consider it as a result about programs rather than machines, taking instruction sequences as programs. The instruction sequences concerned are essentially the finite instruction sequences that can be denoted by closed PGA$_{bij}$ terms. Unlike in the current paper, the basic instructions are not fixed, but their effects are restricted to the manipulation of something that can be understood as the content of the tape of a Turing machine with a specific tape alphabet, together with the position of the tape head. Different choices of basic instructions give rise to different halting problem instances and one of these instances is essentially the same as the halting problem for Turing machines. Because of their orientation to Turing machines, we consider all instances treated in [3] theoretical halting problem instances.

All halting problem instances would evaporate if the instruction sequences concerned would be restricted to the ones without backward jump instructions. This is irrespective of whether the effects of the basic instructions have anything to do with the manipulation of a Turing machine tape. In the case that we have basic instructions to set and get the content of Boolean registers, instruction sequences without backward jump instructions are sufficient to compute all functions $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$ $(n, m \in \mathbb{N})$. This raises the question whether there exists a good reason for not abandoning backward jump instructions altogether in such cases. The function that models the multiplication of natural numbers on their representation in the binary number system offers a good reason: the length of the instruction sequences that compute it according to the long multiplication algorithm can be reduced significantly by the use of backward jump instructions, even more than by going over to one of the multiplication algorithms that are known to yield shorter instruction sequences without backward jump instructions than the long multiplication algorithm such as for example the Karatsuba multiplication algorithm (see e.g. [6]).

Thus, the instruction sequences $\text{LMUL}'_n$ and the instruction sequences $\text{LMUL}''_n$ form a hard witness of the inevitable existence of a halting problem in the practice of imperative programming, where programs must have manageable
size. Because of its orientation to actual programming, we consider the halting problem for the instruction sequences with forward and backward jump instructions, and with only basic instructions to set and get the content of Boolean registers, a practical halting problem. It is unknown to us whether there is a connection between the solvability or unsolvability of the halting problem for these instruction sequences and some form of diagonal argument. It is easy to prove that this halting problem is both NP-hard and coNP-hard. We do not know whether stronger lower bounds for its complexity can be found in the literature. An extensive search for such lower bounds and other result concerning this halting problem or a similar halting problem has been unsuccessful.

7 Long Multiplication and the Concept of an Algorithm

In this section, another point is discussed which was raised by the material in Section 5. This point concerns the concept of an algorithm.

At the end of Section 5, we implicitly state that the instruction sequences $LMUL'_n$ and the instruction sequences $LMUL''_n$ realize the same algorithm. We have asked ourselves the question why this is an acceptable statement and what this says about the definition of an algorithm. We consider it an acceptable statement because all the different views on what characterizes an algorithm lead to the conclusion that we have to do here with different realizations of the same algorithm. We cannot prove this due to the absence of a generally accepted mathematically precise definition of the concept of an algorithm. The cause of this absence seems to be the general acceptance of the exact mathematical concept of a Turing machine and equivalent mathematical concepts as adequate replacements of the intuitive concept of an algorithm.

Unfortunately, Turing machines are quite remote from anything related to actual programming. Moreover, we can construct at least two different Turing machines for the one algorithm realized by both the instruction sequences $LMUL'_n$ and the instruction sequences $LMUL''_n$: one without a counterpart of a for loop and one with a counterpart of a for loop. So Turing machines do not enforce a level of abstraction that is sufficient for algorithms. Therefore, we doubt whether the mathematical concept of a Turing machine is an adequate replacement of the intuitive concept of an algorithm. This means that we consider a generally accepted mathematically precise definition of the concept of an algorithm still desirable. Below, we outline a possible avenue to such a definition.

We restrict ourselves to algorithms for computing functions on bit strings. This has the advantage that data representation is hardly an issue in the realizations of algorithms. Moreover, we adopt the common practice among mathematicians to treat the length of the input of an algorithm as a parameter of the algorithm. In the perspective that a program is in essence an instruction sequence, taking into account the experience gained in this paper with realizing algorithms by instruction sequences, we consider the following to be a first approximation of a mathematically precise definition of the concept of an algo-
rithm: “an algorithm is a mapping from the set of natural numbers to the set of equivalence classes of the instruction sequences with backward jumps used in this paper with respect to an appropriate equivalence relation”. The underlying idea is that for each algorithm, for each \( n \), there is a class of algorithmically equivalent instruction sequences that realize the algorithm for that \( n \). This idea refines an idea that was already put forward by Milner in 1971 (see [19]).

What exactly should be considered algorithmically equivalent instruction sequences is a matter of further study. Some requirements for algorithmic equivalence are:

– each instruction sequence is algorithmically equivalent to each instruction sequence that produces the same behaviour;
– each instruction sequence is algorithmically equivalent to the instruction sequence obtained from it by consistently exchanging 0 and 1;
– each instruction sequence is algorithmically equivalent to each instruction sequence obtained from it by renumbering the auxiliary Boolean registers used;
– each instruction sequence is algorithmically equivalent to each instruction sequence obtained from it by transposing basic instructions that have no influence on each other;
– each instruction sequence is algorithmically equivalent to each instruction sequence obtained from it by replacing subsequences that are the result of the concatenation of an instruction sequence a number of times with itself by an implementation of a for loop of which it is the unwinding.

Of course, there is a possibility that additional requirements are necessary. Note that \( \text{LMUL}^\prime_n \) and \( \text{LMUL}^\prime\prime_n \) are algorithmically equivalent according to the last-mentioned requirement. It is mainly this requirement that makes it difficult to give an exact mathematical definition of an algorithmic equivalence relation satisfying the above-mentioned requirements. We further remark that it is not clear to us whether such a definition is relevant at all if the conceivable viewpoint is taken that there may be different degrees to which an instruction sequence realizes an algorithm.

Above, we have restricted ourselves to algorithms for computing functions on bit strings. We could restrict ourselves further to algorithms for computing projective functions on bit strings, i.e. functions on bit strings for which the restriction to bit strings of any given length can handle each restriction to bit strings of a shorter length if sufficiently many leading zeros are added (see [5]). This means that an instruction sequence that computes the restriction of such a function to bit strings of a certain length can also be used to compute the restriction of the function concerned to bit strings up to that length. The projective functions on bit strings include all functions that model operations on natural numbers on their representation in the binary number system.

8 Concluding Remarks

We have demonstrated that, in the case that the other instructions are only instructions to set and get the content of Boolean registers, forward jump instruc-
tions, and a termination instruction, the function that models the multiplication of natural numbers on their representation in the binary number system can be computed according to a minor variant of the long multiplication algorithm by quadratic-length instruction sequences without backward jump instructions and by linear-length instruction sequences with backward jump instructions. Be aware that we have not shown that this function cannot be computed by linear-length instruction sequences without backward jump instructions. However, the scientific literature on multiplication algorithms (see e.g. [12,16,21,22]) indicates that it is likely that it cannot be computed by linear-length instruction sequences without backward jump instructions.

We have also gone into the observations that the demonstration provides a hard witness of the inevitable existence of a halting problem in the practice of imperative programming and that it makes manifest the lack of a definition of the concept of an algorithm that makes it possible to prove whether two instruction sequences realize the same algorithm.

The viewpoints on what is an algorithm are diverse in character. Milner’s idea that algorithms are equivalence classes of programs can also be found in [25]. A rather strange twist is that constructions of primitive recursive functions are considered to be programs. In [20], algorithms are viewed as isomorphism classes of tuples of recursive functionals that can be defined by repeated application of certain schemes. In [9], which is concerned with algorithms on Kahn-Plotkin’s concrete data structures, algorithms are viewed as pairs of a function and a computation strategy that resolves choices between possible ways of computing the function. In [13], an algorithm is defined as an object that satisfy certain postulates. According to this definition, Gurevich’s abstract state machines capture algorithms. In [18], it is claimed that the only algorithms are those realized by Kolmogorov machines and that therefore the concept of a Kolmogorov machine can be regarded as an adequate formal characterization of the concept of an algorithm (see also [24]).

In [10], it is argued that the intuitive notion of algorithmic equivalence of programs cannot be captured by an equivalence relation. This is also argued in the philosophical discussion of the view that algorithms are mathematical objects presented in [11]. The given arguments are no reason for us to doubt the usefulness of studying equivalence relations that capture algorithmic equivalence to a certain degree.

References