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Publication date
2014

Document Version
Final published version

Published in
Theory and Applications of Categories

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Citation for published version (APA):
van den Berg, B., & Heunen, C. (2014). Extending obstruction to noncommutative functorial spectra. *Theory and Applications of Categories*, *29*, 457-474. Article 17. http://www.tac.mta.ca/tac/volumes/29/17/29-17abs.html

Theory and Applications of Categories, Vol. 29, No. 17, 2014, pp. 457–474.

EXTENDING OBSTRUCTIONS TO NONCOMMUTATIVE FUNCTORIAL SPECTRA

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Abstract. Any functor from the category of C*-algebras to the category of locales that assigns to each commutative C*-algebra its Gelfand spectrum must be trivial on algebras of n-by-n matrices for $n \geq 3$. This obstruction also applies to other spectra such as those named after Zariski, Stone, and Pierce. We extend these no-go results to functors with values in (ringed) topological spaces, (ringed) toposes, schemes, and quantales. The possibility of spectra in other categories is discussed.

1. Introduction

The spectrum of a commutative ring is a leading tool of commutative algebra and algebraic geometry. For example, a commutative ring can be reconstructed using (among other ingredients) its Zariski spectrum, a coherent topological space. Spectra are also of central importance to functional analysis and operator algebra. For example, there is a dual equivalence between the category of commutative C*-algebras and compact Hausdorff topological spaces, due to Gelfand.[1]

A natural question is whether such spectra can be extended to the noncommutative setting. Indeed, many candidates have been proposed for noncommutative spectra. In a recent article [23], M. L. Reyes observed that none of the proposed spectra behave functorially, and proved that indeed they cannot, on pain of trivializing on the prototypical noncommutative rings $M_n(\mathbb{C})$ of n-by-n matrices with complex entries. To be precise: any functor $F\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Set}$ that satisfies $F(C) = \mathrm{Spec}(C)$ for commutative rings C, must also satisfy $F(M_n(\mathbb{C})) = \emptyset$ for $n \geq 3$. [2] This result shows in a strong way why the traditional notion of topological space is inadequate to host a good notion of noncommutative spectrum. Its somewhat elaborate proof is based on the Kochen–Specker Theorem [17]. It is remarkable that a theorem from mathematical physics would have something to say about all possible rings.

Benno van den Berg was supported by the Netherlands Organisation for Scientific Research (NWO). Chris Heunen was supported by the Office of Naval Research under grant N000141010357 and by the Engineering and Physical Sciences Research Council Fellowship EP/L002388/1.

Received by the editors 2014-01-21 and, in revised form, 2014-07-15.

Transmitted by Giuseppe Rosolini. Published on 2014-08-14.

2010 Mathematics Subject Classification: 16B50, 46L85.

Key words and phrases: Ring spectra, Kochen-Specker Theorem.

[1] For convenience, we take all rings and C*-algebras to be unital, although that is not essential.

[2] The rings $\mathbb{M}_n(\mathbb{C})$ and $\mathbb{C}$ are Morita-equivalent, and so behave similarly in many ways. But for our purposes they are very different: for $n = 3$ the theorems hold, for $n = 2$ they do not.

One could hope that less orthodox notions of space are less susceptible to this obstruction. In particular, there are notions of space, such as that of a locale or a topos, in which the notion of point plays a subordinate role. In fact, one of the messages of locale theory and topos theory is that one can have spaces with a rich topological structure, but without any points whatsoever. Indeed, many of the proposed candidate spectra for noncommutative C*-algebras have been, or could be, phrased in such terms.

The main result of this article is that the obstruction cannot be circumvented in this way. We will rule out many candidates for categories of noncommutative Gelfand spectra by deriving various no-go theorems for locales, toposes, ringed toposes, and even quantales. Additionally, we prove similar limitative results for Zariski, Stone and Pierce spectra. These results will all follow from two basic ingredients. The first is the Kochen–Specker Theorem, as in [23]. The second is a general extension theorem, prompted by our work in [6], that allows us both to significantly simplify and extend Reyes' argument.

The basic obstruction is given by the Kochen–Specker Theorem. It relates Boolean algebras to a certain noncommutative notion of Boolean algebra. More precisely, it can be rephrased to say that any morphism of so-called partial Boolean algebras, from the projections in $\mathbb{M}_n(\mathbb{C})$ to a Boolean algebra, must trivialize when $n \geq 3$.

The general extension theorem, as its name suggests, uses some simple category theory to extend this basic obstruction to far more general situations. To see how it works, consider the following commuting diagram of functors and categories.

$$\begin{tikzcd} \mathbf{C} \arrow[r, "S"] \arrow[d, hook] & \mathbf{S} \arrow[d, "?"] \\ \mathbf{R} \arrow[r, "?"'] & ? \end{tikzcd}$$

Here, $\mathbf{R}$ consists of a kind of rings, $\mathbf{C}$ is the full subcategory of commutative ones, the functor S takes the spectrum, and $\mathbf{S}$ consists of the spectral spaces. The goal is to extend S to the noncommutative setting. The extension theorem will state that, as long as the functor on the right-hand side preserves limits, the bottom functor must degenerate. Regarded this way, one could say that what the Kochen–Specker Theorem obstructs, is transporting S along functors whose images have the same limit behaviour.

The paper is structured as follows. First, Section 2 motivates why it is a priori not unreasonable to look to pointless topology for noncommutative spectra. Section 3 recalls the Kochen–Specker Theorem and several variations. Section 4 then sets the stage with the abstract results. After that, Sections 5–7 draw corollaries of interest from these main theorems. This host of impossibility results does not mean that it is hopeless to search for a good notion of noncommutative spectrum. We end the paper on a positive note by discussing ways of circumventing the obstruction in Section 8, that will hopefully serve as a guide towards the 'right' generalization of the notion of space.

2. Pointfree topology

The idea of a form of topology in which the notion of an open (or a region in space) is primary and the notion of a point plays a subordinate role dates back at least to Whitehead [25, 26]. For a long time these ideas remained quite philosophical in nature and belonged to the periphery of mathematics. But this changed with the work of Grothendieck [3]. The notion of a *topos*, which he seems to have regarded as his most profound idea, is really a pointfree concept of a space. By now it is clear that a mathematically viable theory of pointfree spaces is possible and with topos theory this has reached a considerable degree of maturity and sophistication [16, 20].

Within the category of toposes the *localic toposes* play an important role. Here they will be important because toposes that arise as spectra are localic. We will define these toposes in Definition 2.8 below; we will have no need to consider toposes that are not localic. To define these localic toposes, the crucial observation is that in the construction of the category of sheaves over a topological spaces, the points of the space play no role. Indeed, all that matters is the structure of the lattice of opens of the space. So, to define a category of sheaves, one only needs a suitable lattice-theoretic structure. The precise structure required is formalized by the concept of a *locale*, which is an important notion of pointfree space in its own right [14, 15].

2.1. DEFINITION. *A* complete lattice *is a partially ordered set of which arbitrary subsets have a least upper bound. In a complete lattice every subset also has a greatest lower bound. A* locale *is a complete lattice that satisfies the following infinitary distributive law:*

$$\bigvee(x \wedge y_i) = x \wedge \bigvee y_i.$$

The elements of a locale are called opens*. A morphism $K \to L$ of locales is a function $f\colon L \to K$ that satisfies $f(x \wedge y) = f(x) \wedge f(y)$ and $f(\bigvee x_i) = \bigvee f(x_i)$. (Note the change in direction!) This forms a category* **Loc**.

The primary example of a locale is the collection of open subsets of a topological space. Moreover, a continuous function between topological spaces induces a morphism between the corresponding locales (in the same direction). Thus we have a functor **Top** $\to$ **Loc**.

As it happens, this functor has a right adjoint. To construct it, define a *point* of a locale L as a morphism $p\colon 1 \to L$. Here, 1 is the terminal object in the category of locales, which coincides with the set of open sets of a singleton topological space. The set of points of L may be topologized in a natural way, by declaring its open sets to be those of the form $\{p \mid p^{-1}(U) = 1\}$ for opens U in L. This defines the right adjoint **Loc** $\to$ **Top**.

As usual, this adjunction becomes an equivalence if we restrict to the full subcategories of those locales and spaces for which the unit and counit are isomorphisms. These are called the *spatial locales* and *sober spaces*, respectively. For topological spaces, sobriety is really a weak separation property (for example, any Hausdorff topological space is sober). Thus, locales and topological spaces are closely related.

There are, however, a few subtle differences. One of the most important is that in the category of locales, limits are computed differently than in the category of topological spaces. This is one of the reasons why one might suspect that a good pointfree notion of spectrum may be possible. In fact, the following considerations may lead one to hope that a suitable notion of a pointfree space could avoid the obstruction observed by Reyes:

1. Many notions of spectrum lend themselves quite naturally to a pointfree formulation [14].

2. In many cases points correspond to maximal ideals. It is well-known that these behave very poorly functorially.

3. Limits play an important role in Reyes' result, and here as well. But limits are computed differently in topological spaces and locales. (In fact, this aspect of locales is emphasized in [15].)

But, as we will see below, the obstruction to nonfunctorial spectra is so fundamental that it precludes suitable notions of spectra in locales and toposes as well.

The problem is with point (3). Although limits in **Loc** and **Top** are computed differently *in general*, this is not what happens with limits of locales and topological spaces that arise as spectra. There, the constructions move perfectly in tandem. This follows from the fact that locales that arise as spectra are (i) closed under limits and (ii) spatial. In fact, (i) alone already precludes the existence of suitable spectra in the category of locales, as Section 4 below will make clear. For this reason, we will only explain (i) in some detail here.

2.2. Remark. Proving that locales that arise as spectra are spatial relies on nonconstructive principles, such as the Prime Ideal Theorem (a consequence of the axiom of choice). In fact, the arguments in this paper are mostly constructive: only the proofs in Section 6 rely on results that might not be valid constructively. (That the locale-theoretic analogues of nonconstructive results in topology often are constructively valid is another aspect of locale theory emphasized in [15].)

For example, Gelfand duality concerns compact Hausdorff spaces. Being Hausdorff is something which is rather hard to express in localic terms: but, fortunately, for compact spaces being Hausdorff is equivalent to being regular, and regularity is more readily expressed in localic terms [14, page 80].

2.3. Definition. *A locale L is called* compact *if any subset $S \subseteq L$ whose least upper bound is the top element has a finite subset whose least upper bound is also the top element.*

If a and b are two elements of a locale L, then a is well inside *b if $c \wedge a = 0$ and $c \vee b = 1$ for some $c \in L$. A locale L is called* regular *if any $a \in L$ is the least upper bound of the elements well inside it.*

2.4. Lemma. *Compact regular locales are closed under limits in* **Loc**.

Proof. This follows from the fact that the inclusion of the full subcategory **KRLoc** of compact regular locales inside the category of locales has a left adjoint (namely the Stone-Čech compactification, see [14, page 130 and page 88]). ∎

Stone duality is a duality between Boolean algebras and Stone spaces. To define the localic version of Stone spaces, observe that if D is a distributive lattice, then the collection $\mathrm{Idl}(D)$ of ideals on D (ordered by inclusion) is a locale. In fact, this construction is part of a functor

$$\mathrm{Idl}\colon \mathbf{DLat}^{\mathrm{op}} \to \mathbf{Loc}$$

sending ideals to the down closure of their direct images along maps of distributive lattices. This functor is faithful, but not full.

2.5. Definition. *A* coherent locale *is one equivalent to one of the form* $\mathrm{Idl}(D)$*. Any coherent locale is compact; if it is also regular, we call it a* Stone locale*. A map between coherent locales that is isomorphic to one in the image of the functor* Idl *is called* coherent*.*

2.6. Lemma. *If a diagram in* $\mathbf{Loc}$ *consists of coherent locales and coherent morphisms between them, then its limit is again a coherent locale.*

Proof. This follows from the fact that $\mathrm{Idl}\colon \mathbf{DLat}^{\mathrm{op}} \to \mathbf{Loc}$ is faithful and right adjoint to the forgetful functor [14, page 59]. ∎

2.7. Lemma. *Stone locales are closed under limits in* $\mathbf{Loc}$*.*

Proof. This follows from Lemmas 2.4 and 2.6, together with the fact that every map between Stone locales is coherent [14, page 71]. ∎

As mentioned before, these results will preclude the existence of functorial spectra in the category of locales. They will also preclude the existence of functorial spectra in the category of toposes. Before we can explain that, let us first indicate how one can define a category of sheaves on a locale.

2.8. Definition. *A* presheaf *on a locale L is a functor $X\colon L^{\mathrm{op}} \to \mathbf{Sets}$. More concretely, a presheaf consists of a family of sets $(X(p))_{p\in L}$ together with for any $q \leq p$ a restriction operation*

$$(-) \restriction q : X(p) \to X(q)$$

satisfying some natural compatibility conditions.

A presheaf X is a sheaf *when for any family of elements $\{p_i \in L \mid i \in I\}$ and $\{x_i \in X(p_i) \mid i \in I\}$ with $x_i \restriction p_i \wedge p_j = x_j \restriction p_i \wedge p_j$ for all $i, j \in I$ there is a unique element $x \in X(\bigvee p_i)$ with $x \restriction p_i = x_i$ for every $i \in I$.*

For any locale L the sheaves on L, with natural transformations between them, form a topos $\mathrm{Sh}(L)$*. A topos which is equivalent to one of this form is called* localic*.*

The construction of taking sheaves on a locale is functorial. The crucial result that will preclude noncommutative spectra valued in toposes is the following.

2.9. Lemma. *There is a full and faithful functor* $\mathrm{Sh}\colon \mathbf{Loc} \to \mathbf{Topos}$ *that assigns to every locale the category of sheaves over that locale. It preserves limits.*

Proof. For the first statement, see [20, Proposition IX.5.2]. For the second, [16, C.1.4.8]. ∎

3. The Kochen–Specker Theorem

The Kochen–Specker Theorem is a famous and important result from the foundations of quantum mechanics. Its original intention was to preclude the possibility of hidden variable theories, but there are interpretational debates about whether this conclusion is valid. Its mathematical content is important to us as an example of an obstruction, as will be defined in the next section. It was originally stated in terms of partial algebras, which also form a convenient starting point for us.

The idea behind partial algebras is to break an algebra into parts; each part itself is a (sub)algebra with particularly nice properties, but the cohesion between the parts is lost. This lets us, for example, think about a (noncommutative) ring in terms of its commutative parts. In general, of course, the partial algebra contains less information, precisely because the whole algebra does have cohesion between the parts. The Kochen–Specker theorem, and our results based on it, concern partial algebras; they do not analyse how much "more cohesive" an algebra is than the sum of its parts.

A *partial Boolean algebra* consists of a set B with:

- a reflexive and symmetric binary (*commeasurability*) relation $\odot \subseteq B \times B$;
- elements $0, 1 \in B$;
- a (total) unary operation $\neg \colon B \to B$;
- (partial) binary operations $\wedge, \vee \colon \odot \to B$;

such that every set $S \subseteq B$ of pairwise commeasurable elements is contained in a set $T \subseteq B$, whose elements are also pairwise commeasurable, and on which the above operations determine a Boolean algebra structure. A morphism of partial Boolean algebras is a function that preserves commeasurability and all the algebraic structure, whenever defined. More precisely, we have:

- $f(a) \odot f(b)$ whenever $a \odot b$;
- $f(0) = 0$ and $f(1) = 1$;
- $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$ whenever $a \odot b$;
- $f(\neg a) = \neg f(a)$ for $a \in B$.

Examples of partial Boolean algebras are ordinary Boolean algebras, where the commeasurability relation is total (we will also call these *total Boolean algebras* for that reason), and projection lattices of Hilbert spaces. In fact, the collection of projections

$$\mathrm{Proj}(A) = \{p \in A \mid p^* p = p\}$$

carries the structure of a partial Boolean algebra for every C*-algebra A (where we say that two projections are commeasurable when they commute). The Kochen–Specker Theorem now reads as follows.

3.1. THEOREM. [Kochen–Specker Theorem] *Let* $f\colon \operatorname{Proj}(\mathbb{M}_n(\mathbb{C})) \to B$ *be a morphism of partial Boolean algebras for* $n \geq 3$. *If* B *is a (total) Boolean algebra, then it must be the terminal one (in which* $0 = 1$*).*

PROOF. See [17, 22]. ■

If B is a partial Boolean algebra and we write $\mathcal{C}(B)$ for the diagram of its total subalgebras and inclusions between them, then we can rephrase the previous theorem as follows (see also [4]).

3.2. COROLLARY. *If* $n \geq 3$, *then the colimit of* $\mathcal{C}(\operatorname{Proj}(\mathbb{M}_n(\mathbb{C})))$ *in the category of Boolean algebras is the terminal Boolean algebra.*

PROOF. Suppose we have a cocone from $\mathcal{C}(\operatorname{Proj}(\mathbb{M}_n(\mathbb{C})))$ to B in the category of Boolean algebras. Clearly, it can also be considered as a cocone in the category of partial Boolean algebras. But because the colimit of $\mathcal{C}(\operatorname{Proj}(\mathbb{M}_n(\mathbb{C})))$ in the category of partial Boolean algebras exists and is precisely $\operatorname{Proj}(\mathbb{M}_n(\mathbb{C}))$ (see [6]), it follows from Theorem 3.1 that B is trivial. ■

We will also need a variation for C*-algebras. First, we define the appropriate partial notion. A *partial C*-algebra* is a set A with:

- a reflexive and symmetric binary (*commeasurability*) relation $\odot \subseteq A \times A$;
- elements $0, 1 \in A$;
- (partial) binary operations $+, \cdot\colon \odot \to A$;
- a (total) involution $*\colon A \to A$;
- a (total) function $\cdot\colon \mathbb{C} \times A \to A$;
- a (total) function $\|-\| : A \to \mathbb{R}$;

such that every set $S \subseteq A$ of pairwise commeasurable elements is contained in a set $T \subseteq A$, whose elements are also pairwise commeasurable, and on which the above operations determine the structure of a commutative C*-algebra. A morphism of partial C*-algebras is a morphism $f\colon A \to B$ preserving the commeasurability relation and all the algebraic structure, whenever defined. More precisely, we have:

- $f(0) = 0$ and $f(1) = 1$;
- $f(a) \odot f(b)$, $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ whenever $a \odot b$;
- $f(a)^* = f(a^*)$ for $a \in A$;
- $f(za) = zf(a)$ for $z \in \mathbb{C}$ and $a \in A$.

Any commutative C*-algebra is an example of a partial C*-algebra, on which the commeasurability relation is total. Moreover, for any C*-algebra A, the normal elements

$$N(A) = \{a \in A \mid aa^* = a^*a\}$$

carry the structure of a partial C*-algebra (where commeasurability means commutativity). Again, we write $\mathcal{C}(A)$ for the diagram of total subalgebras of a partial C*-algebra A and inclusions between them.

3.3. COROLLARY. *If $n \geq 3$, then the colimit of $\mathcal{C}(\mathbb{M}_n(\mathbb{C}))$ in the category of commutative C*-algebras is the terminal C*-algebra (in which $0 = 1$).*

PROOF. Suppose we have a cocone from $\mathcal{C}(\mathbb{M}_n(\mathbb{C}))$ to A in the category of commutative C*-algebras. Again, we consider this as a diagram in the category of partial C*-algebras, where the colimit of $\mathcal{C}(\mathbb{M}_n(\mathbb{C}))$ is precisely $N(\mathbb{M}_n(\mathbb{C}))$ (see [6]). So we obtain a map $f : N(\mathbb{M}_n(\mathbb{C})) \to A$ of partial C*-algebras. By restricting f to the projections we obtain a map $\mathrm{Proj}(f) : \mathrm{Proj}(\mathbb{M}_n(\mathbb{C})) \to \mathrm{Proj}(A)$ to which Theorem 3.1 applies. Therefore A must be the terminal C*-algebra. ∎

4. Obstructions

This section develops a completely general way to extend obstructions like that of the previous section. We start with the general extension theorem, and then formalize obstructions in suitable abstract terms.

4.1. PROPOSITION. *Suppose given a commuting diagram of categories and functors*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, "H"'] & \mathbf{B} \arrow[d, "K"] \\ \mathbf{C} \arrow[r, "G"'] & \mathbf{D} \end{tikzcd}$$

where $\mathbf{B}$ is complete, and K preserves limits. If

- *$\mathcal{A}$ is a diagram in $\mathbf{A}$,*
- *there is a cone from X to $H\mathcal{A}$ in $\mathbf{C}$,*
- *$Y = \lim F\mathcal{A}$,*

then there exists a morphism $G(X) \to K(Y)$ in $\mathbf{D}$.

PROOF. Because K preserves limits, $K(Y) = K(\lim F\mathcal{A}) = \lim KF\mathcal{A}$. The square above commutes, therefore $K(Y) = \lim GH\mathcal{A}$. By assumption, there is a cone from X to $H\mathcal{A}$ in $\mathbf{C}$. Hence, there is a cone from GX to $GH\mathcal{A}$ in $\mathbf{D}$. But we already saw that $K(Y)$ is the target of the universal such cone. Hence there exists a unique mediating morphism $G(X) \to K(Y)$. ∎

Notice that the assumptions of the previous proposition were stronger than necessary: $\mathbf{B}$ need not be complete, we only really need $\lim F\mathcal{A}$ to exist in $\mathbf{B}$. Here is an illustration of the situation (that will turn out not to be obstructed).

4.2. Example. This illustration works best with colimits instead of limits, so we will work in the opposite setting of the previous proposition. Let $\mathbf{A}$ be the category of finite sets and injective functions, included in the category $\mathbf{C}$ of all sets and injections. Take $\mathbf{D}$ to be the ordered class of cardinal numbers, regarded as a category, and let $\mathbf{B}$ be its subcategory of at most countable cardinals, and K the inclusion. Finally, set F and G to be the functors that take cardinality. Then $\mathbf{B}$ is cocomplete, and K preserves colimits.

Clearly, every set X is the colimit in $\mathbf{C}$ of the directed diagram $\mathcal{A}$ in $\mathbf{A}$ of its finite subsets and inclusions amongst them. If X is finite, then $Y = \operatorname{colim} F\mathcal{A} = \sup_{A\in\mathcal{A}} \operatorname{card}(A) = \operatorname{card}(X)$, giving a morphism $K(Y) \to G(X)$ in $\mathbf{D}$. If X is infinite, then $Y = \sup_{A\in\mathcal{A}} \operatorname{card}(A)$ is at most countable, and therefore there still is a morphism $Y \leq \operatorname{card}(X)$ in $\mathbf{D}$.

We can think of the previous proposition as saying that the existence of (universal) cones to diagrams in $\mathbf{A}$ can be transported along the functors F and G. Next, we turn to formalizing obstructions to such extensions in the language of the previous proposition. (We are using obstruction here in the normal colloquial sense; no analogy with algebraic topology is intended.)

4.3. Definition. *In the situation of Proposition 4.1, an* obstruction *to an object X in $\mathbf{C}$ is a diagram $\mathcal{A}$ in $\mathbf{A}$ together with a cone from X to $H\mathcal{A}$ in $\mathbf{C}$ such that* $\lim F\mathcal{A}$ *is initial in $\mathbf{B}$. The object X is called* obstructed *if an obstruction to it exists.*

As a final abstract result, we now consider what happens when we try to extend obstructed objects using Proposition 4.1. An initial object is *strict* when any morphism into it is an isomorphism.

4.4. Theorem. *In the situation of Proposition 4.1: if K preserves initial objects, and initial objects in $\mathbf{D}$ are strict, then G maps obstructed objects to initial objects.*

Proof. Let X be an obstructed object in $\mathbf{C}$. Then there are a diagram $\mathcal{A}$ in $\mathbf{A}$ and a cone from X to $H\mathcal{A}$ in $\mathbf{C}$ such that $Y = \lim F\mathcal{A}$ is initial. Proposition 4.1 now provides a morphism $G(X) \to K(Y)$ in $\mathbf{D}$. But since K preserves initial objects, $K(Y)$ is initial in $\mathbf{D}$, and in fact strictly so. Hence the morphism $G(X) \to K(Y)$ must be an isomorphism, making $G(X)$ into a (strict) initial object. ■

The previous theorem provides an intuition behind Definition 4.3: whereas X supports a cone to $H\mathcal{A}$, this cone trivialises when transported along G.

5. Gelfand spectrum

This section is the first of several deriving no-go results. It shows that there can be no nondegenerate functor extending Gelfand duality that takes values in locales, topological spaces, toposes, or quantales.

For us, Gelfand duality is best considered as a duality between the category **cCstar** of commutative C*-algebras and the category **KRLoc** of compact regular locales. This duality exhibits every commutative C*-algebra A as isomorphic to one of the form $\{f : X \to \mathbb{C} : f \text{ continuous}\}$ for some compact regular locale X; the opens of the locale X can be chosen to be the closed ideals of the commutative C*-algebra A, ordered by inclusion.

Combining the extension of Section 4 with the obstruction of Section 3, we now immediately find that there can be no nondegenerate functor from C*-algebras to locales that extends the Gelfand spectrum.

5.1. Corollary. *Any functor G: $\mathbf{Cstar}^{\mathrm{op}} \to \mathbf{Loc}$ that assigns to each commutative C*-algebra its Gelfand spectrum trivializes on $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$.*

Proof. We instantiate the setting of Proposition 4.1 by

$$\begin{tikzcd} \mathbf{cCstar}^{\mathrm{op}} \arrow[r, "\mathrm{Spec}"] \arrow[d, hook] & \mathbf{KRLoc} \arrow[d, hook, "K"] \\ \mathbf{Cstar}^{\mathrm{op}} \arrow[r, "G"'] & \mathbf{Loc}. \end{tikzcd}$$

By Lemma 2.4, **KRLoc** is complete and K preserves limits. Considering $X = \mathbb{M}_n(\mathbb{C})$ in **CStar** and $\mathcal{C}(\mathbb{M}_n(\mathbb{C}))$ in **cCStar**, it follows from the fact that Spec is part of a duality, and hence preserves limits, in combination with Corollary 3.3 that X is obstructed when $n \geq 3$. Since the initial object in **KRLoc** and **Loc** is the locale of opens of the empty topological space, which is a strict initial object in both categories, the statement follows from Theorem 4.4. ■

5.2. Remark. In fact, any functor as in the previous corollary must trivialize on many more objects than just $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$. For example, one easily derives that any C*-algebra A allowing a morphism $\mathbb{M}_n(\mathbb{C}) \to A$ for $n \geq 3$ is also obstructed. These are precisely those C*-algebras of the form $\mathbb{M}_n(B)$ for $n \geq 3$ and any C*-algebra B [19, Corollary 17.7]. Therefore, more generally, direct sums $\bigoplus_i \mathbb{M}_{n_i}(B_i)$ are also ruled out when $n_i \geq 3$ for each i. Any von Neumann algebra without direct summands $\mathbb{C}$ or $\mathbb{M}_2(\mathbb{C})$ is obstructed, too [8]. This remark holds for all corollaries to follow.

Because of the aforementioned equivalence between the categories of compact Hausdorff spaces and compact regular locales, the previous corollary holds equally well for topological spaces.

5.3. Corollary. *Any functor G: $\mathbf{Cstar}^{\mathrm{op}} \to \mathbf{Top}$ that assigns to each commutative C*-algebra its Gelfand spectrum trivializes on $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$.*

Since $\mathbb{M}_n(\mathbb{C})$ and all its sub-C*-algebras are von Neumann algebras, the previous two results also holds for von Neumann algebras:

5.4. Corollary. *Any functor* $G\colon \mathbf{Neumann}^{\mathrm{op}} \to \mathbf{Loc}$ *or* $G\colon \mathbf{Neumann}^{\mathrm{op}} \to \mathbf{Top}$ *that assigns to each commutative von Neumann algebra its Gelfand spectrum trivializes on* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$.

Because a locale is a reasonably elementary geometric notion, one might hold out hope for nondegenerate functorial extensions valued in categories of more involved geometric objects. However, we can use Corollary 5.1 as a stepping stone to derive no-go results for the more involved geometric notions of toposes and quantales.

5.5. Corollary. *Any functor* $G\colon \mathbf{Cstar}^{\mathrm{op}} \to \mathbf{Topos}$ *that assigns to each commutative C*-algebra its Gelfand spectrum trivializes on* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$.

Proof. Since both the inclusion $\mathbf{KRLoc} \to \mathbf{Loc}$ and $\mathrm{Sh}\colon \mathbf{Loc} \to \mathbf{Topos}$ preserve limits (see Lemmas 2.4 and 2.9, respectively), their composition does as well. Therefore, the proof of Corollary 5.1 applies when we put **Topos** in the bottom right corner. ∎

The previous corollary might not have come as a surprise after Corollary 5.1. After all, if locales are 'not noncommutative enough' to accommodate a good notion of noncommutative Gelfand spectrum, then why would the 'equally not noncommutative' toposes do so? We will now consider quantales, which were intended to be noncommutative versions of locales. In fact, quite some effort has gone into studying them as candidates for Gelfand spectra of noncommutative C*-algebras [21, 18]. The proof of the previous corollary shows that there is no nondegenerate extension of the Gelfand spectrum with values in any category of which compact regular locales are a subcategory that is closed under limits. We can use the same idea in the following.

A *quantale* is a partially ordered set Q that has least upper bounds of arbitrary subsets, and is equipped with an element $e \in Q$ and an associative multiplication $Q \times Q \to Q$ satisfying the following equations:

$$\bigvee(xy_i) = x(\bigvee y_i), \qquad \bigvee(y_i x) = (\bigvee y_i)x, \qquad ex = x = xe.$$

A morphism $Q \to Q'$ of quantales is a function $f\colon Q' \to Q$ satisfying $f(e) = e'$, $f(\bigvee x_i) = \bigvee f(x_i)$, and $f(xy) = f(x)f(y)$. Any locale is a quantale when we take meet as multiplication and the top element as unit. Hence we can regard the Gelfand spectrum as a functor $\mathbf{cCstar}^{\mathrm{op}} \to \mathbf{Quantale}^{\mathrm{op}}$.

5.6. Lemma. *Compact regular locales are closed under limits in* $\mathbf{Quantale}^{\mathrm{op}}$.

Proof. See [18, Corollary 4.4]. ∎

5.7. Corollary. *Any functor* $G\colon \mathbf{Cstar}^{\mathrm{op}} \to \mathbf{Quantale}^{\mathrm{op}}$ *that assigns to each commutative C*-algebra its Gelfand spectrum trivializes on* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$.

Proof. Using Lemma 5.6 instead of Lemma 2.4, the proof of Corollary 5.1 establishes the statement. ∎

At first sight the previous corollary might seem to contradict results of [18]: one can reconstruct the original C*-algebra from its maximal spectrum, and the assignment which sends a C*-algebra to its maximal spectrum is functorial. However, this functor does not send a commutative C*-algebra to its Gelfand spectrum, but to something from which it may be reconstructed (its so-called spatialization). Therefore the maximal spectrum does not satisfy our specification square of Proposition 4.1.

6. Zariski spectrum

In this section we turn to the Zariski spectrum. This construction underlies algebraic geometry by connecting commutative rings to coherent spaces via the prime ideals of the ring [9, 14]; more precisely, the Zariski spectrum of a commutative ring A is the locale whose opens are the radical ideals of A. Before we go on to extending obstructions to noncommutative generalizations of this duality, we first consider the basic no-go result. The abstract machinery from Sections 3 and 4 does not apply directly, because the Zariski spectrum functor $\mathbf{cRing}^{\mathrm{op}} \to \mathbf{Loc}$ famously does not preserve (products and hence) limits. Fortunately, it suffices to restrict to finite-dimensional complex algebras, where the Zariski spectrum functor does preserve limits, and where our obstructed objects $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$ live.

6.1. Corollary. *Any functor* $G \colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Loc}$ *that assigns to each commutative ring its Zariski spectrum trivializes on* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$.

Proof. When a commutative algebra A over $\mathbb{C}$ is finite-dimensional, it is Artinian as a ring, and therefore any prime ideal is maximal [9, Theorem 2.14]. In particular, every point in $\mathrm{Spec}(A)$ is closed. In turn, maximal ideals correspond bijectively, and functorially, to algebra homomorphisms: a character $f \colon A \to \mathbb{C}$ corresponds to its kernel $f^{-1}(0)$. Thus, when restricted to finite-dimensional commutative complex algebras, the Zariski spectrum functor is naturally isomorphic to a representable functor: $\mathrm{Spec} \cong \mathbf{cRing}(-,\mathbb{C}) \colon \mathbf{fcAlg}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Set}$. Moreover, in this case there are only finitely many maximal ideals [9, Theorem 2.14], so $\mathrm{Spec}(A)$ must be discrete. Clearly discrete locales are closed under limits in $\mathbf{Loc}$ (see also Lemma 2.6), so this restricted functor preserves finite limits, and just as in Corollary 5.1, we see that any functor $\mathbf{fAlg}_{\mathbb{C}}^{\mathrm{op}} \to \mathbf{Loc}$ that assigns to each commutative algebra its Zariski spectrum must trivialize on $\mathbb{M}_n(\mathbb{C})$ for $n \geq 3$. Precomposing with the inclusion $\mathbf{fAlg}_{\mathbb{C}} \hookrightarrow \mathbf{Ring}$ finishes the proof. ∎

Reyes' result [23] now follows directly from the previous, constructive, corollary.

This basic no-go result can be extended to values in categories of which coherent locales are a subcategory that is closed under limits, as in Section 5. For example, we get the following corollary.

6.2. Corollary. *Any functor* $G \colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Topos}$ *that assigns to each commutative ring its Zariski spectrum trivializes on* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$.

In Section 5 we used closure under limits to extend the basic no-go result. Another way is by postcomposing with functors that reflect initial objects, as in the rest of this section. Incidentally, these limitations also apply to functorial extensions of Gelfand duality discussed in Section 5.

Another generalized notion of space is that of a *ringed topological space* or *ringed locale* [11]. These are topological spaces/locales together with a sheaf of commutative rings over them, and are important in algebraic geometry. Every topological space/locale X can be regarded as a ringed space by letting the structure sheaf be the sheaf of continuous functions on opens of X. One can also consider the notion of a *ringed topos*: a topos together with a commutative ring object in it. This notion generalizes those of ringed topological spaces and ringed locales, because the category of sheaves over a ringed space is a ringed topos almost by definition. The import lies in the fact that every commutative ring is isomorphic to the ring of global sections of a sheaf of local rings. Thus we can regard the Zariski spectrum as a functor $\mathbf{cRing}^{\mathrm{op}} \to \mathbf{RingedTop}$, $\mathbf{cRing}^{\mathrm{op}} \to \mathbf{RingedLoc}$, or $\mathbf{cRing}^{\mathrm{op}} \to \mathbf{RingedTopos}$.

6.3. Corollary. *Any functor* $G\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{RingedTopos}$ *that assigns to each commutative ring its Zariski spectrum trivializes on* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$. *The same holds when we replace* $\mathbf{RingedTopos}$ *by* $\mathbf{RingedTop}$ *or* $\mathbf{RingedLoc}$.

Proof. The forgetful functor $U\colon \mathbf{RingedTopos} \to \mathbf{Topos}$ reflects initial objects. Since UG is a functor satisfying the hypotheses of Corollary 6.2, $UG(\mathbb{M}_n(\mathbb{C}))$ is initial when $n \geq 3$. But that means that $G(\mathbb{M}_n(\mathbb{C}))$ is initial. ∎

Actually, the main notion of interest in algebraic geometry is that of a scheme (see [11]). A locally ringed space is a ringed space where each stalk of the structure sheaf is not just a ring but a local ring. An *affine scheme* is a locally ringed space isomorphic to the Zariski spectrum of some commutative ring. A *scheme* is a locally ringed space admitting an open cover, such that the restriction of the structure sheaf to each covering open is an affine scheme.

6.4. Corollary. *Any functor* $G\colon \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Scheme}$ *that assigns to each commutative ring its Zariski spectrum trivializes on* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$.

Proof. The forgetful functor from the category of schemes to **Top** reflects initial objects, so the proof of the previous corollary applies. ∎

7. Stone and Pierce spectra

In this section we will have a further look at some dualities related to the Stone spectrum, where the Kochen–Specker Theorem also provides an obstruction to further extending them to suitably noncommutative structures.

First we consider the Stone spectrum, that provides a duality between Boolean algebras and Stone locales: given a Boolean algebra, the associated Stone locale has as opens the ultrafilters on B; and given a Stone locale L, the original Boolean algebra can be reconstructed by taking the complemented elements in L.

7.1. COROLLARY. *Any functor* $F\colon \mathbf{PBoolean}^{\mathrm{op}} \to \mathbf{Loc}$ *that assigns to each Boolean algebra its Stone spectrum trivializes on* $\mathrm{Proj}(\mathbb{M}_n(\mathbb{C}))$ *for* $n \geq 3$.

PROOF. If one considers the diagram

$$\begin{tikzcd} \mathbf{Boolean}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathbf{Stone} \arrow[d] \\ \mathbf{PBoolean}^{\mathrm{op}} \arrow[r, "F"'] & \mathbf{Loc} \end{tikzcd}$$

and the object $\mathrm{Proj}(\mathbb{M}_n(\mathbb{C}))$ in **PBoolean** (together with its diagram of commutative subalgebras in **Boolean**), we see that they are obstructed for every $n \geq 3$. Therefore they will be sent to the initial object by F. ∎

Traditional quantum logic, by which we mean the approach dating back to Birkhoff and von Neumann [7], considers orthomodular lattices. A lattice L is called *orthocomplemented*, if it comes equipped with a map $\perp\colon L \to L$ satisfying:

- $a \leq b \Rightarrow b^\perp \leq a^\perp$;
- $(a^\perp)^\perp = a$;
- $a \wedge a^\perp = 0$ and $a \vee a^\perp = 1$.

We call $a^\perp$ the *orthocomplement* of a, and say that a is commeasurable with b (and write $a \odot b$), if

$$a = (a \wedge b) \vee (a \wedge b^\perp).$$

This relation is clearly reflexive, but need not be symmetric; if it is, we will call the lattice *orthomodular*.[3] With lattice homomorphisms preserving orthocomplements as morphisms, orthomodular lattices form a category **OrthoLat**.

The previous no-go result extends to orthomodular lattices. This is due to several facts. First of all, every Boolean algebra is an orthomodular lattice. In fact, these are precisely the orthomodular lattices in which every two elements are commeasurable [5, Corollary II.4.6]. Furthermore, projections $\mathrm{Proj}(\mathbb{M}_n(\mathbb{C}))$ in n-dimensional complex Hilbert space can be identified with the subspace of $\mathbb{C}^n$ they project onto, and therefore form an orthomodular lattice [5, Section III.4]: the order comes from subspace inclusion, and $\perp$ comes from orthocomplement. Now, the relation $\odot$ gives every orthomodular lattice the structure of a partial Boolean algebra [5, Theorem II.4.5]. Projection lattices thus obtain partial Boolean algebra structure: projections p and q commute if and only if the subspaces $p(\mathbb{C}^n)$ and $q(\mathbb{C}^n)$ they project onto are commeasurable in the orthomodular lattice of linear subspaces [5, Exercise III.18]. Therefore also the two different notions of total (or commeasurable) subalgebra agree.

[3] This is equivalent to the usual statement of the orthomodular law $a \leq b \Rightarrow b = a \vee (b \wedge a^\perp)$ by [5, Theorem II.3.4].

7.2. Corollary. *Any functor* $\mathbf{OrthoLat}^{\mathrm{op}} \to \mathbf{Loc}$ *that assigns to each Boolean algebra its Stone spectrum trivializes on* $\mathrm{Proj}(\mathbb{M}_n(\mathbb{C}))$ *for* $n \geq 3$*.*

Proof. Proved in the same way as the previous corollary, where this time we put $\mathbf{OrthoLat}^{\mathrm{op}}$ in the bottom left corner. ∎

Next, we turn to the Pierce spectrum, which assigns to a commutative ring the Stone space of its Boolean algebra of idempotents.

7.3. Corollary. *Any functor* $\mathbf{Ring}^{\mathrm{op}} \to \mathbf{Loc}$ *that assigns to each commutative ring its Pierce spectrum trivializes on all* $\mathbb{M}_n(\mathbb{C})$ *for* $n \geq 3$*.*

Proof. Let $F : \mathbf{Ring}^{\mathrm{op}} \to \mathbf{Loc}$ be as in the statement. Let $\mathcal{C}(\mathbb{M}_n(\mathbb{C}))$ be the diagram of commutative self-adjoint subalgebras of $\mathbb{M}_n(\mathbb{C})$. As usual, we will argue that $\lim F\mathcal{C}(\mathbb{M}_n(\mathbb{C}))$ in **Loc** is initial. Consider the restriction $\overline{F}$ of F to **cNeumann**, and denote G for the functor that sends a commutative von Neumann algebra to its Gelfand spectrum. Since every projection is an idempotent, and the Gelfand spectrum of a commutative von Neumann algebra is given by the Stone space on its projections, there is a natural transformation $\overline{F} \Rightarrow G$. So if $\lim G\mathcal{C}(\mathbb{M}_n(\mathbb{C}))$ is the (strict) initial object in **Loc**, the same must be true for $\lim F\mathcal{C}(\mathbb{M}_n(\mathbb{C})) = \lim \overline{F}\mathcal{C}(\mathbb{M}_n(\mathbb{C}))$. ∎

8. Circumventing obstructions

It might be tempting to conclude from the above impossibility results that it is hopeless to look for a good notion of spectrum for noncommutative structures. But we strongly believe that this is the wrong conclusion to draw. What our results show is merely that a category of noncommutative spectra must have different limit behaviour from the known categories of commutative spectra. One of the central messages of category theory is that objects should be regarded as determined by their behaviour rather than by any internal structure. In other words, it is not the internal structure of objects that dictates what morphisms should preserve. It is the other way around: it is the morphisms connecting an object to others that determine that object's characteristics. Ideally, of course, both viewpoints coincide. But the latter viewpoint is better precisely when it is unclear what the objects should be. Historically, noncommutative spectra have almost always been pursued by generalizing the internal structure of commutative spaces (as objects). We believe the right, and optimistic, message to distill from our results is that one should let the search for noncommutative spectra be guided by morphisms instead. Indeed, the few proposals for noncommutative spectra that escape our obstructions have non-standard morphisms between them:

- There is a notion of noncommutative spectrum due to Akemann, Giles and Kummer [1, 10, 2]. It allows one to reconstruct the original C*-algebra, but the correspondence is only functorial for certain morphisms of C*-algebras.

- The so-called process of Bohrification gives a functor from the category of C*-algebras to localed toposes [12]. It involves some loss of information, however: one can only reconstruct the partial C*-algebra structure of the original C*-algebra [6]. Indeed the natural morphisms in this setting are partial *-homomorphisms.
- It is possible to construct a functor from the category of C*-algebras to the category of so-called quantum frames [24]. These structures only take into account the Jordan structure of the original C*-algebra, and this is reflected in the choice of morphisms. Indeed, there is no nondegenerate functor between the categories of quantum frames and that of quantales, so there is no contradiction with our results.
- A recent paper by Heunen and Reyes proposes a new notion of spectrum for arbitrary AW*-algebras [13]. It involves an action of the unitary group on the projection lattice, and therefore the natural morphisms are quite unlike those of topological spaces.

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