Modeling and control of congestion phenomena

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Chapter 8

Minimizing the M/G/1 LCFS-PR externalities variance

8.1 Introduction

Consider a single-server queue with a preemptive last-come first-served (LCFS-PR) service discipline to which customers arrive according to a Poisson process with rate $\lambda > 0$. The service demands of the customers are i.i.d. non-negative random variables with distribution function $G(\cdot)$, which are independent from the arrival process. Assume that the system is stable, i.e., $\rho \equiv \lambda\mu_1 < 1$, with, for each $i \geq 1$, μ_i the i-th moment of $G(\cdot)$. In addition, assume that there exist $n+1 \geq 1$ customers $c_1, c_2, \ldots, c_{n+1}$ in the system at time $t = 0$. Denote, for each $1 \leq i \leq n+1$, the remaining service time of c_i by v_i, and assume, without loss of generality, that c_i arrived before c_j for every $1 \leq i < j \leq n+1$. Notably, this is the same M/G/1 LCFS-PR queue as studied in the previous chapter, except for the fact that, at time $t = 0$, there are $n+1$ customers in the queue (as opposed to n). This adjustment is merely for notational convenience.

Now, assume that there is an additional customer c who arrives at time $t = 0$ and whose service demand is equal to $x > 0$. The externalities created by c can be expressed via a random variable $\mathcal{E}$ with a distribution which is determined by $\lambda, n, v_1, v_2, \ldots, v_{n+1}, x$ and $G(\cdot)$. In the previous chapter, the mean and variance of $\mathcal{E}$ were derived to be (cf. (7.8) and (7.10)):

$$\mathbb{E}\mathcal{E} = \frac{(n+1)x}{1-\rho}\,, \tag{8.1}$$

$$\mathrm{Var}\,(\mathcal{E}) = \frac{\lambda\mu_2}{(1-\rho)^3}\left[(n+1)x + 2\sum_{1\leq k\leq \ell\leq n}\left(x - \sum_{i=k}^{\ell} v_i\right)^+\right]. \tag{8.2}$$

It is very eminent that both of these formulae are independent of v_{n+1}. In addition, it is also a bit surprising that the expected externalities are invariant with respect to the values of $v_1, v_2, \ldots, v_n$.

Problem statement

Assume that the system has a manager who knows the statistical assumptions of the model. In addition, assume that, at time $t=0$, they observe the vector

$$\left(x, v_{n+1}, x+\sum_{i=1}^{n+1} v_i, n{+}1\right), \tag{8.3}$$

which includes four coordinates:

1. x, the remaining service time of the customer c who has just entered the service position.
2. v_{n+1}, the remaining service time of c_{n+1} who has just left the service position.
3. $x+\sum_{i=1}^{n+1} v_i$, the total workload in the system at $t=0$.
4. $n+1$, the number of customers who exist in the system at time $t=0$, i.e., the number of customers directly before the arrival of customer c.

In particular, this information scheme looks reasonable in settings in which the manager has limited memory capacity, and is unable to store the remaining service times of all customers.

With this information, the manager may compute $\mathbb{E}\mathcal{E}$ via (8.1), but they are unable to compute $\mathrm{Var}(\mathcal{E})$ via (8.2). Thus, the manager might be willing to compute the range of possible variance values given their available information at $t=0$, i.e., they need to derive the infimum and supremum of $\mathrm{Var}(\mathcal{E})$ over all parameterizations which are consistent with their available information at time $t=0$. Note that we refer to an infimum (resp. a supremum) problem and not to a minimum (resp. maximum) problem because the manager knows that there are $n+2$ customers in the system (including customer c). Thus, any parameterization which is consistent with this knowledge must be such that $v_i>0$ for every $1\le i\le n$. In this chapter, we focus on the infimum problem, as the supremum is evident. That is, since v_1 and v_n are both contributing to the least number of sums in (8.2), it is clear that $\mathrm{Var}(\mathcal{E})$ reaches its supremum if $(v_1, v_2, \dots, v_n)$ is either $(w, 0, 0, \dots, 0)$ or $(0, 0, \dots, 0, w)$, where $w \equiv \sum_{i=1}^{n} v_i$.

Eminently, when $w \ge nx$ the infimum problem is simple and hence we consider the next setup: let $n\in\mathbb{N}$, $x>0$, and $w\in(0, nx)$ be fixed parameters, and define a function

$$f(\boldsymbol{v}) \equiv \sum_{1\le k\le \ell\le n} \left(x-\sum_{i=k}^{\ell} v_i\right)^{+}, \quad \boldsymbol{v}=(v_1, v_2, \dots, v_n)\in\mathbb{R}_{+}^{n}.$$

Then, the infimum problem is translated to the minimization

$$\min_{\boldsymbol{v}\in\Lambda} f(\boldsymbol{v}), \tag{8.4}$$

where

$$\Lambda \equiv \left\{\boldsymbol{v}\in\mathbb{R}_{\ge 0}^{n}\ ;\ \sum_{i=1}^{n} v_i \le w\right\}. \tag{8.5}$$

Combinatorial version

Now, we present a combinatorial version of the continuous minimization (8.4). Specifically, it is stated as follows: with the parameters of the original problem, denote $m \equiv \left\lfloor \frac{w}{x} \right\rfloor$ and $r \equiv w - mx$. Then, define the domain

$$\Upsilon \equiv \left\{ \boldsymbol{v} \in \mathbb{R}^n_{\geq 0} \; ; \; \sum_{i=1}^{n} v_i \leq w \; , \; |\{i; v_i = x\}| = m \; , \; |\{i; v_i = r\}| \geq 1 \right\} .$$

Verbally, if $r > 0$, then Υ is the collection of all real n-dimensional vectors having m components which equal x, one component that equals r, and for which all other components have value zero. Otherwise, $w = mx$ (i.e., $r = 0$), and Υ is the collection of all real n-dimensional vectors that have m components which equal x, and for which all other components equal zero. Notice that Υ is a finite discrete set in both cases.

The combinatorial minimization to be studied in this chapter is:

$$\min_{\boldsymbol{v} \in \Upsilon} f(\boldsymbol{v}) . \tag{8.6}$$

Furthermore, we will explore its interplay with the continuous minimization (8.4). Also, besides being a straightforward combinatorial version of the continuous minimization (8.4), we note in passing that (8.6) might be associated with three additional interpretations which are explained below. We believe that each of these interpretations sheds some light on a different aspect of the combinatorial minimization (8.6). Especially, notice that the third interpretation is also relevant for the continuous minimization (8.4).

Example 8.1.1 (Card collection problem). Let $\mathcal{C} \equiv \{C_1, C_2, \ldots, C_n\}$ be a collection of n cards, m be an integer which is less than n, and define

$$\mathcal{C}_m \equiv \{\{C_{i_1}, C_{i_2}, \ldots, C_{i_m}\}; 1 \leq i_1 < i_2 < \ldots < i_m \leq n\} .$$

Any subset of successive cards is called a *series*, i.e., the set of all series is

$$\mathcal{S} \equiv \{\{C_k, C_{k+1}, \ldots, C_\ell\}\,; 1 \leq k \leq \ell \leq n\} .$$

In addition, there is a single player whose task it is to choose a sub-collection S of m cards that participates in the maximal number of series, i.e., they need to choose $S \in \mathcal{C}_m$ which maximizes

$$\sum_{S' \in \mathcal{S}} 1\{S \cap S' \neq \emptyset\} \quad , \quad S \in \mathcal{C}_m .$$

Notably, this problem coincides with (8.6) parameterized by $x = 1$, $m = w$ and $r = 0$.

Observe that when $r \in (0, 1)$, (8.6) is describing the following stochastic version of the deterministic card collection problem: the player is required to choose m cards and an additional card. The sub-collection of cards which the player possesses after making their choice is random. It will include the m cards for sure, but the additional card will be included with probability r. Then, the purpose of the player is to choose the m cards and the additional card in order to maximize the expected number of series which have non-empty intersection with their cards sub-collection. ◇

$$
\begin{array}{lllll}
(5,1) & & & & \\
(4,1) & (4,2) & & & \\
(3,1) & (3,2) & (3,3) & & \\
\hline
(2,1) & (2,2) & (2,3) & \boxed{(2,4)} & \\
(1,1) & (1,2) & (1,3) & (1,4) & (1,5)
\end{array}
$$

Figure 8.1: Example of a triangular lattice L with $n = 5$. Station (2,4) is activated, and serves all sites in the lower left rectangle.

Example 8.1.2 (Location problem)**.** Consider the triangular lattice

$$L \equiv \left\{(k, \ell) \in \mathbb{N}^2;\; k + \ell \leq n + 1\right\} ,$$

and assume that each point belonging to L represents a site location. In addition, assume there is a planner who should allocate $1 \leq m < n$ stations on the hypotenuse of the lattice. Assume that for each $1 \leq k \leq n$, the station which is located on (k, ℓ) provides service to all sites on

$$\{(k', \ell') \in L; k' \leq k \,, \ell' \leq \ell\} \,.$$

Spatially, this means that every station provides service to all sites which are neither higher than it nor to the right of it (see, e.g., Figure 8.1). Then, under this assumption, the planner's objective is to pick the allocation of the stations in order to maximize the number of sites where service is available. Note that this problem is a special case of (8.6) with the parameters $x = 1$, $w = m$ and $r = 0$.

For more maximal covering location problems, see a recent overview by García and Marín [79]. In addition, a slightly different class of covering problems is of the budgeted maximal coverage, see, e.g., the work of Khuller et al. [131]. ◇

Example 8.1.3 (Excess demand for public goods)**.** In economic theory, a *public good* is any good which is (i) *non-excludable*, i.e., there is no way to prevent its consumption by customers who are willing to do so, and (ii) *non-rivalrous*, i.e., its consumption by one customer does not limit the ability of other customers to consume it. Some traditional examples of public goods are national security, law enforcement, and clean air. In general, public goods are usually supplied by the government, since the private sector has very limited incentives to produce them (see, e.g., [198]). For more information about the theory and applications of public goods, see, e.g., the survey by Blümel et al. [22].

Now, consider an economy with n public goods $G_1, G_2, \ldots, G_n$, and $n(n+1)/2$ customers who are doubly-indexed by (k, ℓ) such that $1 \leq k \leq \ell \leq n$. The government has a budget of $w = mx$ monetary units and the cost of producing the allocation $\boldsymbol{v} \in \mathbb{R}^n_{\geq 0}$, in which for each $1 \leq i \leq n$ there are v_i units of the i-th good, is $x \sum_{i=1}^n v_i$. In addition, for each $1 \leq k \leq \ell \leq n$, the (k, ℓ)-th customer demands to consume one unit of goods from the set $\{G_k, G_{k+1}, \ldots, G_\ell\}$. In particular, this means that the (k, ℓ)-th customer is indifferent between the goods $G_k, G_{k+1}, \ldots, G_\ell$, i.e., they consider these goods as perfect substitutes. In order to demonstrate the market structure, one may think about $G_1, G_2, \ldots, G_n$ as food products (potentially supplied by the government) which are ordered according to their nutritional value. At the same time, $G_1, G_2, \ldots, G_n$ are reversely ordered with respect to their taste. Thus, the above-described market is such that the demand for the intermediate food products is higher than the demand for either healthy (but non-tasty) or tasty (but unhealthy) food products.

The government's objective is to minimize the total excess-demand of the customers in the economy given its budget constraint. Formally, if $G_1, G_2, \ldots, G_n$ are indivisible (resp. divisible) this problem is translated into (8.6) (resp. (8.4)). Note that, in both cases, the government's problem is how to choose the variety of products which are supplied to its citizens. For some other product variety management problems which appear in the existing literature, see, e.g., [140, 184] and the references therein. $\diamond$

Additional relevant literature

Technically, $f(\cdot)$ is a piecewise-linear convex function on the compact domain Λ, i.e., there exists an optimal solution of (8.4). Furthermore, this means that for any given parameterization of (8.4), standard subgradient methods yield a sequence which converges to the numerical value of an optimal solution with adequate convergence rates. A survey about subgradient methods can be found in [26].

In addition, observe that by using the epigraph of the convex piecewise-linear function $f(\cdot)$, the continuous minimization (8.4) can be rephrased as a linear program (LP). Therefore, it is possible to solve the continuous minimization (8.4) in polynomial time, see, e.g., [122]. In fact, it is clear that adding the constraints $v_i \leq x$ for each $1 \leq i \leq n$ will not cause any difference in the resulting minimization. As a result, (8.4) may also be phrased as an *interval linear program*. Thus, the analysis in [14, 15, 228, 229] includes an explicit expression of each optimal solution of (8.4) in terms of some projection matrix and the generalized inverse of the constraints matrix in the resulting linear program.

When $r = 0$, the combinatorial problem in (8.6) may be considered as a version of the 0-1 knapsack problem *with dependencies*. Notably, even when there are no dependencies between the items, the classic 0-1 knapsack problem is NP-hard [74]. To the best of our knowledge, there is not much work about knapsack problems with dependencies. This strand of literature includes the more traditional quadratic knapsack problem. A survey on that topic is given in [174]. In addition, consider the recent paper of Beliakov [10] about knapsack problems with dependencies through non-additive measures and a Choquet integral. Other non-standard versions of the knapsack problem are summarized in [149].

Recall the interpretation of the combinatorial minimization (8.6) as a card collection problem, and assume that $r = 0$. Then, there is also a relation to the *interval stabbing problems* which appear in the work of Schmidt [190]. This reference contains a discussion about the computational complexity of finding the number of series which have non-empty intersection with a given subset of m cards. However, to the best of our understanding, there has been no discussion about the derivation of the optimal subset of m cards, i.e., the subset which has a non-empty intersection with the maximal number of series.

Contributions and organization

This chapter includes a characterization of a set of highly-structured optimal solutions of (8.6). This yields an applicable numerical procedure to construct such an optimal solution of (8.6). In addition, it is shown that for some families of parameterizations, e.g., when $r = 0$, the continuous minimization (8.4) is reduced to the combinatorial one (8.6). Finally, we put forth a conjecture that this reduction is true for every parameterization.

The organization of this chapter is as follows: Section 8.2 considers the combinatorial minimization (8.6). Results regarding the continuous minimization (8.4) are described in Section 8.3. Finally, Section 8.4 discusses concluding remarks.

8.2 Combinatorial minimization

The main result of this section, which concerns a set of solutions for the combinatorial optimization (8.6), is presented in Section 8.2.1. This subsection additionally discusses some implementation details for the numerical derivation of such solutions. The proof of the main result is provided in Section 8.2.2.

8.2.1 Main result

Before being able to provide the main result (Theorem 8.2.6), we first need to introduce some definitions.

Definition 8.2.1. *Let $0 < a < b$ be some integers and $I_{a,b} \equiv \{a, a+1, \ldots, b\}$. Then, $j \in I_{a,b}$ is a middle point of the set $I_{a,b}$ if and only if*

$$\left\lfloor \frac{a+b}{2} \right\rfloor \le j \le \left\lceil \frac{a+b}{2} \right\rceil .$$

Remark 8.2.2. Note that once $b-a$ is odd, there is a unique middle point of $I_{a,b}$. However, if $b-a$ is even, there are two middle points of $I_{a,b}$. $\diamond$

Definition 8.2.3. *Assume that $w \in (0, nx)$, $m \equiv \lfloor \frac{w}{x} \rfloor$ and $r \equiv w - mx$. Then, for every positive integer $\delta \le n+1$, let $\Gamma_\delta \equiv \Gamma_\delta(x, w)$ be the set of solutions $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ with*

$$v_i \equiv \begin{cases} x & i \in \{\Delta_0, \Delta_0 + \Delta_1, \ldots, \sum_{k=0}^{m-1} \Delta_k\}\,, \\ r & i = j\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{8.7}$$

such that:

- *$\Delta_0, \Delta_1, \ldots, \Delta_m$ are positive integers for which:*
    1. *$\sum_{k=0}^{m} \Delta_k = n + 1$.*
    2. *$\max\{\Delta_k; 0 \le k \le m\} = \delta$.*
    3. *t is some element of $\operatorname{argmax}\{\Delta_k; 0 \le k \le m\}$ and $|\Delta_{k_1} - \Delta_{k_2}| \le 1$, for every $k_1, k_2 \in \{0, 1, \ldots, m\} \setminus \{t\}$.*
- *j is a middle element of the set*

$$\left\{ \sum_{k=0}^{t-1} \Delta_k + 1, \sum_{k=0}^{t-1} \Delta_k + 2, \ldots, \sum_{k=0}^{t-1} \Delta_k + \delta - 1 \right\} .$$

Remark 8.2.4. When $w < x$, $m = 0$, and Γ_δ includes all solutions of the form (8.7) which are parameterized with $\Delta_0 = \delta = n+1$, $t = 0$, and j which is a middle point of $\{1, 2, \ldots, n\}$. $\diamond$

Observe that the conditions $w \in (0, nx)$ and $x > 0$ imply that $n \geq 1$. In particular, when $n = 1$ we know that $w < x$, such that there is a unique optimal solution of (8.6) which equals $\boldsymbol{v} = w$. Observe that, as explained in Remark 8.2.4, it is the only element of Γ_2. Theorem 8.2.6 below refers to the case $n \geq 2$.

Definition 8.2.5. *Let $a < b$ be two positive integers. In addition, assume that $\Delta_0(a,b)$, $\Delta_1(a,b), \ldots, \Delta_{a-1}(a,b)$ are positive integers whose sum is b and that satisfy the condition:*

$$\max_{0 \leq k_1, k_2 \leq a-1} |\Delta_{k_1}(a,b) - \Delta_{k_2}(a,b)| \leq 1 \,.$$

Then, define

$$h(a,b) \equiv \frac{1}{2} \sum_{k=0}^{a-1} \Delta_k(a,b) \left[\Delta_k(a,b) - 1\right]. \tag{8.8}$$

It is a trivial exercise to show that, equivalently,

$$h(a,b) \equiv \frac{1}{2} \left\{ (b \bmod a) \left(\left\lfloor \frac{b}{a} \right\rfloor + 1 \right) \left\lfloor \frac{b}{a} \right\rfloor + \left[a - (b \bmod a)\right] \left\lfloor \frac{b}{a} \right\rfloor \left(\left\lfloor \frac{b}{a} \right\rfloor - 1 \right) \right\}.$$

Now we are ready to state the main result:

Theorem 8.2.6. *Assume that $w < nx$ and $n \geq 2$. In addition, denote $m \equiv \left\lfloor \frac{w}{x} \right\rfloor$ and $r \equiv w - mx$.*

1. *If $m = 0$, the set of optimal solutions of (8.6) is $\mathrm{Conv}(\Gamma_{n+1})$ (see also Remark 8.2.4).*
2. *If $m \geq 1$, define two functions:*

$$\begin{aligned} A(\delta) \equiv A(\delta; r) &\equiv x \left[h\left(m, n{+}1{-}\delta\right) + \frac{1}{2}\delta(\delta{-}1) \right] - r \left\lfloor \frac{\delta}{2} \right\rfloor \left\lceil \frac{\delta}{2} \right\rceil, \\ \phi(\delta) &\equiv (2\delta + 1)\left(x - \frac{r}{2}\right) - x \left(\left\lfloor \frac{n - \delta - 1}{m} \right\rfloor + \left\lfloor \frac{n - \delta}{m} \right\rfloor \right) - \frac{r}{2}. \end{aligned}$$

In addition, denote

$$\delta_1 \equiv \min\{\delta \in \{1, 3, 5, \ldots\}\, ;\, \phi(\delta) > 0\} \tag{8.9}$$

$$\delta_2 \equiv \min\{\delta \in \{2, 4, 6, \ldots\}\, ;\, \phi(\delta) > 0\} \tag{8.10}$$

and set

$$\delta^* \equiv \begin{cases} \delta_1 & \text{if } A(\delta_1) < A(\delta_2)\,, \\ \delta_2 & \text{otherwise}\,. \end{cases} \tag{8.11}$$

Then, δ^* is non-decreasing in r and

$$\mathrm{Conv}\left(\Gamma_{\delta^*}\right) \subseteq \arg\min_{\boldsymbol{v} \in \Upsilon} f(\boldsymbol{v}) \,. \tag{8.12}$$

In particular, if $r = 0$, then $\delta^* = 1 + \left\lfloor \frac{n}{m+1} \right\rfloor = \left\lceil \frac{n+1}{m+1} \right\rceil$.

Notably, when either $m = 0$ or $r = 0$, Theorem 8.2.6 includes a precise construction of an optimal solution of the combinatorial minimization (8.6) which is straightforward to implement. Otherwise, when $m \geq 1$ and $r > 0$, Theorem 8.2.6 implies that the derivation of an optimal solution of (8.6) boils down to the computations of δ_1 and δ_2 which are respectively given in (8.9) and (8.10). In what follows, it is explained how to execute these computations.

To begin with, observe that δ_1 and δ_2 are bounded by the solutions of the next two equations in δ:

$$(2\delta + 1)\left(x - \frac{r}{2}\right) - x\left(\frac{n-\delta-1}{m} + \frac{n-\delta}{m}\right) - \frac{r}{2} = \pm 2\,.$$

Thus, introducing

$$\delta_- \equiv \frac{\frac{r}{2} + x\frac{2n-1-m}{2m} - 1}{x\left(1+\frac{1}{m}\right) - \frac{r}{2}} \quad \text{and} \quad \delta_+ \equiv \frac{\frac{r}{2} + x\frac{2n-1-m}{2m} + 1}{x\left(1+\frac{1}{m}\right) - \frac{r}{2}}\,,$$

it is deduced that $\delta_- \leq \delta_1, \delta_2 \leq \delta_+$. This means that, with

$$\mathcal{D} \equiv \{\lceil\delta_-\rceil, \lceil\delta_-\rceil + 1, \ldots, \lfloor\delta_+\rfloor\}\,,$$

the search for δ_1 (resp. δ_2) may be limited to a set of integers $\mathcal{D}_1 \equiv \mathcal{D} \cap \{1, 3, 5, \ldots\}$ (resp. $\mathcal{D}_2 \equiv \mathcal{D} \cap \{2, 4, 6, \ldots\}$). In particular, since $r \leq x$ and $m \geq 1$, we know

$$\delta_+ - \delta_- = \frac{2}{x\left(1+\frac{1}{m}\right) - \frac{r}{2}} \leq \frac{4}{x}\,.$$

Therefore, we get an upper bound

$$\max_{i=1,2} |\mathcal{D}_i| \leq \frac{1}{2}\left\lceil\frac{4}{x}\right\rceil + 1\,,$$

which is fixed in all parameters except x. Furthermore, observe that

$$\delta \mapsto (2\delta + 1)\left(x - \frac{r}{2}\right) - x\left(\left\lfloor\frac{n-\delta-1}{m}\right\rfloor + \left\lfloor\frac{n-\delta}{m}\right\rfloor\right)$$

is increasing in δ. As a result, δ_1 (resp. δ_2) may be derived via an application of a standard bisection method on the set $\mathcal{D}_1$ (resp. $\mathcal{D}_2$). Observe that the definition $\phi(\cdot)$ does not involve any complicated loops, such that the application of the bisection method is relatively inexpensive.

Lastly, once δ_1 and δ_2 are computed, (8.11) dictates that we need to call $A(\cdot)$ twice to decide whether $\delta^* = \delta_1$ or $\delta^* = \delta_2$. Importantly, by definition (recall also the definition of $h(\cdot)$ in (8.8)), the computation of $A(\cdot)$ involves only standard arithmetical operations. As a result, we may conclude that Theorem 8.2.6 yields an applicable numerical procedure for obtaining an optimal solution of (8.6).

8.2.2 Proof

Before providing the proof of Theorem 8.2.6, we would like to demonstrate some of its guidelines by an example.

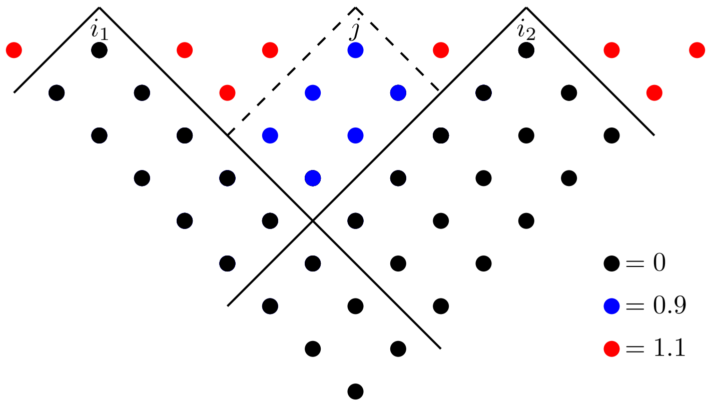

Figure 8.2: Representing $f(\cdot)$ as the sum of the values of the individual balls when the parameters are $n = 9$, $x = 1.1$ and $w = 2.4$.

Example 8.2.7. Consider the special case $n = 9$, $x = 1.1$, and $w = 2.4$. Each solution $\boldsymbol{v} = (v_1, \ldots, v_9)$ is characterized by a set $\{i_1, i_2, j\} \subseteq \{1, \ldots, 9\}$, $i_1 < i_2$, such that $v_{i_1} = v_{i_2} = 1.1$, $v_j = 0.2$ and $v_i = 0$ otherwise.

Observe that by constructing a triangle as in the example displayed in Figure 8.2, in which the value of the k-th ball in row ℓ is given by $x - \sum_{i=k}^{\ell+k-1} v_i$, for each choice of $\{i_1, i_2, j\}$ the value of $f(\boldsymbol{v})$ can be found by summing over the values of the individual balls in such a triangle. Notably, for each choice of i_1, i_2, there are at most three sub-triangles whose mass is unequal to 0, and within one of these triangles there is a rectangle within which the balls attain mass 0.9. Therefore, with $\Delta_1 = i_1, \Delta_2 = i_2 - i_1$ and $\Delta_3 = 10 - i_2$, for $i_1 < j < i_2$, we may write

$$f(\boldsymbol{v}) = 1.1 \cdot \left[\frac{\Delta_1(\Delta_1 - 1)}{2} + \frac{\Delta_2(\Delta_2 - 1)}{2} + \frac{\Delta_3(\Delta_3 - 1)}{2}\right] - 0.2 \cdot (j - i_1)(i_2 - j).$$

Now, given i_1 and i_2, observe that the number of balls within the rectangular area is maximal if j is located at a middle point of the hypotenuse of the largest triangle (note that there are two middle points when the hypotenuse consists of an even number of balls). Thus, i_1, i_2 should minimize

$$f(\boldsymbol{v}) = 1.1 \cdot \sum_{i=1}^{3} \frac{\Delta_i(\Delta_i - 1)}{2} - 0.2 \cdot \left\lfloor \frac{\Delta_t}{2} \right\rfloor \left\lceil \frac{\Delta_t}{2} \right\rceil,$$

with $\Delta_t = \max\{\Delta_1, \Delta_2, \Delta_3\}$. ◇

There are two observations that turn out to be useful in the formal proof of Theorem 8.2.6. First, notice that in Definition 8.2.5, it is always possible to choose $\{\Delta_k(a, b)\}_{k=0}^{a-1}$ with $\Delta_0(a, b) = \left\lfloor \frac{b}{a} \right\rfloor$. Thus, it is possible to choose $\Delta_0(a, b+1), \Delta_1(a, b+1), \ldots, \Delta_{a-1}(a, b+1)$ such that for every $0 \le k \le a - 1$:

$$\Delta_k(a, b+1) \equiv \begin{cases} \Delta_0(a, b) + 1 & k = 0\,, \\ \Delta_k(a, b) & \text{otherwise}\,, \end{cases}$$

and hence

$$\begin{aligned} 2\,[h(a, b+1) - h(a, b)] &= [\Delta_0(a, b) + 1]\,\Delta_0(a, b) - \Delta_0(a, b)\,[\Delta_0(a, b) - 1] \\ &= 2\Delta_0(a, b) = 2 \left\lfloor \frac{b}{a} \right\rfloor. \end{aligned} \tag{8.13}$$

The second useful observation is that for any positive integer δ,

$$4\left\lfloor\frac{\delta}{2}\right\rfloor\left\lceil\frac{\delta}{2}\right\rceil=\begin{cases}\delta^2 & \delta \text{ is even,}\\ \delta^2-1 & \delta \text{ is odd,}\end{cases}$$

and hence

$$\eta(\delta)\equiv\left\lfloor\frac{\delta+1}{2}\right\rfloor\left\lceil\frac{\delta+1}{2}\right\rceil-\left\lfloor\frac{\delta}{2}\right\rfloor\left\lceil\frac{\delta}{2}\right\rceil=\frac{1}{2}\begin{cases}\delta & \delta \text{ is even,}\\ \delta+1 & \delta \text{ is odd.}\end{cases} \tag{8.14}$$

Now, we are ready to provide the proof of Theorem 8.2.6.

Proof of Theorem 8.2.6. For every $\{i_1,i_2,\ldots,i_m,j\}\subseteq\{1,2,\ldots,n\}$ define, for $i\in\{1,\ldots,n\}$,

$$v_i(i_1,i_2,\ldots,i_m,j)\equiv\begin{cases}x & i\in\{i_1,i_2,\ldots,i_m\}\,,\\ r & i=j\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

and the resulting solution

$$\boldsymbol{v}(i_1,i_2,\ldots,i_m,j)=\left(v_i(i_1,i_2,\ldots,i_m,j)\right)_{1\le i\le n}\,.$$

Without loss of generality, assume that $i_1<i_2<\ldots<i_m$ (otherwise sort these indices and rename them properly). In addition, let $(i_0,i_{m+1})\equiv(0,n+1)$ and for each $0\le k\le m$ define $\Delta_k\equiv i_{k+1}-i_k$. Furthermore, denote $t\equiv\max\{k\ge 0;i_k\le j\}$ and observe that the objective function is given by:

$$\begin{aligned}g(i_1,i_2,\ldots i_m,j)&\equiv f\left[\boldsymbol{v}(i_1,i_2,\ldots,i_m,j)\right]\\&=x\sum_{k=0}^{m}\psi\left(\Delta_k\right)-r(j-i_t)(i_{t+1}-j)\,,\end{aligned} \tag{8.15}$$

where $\psi(y)\equiv\frac{y(y-1)}{2}$, $y\ge 1$. Now, given fixed $i_1,i_2,\ldots,i_m$, then (8.15) reveals that the optimal j must be a middle point of $\{i_t+1,i_t+2,\ldots,i_{t+1}-1\}$. Notably, once there are two middle points, a symmetry argument implies that they yield the same objective value and hence they are equivalent. This observation completes the proof for the first theorem's statement due to the convexity of $f(\cdot)$.

In order to complete the proof we need to assume that $m\ge 1$ and it is left to minimize

$$\widetilde{g}(\Delta_0,\Delta_1,\ldots,\Delta_m)\equiv x\sum_{k=0}^{m}\psi(\Delta_k)-r\left\lfloor\frac{\Delta_t}{2}\right\rfloor\left\lceil\frac{\Delta_t}{2}\right\rceil\,.$$

In particular, $\psi(\cdot)$ is strictly convex and increasing on $[1,\infty)$. Consequently, the optimal $\{\Delta_k;0\le k\le m\}$ is characterized by the next conditions:

1. $\Delta_0+\Delta_1+\ldots+\Delta_m=n+1$.
2. $\Delta_t=\max\{\Delta_k;0\le k\le m\}$ and
   $|\Delta_{k_1}-\Delta_{k_2}|\le 1$, for every $k_1,k_2\in\{0,1,\ldots,m\}\setminus\{t\}$.

3. Δ_t is a minimizer of the function $A(\delta)$ (in δ).

Especially, (8.13) and (8.14) imply that

$$\begin{aligned}\mathrm{d}_1 A(\delta) &\equiv A(\delta+1) - A(\delta)\\ &= x\left(\delta - \left\lfloor \frac{n-\delta}{m} \right\rfloor\right) - r\eta(\delta)\\ &= \delta\left(x - \frac{r}{2}\right) - x\left\lfloor \frac{n-\delta}{m} \right\rfloor - \frac{r\mathbf{1}_{\{1,3,5,\ldots\}}(\delta)}{2}\,.\end{aligned}$$

Consequently, deduce that

$$\mathrm{d}_2 A(\delta) \equiv A(\delta+2) - A(\delta) = \mathrm{d}_1 A(\delta+1) + \mathrm{d}_1 A(\delta) = \phi(\delta)\,.$$

Since $\mathrm{d}_2 A(\cdot)$ is increasing in δ, the minimizer of $A(\cdot)$ must be δ^* as given in the statement of the theorem, and hence (8.12) stems from the convexity of $f(\cdot)$.

When $r = 0$, $\mathrm{d}_1 A(\cdot)$ is also increasing in δ, such that δ^* is equivalently characterized by the minimum $\delta > 0$ for which $\mathrm{d}_1 A(\delta) > 0$, which yields Δ_t for this case. It is left to prove that δ^* is non-decreasing in r. To this end, take $0 \le r_1 < r_2 < x$ and notice that for every δ, the difference

$$A(\delta; r_2) - A(\delta; r_1) = -(r_2 - r_1)\left\lfloor \frac{\delta}{2} \right\rfloor \left\lceil \frac{\delta}{2} \right\rceil$$

is decreasing in δ. Thus, Theorem 10.6 in [202] implies the required result. □

8.3 Continuous minimization

Considering the continuous minimization problem posed in (8.4), Section 8.3.1 presents a set of optimal solutions in case $r = 0$, whereas Section 8.3.2 discusses the case $0 < r < x$.

8.3.1 When $r = 0$

Theorem 8.3.1 below reveals that once $r = 0$, the class of optimal solutions of (8.6) stated in Theorem 8.2.6 is actually optimal in the broader (compact) minimization domain Λ defined in (8.5). Before providing its statement, we present the following useful notations:

$$\tau_u \equiv \tau_u(n,m) \equiv \left\lceil \frac{n+1}{m+1} \right\rceil \quad , \quad \tau_l \equiv \tau_l(n,m) \equiv \left\lfloor \frac{n+1}{m+1} \right\rfloor\,.$$

Theorem 8.3.1. *If $r = 0$, i.e., $w = mx$ for some positive integer m, then*

$$\mathrm{Conv}\left(\Gamma_{\tau_u}\right) \subseteq \arg\min_{\boldsymbol{v}\in\Lambda} f(\boldsymbol{v})$$

and

$$\min_{\boldsymbol{v}\in\Lambda} f(\boldsymbol{v}) = (\tau_u - 1)\left(x(n+1) - \frac{1}{2}(w+x)\tau_u\right)\,.$$

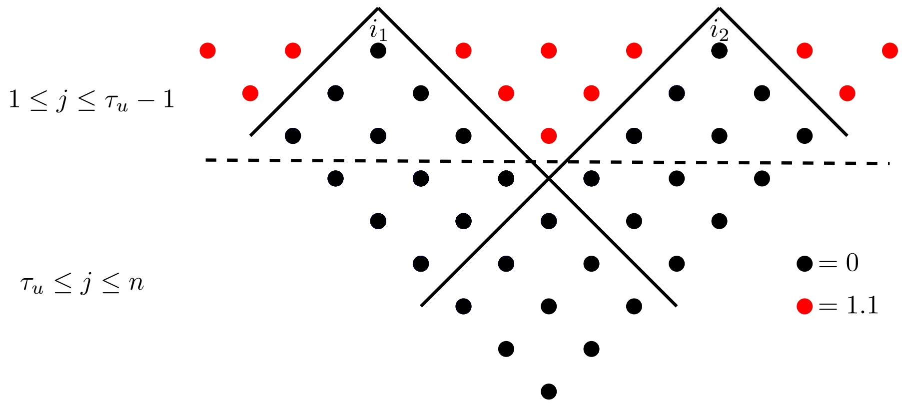

Figure 8.3: Representing $f(\cdot)$ as the sum of the values of the individual balls when the parameters are $n = 9$, $x = 1.1$ and $w = 2.2$. For each $1 \leq j \leq n$, the function $f_j(\cdot)$ is the sum of the balls in row j. The optimal solution spreads the mass in almost equal gaps, i.e., a minimum gap of $\tau_l = 3$, and a maximum gap of $\tau_u = 4$. For each $j \geq \tau_u$, the balls in row j have no mass, whereas for each row $1 \leq j \leq \tau_l$, there is no overlap between the balls affected by the different vector points with mass.

Remark 8.3.2. Note that construction of some vector in Γ_{τ_u} is not computationally demanding. Therefore, when $r = 0$, Theorem 8.3.1 yields a natural numerical procedure which is easy to implement in order to solve the continuous minimization (8.4). ◇

Proof of Theorem 8.3.1. Let $\boldsymbol{v}^*$ be an element of Γ_{τ_u}. In addition, for any $1 \leq j \leq n$ and $\boldsymbol{v} \in \Lambda$, define

$$L_j \equiv \left\{(k, \ell) \in \mathbb{N}^2 \; ; \; k \leq \ell \leq n \; , \; \ell - k + 1 = j\right\}$$

and

$$f_j(\boldsymbol{v}) \equiv \sum_{(k,\ell) \in L_j} \left(x - \sum_{i=k}^{\ell} v_i\right)^+ ,$$

such that

$$f(\boldsymbol{v}) = \sum_{j=1}^{n} f_j(\boldsymbol{v}) \,. \tag{8.16}$$

In what follows, we will prove that $f_j(\boldsymbol{v}) \geq f_j(\boldsymbol{v}^*)$ for any $1 \leq j \leq n$ and $\boldsymbol{v} \in \Lambda$, thereby providing an explicit expression for $f_j(\boldsymbol{v}^*)$. This will end the proof since $f(\cdot)$ is convex, and the minimum objective function is found through (8.16) with input $\boldsymbol{v}^*$.

To begin with, as illustrated in in Figure 8.3, observe that

$$\max\left\{i_2 - i_1; 1 \leq i_1 < i_2 \leq n \;\text{ s.t. } v^*_{i_1} = v^*_{i_2} = x, v^*_{i_1+1} = \ldots = v^*_{i_2-1} = 0\right\}$$

is bounded from above by τ_u. Consequently, $f_j(\boldsymbol{v}^*) = 0$ for every $j \geq \tau_u$, i.e., it is left to consider $1 \leq j \leq \tau_u - 1$. To this end, consider some $1 \leq j \leq \tau_u - 1$ and recall that

$$\min\left\{i_2 - i_1; 1 \leq i_1 < i_2 \leq n \;\text{ s.t. } v^*_{i_1} = v^*_{i_2} = x\right\} = \tau_l \geq \tau_u - 1 \,.$$

Thus, since $\sum_{i=k}^{\ell} v^*_i \leq x$ for every $(k, \ell) \in L_j$, deduce that

$$\left|\left\{(k, \ell) \in L_j \; ; \; \sum_{i=k}^{\ell} v^*_i = x\right\}\right| = jm \,,$$

and hence

$$f_j(\boldsymbol{v}^*) = \sum_{(k,\ell)\in L_j} \left(x - \sum_{i=k}^{\ell} v_i^* \right)$$
$$= (n{+}1{-}j)x - \sum_{(k,\ell)\in L_j} \sum_{i=k}^{\ell} v_i^*$$
$$= (n{+}1{-}j)x - jmx$$
$$= (n{+}1{-}j)x - jw\,.$$

Now, consider some $\boldsymbol{v} \in \Lambda(m) \cap [0,x]^n$ and observe that

$$f_j(\boldsymbol{v}) \geq \sum_{(k,\ell)\in L_j} \left(x - \sum_{i=k}^{\ell} v_i \right) = (n+1-j)x - \sum_{(k,\ell)\in L_j} \sum_{i=k}^{\ell} v_i\,.$$

Note that for each $1 \leq i \leq n$ we have

$$|\{(k,\ell) \in L_j; k \leq i \leq \ell\}| \leq j\,,$$

and hence

$$\sum_{(k,\ell)\in L_j} \sum_{i=k}^{\ell} v_i \leq jw\,,$$

from which the result follows. □

8.3.2 When $0 < r < x$

It is natural to check whether Theorem 8.3.1 can be extended to the case $0 < r < x$. Example 8.3.3 below demonstrates that such a generalization is not straightforward.

Example 8.3.3. Consider the parameterization $n = 7$, $x = 1$, and $w = 2.2$. According to Theorem 8.2.6, one optimal solution of (8.6) is given by $\boldsymbol{v}_1 = (0, 1, 0, 0, 1, 0.2, 0)$, with $f(\boldsymbol{v}_1) = 6.6$. However, $\boldsymbol{v}_1$ is a sub-optimal solution of (8.4), as the vector $\boldsymbol{v}_2 = (0, 0.2, 0.8, 0.2, 0.8, 0.2, 0)$ yields $f(\boldsymbol{v}_2) = 6.4$. ◇

In Example 8.3.3, $\boldsymbol{v}_2$ can be written as a convex combination of $\boldsymbol{v} = (0, 0, 1, 0, 1, 0, 0) \in \Gamma_{\tau_u(n,m)}$ and $\tilde{\boldsymbol{v}} = (0, 1, 0, 1, 0, 1, 0) \in \Gamma_{\tau_u(n,m+1)}$, which are optimal solutions when $w = mx$ and $w = (m+1)x$, respectively. Therefore, we still get a reduction of the continuous minimization (8.4) to the combinatorial one (8.6). The following Theorem 8.3.4 expands this observation to a wider family of parameterizations.

Theorem 8.3.4. *Denote $y \equiv y(x,w) \equiv x - r$, $\tau_{u,1} \equiv \tau_u(n,m)$, $\tau_{l,1} \equiv \tau_l(n,m)$, $\tau_{u,2} \equiv \tau_u(n,m+1)$, and $\tau_{l,2} \equiv \tau_l(n,m+1)$, and consider any two vectors $\boldsymbol{v}_y \in \Gamma_{\tau_{u,1}}(y, my)$ and $\boldsymbol{v}_r \in \Gamma_{\tau_{u,2}}(r, (m+1)r)$. If $\tau_{u,1} = \tau_{u,2}$ or $\tau_{l,1} = \tau_{l,2}$, then*

$$\boldsymbol{v}^* \equiv \boldsymbol{v}^*(\boldsymbol{v}_y, \boldsymbol{v}_r) \equiv \boldsymbol{v}_y + \boldsymbol{v}_r$$

is such that

$$f(\boldsymbol{v}^*) = \min_{\boldsymbol{v}\in\Lambda} f(\boldsymbol{v}) = (\tau_{u,1} - 1)\left(x(n+1) - \frac{1}{2}(w+x)\tau_{u,1} \right)\,.$$

Remark 8.3.5. Prominently, the objective values in Theorems 8.3.1 and 8.3.4 are given by the same formula. Considering a representation as in Figure 8.3, an optimal vector from Theorem 8.3.1 allocates the elements with mass such that, before row τ_u, no elements with mass affect the same balls, and from row τ_u onward, all balls are zero. A similar idea is true for an optimal vector in Theorem 8.3.4: no two elements whose masses sum to x affect the same balls before τ_u, and from τ_u onward, all balls have zero mass. $\diamond$

Example 8.3.6. Theorem 8.3.4 is *not* an extension of Theorem 8.3.1. For example, if $n = 7, x = 1$, and $w = 2.2$, then *only* Theorem 8.3.4 can be applied. To see this, observe that $m = \left\lfloor \frac{2.2}{1} \right\rfloor = 2$ and $r = 2.2 - 2 \cdot 1 > 0$, i.e., it is impossible to apply Theorem 8.3.1. On the other hand, notice that

$$\tau_l(n,m) = \left\lfloor \frac{8}{3} \right\rfloor = 2 = \left\lfloor \frac{8}{4} \right\rfloor = \tau_l(n,m+1).$$

and hence Theorem 8.3.4 may be applied.

Similarly, Theorem 8.3.1 is *not* an extension of Theorem 8.3.4. For example, if $n = 8, x = 1$, and $w = 3$, then *only* Theorem 8.3.1 can be applied. To see this, observe that $m = \left\lfloor \frac{3}{1} \right\rfloor = 3$ and $r = 3 - 3 \cdot 1 = 0$, i.e., Theorem 8.3.1 may be applied. On the other hand, notice that

$$\tau_l(n,m) = \left\lfloor \frac{9}{4} \right\rfloor \neq \left\lfloor \frac{9}{5} \right\rfloor = \tau_l(n,m+1) \quad , \quad \tau_u(n,m) = \left\lceil \frac{9}{4} \right\rceil \neq \left\lceil \frac{9}{5} \right\rceil = \tau_u(n,m+1)$$

and hence it is impossible to apply Theorem 8.3.4. $\diamond$

Proof of Theorem 8.3.4. Define L_j and $f_j(\boldsymbol{v})$ as in the proof of Theorem 8.3.1. Then, observe that for each $z \in \{y, r\}$

$$\max \{i_2 - i_1; 1 \leq i_1 < i_2 \leq n \ \text{ s.t. } v_{z,i_1} = v_{z,i_2} = z, v_{z,i_1+1} = \ldots = v_{z,i_2-1} = 0\}$$

is bounded from above by $\tau_{u,1}$. Therefore, the fact that $y + r = x$ implies that $f_j(\boldsymbol{v}^*) = 0$ for every $j \geq \tau_{u,1}$. Moreover, observe that for each $z \in \{y, r\}$,

$$\min \{i_2 - i_1; 1 \leq i_1 < i_2 \leq n \ \text{ s.t. } v_{z,i_1} = v_{z,i_2} = z\} \geq \tau_{l,2}\,.$$

In case that $\tau_{u,1} = \tau_{u,2}$, then $\tau_{l,2} \geq \tau_{u,2} - 1 = \tau_{u,1} - 1$. In addition, if $\tau_{l,1} = \tau_{l,2}$, then we get $\tau_{l,2} = \tau_{l,1} \geq \tau_{u,1} - 1$. Thus, for $1 \leq j \leq \tau_{u,1} - 1$ we shall apply the same arguments as in the proof of Theorem 8.3.1, together with the fact that $y + r = x$. Altogether, this implies that for any $\boldsymbol{v} \in \Lambda \cap [0, x]^n$

$$\begin{aligned}
f_j(\boldsymbol{v}^*) &= \sum_{(k,\ell)\in L_j} \left(x - \sum_{i=k}^{\ell} v_i^* \right) \\
&= (n+1-j)x - \sum_{(k,\ell)\in L_j} \sum_{i=k}^{\ell} v_{y,i} - \sum_{(k,\ell)\in L_j} \sum_{i=k}^{\ell} v_{r,i} \\
&= (n+1-j)x - jmy - j(m+1)r \\
&= (n+1-j)x - jw \leq f_j(\boldsymbol{v})\,.
\end{aligned}$$

Finally, the objective value of $\boldsymbol{v}^*$ is found through (8.16). $\square$

It is natural to look for a unifying generalization of Theorems 8.3.1 and 8.3.4. The following conjecture has been verified by many numerical examples in Mathematica.

Conjecture 8.3.7. *Denote* $y \equiv y(x,w) \equiv x - r$, $\tau_1 \equiv \tau_u(n,m)$, $\tau_2 \equiv \tau_u(n,m+1)$, *and consider any two vectors* $\boldsymbol{v}_y \in \Gamma_{\tau_1}(y, my)$ *and* $\boldsymbol{v}_r \in \Gamma_{\tau_2}(r,(m+1)\,r)$. *Then,*

$$\boldsymbol{v}^* \equiv \boldsymbol{v}^*(\boldsymbol{v}_y, \boldsymbol{v}_r) \equiv \boldsymbol{v}_y + \boldsymbol{v}_r$$

is an optimal solution of (8.4).

Remark 8.3.8. Observe that when $r = 0$, Conjecture 8.3.7 becomes Theorem 8.3.1. Moreover, note that the case $m \geq \lfloor \frac{n}{2} \rfloor$ is trivial, as $f_j(\boldsymbol{v}^*) = 0$ for any $j \geq 2$, and $f_1(\boldsymbol{v})$ has the same value for each $\boldsymbol{v} \in \Lambda \cap [0,x]^n$. Therefore, Theorem 8.3.4 implies that it is left to prove Conjecture 8.3.7 under the assumptions $0 < r < x$, $m < \lfloor \frac{n}{2} \rfloor$, and $\tau_u(n,m) \neq \tau_u(n,m+1)$. $\diamond$

Remark 8.3.9. Note that if Conjecture 8.3.7 is true, the convexity of both $f(\cdot)$ and Λ imply that the resulting convex hull is a subset of optimal solutions. $\diamond$

Remark 8.3.10. Observe that the construction of $\boldsymbol{v}^*$ is based on optimal solutions of the combinatorial minimization (8.6). Therefore, we conjecture that the reduction of the continuous minimization (8.4) to the combinatorial minimization (8.6) is valid for any parameterization. $\diamond$

8.4 Concluding remarks

This chapter has been dedicated to the study of the minimization of the externalities in the LCFS-PR M/G/1 queue on the discrete domain Υ (see (8.6)), as well as on the compact domain Λ (see (8.4)). In the combinatorial case we were successful in characterizing a set of optimal solutions. For the continuous setting, we were able to prove the minimal variance value under some specific parameterizations.

Prominently, the incomplete part of this chapter is the proof of Conjecture 8.3.7. We anticipate that its proof should follow from (at least) one of the following directions: (i) phrasing the problem as a dynamic program, which can be solved by an application of Theorem 8.3.1, or (ii) applying the theory of Lagrange multipliers in the context of sub-differential calculus (see, e.g., [187]). Our experience teaches that while both of these approaches look reasonable, they are not straightforward to apply. Therefore, providing the proof of Conjecture 8.3.7 is still a problem for future research.