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### Information processing in complex networks

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## Appendix A

# Appendix to Chapter 2

### A1 Methodology of computer simulations

In addition to the numerical calculation of  $D(s_i)$  we measure it experimentally using computer simulations. We simulate 6000 ferromagnetic Ising spins connected by a heavy-tailed network of interactions, where the number of connections of a spin, or degree, is distributed as  $p(k) \propto k^{-\gamma}$ . An Ising spin can be in one of the two states  $\{-1, +1\}$ . Each spin  $s$  only interacts with its nearest-neighbors, and the local energy of a spin state is

$$\text{En}(s^t) = - \sum_{\{r:(s,r) \in E\}} s^t \cdot r^t, \quad (5)$$

i.e., there is no external magnetization. The global energy function is

$$\text{En}(S^t) = - \sum_{(s,r) \in E} s^t \cdot r^t. \quad (6)$$

The network topology is generated randomly at the start of a new simulation, but is kept constant during a simulation. On average, the topology is uncorrelated except for the degree distribution, i.e., there are no other topological features imposed such as additional assortativity, community structure, or transitivity(7).

The unit states are updated according to the Metropolis-Hastings algorithm. That is, at each time step, one random unit is selected which attempts to flip its state according to the energy of the two possible states at that time. If the

alternative spin state has a lower energy then it is always selected; otherwise the spin flips its state according to the probability  $e^{(\text{En}(s^t) - \text{En}(-s^t))/T}$ .

The purpose of the simulations is to estimate for each spin  $s_i$  the amounts of mutual information  $I(s_i^{t-\Delta} | S^t)$  between its instantaneous state at a previous time  $t - \Delta$  and that of the entire network at time  $t$ , for  $\Delta \geq 0$ . The rate at which this information decays to zero for increasing  $\Delta$  is a measure of the information dissipation time  $D(s_i)$  of unit  $s_i$ . In other words, it is the time that the system state ‘remembers’ the unit’s state.

To do this we first equilibrate the system by performing  $10^6$  unit state updates: the system state at this point is selected as the ‘snapshot’ system state  $S^t$ . Then we sample  $9 \cdot 10^4$  system state trajectories  $S^{t-90,000}, \dots, S^t$  that lead to  $S^t$ , each consisting of  $9 \cdot 10^4$  spin flip attempts. Since it is too time-consuming to generate random state trajectories and select the ones that lead to  $S^t$ , we ‘back-track’ from  $S^t$  by generating a precedent state  $S^{t-1}$ , and from that state we generate again a precedent state  $S^{t-2}$ , and so on. In this manner we ensure that each generated state trajectory leads to  $S^t$ . Each precedent state to  $S^t$  for any  $t$  is generated according to the probability density  $p(S^{t-1} | S^t) = p(S^t | S^{t-1}) \cdot \frac{p(S^{t-1})}{p(S^t)}$ , i.e., we use Bayes’ rule.

We use these trajectories to estimate the probability densities  $p(s_i)$  and  $p(s_i^{t-\Delta} | S^t)$  for each spin  $s_i$ , which are needed to compute  $I(s_i^{t-\Delta} | S^t)$ . We ignore the units which are in their ‘unexpected’ state, i.e., if the equilibrium probability  $p(s_i = +1)$  that unit  $s_i$  has state  $+1$  is more than  $1/2$  then we do not estimate its IDT if at time  $t$  its state  $s^t = -1$ , and vice versa. This does not affect our results significantly because the more connections that a spin has, the less probable it is to be found in its unexpected state. The reason for leaving out spins in their unexpected state is that the probability curve

$p(s_i^{t-\Delta} | S^t)$  passes through  $1/2$ , and consequently the information curve  $I(s_i^{t-\Delta} | S^t)$  as function of  $\Delta \geq 0$  drops to zero and then ‘bounces’ up again towards its equilibrium value. This makes estimating its IDT ambiguous: searching for a  $\Delta$  such that a spin’s mutual information equals an arbitrary small constant  $\varepsilon$  returns two values. To avoid the problem of estimating its IDT in a representative manner we ignore these spins, since the probability of finding a spin in its unexpected state decreases exponentially as function of its degree in our model description.

For all other spins we estimate the  $\Delta$  such that a spin’s mutual information  $I(s_i^{t-\Delta} | S^t)$  equals an arbitrary small constant  $\varepsilon$ . We choose  $\varepsilon = 10^{-3}$ ; a different multiplicative value would only add a constant to the IDT values for all  $k$ . To avoid evaluating the formula of mutual information many times, which is time consuming due to the logarithms, we first find the probability value  $p_\varepsilon$  such that  $H(s) - H(s | p(s = +1) = p_\varepsilon) = \varepsilon$  for unit  $s$ . Then we can find the  $\Delta$  by regression directly from the unit’s state probability curve  $p(s_i^{t-\Delta} | S^t)$  by searching when it equals  $p_\varepsilon$ . In order to cope with the sampling noise we first fit an exponential curve  $a + b \cdot \exp(-c \cdot \Delta)$  to the state probability curve and then solve the equation  $a + b \cdot \exp(-c \cdot \Delta) = p_\varepsilon$ . The result of one simulation is then 6000 estimated IDTs, or  $D(s_i)$ , for each unit  $s_i$ , for the state trajectories that lead up to one ‘snapshot’ system state. The results in Figure 3 were created with six simulations, i.e., almost  $6 \cdot 6000 = 36000$  data points.

These estimated IDTs from the simulations’ time series corroborate the numerical calculation of  $D(s_i)$  and the proof of diminishing IDT for highly connected units. In the methodology we do not assume a locally tree-like network structure. Also we do not make the random-mixing assumption, i.e., the assumption that there are no degree-degree correlations or other structural biases in the network. Although we do generate each network to be random except for the degree distribution, each network realization likely

contains structural biases due to its relatively small size of 6000 units. That is, there will be lowly-connected units that incidentally connect only to highly-connected units (dis-assortativity), and vice versa; there will also be regions in the network with a higher or lower transitivity than expected.

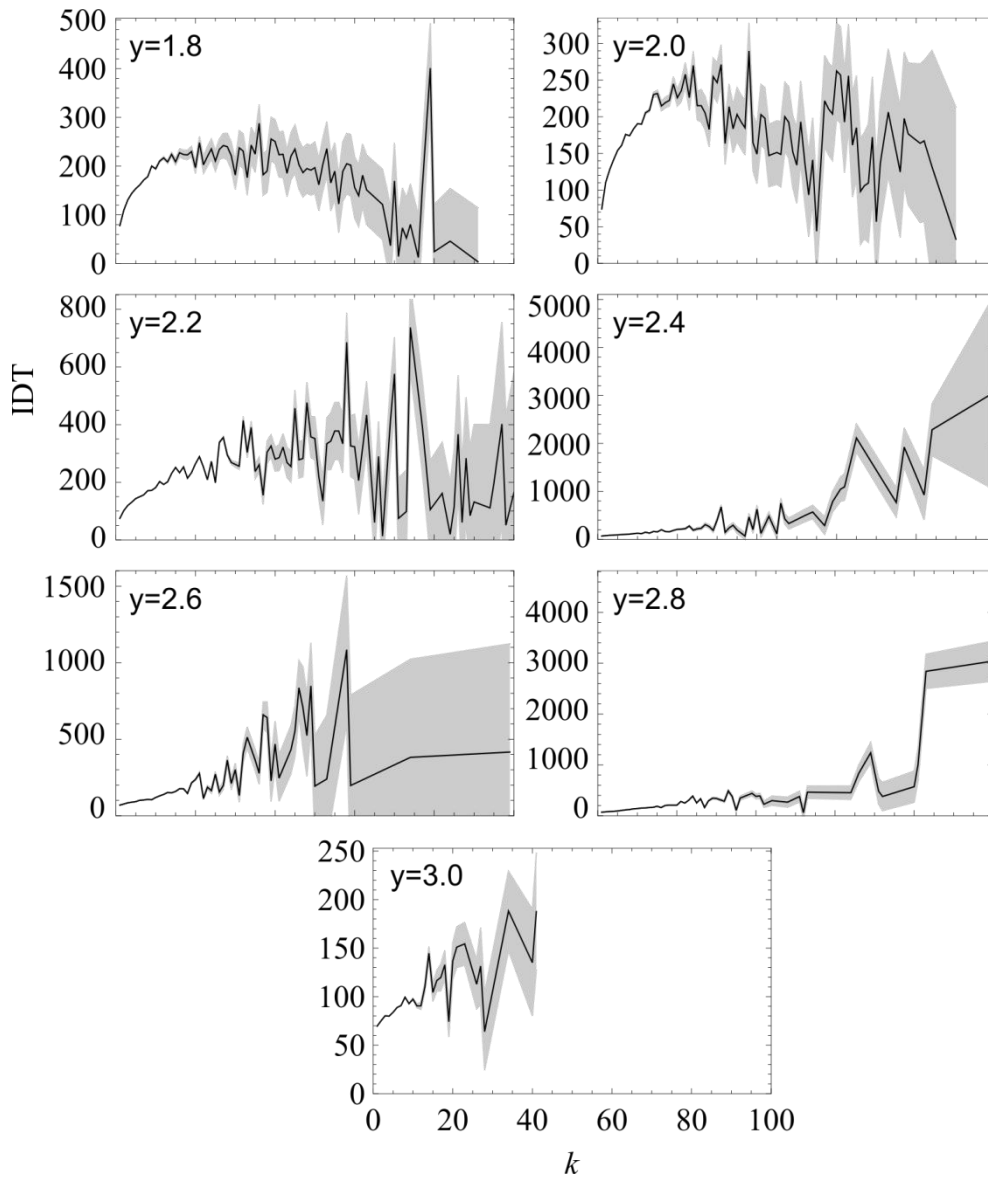
## **A2 Results of simulations for other $\gamma$ values**

We have performed simulations of Ising spins in random networks with values for the network parameter  $\gamma$  (exponent) ranging from 1.6 to 3.0, inspired by the range of exponents typically found in the literature. The higher the value of  $\gamma$ , the fewer highly-connected units will be generated. Consequently, for increasing  $\gamma$  it becomes increasingly difficult to obtain a statistically significant IDL curve for highly connected units, which is the focus of this work. Therefore, to create the simulation results in the main text we used  $\gamma=1.6$ , the lowest value in the range, with a network consisting of 6000 units. Estimating the IDL of each unit in a single such network takes about three days on a regular desktop. Each single plot is created by combining the results of six simulations.

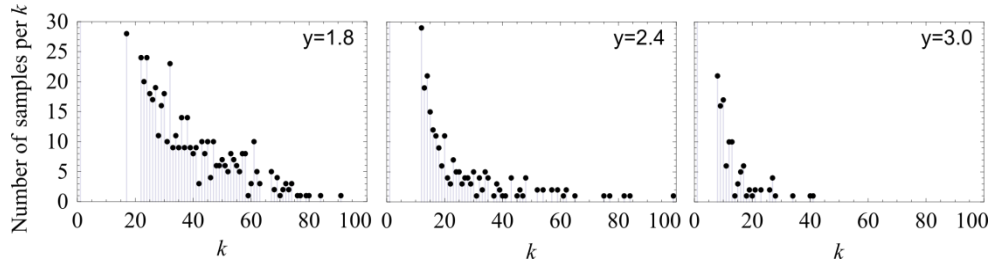
Here we show the IDL curves for the remaining values for  $\gamma$ . In order to make the computation more feasible we use smaller networks consisting of 2000 units. In addition, we simulate state trajectories of 40000 time steps instead of 90000, which means that there are fewer data points for the exponential curve fitting procedure. A single simulation now takes about five hours on a regular desktop computer. The results are summarized in Figure A15.

For the network parameter values up to and including  $\gamma=2.2$  we still observe that the dynamical impact (IDT) of highly connected units diminishes as function of  $k$ . Starting from  $\gamma=2.4$  this phenomenon disappears; however, at the same time, the plots lose their statistical significance. In Figure A16 we show for  $\gamma=\{1.8,2.4,3.0\}$  the number of units per connectivity  $k$  that was used to produce the corresponding plots in

Figure A15. It is evident that starting from around  $\gamma = 2.4$  we are able to obtain only a few highly-connected units in our simulations. Not only are highly connected units relatively rare in our random networks compared to peripheral units, they are often omitted from the analysis because of insufficient statistics. That is, in the simulations many highly connected units change their state too seldom in order to reliably estimate the state probability curve over time. For this reason, the simulation results in the main text were produced with the value  $\gamma = 1.6$ , the lowest in our range, and for this value we increased the network size, number of time steps, and number of state trajectories in order to more reliably estimate the IDT of highly-connected units. Although from Figure A15 one could also suspect that high values for  $\gamma$  change the shape of the IDT curve in the high- $k$  regime, based on the developed theory in Chapter 2 we expect that the IDT should diminish for all  $\gamma$ .



**Figure A15: Estimated IDT of a unit that has  $k$  connections for  $\gamma$  ranging from 1.8 to 3.0. Each plot is created using six simulations. The solid black line connects the mean IDT per connectivity; the grey area is within two standard errors of the mean.**



**Figure A16:** Number of units per connectivity  $k$  that was used to produce the corresponding plots in Figure A15, focused on the range  $[0, 30]$ . Bars that would exceed the limit of 30 or with a magnitude of zero are not drawn.