Measurement of the atmospheric neutrino energy spectrum

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CHAPTER 3

ENERGY RECONSTRUCTION

In this chapter, a method to reconstruct the energy of muons traversing the detector is presented. The method attempts to maximize the agreement of the expected amount of light in the optical modules with the amount of light that is actually observed. We construct a maximum likelihood function modeling the muon traversal through the detector and keeping the energy of the muon as the free parameter.

In section 3.1, the energy-loss processes that take place when a charged particle passes through matter as well as the propagation of light in the same medium are discussed. The Monte Carlo simulation tools used in this work are summarized in section 3.2. Section 3.3 contains a description of the track reconstruction method which is used in the present analysis. The maximum likelihood energy reconstruction is described in section 3.4 and the performance of the method is examined in section 3.5.

3.1 Muon energy loss

Charged particles lose energy while traversing matter. The mean rate of energy-loss is given by the Bethe-Bloch equation [Revi 06],

\[- \frac{dE}{dx} = K z^2 Z A \beta^2 \left[ \frac{1}{2} \ln \frac{2 m_e c^2 \beta^2 \gamma^2 T_{\text{max}}}{I} - \beta^2 - \frac{\delta (\beta \gamma)}{2} \right].\]

(3.1.1)

Here, \( K = 4\pi N A r_e^2 m_e c^2 \) is a constant, \( z \) is the charge of the incident particle, \( Z \) and \( A \) are the atomic number and atomic mass of the absorber respectively, \( \beta \) is the velocity of the incident particle, \( \gamma \) is the Lorentz factor, \( I \) is the mean excitation potential of the atoms in the medium and \( T_{\text{max}} \) is the maximum kinetic energy which can be transferred to a free electron in a single collision. The potential \( I \) is essentially the average orbital frequency \( \hbar \nu \) of bound electron states in the atoms, where \( \hbar \) is Planck’s constant. The calculation of the mean excitation potential is a challenging task and its values for various absorption materials are deduced by energy-loss measurements. The value of \( I \) for water is 75 eV [Leo 94]. The term \( \delta \) corresponds to the density effect correction [Term 40]. It is included in order to take into account the effect on the
ionization energy-loss, of the polarization of the medium induced by the passage of the charged particle. This effect results in a reduced energy-loss due to ionization especially for electrons further away from the track. The mean energy-loss for positively charged muons in copper is shown in figure 3.1. For materials with decreasing $Z$, there is a slow increase in the rate of energy-loss, because the ratio $Z/A$ tends to increase. Relativistic particles with mean energy-loss rates close to the minimum of the Bethe-Bloch curve are called minimum ionizing particles (MIP). The energy range of interest in the present work is between a few hundreds of GeV to a few hundreds of TeV.

A muon traveling through rock or water will lose energy due to ionization, pair production, photo-nuclear interactions and Bremsstrahlung radiation. The average energy-loss per unit path length can be expressed in the following way:

$$-\frac{dE}{dx} = (\frac{dE}{dx})_I + (\frac{dE}{dx})_P + (\frac{dE}{dx})_N + (\frac{dE}{dx})_B,$$

(3.1.2)

where the four indices I,P,N and B correspond to the above mentioned processes respectively. The energy-loss processes can be divided into continuous and stochastic. When the number of discrete collisions over a macroscopic path length is very large and additionally each collision contributes a small fraction of the total energy-loss, the process is considered continuous. On the other hand, stochastic processes occur rarely.
and a single instance can be responsible for a large fraction of the total energy-loss leading to large energy-loss fluctuations. Distinguishing between continuous and stochastic processes, equation (3.1.2) can be written as:

\[-\frac{dE}{dx} = a(E) + b(E)E.\]  

(3.1.3)

The first term is due to the ionization, a quasi continuous process, while the second term includes the contributions from stochastic processes. These terms are energy-dependent as can be seen in figure 3.2.

The energy-loss per unit track length $\frac{dE}{dx}$ in water as a function of the muon energy is shown in figure 3.3. The curve shown here corresponds to the right part of the Bethe-Bloch curve shown in figure 3.1. One sees that up to a few hundred GeV the energy-loss is almost constant while for higher energies it rises linearly with the energy. This is due to the fact that ionization losses increase logarithmically with energy and linearly with the atomic number $Z$, while radiative losses increase approximately linearly with energy and quadratically with $Z$, therefore causing the stochastic processes to dominate above approximately 1 TeV. This is called the critical energy, indicated as $E_\mu c$ in figure 3.1 and it is defined as the energy at which the ionization energy-losses are equal to the radiative energy-losses i.e. $a(E_\mu c) = b(E_\mu c) \cdot E_\mu c$. The quadratic dependence of radiative losses on $Z$ is responsible for the lower critical energy in the case of copper. Assuming that $a(E)$ and $b(E)$ are constant, a frequently used approximation, equation (3.1.3) can be written as:

\[-\frac{dE}{dx} \simeq a + b \cdot E,\]  

(3.1.4)
where for water, \( a \simeq 2.67 \text{ MeV/cm} \) and \( b = b_P + b_B + b_N \simeq (1.7 + 1.2 + 0.5) \cdot 10^{-6} \text{ cm}^{-1} \).

Integrating (3.1.4) one can calculate the mean range of a muon with initial energy \( E_0 \),

\[
R = \frac{1}{b} \ln(1 + \frac{b}{a} E_0). \tag{3.1.5}
\]

This is the average distance travelled by a muon with initial energy \( E_0 \) before it loses all of its energy.

The challenge in reconstructing the muon energy lies in the fact that for lower energies the light yield of the muon is almost constant, making it difficult to distinguish between e.g., a 100 GeV and a 500 GeV muon. In addition, light from potassium decay and bioluminescence contribute a significant amount of background light for such low energy events. For higher energies, the difficulty in reconstructing the muon energy arises from the stochastic nature of the energy-loss processes. A muon may lose a significant fraction of its energy within a short track length by a very bright shower. The fraction of the muon energy that is carried away by Bremsstrahlung photons increases as the energy of the muon and the atomic number of the absorber increase. The photon energy spectrum is continuous and it is possible for a muon to lose all of its energy in a single shower. If a very large shower happens to be outside the detector, its light will not be detected leading to an underestimate of the muon energy at the point of its creation and therefore of the parent neutrino energy. The energy we attempt to estimate in the present work is the energy of the muon in the vicinity of the detector. This energy is not the same as the parent neutrino energy, a problem that we address in chapter 4.
3.1 Muon energy loss

3.1.1 Energy loss processes

The most relevant energy-loss process for energies less than $\sim 1$ TeV is ionization. When a muon scatters with atoms from the surrounding medium it transfers energy to atomic electrons. This process is described well by the Bethe-Bloch formula and it corresponds to the almost flat part of the curve, before radiative effects dominate the energy-losses. The total energy-loss due to ionization caused by a minimum ionizing particle is about 2 MeV/cm. Energy is also carried away by the Čerenkov photons but it is negligible compared to the other processes. The energy-loss due to Čerenkov photons from a relativistic muon traveling in water can be expressed as \[\text{Leo 94}]:

$$-\frac{dE}{dx} = \frac{\alpha \hbar}{c} \int \omega d\omega \sin^2 \Theta_C,$$

where $\alpha$ is the fine structure constant. Integrating from a minimal wavelength of 100 nm ($n(100 \text{ nm}) \simeq 1$) we find a contribution of the order of $\sim 0.05$ MeV/cm, much smaller than the ionization energy-loss.

Above $\sim 1$ TeV, the stochastic processes explained below start to dominate over the ionization energy-losses. These stochastic processes are characterized by large energy fluctuations and the generation of electromagnetic and hadronic showers. The energy-loss $dE/dx$ is not continuous, especially for higher muon energies. For that reason, the energy-loss rate (3.1.2) or (3.1.4) is interpreted as the average muon energy-loss along $dx$. The main energy-loss process at such energies is pair production. In the presence of an atom, the muon can produce an electron-positron pair transferring part of its energy to the nucleus. A diagram illustrating this process is shown in figure 3.4. In addition to pair production, energy is lost via photo-nuclear interactions. These are processes where a muon interacts inelastically with a nucleon inside the medium (figure 3.5). Finally, Bremsstrahlung radiation (figure 3.6), which takes place when a muon decelerates due to electromagnetic interactions with atoms from the medium,

Figure 3.4: Pair-production Feynman diagram. The incoming (outgoing) muon is indicated as $\mu_i (\mu_f)$ while the target nucleus is $p_i$. A hadronic shower is included in the final state when the momentum transfer is large enough, i.e. above the pion production threshold.
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Figure 3.5: Nuclear interaction Feynman diagram. The incoming (outgoing) muon is indicated as $\mu_i$ ($\mu_f$) and the target nucleus as $p_i$. 

Figure 3.6: Bremsstrahlung Feynman diagram. The initial (final) states are indicated with a subscript $i$ ($f$).

contributes to the energy-loss. This deceleration may cause the emission of very high energy photons. The radiative cross section as a function of the fractional energy-loss ($x = E\text{loss}/E_\mu$) for pair production goes roughly as $1/x^2$ to $1/x^3$ while for bremsstrahlung the dependence is less steep, i.e. $1/x$. This is the reason why “hard” energy-losses are more probable in bremsstrahlung while pair production losses are closer to being treated as continuous [Keln 67; Mo 69; Ginn 86].

3.1.2 Light propagation and detection

Direct Čerenkov light

A charged particle traveling through a medium faster than the speed of light in that medium emits Čerenkov radiation (see section 2.2.1). The energy carried away by these photons is negligible compared to the energy lost due to stochastic processes or ionization losses as discussed in the previous section. Čerenkov light is emitted under a fixed angle $\cos \Theta_C = \frac{1}{\beta n}$, where $n$ is the index of refraction for that medium, creating a cone of light around the particle’s track. The number of Čerenkov photons emitted per unit track length $\frac{d^2N}{dx d\lambda}$ is given in equation (2.2.5). The number of detectable photons per unit wavelength and per unit area at a distance $R$ from the muon track is given by the number of photons emitted by a track segment $L$, divided by the circular disk area created by the photons from that segment,

$$\Phi_0(R, \lambda) = \frac{d^2N}{dx d\lambda} \frac{1}{2\pi R \sin \Theta_C}, \quad (3.1.7)$$

illustrated in figure 3.7. $L$ is defined as the track segment that is viewed by the optical module under the Čerenkov angle $\Theta_C$. 
3.1 Muon energy loss

Figure 3.7: The number of detectable photons that reach an optical module is related to the area $A$, which is the area covered by the photons emitted under the Čerenkov angle $\Theta_C$ by the track segment $L$.

Light from electromagnetic showers

Along with Čerenkov radiation from the muon, there is also light originating from electromagnetic and hadronic cascades. Bremsstrahlung and pair production are the two processes responsible for the electromagnetic cascades. Hadronic showers occur when a nuclear interaction takes place, producing more hadrons that in turn re-interact and give rise to a hadronic cascade. Electromagnetic cascades consist exclusively of electrons, positrons and photons, and point mainly in the forward direction. Photons create electron-positron pairs that emit Bremsstrahlung radiation. These photons in turn produce more $e^+ - e^-$ pairs. The process continues until all energy is dissipated. Hadronic cascades have a more variable composition and a larger transverse spread. They consist predominantly of pions, muons as well as an electromagnetic component due to the cascades initiated by the photons produced by $\pi^0$ decays.

As discussed in section 3.1, the term $b(E)E$ of equation (3.1.3) takes into account the stochastic processes. The number of detectable photons per unit wavelength and per unit shower energy is given by:

$$
\frac{d^2 N}{dE d\lambda} = \frac{dx}{dE} \frac{d^2 N}{dx d\lambda},
$$

(3.1.8)

where $\frac{dx}{dE}$ is the equivalent bare muon track length per unit of shower energy. Based on Geant [Agos 03; Amak 06], a toolkit to simulate the passage of particles through matter, its value is estimated and found to be about $4 \text{ m GeV}^{-1}$ for water. Since light from showers is not emitted under a fixed angle, the Čerenkov cone is distorted and the angular distribution of shower light emission,

$$
\frac{d^2 P}{d\cos \theta d\phi} = c e^{b \cos \theta - c \cos \Theta_C} |^\alpha,
$$

(3.1.9)

needs to be taken into account [Mira 02]. The values of the three parameters are $\alpha = 0.35$, $b = -5.4$ and $c = \frac{1}{2\pi} \frac{1}{0.06667}$. This leads us to the number of detectable
photons originating from electromagnetic showers per unit wavelength, track length and solid angle,

$$\Phi_1(\cos\theta, E, \lambda) = \frac{d^2 N(E)}{dx d\lambda} \frac{d^2 P_\star}{d\cos\theta d\phi},$$  \hspace{1cm} (3.1.10)

where the number of detectable photons per unit wavelength and per unit track length due to electromagnetic showers is:

$$\frac{d^2 N(E)}{dx d\lambda} = b(E)E \frac{d^2 N}{dEd\lambda}.$$  \hspace{1cm} (3.1.11)

The probability distribution of the angle of emission is shown in figure 3.8.

There are two processes that affect light propagation in water, light absorption and light scattering. The way they are introduced into the calculation of the detected light in the optical modules is examined in the following two paragraphs.

**Light absorption**

Light is attenuated as it traverses a certain medium. Individual photons are being absorbed by atoms in the medium via the photoelectric effect. The atomic electron that absorbs the photon is subsequently ejected from the atom carrying away energy equal to the energy of the photon minus the electron’s binding energy. This results in a reduced intensity of the initial photon flux. Absorption can be accounted as follows.

If the initial flux is $\Phi$, then after a distance $d$ this flux will be $\Phi'$:

$$\Phi' = \Phi \cdot e^{-\frac{d}{\lambda_{\text{abs}}}},$$  \hspace{1cm} (3.1.12)

where $\lambda_{\text{abs}}$ is the absorption length, i.e. the distance at which $\sim 63\%$ of the photons will have been absorbed. The absorption length as a function of the photon wavelength is shown in figure 2.10.

**Light scattering**

Light scattering off particles in the medium affects the number of detectable photons that reach the optical modules. Incident electromagnetic radiation deforms the charge
distribution of the scattering particle, forming an oscillating dipole that radiates light. The scattering angle, effectively the direction of light after the scattering, has to be taken into account. Particles with size much larger than the incident light wavelength contribute to small angle scattering, known as Mie scattering [Mie 08], due to destructive interference from light radiated from different parts of the scattering particle. Rayleigh scattering [Rayl 71] refers to scattering from particles with sizes much smaller than the wavelength that contribute to large scattering angles since the particle is subject to a uniform electric field. Rayleigh scattering is the limiting case of the Mie solution to Maxwell’s equations for particles that satisfy:

$$\alpha \ll \frac{\lambda}{2\pi n},$$

(3.1.13)

where \(\alpha\) is the radius of the scattering particle, \(\lambda\) is the incident light wavelength and \(n\) the refractive index. The flux of direct photons reaching the OM’s will be reduced by the number of photons that are scattered along the way. A term that will effectively reduce the number of direct photons needs to be included when the contribution of the direct photon flux is calculated. Analogously to light absorption, the flux is reduced as:

$$\Phi' = \Phi \cdot e^{-\frac{d}{\lambda_s}},$$

(3.1.14)

where \(\lambda_s\) is the wavelength-dependent scattering length, which is shown in figure 2.11.

Various models have been developed to describe the scattering angle probability density function \(dP/d\Omega_s(\theta_s)\). The scattering probability depends only on the scattering angle \(\theta_s\), defined as the angle between the photon direction before and after scattering. This is due to the rotational symmetry of light scattering. Two different scattering models’ parametrizations are considered in the present work [Bail 00]. The \(f_4\)-model is based on the so-called “medsea” parametrization which is a combination of two Henyey-Greenstein functions. They are defined as:

$$f_{HG}(g; \cos \theta_s) = \frac{1}{4\pi} \frac{1 - g^2}{(1 + g^2 - 2g \cos \theta_s)^{\frac{3}{2}}},$$

(3.1.15)

where \(g\) is the average cosine of the scattering angle. The parametrization for the probability density function of the scattering angle is defined as:

$$\frac{dP_s}{d\Omega_s} = p \cdot f_{HG}(g_1; \cos \theta_s) + (1 - p) \cdot f_{HG}(g_2; \cos \theta_s).$$

(3.1.16)

The Henyey-Greenstein functions are normalized to unity for the full solid angle. In the \(f_4\)-model, the following values are used:

\[
\begin{align*}
p & = 1, \\
g_1 & = 0.77, \\
g_2 & = 0.
\end{align*}
\]

For \(g = 0\) the probability density is uniform for all scattering angles while as \(g\) approaches unity the density peaks strongly in the forward direction, i.e. \(\theta_s = 0^\circ\).
The second model under consideration is the p0.0075-model which is a combination of Rayleigh and Mie scattering. The parametrization used for this model is:

\[
\frac{dP_s}{d\Omega_s} = p \cdot f_R(\beta; \cos \theta_s) + (1 - p) \cdot f_{HG}(g; \cos \theta_s),
\]

(3.1.17)

where \( g = 0.924 \) and \( p = 0.17 \), i.e. 17% contribution from Rayleigh scattering. The Rayleigh term \( f_R \) is given by:

\[
f_R(\beta; \cos \theta_s) = \frac{1}{4\pi} \frac{1}{1 + \frac{\beta}{3}} (1 + \beta \cos^2 \theta_s).
\]

(3.1.18)

The \( \beta \) term is a measure of the spherical symmetry of the particles on which scattering takes place and is equal to one for a perfect sphere. For water molecules \( \beta = 0.853 \). The distribution of the scattering angle due to large particles in the p0.0075-model is obtained from in situ measurements. The probability distributions of the scattering angle for the \( f_4 \) and \( p0.0075 \) models are shown in figure 3.9. The higher contribution to large angle scattering in the p0.0075-model is due to the Rayleigh term that is not present in the \( f_4 \)-model as well as the fact that the term \( g = 0.77 \) in the Henyey-Greenstein function favors forward scattering.

Light detection

After taking into account photon emission and propagation, the only term missing to have an estimate of the expected charge that will be measured by the optical modules is the response of the PM tube. Therefore, the charge that we expect to measure is a convolution of three factors. They describe how much light is emitted from the muon, how photons are propagated and attenuated in the medium before they finally reach the OM’s and how the PM tubes translate this light into a measured charge pulse. The detection efficiency depends on the wavelength of light as well as the angle of incidence of the photon on the PM tube. The angular acceptance accounts for the angle of incidence, defined as the angle between the photomultiplier axis and the direction of the incident photon. The PM tube quantum efficiency (see fig. 2.16), together with the glass and gel opacity to light, is taken into account.
3.2 Monte Carlo simulation tools

In this section we give an overview of the simulation chain and the software packages used to generate the neutrino and muon events in the detector. The standard ANTARES simulation tools are used for the generation of muons and neutrinos, the propagation of muons and other secondary particles towards and through the detector along with photon emission and propagation, and finally the simulation of the optical modules’ response. The simulation we use corresponds to the best available description of the true data taking conditions, using information about the rates, the condition of the optical modules and the run duration from the corresponding data runs. In this way, a realistic run-by-run simulation of the physics and data taking process is achieved [Rivi 12].

Assuming an initial flux of neutrinos at the surface of the Earth, the rate of detected neutrino events is expressed as:

$$R = \int \int \int \frac{d^2 \Phi(E, \hat{d})}{dEd\Omega} P_\odot(E, \hat{d}) \rho(\vec{x}) N_A \sigma_{CC}(E) P_{det}(E, \hat{d}, \vec{x}) dE d\Omega d\vec{x}. \quad (3.2.1)$$

The differential neutrino flux at the surface of the Earth $d^2 \Phi(E, \hat{d})/dEd\Omega$ is given in units of $\text{GeV}^{-1}\text{sr}^{-1}\text{m}^{-2}\text{s}^{-1}$. $P_\odot(E, \hat{d})$ is the probability of a neutrino traversing the Earth without undergoing an interaction, $\rho(\vec{x}) N_A$ is the number of nucleons per unit volume in m$^{-3}$ and $\sigma_{CC}(E)$ is the total charged current neutrino-nucleon interaction cross section in m$^2$. The detection probability is given by $P_{det}$ and depends on:

- the energy and direction of the muon at the neutrino interaction vertex,
- the muon energy, position and direction when it reaches the detector and the probability for this to happen,
- the muon light yield and the detector’s response to this signal, and
- the reconstruction and event selection process.

The first step in the simulation chain is to generate a flux of neutrino events in the vicinity of the detector, that have a chance of producing a detectable muon signal. This is done with Genhen v6r3 [Bail 02a]. Neutrino events are generated isotropically inside a large cylinder around the detector. The size of this cylinder is determined in such a way so that all neutrinos that are able to produce a detectable muon inside the detector will be simulated, and it is based on the maximum range that a muon can travel. A muon can be detected if it reaches the can, defined as the area surrounding the ANTARES instrumented volume extending typically up to 2-3 light attenuation lengths away. Muons outside the can are too far away to produce detectable light, therefore only the propagation of particles from their generation point to the can is simulated and not the photon emission and development of electromagnetic and hadronic showers. The propagation of the muon from the neutrino interaction vertex to the can is simulated with the MUSIC package [Anto 97]. The energy losses of the muon as well as the changes
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on its direction due to multiple Coulomb scattering are included in the simulation. The charged current neutrino interactions are simulated in the LEPT0 package [Inge 97] using the CTEQ6-DIS parton distribution functions [Pumpl 02b]. The uncertainty in the cross section is estimated at approximately 3% in the energy range between $10^{-8}$ GeV.

The flux of neutrinos is attenuated as they propagate inside the Earth. This information is included in $P_\odot(E, \hat{d})$ of equation (3.2.1) and is given by:

$$P_\odot(E, \hat{d}) = e^{-\rho(\hat{d})N_A\sigma(E)}, \quad (3.2.2)$$

where $\rho(\hat{d})$ is the amount of matter traversed by the neutrino on its way to the detector and depends on the direction of the neutrino. Neutrinos with very high energies are more probable to undergo an interaction due to the increased cross section. Additionally, neutrinos that travel in the vertical direction traverse more matter, including the denser core of the Earth, which leads to a significant suppression of the very high energy vertical neutrino flux.

The atmospheric muon background is simulated with MUPAGE v3r5 [Carm 09]. It is based on parametrizations describing the muon flux as a function of the muon energy and angular distribution on the surface of the can. The generated number of events correspond to one tenth of the livetime, therefore the atmospheric muon Monte Carlo distributions shown in later chapters are scaled up by a factor of ten.

The muon propagation inside the can and the light that reaches the optical modules is simulated with the KM3 v3r7 package [Bail 02b]. Since the tracking of every single photon emitted is computationally very inefficient, a set of tables is constructed taking into account the absorption and scattering of light and storing the average photon fields produced by the muon for different distances, positions and orientations of the OMs with respect to the track. The number and times of hits on the optical modules are then sampled from these tables. For the tracking of particles other than muons, a Geant based simulation is used in the package Geasim v4r10 [Brun]. In this, only the attenuation of light is considered, while photon scattering is not simulated. The effect of the OM angular acceptance and efficiency is included in this step.

Finally, the simulation of the electronics response such as the charge integration and the dead time is performed with the TriggerEfficiency program [Jong 09]. In this step optical background hits are added and the online triggers used in real data are simulated. Optical background hits are generated according to a Poisson distribution based on real measured rates in order to reproduce the specific run’s data taking conditions. Signal as well as background hits are generated after simulating the electronics response such as the charge threshold, time integration and dead time. The trigger logics that were used during the corresponding data run are finally applied. The events used in the present analysis are triggered with either the 3N or T3 triggers and reconstructed with the methods described in the following sections.
3.3 Track reconstruction

Before proceeding to the details of the energy reconstruction, a brief description of the track reconstruction algorithm is given since it is an essential ingredient for the energy reconstruction algorithm. It is also important for the understanding and discussion of the results of the energy reconstruction. Track reconstruction is performed in four consecutive fitting procedures [Heij 04]. The first stages provide a starting point for the last fit that gives the best results. The reconstruction algorithm looks for parameters defining the geometry of the track, i.e. the direction \( \vec{d} \equiv (d_x, d_y, d_z) \) and the position \( \vec{p} \equiv (p_x, p_y, p_z) \) of the muon at some fixed time \( t_0 \), that maximize the probability of them being compatible with the observed hits. It is based on the time residuals of the hits in an event:

\[
    r_i = t_i - t_{th}^i,
\]

where \( t_i \) is the time of the hit and \( t_{th}^i \) is the expected time of hit coming from a Čerenkov photon given the input track. At first a hit pre-selection based on the time and amplitude of the hits is performed to remove the majority of background hits. The first stage of the track reconstruction procedure is a linear prefit. Hits with an amplitude of more than 3 photoelectrons or clusters of hits on a floor within 25 ns are used with the additional assumption that they occur on points along the muon track. An OM recording a hit with a high amplitude is more likely to be located closer to the track. Using the amplitude information and the orientation of the PMT, the distance of the muon track from the OM is estimated and therefore the hit position that is most likely to lie on the muon track. This leads to a linear relation of the form:

\[
    y = H\Theta,
\]

where \( y = [x_1, y_1, \ldots, z_n] \) is the vector with the hit positions, \( \Theta = [p_x, d_x, p_y, d_y, p_z, d_z]^T \) is the vector containing the track parameters i.e. position and direction, and \( H \) is a matrix containing the hit times:

\[
    H = \begin{bmatrix}
        1 & ct_1 & 0 & 0 & 0 & 0 \\
        0 & 0 & 1 & ct_1 & 0 & 0 \\
        0 & 0 & 0 & 0 & 1 & ct_1 \\
        1 & ct_2 & 0 & 0 & 0 & 0 \\
        0 & 0 & 1 & ct_2 & 0 & 0 \\
        0 & 0 & 0 & 0 & 1 & ct_2 \\
        \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
        0 & 0 & 0 & 0 & 1 & ct_n 
    \end{bmatrix}.
\]

The first estimate of the track parameters in vector \( \Theta \) is obtained through a \( \chi^2 \)-minimization,

\[
    \chi^2 = [y - H\Theta]^T F [y - H\Theta],
\]

which at this point is not a very accurate result but serves as the starting point for the steps that follow. The matrix \( F \) is the inverse of the hit position error-covariance matrix, which is assumed to be diagonal.
The second stage is based on an M-estimator fit. M-estimators work in a similar fashion to maximum likelihood or least squares estimators, by maximizing some function $g$. It is not as accurate as a maximum likelihood fit but it has the benefit of not being too sensitive to the start value. A $\chi^2$ estimator of the form $g(r) = -r^2$ on the other hand is not suitable since it does not take into account background hits. The function of time residuals $r$ chosen in this case is:

$$g(r_i) = -2\sqrt{1 + r_i^2/2} + 2,$$

and the resulting estimator is called “L1-L2” [Zhan 97]. Function (3.3.5) was chosen in such a way to exhibit a linear and quadratic behavior for large and small values of $r$ respectively. Studies have shown that the use of additional information on the amplitude of the hits $A_i$, as well as information on the angular acceptance of the PMT’s lead to an improved performance of the M-estimator. Finally the function to be maximized is:

$$g = \sum_i \kappa \left( -2\sqrt{1 + A_i r_i^2/2} \right) - (1 - \kappa) f_{ang}(a_i),$$

where the relative contribution of each term is expressed through the parameter $\kappa = 0.05$, optimized using simulated events. The angular response of the PM tube as a function of the photon angle of incidence, calculated under the Čerenkov hypothesis for the photon emission angle, is described by $f_{ang}$. The hit selection for this stage accepts hits with time residuals from -150 ns to 150 ns and distances smaller than 100 m from the first track fit result. The reason for this is to keep hits that have the highest probability of being due to photons emitted from the muon track. Discarding hits far away from the track or with very large time residuals, hits that are most likely due to optical background are eliminated. Hits with amplitudes of more than 2.3 photoelectrons are also kept, since they are unlikely to be due to potassium decay.

The third step is a maximum-likelihood fit where the hit selection is based on the track output of the M-estimator fit. The likelihood of the event is the product of the likelihoods of each hit:

$$P(\text{event}|\text{track}) = \prod_i P(t_i|t_i^{th}, a_i, b_i, A_i),$$

where “event” refers to the collection of hits and “track” to the parameters defining the position and direction of the muon that is being reconstructed. The cosine of the angle of incidence of the photon on the OM is denoted as $a_i$, $b_i$ is the photon path length and $A_i$ the amplitude of the hit. Hits that are part of a coincidence, have an amplitude larger than 2.5 p.e. or have a residual within $-0.5 \times R$ and $R$, where $R$ is the root-mean-square of the residuals used in the previous step, are selected. This selection ensures that the majority of hits due to optical background are rejected. The size of the interval $[-0.5 \times R, R]$ depends on how close the track from the previous step is to the true track. The PDF used in this step was developed in the work of Hubaut [Huba 99] and does not include the contribution of optical background hits. Additionally, this
probability density function depends only on the time residuals. The M-estimator and the log-likelihood as a function of the time residuals are shown in figure 3.10 together with the least-squares estimator \( g(r) = -r^2 \).

The last two steps are repeated using different starting points as input tracks. For this reason a series of rotations and translations are performed to the result of the prefit which are in turn used again as the starting point to the M-estimator stage. The track with the best likelihood per degree of freedom from the maximum likelihood fit is kept and used as a starting point for the last stage of the reconstruction. This is a maximum likelihood fit with an improved PDF which replaces the one in equation (3.3.7). The new PDF is the sum of the probability densities for signal hits and optical background hits:

\[
P(t_i | t_{i}^{th}, a_i, b_i, A_i) = \frac{1}{N_{\text{total}}} \left[ P_{\text{signal}}N_{\text{signal}} + R_{\text{background}} \right],
\]

(3.3.8)

where \( N_{\text{total}} \) and \( N_{\text{signal}} \) are the total number of hits and the number of signal hits, i.e. not background, respectively and \( R_{\text{background}} = \frac{N_{\text{bg}}}{T} = N_{\text{bg}}P_{\text{bg}} \) is the optical background rate. \( T \) is the time window of the selection of hits in the event and \( N_{\text{bg}} \) the expected number of background hits. The parametrization of \( P_{\text{signal}} \) is obtained from the work of Heijboer [Heij 04]. A hit selection is also performed here, keeping hits in local coincidences or hits with a higher amplitude than 2.5 photoelectrons within a time residual window of -250 to 250 ns.

A dedicated variable \( \Lambda \) is defined to characterize the quality of the fit as:

\[
\Lambda \equiv \frac{\log L}{N_{\text{DOF}}} + 0.1(N_{\text{comp}} - 1),
\]

(3.3.9)

where the first term is the log-likelihood value per degree of freedom at the maximum and \( N_{\text{comp}} \) is the number of compatible solutions found by the reconstruction algorithm. This number is equal to the number of M-estimator starting points that result in a track direction within \( 1^\circ \) from the result with the best likelihood per degree of freedom. In general, \( \Lambda \) has a higher value for well-reconstructed events. This is shown in figure 3.11 where the distribution of the reconstruction error \( \beta \), defined as the difference between the generated and reconstructed muon direction, is plotted before and after a cut of \( \Lambda \) at
Figure 3.11: Distribution of the angular track reconstruction error $\beta$ of all tracks (white). The shaded area is the distribution after performing a $\Lambda$ cut, rejecting values lower than -5.3. The left plot corresponds to upward going tracks with a spectrum following the atmospheric neutrino spectrum. The right plot corresponds to an $E^{-2}$ flux. See text for explanation.

-5.3, a commonly used value in many ANTARES analyses. A large fraction of the badly reconstructed events is rejected in this way. The left plot corresponds to a spectrum weighted with the atmospheric neutrino flux, while the plot on the right corresponds to a flux weighted with $E^{-2}$. The differences on the two plots are attributed to the fact that the $E^{-2}$ flux, being less steep than the atmospheric neutrino flux, consists of more high energy neutrino events and therefore has on average more signal hits in the event. After the $\Lambda$ quality cut, the median of the angular error distribution for the atmospheric and $E^{-2}$ fluxes is 0.72° and 0.23°, with an efficiency of 21% and 51% respectively. The angular resolution achieved as a function of the neutrino energy, after a quality cut of $\Lambda > -5.3$, is shown in figure 3.12.

Figure 3.12: Median angular difference between the generated and reconstructed muon as a function of the parent neutrino energy for upward-going neutrinos reconstructed with $\Lambda > -5.3$. 
3.4 Maximum likelihood energy reconstruction

Several energy reconstruction methods based on maximum likelihood have been developed in other experiments [Miov 01; Wins 08]. The energy reconstruction method developed here is outlined as follows. The expected number of photoelectrons in a certain optical module coming from a given track is calculated as a function of energy. A function which gives the likelihood that the measured amount of light in the optical modules is the result of a given muon track is defined. The probability density functions for the arrival times of the photons are introduced and calculated in [Jong 10] and will be discussed in the following section. The directional information obtained after track reconstruction is considered \textit{a priori} correct. In what follows we will first use the directional information from the generated Monte Carlo muon. In later sections the effect of directional reconstruction on the energy estimation will be examined. A standard one-dimensional minimizer is used to find the energy for which the likelihood function has a maximum.

3.4.1 Probability density functions

The arrival times of the photons on the optical modules are important in the estimation of the energy. The expected arrival time of the photons \( t_0 \), assuming a Čerenkov cone hypothesis, corresponds to the shortest optical path from the point of emission to the position of the PM tube. This means that the distribution of the arrival time of light will peak at \( t = t_0 \). If there was no scattering or dispersion, the arrival time would be completely determined by the emission angle of the emitted photon with respect to the muon track. However, due to dispersion the dependence of the arrival time on the wavelength must also be considered. In addition, since light can be emitted in all directions during an electromagnetic shower the arrival time depends on the position of the shower on the track. Scattering affects the photon path from the emission point to the optical module which is now not uniquely defined anymore. This is an important point since the direction of the photon after scattering determines the angle of incidence on the optical module and consequently the angular acceptance value for this angle. The solid angle under which the photomultiplier is seen is evaluated from the scattering point and not from the point of emission.

**Direct light**

When considering direct light from the muon, the arrival time distribution is still affected by dispersion, and consequently the photon arrival times depend on the wavelength. The angle of photon emission is fixed and there is no scattering. The probability density function can be expressed as:

\[
dP_{dm} \frac{dt}{dt} = \Phi_0(R, \lambda) A \left( \frac{\partial t}{\partial \lambda} \right)^{-1} \epsilon(\cos \theta_{inc}) QE(\lambda) e^{-\frac{d}{\lambda_{abs}}} e^{-\frac{d}{\lambda_s}},
\]

where \( \Phi_0 \) is the detectable photon flux given in equation (3.1.7), \( A = 0.044 \text{m}^2 \) is the PMT photocathode area, \( \epsilon \) is the angular acceptance as a function of the incidence angle.
angle $\theta_{\text{inc}}$, $QE(\lambda)$ is the quantum efficiency as a function of the wavelength and $R$ is the vertical distance of the PMT from the track. For a direct photon from the muon, the arrival time depends only on the wavelength of the light which is taken into account through the $\frac{\partial t}{\partial \lambda}$ term.

For direct light from electromagnetic showers, since the photons are emitted in all directions, the arrival time depends on the point $z$ on the muon track where the light was emitted. Moreover, there are two points on the track $z_1$ and $z_2$ that can have the same timing characteristics as illustrated in figure 3.13. The probability density function for the arrival times of light is:

$$\frac{dP_{d\text{EM}}}{dt} = \int d\lambda \sum_{z_1,z_2} \left( \frac{dt}{dz} \right)^{-1} \Phi_1(\cos \theta, E, \lambda) d\Omega \epsilon(\cos \theta_{\text{inc}}) QE(\lambda) e^{-\frac{\lambda}{\lambda_{\text{abs}}}} e^{-\frac{\lambda}{\lambda_s}},$$  

(3.4.2)

where $\Phi_1$ is the flux given by equation (3.1.10) and $d\Omega$ is the solid angle of the PMT, as viewed from the photon emission point.

**Scattered light**

For scattered light from the muon one has to integrate over the azimuthal photon emission angles and positions. In addition, the optical path is not uniquely defined and depends on the photon scattering point along its path as shown in figure 3.14. For light that scattered once, the arrival time $t$ of the photon on the OM is given by:

$$ct = z + n_g(|\vec{u}| + |\vec{v}|),$$  

(3.4.3)

where $|\vec{u}|$ and $|\vec{v}|$ are the distances travelled by the photon before and after scattering, $z$ is the photon emission point along the muon track and $n_g$ is the index of refraction corresponding to the group velocity of light. The corresponding probability density function for scattered muon light is:

$$\frac{dP_{s\text{m}}}{dt} = \iint d\lambda dz d\phi \frac{1}{2\pi} \frac{dN}{dx} \frac{1}{\lambda_s} \left( \frac{\partial t}{\partial u} \right)^{-1} \epsilon(\cos \theta_{\text{inc}}) QE(\lambda) e^{-\frac{\lambda}{\lambda_{\text{abs}}}} \frac{dP_s}{d\Omega_s} d\Omega_s.$$  

(3.4.4)
For a photon that scattered only once, the arrival time depends only on the position of the scattering along its path which is taken into account through the $\partial t/\partial u$ term.

In the case of scattered light from electromagnetic showers, the probability density function is

$$
\frac{dP_{EM}}{dt} = \int \int \int d\lambda dz d\phi d\cos \theta \Phi_1(\cos \theta, E, \lambda) \frac{1}{\lambda_s} \times \\
\times \left( \frac{\partial t}{\partial u} \right)^{-1} \epsilon(\cos \theta_{inc}) Q E(\lambda) e^{-\frac{4}{d\Omega_s}} \frac{dP_s}{d\Omega_s} d\Omega,
$$

where $\Phi_1$ is the flux in equation (3.1.10) and $P_s$ is the probability density function for the light scattering angle (equations (3.1.16) and (3.1.17)) discussed in section 3.1.2.

In what follows, the time integrated values of the probability density functions are used in order to obtain the number of expected photoelectrons in a given time window. The optical background is considered constant and equal to 60 kHz per optical module. This is not the case for real data. When events in data are reconstructed, the average measured rate over all OM’s is used instead. These hits are added in the calculation of the expected number of photoelectrons by multiplying that rate with the time window under consideration.

### 3.4.2 Likelihood function

We define the likelihood function as follows:

$$
\mathcal{L}(E) = \frac{1}{N_{OM}} \prod_{i} L_{i}(E).
$$

The product is taken over all optical modules, whether there was a hit recorded or not. Optical modules that are further than 300 m away from the track are not taken into account, since it is not expected to have any significant amount of light so far away from the track. This corresponds to an ideal situation where all phototubes are active. When this method is applied to real data, dead phototubes are excluded. $L_{i}(E)$ is the
probability of observing a pulse amplitude $A_i$ given a certain expected amplitude on the $i^{th}$ optical module. These individual likelihood functions $L_i(E)$ are constructed as:

$$L_i(E) = \begin{cases} 
    P(A; \langle n_{pe} \rangle) = \sum_{n_{pe}=1}^{n_{pe\max}} P(n_{pe}; \langle n_{pe} \rangle) \cdot P(A; n_{pe}) & \text{hit} \\
    P(0; \langle n_{pe} \rangle) = e^{-\langle n_{pe} \rangle} + P_{\text{threshold}}(\langle n_{pe} \rangle) & \text{no hit}
\end{cases} \quad (3.4.7)$$

The first equation of (3.4.7) consists of two terms, the Poisson probability of having $n_{pe}$ photoelectrons given that the expectation is $\langle n_{pe} \rangle$,

$$P(n_{pe}; \langle n_{pe} \rangle) = \frac{\langle n_{pe} \rangle^{n_{pe}} e^{-\langle n_{pe} \rangle}}{n_{pe}!}, \quad (3.4.8)$$

as well as a Gaussian term which expresses the probability that $n_{pe}$ photoelectrons in the photocathode will yield an amplitude $A$,

$$P(A; n_{pe}) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{1}{2} \left( \frac{A-N(n_{pe})}{\sigma} \right)^2}. \quad (3.4.9)$$

For this term, $N$ and $\sigma$ are derived from the known calibration values of the mean and width of the single photoelectron peak (see equations (2.8.4) and (2.8.5)):

$$N = n_{pe} \cdot \mu_{SPE}, \quad (3.4.10)$$

$$\sigma^2 = n_{pe} \cdot \sigma_{SPE}^2. \quad (3.4.11)$$

The values used in this work are $\mu_{SPE} = 1$ and $\sigma_{SPE}^2 = 0.3$. The energy dependence of the likelihood function is expressed in calculating the expected number of photoelectrons $\langle n_{pe} \rangle$ as will be discussed shortly. The summation in equation (3.4.7) is performed with $n_{pe\max} = 80$. A cut at 40 photoelectrons was imposed when either the charge of the hit or the expected number of photoelectrons was above this value to account for saturation effects. The probability of having a hit is renormalized in order to take into account the tail of the distribution $P(n_{pe}; \langle n_{pe} \rangle) \cdot P(A; n_{pe})$ that falls below the threshold value (and even into negative values) of the observed charge, which is in particular relevant for low values of $\langle n_{pe} \rangle$.

The second equation of (3.4.7) is used when there is no hit recorded on the optical module under consideration. The first term is the Poisson probability of observing zero photoelectrons when the expected value is $\langle n_{pe} \rangle$. The second term takes into account the threshold effect, i.e. the probability that a photon conversion in the optical module will give an amplitude below the threshold level. This is especially relevant for low energy events that produce only a small amount of light. The threshold probability is given by:

$$P_{\text{threshold}}(\langle n_{pe} \rangle) = \sum_{n_{pe}=1}^{n_{pe\max}} P(n_{pe}; \langle n_{pe} \rangle) \int_0^{A_{th}} P(A; n_{pe}) \, dA, \quad (3.4.12)$$
where $A_{th}$ is the threshold amplitude. In general, this is different for every optical module. In the simulation its value is set to 0.3 photoelectrons and this is the value that is used in the energy estimation as well, both in the Monte Carlo for performance studies as well as in real data.

The goal of the energy reconstruction is of course to find the most probable energy of the event, given the light signature of the passing muon in the detector. As mentioned earlier, the energy dependence is contained in the expected number of photoelectrons $\langle n_{pe} \rangle$. The expected number of photoelectrons in a given time interval $(t_{\text{min}}, t_{\text{max}})$ is the time integral of the probability densities:

$$
\langle n_{pe} \rangle = \int_{t_{\text{min}}}^{t_{\text{max}}} dt \left( P_{dm} + P_{sm} + P_{dEM} + P_{sEM} \right) + R_{bg} (t_{\text{max}} - t_{\text{min}}),
$$

(3.4.13)

where the probability densities $P_x$ are:

$$
P_{dm}(R, \theta, \phi, t) = \frac{dP_{dm}(R, \theta, \phi, t)}{dt},
$$

(3.4.14)

$$
P_{sm}(R, \theta, \phi, t) = \frac{dP_{sm}(R, \theta, \phi, t)}{dt},
$$

(3.4.15)

$$
P_{dEM}(R, \theta, \phi, t) = \frac{dP_{dEM}(R, \theta, \phi, t)}{dt} \cdot E',
$$

(3.4.16)

$$
P_{sEM}(R, \theta, \phi, t) = \frac{dP_{sEM}(R, \theta, \phi, t)}{dt} \cdot E'.
$$

(3.4.17)

The subscripts used above stand for direct light from the muon ($dm$), scattered light from the muon ($sm$), direct light from electromagnetic showers ($dEM$) and scattered light from electromagnetic showers ($sEM$). Here, $R$ is the distance of the track from the optical module, and $\theta$ and $\phi$ are the relative zenith and azimuth angles of the PMT with respect to the track. The probability densities for electromagnetic showers have been normalized to 1 GeV. This is the reason why the last two equations are scaled by the energy $E'$ (see equation 3.1.11).

In order to take into account the average energy-loss of the muon, from the moment of entering the vicinity of the detector creating the first hit until the moment it reaches the optical module under consideration, energy $E'$ is expressed as:

$$
E' = \left( \frac{a}{b} + E \right) e^{-bL} - \frac{a}{b},
$$

(3.4.18)

where $E$ is the free parameter of the likelihood function. Equation (3.4.18) is obtained after an integration of equation (3.1.4), answering the question on how much is the energy $E'$ of a muon of initial energy $E$ after traveling a path length $L$. Distance $L$ is calculated under the assumption that the hits are created by photons emitted under the Čerenkov hypothesis. The constants $a = 0.27$ GeV/m and $b = 3.4 \cdot 10^{-4}$ m$^{-1}$ are the ones in the average energy-loss equation (3.1.4) and their exact values are taken from [Klim 01].

The algorithm searches for hits on the PM tubes within a time window of $(t_0 - 10 \text{ ns}, t_0 + 390 \text{ ns})$, where $t_0$ is the expected arrival time of the photon under the Čerenkov
hypothesis, given a certain track geometry. When a hit is recorded on the PMT at time $t_{hit}$, the time integration window of equation (3.4.13) is taken to be $(t_{hit}, t_{hit} + 40 \text{ ns})$. This corresponds to the integration time of the ARS. When no hit is observed the probability densities are integrated for the entire period of $(t_0 - 10 \text{ ns}, t_0 + 390 \text{ ns})$. Finally, the term $R_{bg}(t_{max} - t_{min})$ corresponds to the optical background contribution to the expected number of photoelectrons in the time window $(t_{min}, t_{max})$.

The next step is to find the energy for which $-\log \mathcal{L}$, i.e. the natural logarithm of equation (3.4.6), is minimum. The shape of the likelihood function as a function of the energy (i.e. the free parameter of the likelihood function) is shown in figure 3.15 for a few example events of different energies for illustration purposes. The most probable energy is given by the energy value which minimizes $-\log \mathcal{L}(E)$. The gradual underestimation observed for increasing energies will be discussed later in this chapter when the overall performance of the method is examined in more detail.

Figure 3.15: Natural logarithm of the likelihood function, $-\log \mathcal{L}$, as a function of energy for random simulated muons of energy $10^{3.09}\text{ GeV}$, $10^{4.24}\text{ GeV}$, $10^{5.33}\text{ GeV}$ and $10^{6.33}\text{ GeV}$ (in reading order). The minima correspond to the energy estimate found by the energy reconstruction algorithm shown by the solid line. The dashed line corresponds to the generated Monte Carlo muon energy.
3.5 Energy reconstruction performance

The results of the energy reconstruction as well as the performance will be discussed in the following sections. The Monte Carlo sample that is used consists of upward-going neutrino events. Realistic optical background hits are generated according to a Poisson distribution, with the mean rate determined from the average rate over all PMT’s taken from real data. A flat, i.e. constant, background of $R_{bg} = 60 \text{kHz}$ is used at the energy reconstruction level (equation 3.4.13). For the photomultiplier simulation, 2 ARS’s per optical module were considered, each with an integration time of 40 ns and a dead time of 250 ns. The events were triggered with both the 3N and T3 triggers, described in section 2.7.

3.5.1 Energy resolution

The correlation between the true Monte Carlo muon energy and the reconstructed energy is shown in figure 3.16. The Monte Carlo sample consists of upward-going neutrinos and the true Monte Carlo direction has been used as the input to the energy reconstruction to decouple the performance of the energy reconstruction from that of the track reconstruction. Figure 3.16 shows that there is a good correlation between the true muon energy and the output of the energy reconstruction. However a progressive underestimation of the energy is visible for higher muon energies. The energy of a muon event with energy close to $10^6 \text{GeV}$, for example, will be underestimated by almost one order of magnitude. From now on we define:

$$\delta \log E \equiv \log_{10} \frac{E_{\text{Reco}}}{E_{\text{MC}}}.$$  \hspace{1cm} (3.5.1)

The underestimation is more visible in figure 3.17 where $\delta \log E$ is plotted against the logarithm of the muon Monte Carlo energy. For a perfect energy reconstruction method this plot would be a straight line at $\delta \log E = 0$. The deviations from zero indicate the bias of the method while the spread is a measure of the resolution. The quality of the energy reconstruction depends on the muon energy. A typical measure of the energy resolution is the standard deviation from a Gaussian fit of the $\delta \log E$, assuming that a Gaussian fit describes this distribution well. A few distributions of $\delta \log E$ for different ranges in energy are shown in figure 3.18. The distributions of $\delta \log E$ are reasonably well described by Gaussian distributions. There are still tails present and the mean of the Gaussian fit is slightly shifted from the mean of the distributions. The deviations from a Gaussian shape can be expected since the distributions shown in figure 3.18 are essentially projections of slices in $\log_{10} E_{\text{MC}}$ on the $\delta \log E$ axis (see fig. 3.17). The fact that the slices in Monte Carlo energy have a finite width and are not mono-energetic, as well as the presence of some mis-reconstructed events, lying far away from the mean creating the tails of these distributions, are responsible for the observed deviations.

To evaluate the overestimation or underestimation of the energy we observe the behavior of the mean of the Gaussian fit. The mean and standard deviation are shown
Figure 3.16: $\log_{10} E_{\text{Reco}}$ against $\log_{10} E_{\text{MC}}$ for an upward-going muon neutrino sample. The true Monte Carlo direction of the muon is used. The dashed line corresponds to $E_{\text{Reco}} = E_{\text{MC}}$.

Figure 3.17: $\delta \log E$ as a function of $\log_{10} E_{\text{MC}}$ for an upward-going muon neutrino sample. An ideal energy reconstruction performance would result in a straight $\delta \log E = 0$ line.
3.5 Energy reconstruction performance

Figure 3.18: Distributions of $\delta \log E$ for muon tracks with true Monte Carlo energy in six different energy ranges (in reading order $10^{1.95} - 10^{2.05}$ GeV, $10^{2.95} - 10^{3.05}$ GeV, $10^{3.95} - 10^{4.05}$ GeV, $10^{4.95} - 10^{5.05}$ GeV, $10^{5.95} - 10^{6.05}$ GeV and $0 - 10^7$ GeV). The solid line represents a Gaussian fit.

For higher muon energies the reconstructed energy is systematically below the true energy. This is partially due to the saturation of the optical module signal at 20 photoelectrons. For lower energies, especially below 1 TeV, the light that is emitted by the muon is weakly dependent on its energy, as discussed in section 3.1, resulting in an overestimation. The resolution remains fairly stable over the whole energy range and the value of the standard deviation of the Gaussian fits is well below 0.4. Since the Gaussian fits only approximately describe the distribution of $\delta \log E$, an additional check is performed to examine the percentage of the events that belong inside one, two and three standard deviations from the mean of the Gaussian fit. If the fit were to describe the distribution perfectly the fraction of events should...
be 68%, 95% and 99.7% respectively. The fraction of events for one, two and three standard deviations is shown in figure 3.21 as a function of the energy. The agreement is fairly reasonable with the tails of the distributions created by outlier events causing a minor discrepancy in the area between two and three standard deviations.

The energy reconstruction quality shows no dependence on the muon direction as shown in figure 3.22 where the value of $\delta \log E$ is plotted against both the zenith and the azimuth angle of the track’s true Monte Carlo direction for upward going events.

### 3.5.2 Offset correction

The bias that is present, gradually underestimating the true energy as we move to higher values, can be corrected by applying a simple linear correction. This is relevant if one wants to examine the energy of a single event. For a collective study, i.e. examining the energy spectrum, such a correction is not necessary since it will be accounted for in the unfolding procedure. We address the issue of applying this event-by-event correction here. The mean true energy is plotted against slices of reconstructed energy and a linear fit is applied as shown in figure 3.23. This line gives the most probable true muon
3.5 Energy reconstruction performance

Figure 3.22: Dependence of energy reconstruction on the zenith and azimuth angle of the muon direction.

Figure 3.23: Correlation between the true Monte Carlo energy ($E_{MC}$) and the output of the energy reconstruction ($E_{Reco}$). A linear fit is applied to determine the relation between these two quantities, $\log_{10}E_{MC} = a + b \cdot \log_{10}E_{Reco}$, where $a = -0.58$ and $b = 1.27$.

energy for a given reconstructed energy value. The fit is applied in the energy range of $10^{2.5}$ GeV to $10^{5.5}$ GeV of the reconstructed energy where the correlation exhibits a linear behavior. For lower energies the limited contribution of energy-losses compared to background light leads to an overestimation of the true energy, while for higher energies the electronics saturation leads to an underestimation. This is visible in figures 3.16 and 3.17 where one sees the flattening of the distribution for energies below $\sim 10^{2.5}$ GeV and above $\sim 10^{5.5}$ GeV. The mean and $\sigma$ of the Gaussian fits per slice of reconstructed energy are shown in figures 3.24 and 3.25. The mean appears to be close to zero over the whole energy range and the resolution achieved is at the level of 0.4 in the logarithm of the energy for energies above around 2-4 TeV. As mentioned earlier, such a correction is useful when we need the energy estimate of a single event. This correction will not be used in the work presented here, unless explicitly stated otherwise.

3.5.3 Dependence on track reconstruction quality

In this section we will examine the dependence of the energy resolution on the quality of the track reconstruction. The performance of the energy estimation is expected to be strongly dependent on the quality of the track reconstruction. An error in the
directional information of the track can result in different expected photon fluxes on the optical modules which in turn will lead to a reduced energy reconstruction performance. In this case, the direction that was the result from the track reconstruction has been used instead of the generated muon direction. The output of the track reconstruction is used to calculate parameters such as the distance of the track from the optical modules or the orientation of the modules with respect to the track that in turn are fed into the likelihood function of the energy reconstruction. Large deviations from the correct values, attributed to mis-reconstructed events, are propagated to large deviations from the true muon energy.

The average deviation from the mean $\delta \log E$ for a known Monte Carlo energy as a function of the angular error from the track reconstruction $\beta$, defined as the angle between the true Monte Carlo and reconstructed muon tracks, is shown in figure 3.26. This quantity is a measure of the energy resolution. It is clear from the figure that the quality of the energy reconstruction worsens as the error from the track reconstruction becomes larger. The same conclusion can be drawn by looking at the dependence of the
average deviation from the mean $\delta \log E$ on the track reconstruction quality parameter $\Lambda$, shown in figure 3.27. Lower values of $\Lambda$ that correspond to poorly reconstructed tracks also characterize events with poorly reconstructed energies. This is shown in figure 3.28 where a cut at $\Lambda < -5.3$ improves the resolution, indicated by a reduction of the spread of the $\delta \log E$ distribution. The $\sigma$ of the Gaussian fits is decreased by more than 0.05, reaching the same level of performance as using the true Monte Carlo direction and position. The effect of poorly track-reconstructed events is manifested as long tails in the $\delta \log E$ distributions, especially for higher energy events. These tails disappear when a quality cut is applied, providing a more reliable Gaussian shape.

In order to evaluate how efficient the energy reconstruction is, one needs to see what fraction of the number of the events, where the track fit was successful, ended up having their energy reconstructed. In few cases, the shape of the log-likelihood function is monotonically decreasing and the maximum is at the lower end of the maximization range of energies. We consider that the energy reconstruction of these events failed. As we will soon see, this fraction is negligible. We define the efficiency of the energy

![Figure 3.27: Average deviation from the mean $\delta \log E$ as a function of the track reconstruction quality parameter $\Lambda$.](image)

![Figure 3.28: Energy resolution for all reconstructed events as well as events after a $\Lambda$ quality cut of $\Lambda > -5.3$.](image)
reconstruction as:

\[ \text{Efficiency} = \frac{\# \text{ of tracks with reconstructed energy}}{\# \text{ of tracks with reconstructed direction}}. \quad (3.5.2) \]

The reconstruction efficiency for different \( \Lambda \) values as a function of the true Monte Carlo energy is shown in figure 3.29. The more events with poor track reconstruction we use in the sample the more the efficiency drops. The efficiency of the energy reconstruction algorithm is fairly high, reconstructing successfully more than 90% of the events in all cases. There is of course a dependence on the \( \Lambda \) quality parameter as shown in the figure. For very well track-reconstructed events (\( \Lambda > -5.3 \)) the efficiency is above 96%, increasing to 100% for higher energies. The efficiency drops when more poorly reconstructed tracks are included in the sample. Imposing no quality cut on the \( \Lambda \) variable one sees that the efficiency is decreased, especially for events with low or high energy, indicating that the energy reconstruction fails to estimate the energy of poorly reconstructed tracks. Similarly to equation (3.5.2), we define the efficiency related to the choice of the track reconstruction quality cut \( \Lambda \) as:

\[ \text{Efficiency} = \frac{\# \text{ of tracks selected after } \Lambda \text{ cut}}{\# \text{ of tracks with reconstructed direction}}. \quad (3.5.3) \]

The efficiencies for different values of \( \Lambda \) cut are shown in figure 3.30.
3.5 Energy reconstruction performance

3.5.4 Effect of scattering model

In this section we examine the effect of the scattering model that is assumed in the probability density functions in order to calculate the expected number of photoelectrons in the optical modules. The energy reconstruction method described in this chapter is applied using the $f_4$ and $p_{0.0075}$ models described in section 3.1.2. In the Monte Carlo simulations, the $p_{0.0075}$-model was used to describe light scattering. The ratio between the output of the energy reconstruction $E_{\text{Reco}}(f_4)$ using the $f_4$-model and the energy $E_{\text{Reco}}(p_{0.0075})$ using the $p_{0.0075}$-model is shown in figure 3.32. Using the $f_4$-model systematically leads to lower energy values than the ones obtained with the $p_{0.0075}$-model. The effect is more prominent at lower energies (below a few TeV), but always remains less than 4%. This can be attributed to the fact that the majority of light emitted from the muon and the showers along the track is towards the forward direction, peaking at the Čerenkov angle. The $f_4$-model favors forward scattering on a broader range of angles than the $p_{0.0075}$-model (see figure 3.9). If the assumption of the $f_4$-model used to reconstruct the energy produces more hits for a certain muon

Figure 3.30: Track reconstruction efficiency for different values of $\Lambda$ cut, indicated on the figure.

Figure 3.31: Distribution of $\delta \log E$ for different scattering models used in the calculation of the PDFs. The reconstructed track has been used instead of the true muon direction and no additional cuts are applied.
energy than the $p0.0075$-model used in the simulation, the reconstructed energy will end up lower than the true muon energy, which is what is illustrated in the figures. Finally, the uncertainty of the light absorption length in sea water is expected to have a more significant impact to the energy reconstruction result than the one due to the use of different scattering models. This effect will be examined in more detail in the following chapter.