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Stochastic equilibria of an asset pricing model with heterogeneous beliefs and random dividends

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Abstract

We investigate dynamical properties of a heterogeneous agent model with random dividends and further study the relationship between dynamical properties of the random model and those of the corresponding deterministic skeleton, which is obtained by setting the random dividends as their constant mean value. Based on our recent mathematical results, we prove the existence and stability of random fixed points as the perturbation intensity of random dividends is sufficiently small. Furthermore, we prove that the random fixed points converge almost surely to the corresponding fixed points of the deterministic skeleton as the perturbation intensity tends to zero. Moreover, simulations suggest similar behaviors in the case of more complicated attractors. Therefore, the corresponding deterministic skeleton is a good approximation of the random model with sufficiently small random perturbations of dividends. Given that dividends in real markets are generally very low, it is reasonable and significant to some extent to study the effects of heterogeneous agents' behaviors on price fluctuations by the corresponding deterministic skeleton of the random model.

Keywords: Heterogeneous beliefs; Random dividends; Random fixed points; Stability; Bifurcation

JEL classification: C0, C6, D84, E3, G12

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1 Introduction

Heterogeneous agent models have received more and more interest in the past decade, providing a wide range of explanations for many stylized facts in financial markets. Related studies mainly include computational oriented work, see Lebaron (2006) for an overall recent survey of the literature, and analytically tractable work, see Hommes (2006) and Chiarella et al. (2009) for a recent survey on research along this direction. In the latter case, one of the most important models is given by Brock and Hommes (1997, 1998). Brock and Hommes (1997) provided a simple evolutionary framework for endogenous strategy selection, where the strategies that have performed well attract more followers. Brock and Hommes (1998) applied this framework to a simple present discounted value asset pricing model with heterogeneous beliefs (called BH model hereafter). That is, in each period, agents choose from a finite set of different predictors of the future price of a risky asset. Predictor selection is based on a performance measure such as past realized profits. Based on bifurcation theory of discrete dynamical systems, one of their main findings is that as the intensity of choice to switch prediction strategies increases, the fundamental price becomes unstable and bifurcation routes to complicated dynamics such as quasi-periodic and even chaotic asset price fluctuations occur. That is to say, the high intensity of choice to switch strategies is one of the most important reasons for asset price fluctuations.

During the past decade, there have been many papers extending the BH model from different perspectives, see for example, Gaunersdorfer (2000), Chiarella and He (2002, 2003), Brock et al. (2005, 2009), Hommes et al. (2005), Chiarella et al. (2006), Anufriev and Panchenko (2009), Dieci and Westerhoff (2010). The above models aim at studying the effects of agents' behaviors on price fluctuations from the interior of financial markets. They provide a wide range of explanations for many stylized facts of asset prices such as volatility clustering, long memory, etc. by a mixture of theoretical studies and numerical simulations. The models they provided are discrete dynamical systems with random dividends. A common method they applied to the analytical studies is to set the dividends as constants (their means) such that the original systems become deterministic discrete dynamical systems and then to study the deterministic skeletons instead of the original models. Then based on the theoretical results obtained from the deterministic dynamical systems, they

made numerical simulations to find out the properties of the models obtained from adding random perturbations to the deterministic dynamical systems. A natural and important question is whether the deterministic skeletons obtained by setting random dividends as a constant, the mean, are good approximations of the original random models. That is, is it reasonable to study the deterministic skeletons instead of the original random models? In this paper we try to give a partial answer to this question.

In order to focus on the stochastic analysis for the random models and compare it with the corresponding deterministic skeletons more conveniently, here we take the model given by Hommes et al. (2005) as our base model (called HHW model hereafter). Hommes et al. (2005) extended the BH model by considering a market maker scenario, allowing for asynchronous updating of beliefs or strategies and considering a positive supply of outside shares of the risky asset. Despite these differences from the BH model, Hommes et al. (2005) found out that the global picture of asset price dynamics is surprisingly similar to that of the BH model, showing that many dynamic features are robust with respect to details of modeling market institutions and evolutionary strategy switching. Therefore in this paper we take the HHW model as a simple representation of agent based asset pricing models, where dividends follow a stochastic process, trying to investigate whether dynamic features are also robust with respect to random perturbations of dividends.

There have been some results on asset pricing models with random dividends. For example, via dynamic programming Lucas (1978) analytically examined the stochastic behavior of asset prices under the traditional framework with homogeneous and rational traders, where dividends follow a stochastic process; by applying methods of Tong (1990) and Arnold and Boxler (1991), Pötzelberger and Sögner (2003) theoretically investigated the problem of stability and the impact of learning in a setup with random dividends under the framework of boundedly rational traders; based on Arnold (1998), by studying the existence and stability of random fixed points Böhm and Chiarella (2005) theoretically and numerically analyzed the dynamics of prices and price expectations in an economy with overlapping generations and random dividends. However, the methods used in the above studies can not be applied directly to the HHW model with random dividends to show the existence and stability of random fixed points. Also these studies did not talk about the relationship between random systems and their corresponding deterministic skeletons.

In order to study the existence and stability of random fixed points of the HHW model with random dividends and their relationship to the fixed points of the corresponding deterministic skeleton, we found that most of the available mathematical results are related to Lyapunov exponents and random norms, which make the results difficult to be applied directly to practical asset pricing models. See for example, Arnold and Boxler (1991) and Arnold (1998). Therefore we first developed a new mathematical tool in Zhu et al. (2009), in which we gave easy-to-use conditions to guarantee the continuity of Lyapunov exponents for a special case under small random perturbations by using some ideas of Young (1986). In fact, Young (1986) studied a more general case. However, the HHW model does not satisfy the conditions given by Young (1986). In order to deal with the special cases related to the HHW model, we provided some assumptions that are weaker than those given by Young (1986). On the other hand, Zhu et al. (2009) also for the first time provided the conditions to guarantee that the ratio of the random norm and the standard Euclidean norm has deterministic bounds, which plays a crucial role in obtaining the existence, stability and convergence of stationary solutions (i.e. random fixed points) of nonlinear random difference equations in the almost sure sense. Based on the idea of Arnold and Boxler (1991), Zhu et al. (2009) proved existence of stationary solutions of nonlinear random difference equations under new easy-to-use conditions. In addition, Zhu et al. (2009) further studied stability of the random fixed points and their relationship to the fixed points of the corresponding deterministic skeleton in the almost sure sense. In this paper we will apply our recent mathematical results to the HHW model with random dividends.

There is also some literature contributing to stochastic stability and the relationship between stochastic systems and their corresponding deterministic skeletons from a probabilistic point of view. For example, Meyn and Tweedie (1993) studied the existence and stability of stationary distributions. Kifer (1974, 1988) and Medio (2004) investigated the relationship between stationary distributions of random systems and attractors of their corresponding deterministic skeletons. Nishimura and Stachurski (2005) applied stochastic stability theory from Meyn and Tweedie (1993) to study the stability of stochastic optimal growth models. From a general point of view, Diks and Wagener (2008) further found out that in the compact case, the set of stable (non-bifurcating in the weak sense) systems is open and dense. However, what we are interested in is the dynamical properties of the HHW model with

random dividends. In particular, we analytically study the existence and stability of random fixed points and the relationship between random fixed points and their corresponding fixed points of the deterministic skeleton. As far as random fixed points are concerned, existence, stability and convergence of random fixed points in the almost sure sense are stronger, implying existence, stability and convergence of stationary distributions in the weak sense. As shown in Arnold (1998), dynamical study of random models provides a much richer structure than just a family of stochastic processes.

The rest of this paper is set out as follows. Section 2 revisits the HHW model. Section 3 analytically provides existence, stability and convergence of random fixed points of HHW model. The numerical simulations are also given to illustrate the theoretical results. Section 4 presents some further numerical simulations with respect to (quasi-)periodic or strange attractors under random perturbations of dividends. Section 5 concludes the paper with some discussion. Proofs of all propositions in Section 3 are given in Appendix A and some related basic concepts are given in Appendix B.

2 A simple heterogeneous agent asset pricing model

In the following we briefly introduce the HHW model given by Hommes et al. (2005). The HHW model is constructed under the following major assumptions:

- 1) There are two groups of speculators in the market: fundamentalists and trend followers. They are myopic mean-variance maximizer of next-period wealth and have same risk aversion coefficient $a > 0$, same belief about conditional variance σ^2 of excess return (without loss of generality $a\sigma^2 = 1$ is assumed).
- 2) There are only two assets: a risk free asset, whose gross return is $R > 1$, and a risky asset, whose price p_n is given by a market maker who adjusts the price proportional to the observed excess demand.
- 3) The dividend $\{y_n\}$ of the risky asset is i.i.d. with constant mean value $\bar{y}$.
- 4) Traders are boundedly rational and change their strategy asynchronously (a fraction α of traders sticking to their previous strategies). The updated population fractions $\bar{n}_{1n}, \bar{n}_{2n}$ are formed on the basis of discrete choice probability.

- 5) The fundamental price p^* which is a constant, is known by both kinds of speculators.
- 6) The period information gathering cost C_1 for fundamentalists is bigger than C_2 for trend followers.

Let $x_n = p_n - p^*$, $m_n = \bar{n}_{1n} - \bar{n}_{2n}$, $C = C_1 - C_2$. Then the HHW model with stochastic dividends is written as following

$$\begin{cases} x_{n+1} = (1 - \mu R + \frac{\mu g}{2}(1 - m_n))x_n, \\ m_{n+1} = \alpha m_n + (1 - \alpha) \tanh \left[\frac{\beta}{2} \left(g(R - (1 - \mu R + \frac{\mu g}{2}(1 - m_n)))x_n^2 \right. \right. \\ \left. \left. - g z_s x_n - C - g \epsilon \epsilon_{n+1} x_n \right) \right], \end{cases} \quad (2.1)$$

where $\mu > 0$ denotes the speed of price adjustment by the market maker, $g > 0$ is the intensity of expectation about price change by trend followers, $z_s \geq 0$ is the outside supply of shares, β is the intensity of choice measuring how fast agents switch between different prediction strategies and $\{\epsilon \epsilon_{n+1}\} = \{y_{n+1} - \bar{y}\}$ with $\epsilon \in [0, 1]$.

Different from Hommes et al. (2005) in which $\epsilon_{n+1} = y_{n+1} - \bar{y}$ denotes the noise term on dividends, but here we use $\epsilon \epsilon_{n+1} = y_{n+1} - \bar{y}$ to denote the noise term and ϵ is a parameter revealing the corresponding noise (perturbation) intensity. The expression $\epsilon \epsilon_{n+1} = y_{n+1} - \bar{y}$ implies that the mean of ϵ_{n+1} is zero for any n and $\epsilon \in (0, 1]$. Hommes et al. (2005) studied theoretically the existence and stability of fixed points and bifurcation route to randomness in the deterministic skeleton, i.e. the case $\epsilon = 0$ in detail. The motivation of our setting in this way is that we will study the model with stochastic perturbations directly, trying to find out the asymptotical relation (as $\epsilon \rightarrow 0$) between the dynamical properties of the model with stochastic noise and those of its corresponding deterministic skeleton. Then we will know whether and to what extent the study of the corresponding deterministic skeleton is reasonable and significant for the original stochastic model.

3 Theoretical analysis of random fixed points

In order to make a theoretical analysis, we add an assumption to the noise term $\epsilon \epsilon_n$. Assume

(A). Each ε_n takes values in a bounded interval $I := [-M, M]$ and its probability measure ν on I is absolutely continuous with respect to Lebesgue measure with a density φ . In addition, ε_n is independent of the deviations of the past prices from the fundamental values and the differences of past fractions, i.e. $\{(x_k, m_k), k < n\}$.

Remark 1 Here we do not specify the interval I and the density φ . Furthermore, we also do not require that φ should be positive at each point in the interval I . This is because dividends of different stocks in real asset markets may vary in different ranges and their distributions may also be different. Here we try to study the random model (2.1) from a general point of view. Besides, dividends in real asset markets are mainly determined by the performance of firms and always vary in a bounded range. Hence the assumption (A) is natural.

With this assumption, we now begin to study the properties of the random model (2.1). For simplicity of expression, we focus on the symmetric case $z_s = 0$. The main ideas underlying the proofs suggest that similar results also hold in the asymmetric case $z_s > 0$ ¹.

Just as in the deterministic case, we are mainly concerned with equilibrium states of the random dynamical system. In analogue with deterministic fixed points, random fixed points (this concept and related theory are introduced in Appendix B) reveal equilibrium behavior of random dynamical systems. Hence we will first study existence and stability of random fixed points of the random system (2.1) and their relationship to the corresponding fixed points of the deterministic skeleton. Generally speaking, it is difficult to find out random fixed points for such complicated nonlinear random dynamical system. It is natural to check first whether fixed points of the deterministic skeleton are also fixed points after random perturbations are added to the system. For the fundamental fixed point $E = (x^*, m^*) = (0, -\tanh(\frac{\beta C}{2}))$, we have the following proposition.

¹According to Hommes et al. (2005), symmetry breaking occurs for $z_s > 0$: for $z_s = 0$, the bifurcation diagram is symmetric; for $z_s > 0$ and small, the bifurcation diagram is asymmetric. The global dynamical features of HHW model are quite robust with respect to changes in modelling assumptions (including assumption for the supply z_s). For detailed explanations we refer readers to Hommes et al (2005). On the other hand, the computation for the asymmetric case $z_s > 0$ would be more complicated without having much insight. Hence it is good enough to consider the symmetric case $z_s = 0$ for understanding the dynamical properties of the model with small $z_s > 0$.

Proposition 3.1 *For any $\epsilon \in [0, 1]$ and any sample path of $\{\varepsilon_n\}$, $E = (x^*, m^*) = (0, -\tanh(\frac{\beta C}{2}))$ is always a fixed point. Furthermore, in the case of hyperbolicity, its stability in the random case is the same as that in the deterministic case.*

This proposition indicates that the fundamental fixed point E of the deterministic HHW model and its stability can be preserved from any random perturbations of dividends. That is, as long as the fundamental steady state is asymptotically stable, the price p_n of the risky asset will eventually tend to the fundamental price p^* under any random perturbations of dividends provided p_0 is close to p^* . The numerical simulations shown in Figure 1² are consistent with the theoretical result of Proposition 3.1.

Figure 1: Time series of x of the first 200 steps for $\beta = 2.8$ and $\epsilon = 0$ (a), 0.1 (b), 0.6 (c) and 1 (d). In each figure, five initial values are considered: $(x_0, m_0) = (1.0, -0.8), (1.2, -0.6), (-1.0, -0.8), (-1.2, -0.6), (0, -0.9)$.

Here in order to compare the random model with the corresponding deterministic

²Here the same sample paths of noise are used in the four figures of Figure 1.

HHW model, we choose the same parameter values as in Hommes et al. (2005) except for z_s and $\epsilon\epsilon_n$. That is, except for the intensity of choice β and the noise intensity ϵ , in all figures the other parameters are fixed at

$$R = 1.1, g = 1.15, \mu = 1.6, \alpha = 0.5, C = 1.0, z_s = 0, \epsilon_n \sim U(-1, 1),$$

where $U(-1, 1)$ denotes uniform distribution on $[-1, 1]$. It is emphasized that numerous simulations suggest that all numerical results in this paper are independent of the selection of the sample path of $\{\epsilon_n\}$, the distribution density φ and support I . Furthermore, the theoretical results in this section also imply this kind of independence in some cases. Moreover, see also Hommes et al. (2005) for the reasons of choosing the particular values of the other parameters. Besides, in Figure 1 we set $\beta = 2.8$ and consider five different initial points to see that the market prices converge to the stable fundamental price under random perturbations of dividends.

Besides the fundamental fixed point, we further study the non-fundamental fixed points. Just as what Hommes et al. (2005) stated, the most interesting case occurs for a strong trend extrapolation parameter ($R < g < 2R$) and a low speed of adjustment parameter ($0 < \mu < \frac{2}{R}$): the fundamental fixed point becomes unstable through a supercritical pitchfork bifurcation. That is, there exists $\beta^* = \frac{2}{C} \tanh^{-1}(\frac{2R}{g} - 1) > 0$ such that the fundamental fixed point E is stable as $\beta < \beta^*$. However, as $\beta^* < \beta < \beta_{NS} = \frac{2}{C} \tanh^{-1}(\frac{2R}{g} - 1) + \frac{g(R-1)}{\mu R(g-R)(2R-1)C}$, E becomes unstable and there appear another two stable non-fundamental fixed points, denoted by E_r and E_l , where $E_r = (x_r, m_r) = (\sqrt{\frac{C(\beta-\beta^*)}{g(R-1)\beta}}, 1 - \frac{2R}{g})$ and $E_l = (x_l, m_l) = (-x_r, m_r) = (-\sqrt{\frac{C(\beta-\beta^*)}{g(R-1)\beta}}, 1 - \frac{2R}{g})$. See Hommes et al. (2005) for the details. Hence we first focus on the case: $R < g < 2R$, $0 < \mu < \frac{2}{R}$, $\beta^* < \beta < \beta_{NS}$, where there exist another two stable non-fundamental fixed points besides the unstable fundamental fixed point in the deterministic case. However, unlike the fundamental fixed point, the two non-fundamental fixed points are no longer fixed points of the random system (2.1). The following propositions show that the two stable non-fundamental fixed points become two almost surely stable random fixed points in the neighborhoods of the two corresponding stable fixed points of the deterministic skeleton for sufficiently small noise intensity, respectively.

Proposition 3.2 *For $R < g < 2R$, $0 < \mu < \frac{2}{R}$, $\beta^* < \beta < \beta_{NS}$, there exists $\epsilon^* > 0$*

satisfying

$$\epsilon^* < \min \left\{ \frac{2(R-1)x_r}{M}, \frac{\beta^* C}{\beta g x_r M}, 1 \right\}$$

such that there are three random fixed points with their corresponding stationary measures supported on disjoint sets as $\epsilon < \epsilon^*$. That is, besides the fundamental fixed point $E = (x^*, m^*) = (0, -\tanh(\frac{\beta C}{2}))$, there exist two other nontrivial random fixed points, respectively, in the neighborhoods of the two corresponding stable non-fundamental fixed points of the deterministic skeleton provided the noise intensity ϵ is sufficiently small.

The result of Proposition 3.2 is shown in Figure 2. In conjunction with Figure 1 (a), Figure 2 (a) illustrates that the fundamental fixed point undergoes a supercritical pitchfork bifurcation in the deterministic case as the intensity of choice β increases from 2.8 to 3.8. According to the model (2.1) and the proof of Proposition 3.1, m -axis is always a stable manifold of the fundamental fixed point E . That is to say that the orbit will stay on the m -axis forever and eventually tend to the fundamental fixed point once the initial point lies on the m -axis. Hence for initial point $(x_0, m_0) = (0, -0.9)$, $x_n \equiv 0$, which corresponds to the middle solid horizontal line as illustrated in Figure 2. In addition, Figure 2 also suggests that the two locally stable non-fundamental fixed points E_r and E_l become two locally stable random fixed points for sufficiently small noise intensity, respectively. If initial value $x_0 > 0 (< 0)$, then the orbit eventually converges to the random fixed point corresponding to $E_r(E_l)$, which fluctuates around $E_r(E_l)$. Furthermore, the volatility of the two random fixed points becomes larger as the noise intensity ϵ increases³. Besides, in order to indicate the symmetry with respect to m -axis and sample path of $\{\varepsilon_n\}$, in each figure of Figure 2 we let $\{\varepsilon_n\}$ for the system with initial values $(x_0, m_0) = (1.0, -0.8), (1.2, -0.6), (0, -0.9)$, but the opposite value $\{-\varepsilon_n\}$ of $\{\varepsilon_n\}$ for the system with initial values $(x_0, m_0) = (-1.0, -0.8), (-1.2, -0.6)$. Each figure in Figure 2 implies the symmetry, which can also be seen from the model (2.1).

In addition, the following several propositions further show that the random fixed points obtained in Proposition 3.2 have several other properties.

Proposition 3.3 *The random fixed points obtained by Proposition 3.2 are measurable.*

³Here the same sample paths of noise are used in the four figures of Figure 2.

Figure 2: Time series of x of the first 200 steps for $\beta = 3.8$ and $\epsilon = 0$ (a), 0.01 (b), 0.06 (c) and 0.1 (d). In each figure, five initial values are considered: $(x_0, m_0) = (1.0, -0.8), (1.2, -0.6), (-1.0, -0.8), (-1.2, -0.6), (0, -0.9)$; the other two solid horizontal lines: the two non-fundamental fixed points x_r and x_l of the deterministic skeleton.

Proposition 3.4 *As $\epsilon \rightarrow 0$, the two nontrivial random fixed points almost surely converge to the corresponding fixed points E_r and E_l of the deterministic skeleton, respectively. Furthermore, as $\epsilon \rightarrow 0$, their corresponding stationary measures also weakly converge to the corresponding Dirac measures supported on E_r and E_l , respectively.*

The result of Proposition 3.4 is illustrated in Figures 2 and 3⁴. Figure 3 suggests that the support of the stationary distribution corresponding to the random fixed

⁴Because of symmetry of the two nontrivial random fixed points, here we just select one initial point $(x_0, m_0) = (1.0, -0.8)$ to carry out 20,000 iteration steps, where the last 18,000 points are used to calculate the stationary distributions.

point near E_r is in the very small neighborhood of E_r as the noise intensity ϵ is close to zero, which is also consistent with Figure 2. Further, Figures 3 also suggest that the supports of the stationary measures become smaller as the noise intensity ϵ decreases. That is to say that the stationary measures weakly converge to the Dirac measure supported on the fixed point E_r of the corresponding deterministic skeleton as the noise intensity ϵ tends to zero. The same results hold for the random fixed point corresponding to E_l .

In fact, Figures 2 and 3 also imply the stability of the two nontrivial random fixed points. The following proposition shows this from a theoretical point of view.

Figure 3: Stationary distribution for $\beta = 3.8$ and $\epsilon = 0$ (a), 0.01 (b), 0.06 (c) and 0.1 (d). In each figure, the initial point $(x_0, m_0) = (1, -0.8)$.

Proposition 3.5 *The two nontrivial random fixed points corresponding to the two locally exponentially stable non-fundamental fixed points of the deterministic skeleton are locally exponentially stable almost surely.*

Since stability of equilibrium states can be characterized by Lyapunov exponents, Proposition 3.5 can be illustrated by Figure 4⁵. Combined with Hommes et al. (2005), Figure 4 (a) suggests that for $\beta < 3.1(\beta^*)$, the maximal Lyapunov exponent of the fundamental fixed point E increases up to zero as β increases; then for $\beta \in (3.1, 4.15)(\beta_{NS} = 4.15)$, at first the two negative Lyapunov exponents of the non-fundamental fixed point E_r are not equal, but eventually become equal as β increases, which corresponds to the pair of conjugate complex eigenvalues. Further, Figures 4 (b), (c) and (d) also imply that the two Lyapunov exponents of the random fixed point are very close to those of the corresponding fixed point of the deterministic skeleton for sufficiently small noise intensity ϵ in the case that $\beta < 4.15$. Thus the two Lyapunov exponents in the random case are still negative as in the deterministic case if $\beta < 4.15$. That is to say that the random fixed point corresponding to the locally exponentially stable fixed point $E_r(E_l)$ is still locally exponentially stable for $3.1 < \beta < 4.15$ provided ϵ is sufficiently small.

Therefore, from the above theoretical and numerical analyses, it can be seen that the system under small random perturbations possesses the similar properties to the corresponding deterministic skeleton in the case that the equilibrium states of the deterministic skeleton are locally stable fixed points. From the economic point of view, in the case that the exogenous random perturbations of dividends are sufficiently small, the behaviors of agents play a dominant role in affecting dynamical behaviors of asset prices.

4 Further numerical analysis

In order to see the effects of random dividends from a global point of view, we first provide the bifurcation diagrams in Figure 5 for $\epsilon = 0$ and 0.01, which correspond to the Lyapunov exponent diagrams in Figures 4 (a) and (b). Figure 5 (a) suggests that in the deterministic case there exist (quasi-)periodic or strange attractors for $\beta > 4.15$, as shown in Hommes et al. (2005). Correspondingly, Figure 5 (b) implies that the random model for $\epsilon = 0.01$ seems to have the similar attractors. Hence in order to get a deeper insight into the phenomena, in the following we study the case

⁵Because of symmetry of the two nontrivial random fixed points, here we just select one initial point $(x_0, m_0) = (1.0, -0.8)$ to carry out 10,000 iteration steps, where the last 2,000 points are used to calculate the Lyapunov exponents.

Figure 4: Lyapunov exponent corresponding to the attractors in the neighborhood of E_r for $\epsilon = 0$ (a), 0.01 (b), 0.06 (c) and 0.1 (d). In each figure, the initial point $(x_0, m_0) = (1, -0.8)$; the above line $- * - * -$: the maximal Lyapunov exponents; the below line $- \circ - \circ -$: the minimal Lyapunov exponents.

that $\beta > 4.15$ in detail. The adopted approaches are numerical simulations. The related theoretical work is left as future research.

4.1 Effects of random dividends on invariant closed curves

For a clear illustration of dynamical behaviors on invariant closed curves under random perturbations of dividends, in Figure 6⁶ we give time series and phase diagrams both in the deterministic case and in the random case that $\epsilon = 0.01$. In Figure 6 (a), for the case that $\epsilon = 0$, the values of x do not tend to be consistent no matter

⁶Here we carry out 10,000 iteration steps with five initial points in each case, where the first 500 and the last 1,000 steps are used to plot the time series and phase diagrams, respectively. Besides, the selection of sample paths of noise is the same as in Figure 2.

Figure 5: Bifurcation diagram for $\epsilon = 0$ (a), 0.01 (b). In each figure, three initial points $(x_0, m_0) = (1, -0.8), (0, 0.9), (-1, -0.8)$.

how many iteration steps are carried out for different initial values near E_r or E_l . In contrast to the deterministic case, in Figure 6 (b), for the case that $\epsilon = 0.01$, the values of x become consistent after 200 steps for different initial values near E_r or E_l , from which we guess that there exist locally stable random fixed points in the neighborhoods of the invariant closed curves of the deterministic skeleton. That is to say that the two attracting invariant closed curves in the deterministic case become two locally stable random fixed points under random perturbations⁷. However, Figures 6 (c) and (d) suggest that the two locally stable random fixed points for $\epsilon = 0.01$ take values in the neighborhoods of the corresponding invariant closed curves of the deterministic skeleton respectively, which can also be seen from bifurcation diagram Figure 5 (b). Hence the dynamical behaviors of the model with small random perturbations of dividends are similar to the corresponding deterministic skeleton near the invariant closed curves.

4.2 Effects of random dividends on chaotic attractors

Finally we try to indicate the random properties for chaotic attractors subject to random perturbations in Figures 7 and 8. For chaotic attractors of deterministic systems, it is known that from two close but different initial points, the two trajectories may deviate greatly after some evolution steps because of the property

⁷This phenomenon is similar to that for limit cycles of differential equations subject to random perturbations, see Arnold (1998) and Schenk-Hoppé (1996) for details.

Figure 6: Time series of x and phase diagram for $\beta = 4.3$ and $\epsilon = 0$ (a), (c) and 0.01 (b), (d). In each figure, five initial points are considered: $(x_0, m_0) = (1.0, -0.8), (1.2, -0.6), (-1.0, -0.8), (-1.2, -0.6), (0, -0.9)$.

of sensitive dependence on initial values, as shown in Figure 7 (a). Different from Figure 6 (b), the values of x do not tend to coincide even over 10,000 steps for two different initial values in Figure 7 (b)⁸. The comparison of Figure 7 (a) with Figure 7 (b) further suggests that the properties of the attractor under small random perturbations are very similar to those of the attractor of the deterministic skeleton, such as the property of sensitive dependence on initial values. In order to see the dynamical behaviors more clearly, the first 1,000 points of the two time series in Figures 7 (a) and (b) are shown in Figures 7 (c) and (d), respectively.

Furthermore, the last 3,000 points in Figures 7 (a) and (b) are used to plot phase diagrams shown in Figure 8. From Figure 8, it can be seen that the attractor in the random case with $\epsilon = 0.01$ looks very close to that in the deterministic case, which

⁸Here the same sample path of $\{\varepsilon_n\}$ is used for two different initial values.

can also be seen from bifurcation diagram Figure 5 (b). As a matter of fact, these properties are consistent with characteristics of Lyapunov exponents with respect to strange attractors in Figures 4, one of which still remains positive and the other still negative under small random perturbations. Hence the numerical simulations imply that the dynamical properties of chaotic attractors of the deterministic skeleton can remain in some sense under sufficiently small random perturbations of dividends.

Figure 7: Time series for $\beta = 4.5$ and $\epsilon = 0$ (a), (c) and 0.01 (b), (d). In each figure, two initial points are considered, where point line: $x_0 = 1.0, m_0 = -0.8$; solid line: $x_0 = 1.2, m_0 = -0.6$.

Therefore, from the above numerical analysis, it can be seen that the system under sufficiently small random perturbations generally possesses some similar dynamical properties to the corresponding deterministic skeleton in the case that the deterministic skeleton has stable invariant closed curves or strange attractors. That is to say, even in the case that there are invariant close curves or strange attractors, the behavior of agents is still an important factor of price fluctuations if dividends

Figure 8: Phase diagram for $\beta = 4.5$ and $\epsilon = 0$ (a) and 0.01 (b). In each figure, two initial points are considered, where point: $x_0 = 1.0, m_0 = -0.8$; star: $x_0 = 1.2, m_0 = -0.6$.

are sufficiently low.

5 Conclusion

In this paper we mainly study the dynamical behaviors of the HHW model with random dividends. Applying our recent mathematical results, we prove that the stable fixed points of its deterministic skeleton become almost surely stable random fixed points of the stochastic system under sufficiently small random perturbations of dividends; furthermore the stable random fixed points converge to the corresponding stable fixed points of the deterministic skeleton almost surely as the perturbation intensity tends to zero. Besides, the numerical simulations also suggest similar behavior in the case that the deterministic skeleton has stable invariant closed curves or strange attractors. From an economic point of view, these results mean that the behaviors of agents play a dominant role in affecting fluctuations of asset prices in the case that the exogenous random perturbations of dividends are sufficiently small. Since the dividends in real markets are generally very low, it is reasonable to study the approximate deterministic skeleton instead of the complicated model with random dividends.

In future research, firstly, it is very important to study theoretically dynamical behaviors on more general attractors such as stable invariant closed curves and strange attractors under sufficiently small random perturbations. Secondly, it is

interesting to apply the ideas and approaches used in this paper to other asset pricing models with i.i.d. random perturbations. Finally, it is significant to study asset pricing models with more general noises, such as AR(1) noise.

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Appendix

A Proof of propositions

Proof of Proposition 3.1. First, it is clear that $E = (x^*, m^*) = (0, -\tanh(\frac{\beta C}{2}))$ is a fixed point of the system (2.1) for all values of $\{\epsilon \epsilon_n\}$.

Second, the Jacobian matrix at the fundamental fixed point E is

$$J_i = \begin{pmatrix} 1 - \mu R + \frac{\mu g}{2}(1 + \tanh(\frac{\beta C}{2})) & 0 \\ -\frac{(1-\alpha)\frac{\beta g}{2}\epsilon \epsilon_i}{\cosh^2[\frac{\beta C}{2}]} & \alpha \end{pmatrix}, \quad i = 1, 2, \dots$$

Set $\gamma := 1 - \mu R + \frac{\mu g}{2}(1 + \tanh(\frac{\beta C}{2}))$, $c_i := -\frac{(1-\alpha)\frac{\beta g}{2}\epsilon \epsilon_i}{\cosh^2[\frac{\beta C}{2}]}$. Thus

$$J_n \cdots J_2 J_1 = \begin{pmatrix} \gamma^n & 0 \\ \sum_{i=1}^n c_i \gamma^{i-1} \alpha^{n-i} & \alpha^n \end{pmatrix}.$$

Since $\log^+ |c_1| \leq \log^+(\beta g \epsilon M)$, then $\log^+ \|J_1\| \in L^1$. Based on the property of Lyapunov exponents,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |\det(J_n \cdots J_1)| = \lambda_1 + \lambda_2 = \log |\gamma| + \log \alpha.$$

Furthermore, since $|\sum_{i=1}^n c_i \gamma^{i-1} \alpha^{n-i}| \leq n \beta g \epsilon M \cdot (\max\{|\gamma|, \alpha\})^{n-1}$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|J_n \cdots J_1\| = \lambda_1 = \max\{\log |\gamma|, \log \alpha\},$$

where λ_1 denotes the maximal Lyapunov exponent (Arnold, 1998, Lemma 3.2.1 and Example 3.3.9). Therefore, the Lyapunov exponents with respect to the fixed point E are

$$\log \left| 1 - \mu R + \frac{\mu g}{2} \left(1 + \tanh \left(\frac{\beta C}{2} \right) \right) \right|, \quad \log \alpha \quad (< 0).$$

That is, in the case of hyperbolicity, the stability of the fixed point in the random case is the same as that in the deterministic case. ■

The proofs of Propositions 3.2, 3.3, 3.4 and 3.5 are based on the theoretical results given by Zhu et al. (2009). In the following we first present the main results given by Zhu et al. (2009).

Consider nonlinear difference equation

$$Y_n = A(\epsilon \varepsilon_n) Y_{n-1} + H(\epsilon \varepsilon_n, Y_{n-1}) + h(\epsilon \varepsilon_n), \quad n \in \mathbb{Z}, \quad (\text{A.1})$$

where $Y_n \in \mathbb{R}^2$, $A(\cdot) \in C^2([-M, M], Gl(2, \mathbb{R}))$, $H(\cdot, \cdot) \in C^2([-M, M] \times \mathbb{R}^2, \mathbb{R}^2)$, $h(\cdot) \in C^2([-M, M], \mathbb{R}^2)$. $\epsilon \varepsilon_n$ denotes the random perturbation, where $\epsilon (\in [0, 1])$ is a constant revealing the corresponding perturbation intensity. Suppose that $\{\varepsilon_n\}$ is independent and identically distributed (i.i.d.) with noise ε_n independent of $\{Y_m, m < n\}$. Further, let ν denote the probability measure of ε_n on $[-M, M]$. Let $\Omega := [-M, M]^{\mathbb{Z}}$ be the bi-infinite product of the interval $[-M, M]$ with itself. For each $\omega \in \Omega$, let its n 'th coordinate $(\omega)_n$ equal some value of ε_n . That is, $\omega = (\cdots \varepsilon_{-1}, \varepsilon_0, \varepsilon_1 \cdots)$.

For convenience of expression, set

$$\begin{aligned} Y_n &= A(\epsilon \varepsilon_n) Y_{n-1} + H(\epsilon \varepsilon_n, Y_{n-1}) + h(\epsilon \varepsilon_n) \\ &=: A_\epsilon(\varepsilon_n) Y_{n-1} + H_\epsilon(\varepsilon_n, Y_{n-1}) + h_\epsilon(\varepsilon_n), \quad n \in \mathbb{Z}. \end{aligned}$$

Define the linear cocycle

$$\Phi_\epsilon(n, \omega) := \begin{cases} A_\epsilon(\varepsilon_n) \cdots A_\epsilon(\varepsilon_2) A_\epsilon(\varepsilon_1), & \text{for } n \geq 1, \\ \text{id}, & \text{for } n = 0, \end{cases}$$

where id denotes a 2×2 identity matrix. Assume that Lyapunov exponents $\{\lambda_i^\epsilon\}$ of Φ_ϵ satisfy $\lambda_2^\epsilon \leq \lambda_1^\epsilon < 0$. Fix a $\kappa(\epsilon) > 0$ such that $\lambda_1^\epsilon + \kappa(\epsilon) < 0$. For any $Y \in \mathbb{R}^2$, let

$$\|Y\|_\omega := \sum_{i=0}^{\infty} e^{-(\lambda_1^\epsilon + \kappa(\epsilon))i} \|\Phi_\epsilon(i, \omega)Y\|$$

denote the random norm of Y defined for the cocycle $\Phi_\epsilon(n, \omega)$, where $\|\cdot\|$ is the standard Euclidean norm in $\mathbb{R}^2$. Furthermore, set $|||Y||| := \text{ess sup} \|Y(\omega)\|_\omega$.

In addition, let $\mathbb{P}^1$ denote the projective space on $\mathbb{R}^2$. The real one-dimensional projective space $\mathbb{P}^1$ is obtained from $\mathbb{R}^2 - \{0\}$ by identifying two vectors if each is a scalar multiple of the other. Each $\bar{u} \in \mathbb{P}^1$ denotes the direction represented by the unit vector $u \in \mathbb{R}^2$. Given $t \in I = [-M, M]$, define a map $\overline{A_\epsilon(t)}(\bar{u}) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ by $\overline{A_\epsilon(t)}(\bar{u}) = \overline{A_\epsilon(t)u}$. Then $\{\overline{A_\epsilon(\varepsilon_n)}\}$ induces a Markov process on $\mathbb{P}^1$. Let E be a Borel subset of $\mathbb{P}^1$. Then the transition probabilities are defined by

$$P^\epsilon(\bar{u}, E) = \nu\{t \in I : \overline{A_\epsilon(t)}(\bar{u}) \in E\}.$$

Zhu et al.(2009) proved the following Theorem A.1 and Propositions A.2 – A.4.

Theorem A.1 *For random difference equation (A.1), assume*

- (i) *The two eigenvalues of $A(0)$ are both nonzero and less than one in modulus,*
- (ii) *For any given $\bar{u} \in \mathbb{P}^1$, the probability measure $P^\epsilon(\bar{u}, \cdot)$ induced by $\overline{A_\epsilon(\varepsilon_1)}(\bar{u})$ is absolutely continuous with respect to Lebesgue measure $m(\cdot)$ on $\mathbb{P}^1$,*
- (iii) *$H(\epsilon\varepsilon_n, 0) = 0$, $DH_Y(\epsilon\varepsilon_n, 0) = 0$ for any $\epsilon \in [0, 1]$ and any $n \in \mathbb{Z}$,*
- (iv) *$h(0) = 0$.*

Then for sufficiently small ϵ , there exists exactly one ergodic stationary solution (random fixed point) in the neighborhood of the origin whose initial value $\xi^\epsilon(\omega)$ satisfies with probability one

$$|||\xi^\epsilon||| \leq K'\epsilon,$$

where $K' > 0$ is a constant independent of ϵ .

Proposition A.2 *The stationary solution (random fixed point) obtained in Theorem A.1 is measurable.*

Proposition A.3 *As $\epsilon \rightarrow 0$, the stationary solution (random fixed point) obtained in Theorem A.1 converges almost surely to zero, which is an isolated fixed point in the deterministic case.*

Proposition A.4 *The stationary solution (random fixed point) obtained in Theorem A.1 is also locally exponentially stable with probability one.*

Remark 2 Note that condition (ii) of Theorem A.1 does not hold for HHW model since there is an exceptional point in $\mathbb{P}^1$. But based on the proofs of Theorem A.1 given by Zhu et al. (2009), the condition (ii) is only used to prove the absolute continuity of stationary measures on projective space $\mathbb{P}^1$ with respect to Lebesgue measure. The following proof of Proposition 3.2 indicates how stationary measures on $\mathbb{P}^1$ are proved to be absolutely continuous with respect to Lebesgue measure even though there is an exceptional point in $\mathbb{P}^1$. Therefore all results of Theorem A.1 and Propositions A.2, A.3, A.4 still hold.

Now we turn to the proofs of Proposition 3.2, Proposition 3.3, Proposition 3.4 and Proposition 3.5.

Proof of Proposition 3.2. In the following we try to show that the two stable non-fundamental fixed points of the deterministic skeleton become two almost surely stable random fixed points of the corresponding random system under small random perturbations of dividends using Theorem A.1. We first focus on the study of the random fixed point corresponding to the non-fundamental fixed point $E_r = (x_r, m_r) = (\sqrt{\frac{C(\beta-\beta^*)}{g(R-1)\beta}}, 1 - \frac{2R}{g})$. In order to apply Theorem A.1, we need to make some transformations to the model (2.1).

Set $X_n := \begin{pmatrix} x_n \\ m_n \end{pmatrix}$. Suppose that model (2.1) is expressed by

$$X_{n+1} = F(\epsilon \varepsilon_{n+1}, X_n). \quad (\text{A.2})$$

As $\epsilon = 0$, the model becomes a deterministic system, which is the same as Hommes et al. (2005). Since $X_r := \begin{pmatrix} x_r \\ m_r \end{pmatrix}$ is a steady state of the deterministic system, then

$$X_r = F(0, X_r). \quad (\text{A.3})$$

Thus based on the expressions (A.2) and (A.3), we have

$$\begin{aligned} X_{n+1} - X_r &= F(\epsilon \varepsilon_{n+1}, X_n) - F(0, X_r) \\ &= A(\epsilon \varepsilon_{n+1})(X_n - X_r) + G(\epsilon \varepsilon_{n+1}, X_n) + (F(\epsilon \varepsilon_{n+1}, X_r) - F(0, X_r)), \end{aligned}$$

where $A(\epsilon \varepsilon_{n+1}) = DF(\epsilon \varepsilon_{n+1}, X_r)$, $G(\epsilon \varepsilon_{n+1}, X_n) = F(\epsilon \varepsilon_{n+1}, X_n) - F(\epsilon \varepsilon_{n+1}, X_r) - A(\epsilon \varepsilon_{n+1})(X_n - X_r)$. Set $Y_n := X_n - X_r$. Then

$$Y_{n+1} = A(\epsilon \varepsilon_{n+1})Y_n + H(\epsilon \varepsilon_{n+1}, Y_n) + h(\epsilon \varepsilon_{n+1}),$$

where $H(\epsilon\varepsilon_{n+1}, Y_n) = G(\epsilon\varepsilon_{n+1}, Y_n + X_r) = F(\epsilon\varepsilon_{n+1}, Y_n + X_r) - F(\epsilon\varepsilon_{n+1}, X_r) - A(\epsilon\varepsilon_{n+1})Y_n$, $h(\epsilon\varepsilon_{n+1}) = F(\epsilon\varepsilon_{n+1}, X_r) - F(0, X_r)$.

For simplification of expressions, further set

$$a := 1 - \mu R + \frac{1}{2}\mu g, \quad b := -\frac{1}{2}\mu g, \quad c := -\frac{1}{2}\beta C, \quad e := \frac{1}{2}\beta g(R - 1 + \mu R - \frac{1}{2}\mu g), \quad f := \frac{1}{4}\beta\mu g^2 > 0.$$

Then the system (2.1) can be expressed by

$$\begin{cases} x_{n+1} = ax_n + bx_n m_n, \\ m_{n+1} = \alpha m_n + (1 - \alpha) \tanh\left(c + ex_n^2 + fx_n^2 m_n - \frac{\beta g}{2} x_n \epsilon\varepsilon_{n+1}\right). \end{cases}$$

Since $x_r = \sqrt{\frac{C(\beta - \beta^*)}{g\beta(R-1)}}$, $m_r = 1 - \frac{2R}{g}$, then

$$2ex_r + 2fm_r x_r = 2x_r \left[\frac{1}{2}\beta g(R - 1 + \mu R - \frac{1}{2}\mu g) + \frac{1}{4}\beta\mu g^2 \left(1 - \frac{2R}{g}\right) \right] = \beta g(R - 1)x_r > 0,$$

$$c + ex_r^2 + fm_r x_r^2 = -\frac{1}{2}\beta C + x_r^2(e + fm_r) = -\frac{1}{2}\beta C + \frac{1}{2}(\beta - \beta^*)C = -\frac{1}{2}\beta^* C < 0.$$

Therefore,

$$\begin{aligned} A_\epsilon(\varepsilon_{n+1}) &:= A(\epsilon\varepsilon_{n+1}) = DF(\epsilon\varepsilon_{n+1}, X_r) \\ &= \begin{pmatrix} a + bm_r & bx_r \\ (1 - \alpha) \frac{2ex_r + 2fm_r x_r - \frac{\beta g}{2} \epsilon\varepsilon_{n+1}}{\cosh^2(c + ex_r^2 + fm_r x_r^2 - \frac{\beta g}{2} x_r \epsilon\varepsilon_{n+1})} & \alpha + (1 - \alpha) \frac{fx_r^2}{\cosh^2(c + ex_r^2 + fm_r x_r^2 - \frac{\beta g}{2} x_r \epsilon\varepsilon_{n+1})} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{1}{2}\mu g x_r \\ (1 - \alpha) \frac{\beta g(R-1)x_r - \frac{\beta g}{2} \epsilon\varepsilon_{n+1}}{\cosh^2(\frac{1}{2}\beta^* C + \frac{\beta g}{2} x_r \epsilon\varepsilon_{n+1})} & \alpha + (1 - \alpha) \frac{fx_r^2}{\cosh^2(\frac{1}{2}\beta^* C + \frac{\beta g}{2} x_r \epsilon\varepsilon_{n+1})} \end{pmatrix} \\ &=: \begin{pmatrix} 1 & -\frac{1}{2}\mu g x_r \\ a_\epsilon^3(\varepsilon_{n+1}) & a_\epsilon^4(\varepsilon_{n+1}) \end{pmatrix}, \end{aligned}$$

where $a_\epsilon^3(\varepsilon_{n+1}) = \frac{(1-\alpha)(\beta g(R-1)x_r - \frac{\beta g}{2} \epsilon\varepsilon_{n+1})}{\cosh^2(\frac{1}{2}\beta^* C + \frac{\beta g}{2} x_r \epsilon\varepsilon_{n+1})}$, $a_\epsilon^4(\varepsilon_{n+1}) = \alpha + \frac{(1-\alpha)fx_r^2}{\cosh^2(\frac{1}{2}\beta^* C + \frac{\beta g}{2} x_r \epsilon\varepsilon_{n+1})}$.

If $\epsilon = 0$, then $A_0(\varepsilon_{n+1}) (= A(0))$ is a deterministic matrix, which is the same as $J_r(\beta)$ in Hommes et al. (2005).

Moreover,

$$\begin{aligned} H(\epsilon\varepsilon_{n+1}, 0) &= G(\epsilon\varepsilon_{n+1}, 0 + X_r) = F(\epsilon\varepsilon_{n+1}, 0 + X_r) - F(\epsilon\varepsilon_{n+1}, X_r) - A(\epsilon\varepsilon_{n+1}) \cdot 0 = 0, \\ DH(\epsilon\varepsilon_{n+1}, 0) &= DG(\epsilon\varepsilon_{n+1}, 0 + X_r) = DF(\epsilon\varepsilon_{n+1}, 0 + X_r) - 0 - A(\epsilon\varepsilon_{n+1}) = 0, \\ h(0) &= F(0, X_r) - F(0, X_r) = 0. \end{aligned}$$

In addition, under the conditions of Proposition 3.2, according to Hommes et al. (2005, p.1064), the two eigenvalues of the deterministic matrix $A(0)$ are both

nonzero and lie inside the unit circle in the complex plane. In the following we will show that transition probability $P^\epsilon(\bar{u}, \cdot)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{P}^1$ except for an exceptional point, and in this case stationary measures on $\mathbb{P}^1$ are still absolutely continuous with respect to Lebesgue measure. Thus just as shown in Remark 2, the corresponding results hold.

Let θ denote the angle corresponding to the unit vector u on the plane. Based on the elementary theory of topology, $\mathbb{P}^1 \cong S^1/\{\theta, \theta + \pi\}$, where S^1 denotes the unit circle in $\mathbb{R}^2$ and $\cong$ denotes topological equivalence. For convenience, we identify $\mathbb{P}^1$ with $S^1/\{\theta, \theta + \pi\}$ and use θ instead of $\bar{u}$ to denote elements of $\mathbb{P}^1$ in the following. Note that in space $S^1/\{\theta, \theta + \pi\}$ angles θ and $\pi + \theta$ are identical denoting one element. Thus if $\theta \neq \arctan \frac{2}{\mu g x_r} =: \theta^*$, then

$$\overline{A_\epsilon(t)}(\theta) = \overline{\begin{pmatrix} 1 & -\frac{1}{2}\mu g x_r \\ a_\epsilon^3(t) & a_\epsilon^4(t) \end{pmatrix}} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \arctan \frac{a_\epsilon^3(t) \cos \theta + a_\epsilon^4(t) \sin \theta}{\cos \theta - \frac{\mu g x_r}{2} \sin \theta} =: \psi_\epsilon(t, \theta).$$

If $\theta = \theta^*$, then for any $\epsilon \in [0, \frac{2(R-1)x_r}{M})$ and any $t \in I$,

$$\overline{A_\epsilon(t)}(\theta^*) = \overline{\begin{pmatrix} 1 & -\frac{1}{2}\mu g x_r \\ a_\epsilon^3(t) & a_\epsilon^4(t) \end{pmatrix}} \begin{pmatrix} \cos \theta^* \\ \sin \theta^* \end{pmatrix} = \overline{\begin{pmatrix} 0 \\ a_\epsilon^3(t) \cos \theta^* + a_\epsilon^4(t) \sin \theta^* \end{pmatrix}} = \frac{\pi}{2}.$$

Note that as $\epsilon < \frac{2(R-1)x_r}{M}$, for any $t \in I = [-M, M]$,

$$\begin{aligned} a_\epsilon^3(t) &= \frac{(1-\alpha)(\beta g(R-1)x_r - \frac{\beta g}{2}\epsilon t)}{\cosh^2[\frac{\beta^* C}{2} + \frac{\beta g}{2}x_r \epsilon t]} > 0, \\ a_\epsilon^4(t) &= \alpha + \frac{(1-\alpha)f x_r^2}{\cosh^2[\frac{\beta^* C}{2} + \frac{\beta g}{2}x_r \epsilon t]} > 0, \\ a_\epsilon^3(t) \cos \theta^* + a_\epsilon^4(t) \sin \theta^* &> 0. \end{aligned}$$

Furthermore, as $0 < \epsilon < \min\{\frac{2(R-1)x_r}{M}, \frac{\beta^* C}{\beta g x_r M}, 1\} =: \epsilon_1$, for any $t \in I$,

$$\begin{aligned} (a_\epsilon^3)'(t) &= \frac{-(1-\alpha)\frac{\beta g}{2}\epsilon[1 + 2x_r(\beta g(R-1)x_r - \frac{\beta g}{2}\epsilon t) \tanh(\frac{\beta^* C}{2} + \frac{\beta g}{2}x_r \epsilon t)]}{\cosh^2[\frac{\beta^* C}{2} + \frac{\beta g}{2}x_r \epsilon t]} < 0, \\ (a_\epsilon^4)'(t) &= \frac{-(1-\alpha)f\beta g\epsilon x_r^3 \tanh(\frac{\beta^* C}{2} + \frac{\beta g}{2}x_r \epsilon t)}{\cosh^2[\frac{\beta^* C}{2} + \frac{\beta g}{2}x_r \epsilon t]} < 0. \end{aligned}$$

Hence if $\theta \in [0, \theta^*) \cup (\theta^*, \frac{\pi}{2}]$, then for any $t \in I$,

$$\frac{\partial \psi_\epsilon}{\partial t} = \frac{(\cos \theta - \frac{\mu g x_r}{2} \sin \theta) \cdot ((a_\epsilon^3)'(t) \cos \theta + (a_\epsilon^4)'(t) \sin \theta)}{(\cos \theta - \frac{\mu g x_r}{2} \sin \theta)^2 + (a_\epsilon^3(t) \cos \theta + a_\epsilon^4(t) \sin \theta)^2} \neq 0.$$

Further, since $\frac{1}{\tanh(\frac{\beta^*C}{2} + \frac{\beta g}{2}x_r\epsilon t)}$ and $2x_r(\beta g(R-1)x_r - \frac{\beta g}{2}\epsilon t)$ are both decreasing functions with respect to t , for any given θ there is at most one point $t_\epsilon^*(\theta)$ such that

$$\frac{(a_\epsilon^3)'(t)}{(a_\epsilon^4)'(t)} + \tan \theta = \frac{1}{2fx_r^3} \left[\frac{1}{\tanh(\frac{\beta^*C}{2} + \frac{\beta g}{2}x_r\epsilon t)} + 2x_r(\beta g(R-1)x_r - \frac{\beta g}{2}\epsilon t) \right] + \tan \theta = 0.$$

Thus if $\theta \in (-\frac{\pi}{2}, 0)$, then

$$\frac{\partial \psi_\epsilon}{\partial t} = \frac{(\cos \theta - \frac{\mu g x_r}{2} \sin \theta) \cdot (a_\epsilon^4)'(t) \cos \theta}{(\cos \theta - \frac{\mu g x_r}{2} \sin \theta)^2 + (a_\epsilon^3(t) \cos \theta + a_\epsilon^4(t) \sin \theta)^2} \left(\frac{(a_\epsilon^3)'(t)}{(a_\epsilon^4)'(t)} + \tan \theta \right)$$

has at most one zero point $t_\epsilon^*(\theta)$. That is, as $0 < \epsilon < \epsilon_1$, $\psi_\epsilon(t, \theta)$ is a monotone or piecewise monotone function with two pieces with respect to t for any given $\theta \neq \theta^*$ in $\mathbb{P}^1$.

If $\psi_\epsilon(t, \theta)$ is monotone with respect to t , then its inverse mapping exists, denoted by $\psi_\epsilon^-(\cdot, \theta)$. Set $U_\theta = \{\theta' | \theta' = \psi_\epsilon(t, \theta), t \in I\}$. For such θ and any bounded continuous function ϕ on $\mathbb{P}^1$,

$$\begin{aligned} \int_{U_\theta} \phi(\theta') dP^\epsilon(\theta, \theta') &= \int_I \phi(\overline{A_\epsilon(t)}(\theta)) d\nu(t) \\ &= \int_I \phi(\overline{A_\epsilon(t)}(\theta)) \varphi(t) dt \\ &= \int_{U_\theta} \phi(\theta') \varphi(\psi_\epsilon^-(\theta', \theta)) \left| \frac{\partial \psi_\epsilon^-}{\partial \theta'}(\theta', \theta) \right| d\theta'. \end{aligned}$$

Hence for any Borel set E in U_θ ,

$$P^\epsilon(\theta, E) = \int_E \varphi(\psi_\epsilon^-(\theta', \theta)) \left| \frac{\partial \psi_\epsilon^-}{\partial \theta'}(\theta', \theta) \right| d\theta'.$$

Besides, based on the definition of $P^\epsilon(\theta, \cdot)$, $P^\epsilon(\theta, E) = 0$ for any Borel set $E \subset \mathbb{P}^1 - U_\theta$. That is to say, $P^\epsilon(\theta, \cdot)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{P}^1$ with density function $\varphi(\psi_\epsilon^-(\theta', \theta)) \left| \frac{\partial \psi_\epsilon^-}{\partial \theta'}(\theta', \theta) \right|$ vanishing for θ' outside U_θ . Furthermore, if $\psi_\epsilon(t, \theta)$ is piecewise monotone with respect to t with two pieces, then in each monotone interval $P^\epsilon(\theta, \cdot)$ is absolutely continuous with respect to Lebesgue measure with a density function. Based on the two density functions, a new density function over the whole interval I can be obtained. Thus in this case $P^\epsilon(\theta, \cdot)$ is also absolutely continuous with respect to Lebesgue measure. Therefore, as $0 < \epsilon < \epsilon_1$, for any given $\theta \neq \theta^*$, $P^\epsilon(\theta, \cdot)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{P}^1$.

Further we show that stationary measures on $\mathbb{P}^1$ exist and are also absolutely continuous with respect to Lebesgue measure. Just as stated in Zhu et al. (2009),

since $\mathbb{P}^1$ is compact, stationary measures on $\mathbb{P}^1$ always exist; see also Arnold (1998). Let μ_ϵ denote any stationary measure on $\mathbb{P}^1$. That is, for any Borel set $E \subset \mathbb{P}^1$,

$$\mu_\epsilon(E) = \int_{\mathbb{P}^1} P^\epsilon(\theta, E) d\mu_\epsilon(\theta) = \int_{\mathbb{P}^1 - \{\theta^*\}} P^\epsilon(\theta, E) d\mu_\epsilon(\theta) + \mu_\epsilon(\{\theta^*\}) P^\epsilon(\theta^*, E). \quad (\text{A.4})$$

For one thing, since $P^\epsilon(\theta, \cdot)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{P}^1$ for any given $\theta \neq \theta^*$ as $0 < \epsilon < \epsilon_1$, then $P^\epsilon(\theta, \{\theta^*\}) = 0$ for any $\theta \neq \theta^*$. However, since $\overline{A_\epsilon(t)}(\theta^*) = \frac{\pi}{2}$ for any t as $\epsilon < \epsilon_1$, then $P^\epsilon(\theta^*, \{\theta^*\}) = 0$. Thus for any initial probability measure $\hat{\mu}_0$, $\hat{\mu}_1(\{\theta^*\}) = \int_{\mathbb{P}^1} P^\epsilon(\theta, \{\theta^*\}) d\hat{\mu}_0(\theta) = 0$. Recursively, for any $k = 1, 2, \dots$, $\hat{\mu}_k(\{\theta^*\}) = \int_{\mathbb{P}^1} P^\epsilon(\theta, \{\theta^*\}) d\hat{\mu}_{k-1}(\theta) = 0$. Hence any stationary measure $\mu_\epsilon(\{\theta^*\}) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \hat{\mu}_k(\{\theta^*\}) = 0$. For another thing, for any $\theta \neq \theta^*$ in $\mathbb{P}^1$, $P^\epsilon(\theta, \cdot)$ is absolutely continuous with respect to Lebesgue measure if $0 < \epsilon < \epsilon_1$. That is, if $\theta \neq \theta^*$, then for any Borel set E on $\mathbb{P}^1$, $P^\epsilon(\theta, E) = 0$ whenever Lebesgue measure $m(E) = 0$. Hence from the expression (A.4), if $m(E) = 0$, then we have $\mu_\epsilon(E) = 0$. That is to say, as $0 < \epsilon < \epsilon_1$, any stationary measure μ_ϵ on $\mathbb{P}^1$ is absolutely continuous with respect to Lebesgue measure.

Thus according to Theorem A.1 and Remark 2, there exists $\epsilon^* > 0$ satisfying $\epsilon^* < \epsilon_1$ such that as $\epsilon < \epsilon^*$, there exists exactly one random fixed point of (A.2) in the neighborhood of the non-fundamental fixed point E_r in the deterministic case whose initial value $\xi^\epsilon(\omega)$ satisfies

$$|||\xi^\epsilon - X_r||| \leq K'\epsilon,$$

where $K' > 0$ is a constant independent of ϵ .

Consequently, for almost all ω ,

$$\|\xi^\epsilon(\omega) - X_r\| \leq \|\xi^\epsilon(\omega) - X_r\|_\omega \leq |||\xi^\epsilon - X_r||| \leq K'\epsilon.$$

That is, as the noise intensity ϵ tends to zero, the radius of the neighborhood containing the random fixed point also converges to zero.

Based on symmetry of the system with respect to m -axis and random perturbations, the system has another random fixed point in the neighborhood of the non-fundamental steady state E_l in the deterministic case as the noise intensity $\epsilon < \epsilon^*$. Furthermore, as ϵ tends to zero, the radius of the neighborhood containing the random fixed point also converges to zero.

Therefore, there exists $\epsilon^* > 0$ satisfying

$$\epsilon^* < \min \left\{ \frac{2(R-1)x_r}{M}, \frac{\beta^* C}{\beta g x_r M}, 1 \right\}$$

such that the supports of their corresponding stationary measures do not intersect. That is, besides the fixed point $E = (x^*, m^*) = (0, -\tanh(\frac{\beta C}{2}))$, there exist another two nontrivial random fixed points respectively in the neighborhoods of the two stable non-fundamental fixed points for sufficiently small noise intensity. ■

Proof of Proposition 3.3. See Proposition A.2 above, also Proposition 5.1 in Zhu et al. (2009). ■

Proof of Proposition 3.4. See Proposition A.3 above, also Proposition 5.2 in Zhu et al. (2009). ■

Proof of Proposition 3.5. See Proposition A.4 above, also Proposition 5.3 in Zhu et al. (2009). ■

B Related basic concepts

Here we provide a brief introduction to the framework used to study random dynamical systems described by random difference equations. The reader is referred to Arnold (1998) for more information of the related theory.

A random dynamical system in the sense of Arnold (1998) consists of two building blocks: a model of the exogenous noise and a model of the system perturbed by noise. The exogenous noise is modeled as a so-called metric dynamical system known from ergodic theory. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space, and let $\sigma : \Omega \mapsto \Omega$ be a measurable invertible mapping which is measure-preserving with respect to $\mathbb{P}$ and whose inverse σ^{-1} is again measurable. Assume that σ is ergodic, i.e. $\mathbb{P}$ is ergodic with respect to σ and let σ^n denote the n th iterate of the mapping σ . The collection $(\Omega, \mathcal{F}, \mathbb{P}, \{\sigma^n\}_{n \in \mathbb{Z}})$ is called an ergodic metric dynamical system, denoted by σ for short.

It is well known that any stationary ergodic process $\{\varepsilon_n\}_{n \in \mathbb{N}}$, $\varepsilon_n : \omega \mapsto \mathbb{R}^m$ can be represented by an ergodic dynamical system. For example, let $\{\varepsilon_n\}_{n \in \mathbb{Z}}$ be a sequence of i.i.d. random variables with probability measure ν on some subset $\Sigma \subset \mathbb{R}^m$. Then $(\Omega, \mathcal{F}, \mathbb{P}) = (\Sigma^{\mathbb{Z}}, \mathcal{B}(\Sigma)^{\mathbb{Z}}, \nu^{\mathbb{Z}})$, where $\Sigma^{\mathbb{Z}}$ is the space of all sample paths of the process, $\mathcal{B}(\Sigma)^{\mathbb{Z}}$ is the Borel σ -algebra of all cylinder sets, and $\nu^{\mathbb{Z}}$ is

the product measure. Each element $\omega = \{\omega(n)\}_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ is a doubly infinite series describing a sample path of the process. The map $\sigma : \Omega \mapsto \Omega, \omega \mapsto \sigma\omega$ is defined by $(\sigma\omega)(n) = \omega(n+1)$ and is called the left shift. Define the map $\varepsilon : \Omega \mapsto \Sigma$ by $\varepsilon(\omega) = \omega(1)$. Then we have $\varepsilon_n(\omega) = \varepsilon(\sigma^{n-1}\omega)$ and thus obtain a representation of the original i.i.d. process as an ergodic metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\sigma^n\}_{n \in \mathbb{Z}})$. Such a process is often referred to as a real noise process.

The second ingredient consists of a parameterized family of time-one maps of topological dynamical systems $F : \Sigma \times X \mapsto X, X \subset \mathbb{R}^K$ inducing the random difference equation

$$x_{n+1} = F(\varepsilon(\sigma^n \omega), x_n) =: \mathcal{F}(\sigma^n \omega)x_n,$$

which governs the evolution of the system. Note that in this appendix we use x to denote any vector in a general space X , different from that in the model (2.1). With x_0 as initial value, the iteration of the mapping $\mathcal{F}$ under the perturbation ω induces a measurable map $\varphi : \mathbb{N} \times \Omega \times X \mapsto X$ defined by

$$\varphi(n, \omega, x_0) = \begin{cases} (\mathcal{F}(\sigma^{n-1}\omega) \cdots \mathcal{F}(\omega))x_0, & \text{if } n \geq 1, \\ x_0, & \text{if } n = 0, \end{cases}$$

such that $x_n = \varphi(n, \omega, x_0)$ is the state of the system at time n . For any initial value $x_0 \in X$ and any perturbation ω , the sequence of points $\{x_n\}_{n \in \mathbb{N}}$ is called an orbit of the random dynamical system φ .

As the random analogue of a fixed point of deterministic dynamical systems, a random fixed point describes the long-run behavior of random dynamical systems.

Definition B.1 (Random Fixed Point) *A random fixed point of a random dynamical system φ is a random variable $x_* : \Omega \mapsto X$ on $(\Omega, \mathcal{F}, \mathbb{P}, \{\sigma^n\})$ such that*

$$x_*(\sigma\omega) = \varphi(1, \omega, x_*(\omega)) \quad \text{for all } \omega \in \Omega',$$

where $\Omega' \subset \Omega$ is a σ -invariant set of full measure, $\mathbb{P}(\Omega') = 1$. In particular, in the case of i.i.d. real noise process, a random fixed point of F is a random variable $x_* : \Omega \mapsto X$ on $(\Omega, \mathcal{F}, \mathbb{P}, \{\sigma^n\})$ such that

$$x_*(\sigma\omega) = F(\varepsilon_1, x_*(\omega)) \quad \text{for all } \omega \in \Omega',$$

where $\Omega' \subset \Omega$ is a σ -invariant set of full measure, $\mathbb{P}(\Omega') = 1$.

If F is independent of the perturbation ω , then the Definition B.1 is consistent with that of a fixed point of deterministic dynamical systems. Definition B.1 implies that $x_*(\sigma^{n+1}\omega) = F(\varepsilon_{n+1}, x_*(\sigma^n\omega))$ for all times n . Therefore, the orbit $\{x_*(\sigma^n\omega)\}_{n \in \mathbb{N}}, \omega \in \Omega$ generated by x_* solves the random difference equation

$$x_{n+1} = F(\varepsilon_{n+1}, x_n).$$

The random fixed point x_* induces an invariant measure $x_*\mathbb{P}$ on $\mathbb{R}^K$ defined by

$$(x_*\mathbb{P})(B) := \mathbb{P}((x_*)^{-1}(B)) = \mathbb{P}\{\omega \in \Omega | x_*(\omega) \in B\}.$$

In fact, the stationarity and ergodicity of the measure $\mathbb{P}$ under the shift σ (i.e., $\mathbb{P} = \sigma\mathbb{P} = \mathbb{P} \circ \sigma^{-1}$) implies the stationarity and ergodicity of $x_*\mathbb{P}$ since

$$((x_*\sigma)\mathbb{P})(B) = \mathbb{P}(\sigma^{-1} \circ (x_*)^{-1}(B)) = \mathbb{P}((x_*)^{-1}(B)) = (x_*\mathbb{P})(B).$$

In addition, if $\mathbb{E}\|x_*\| < \infty$, then

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{n=0}^{T-1} 1_B(x_*(\sigma^n\omega)) = x_*\mathbb{P}(B)$$

for every $B \in \mathcal{B}(X)$. That is, the stationarity and ergodicity of the measure $\mathbb{P}$ under the shift σ imply the stochastic process $\{x_*(\sigma^n\omega)\}_{n \in \mathbb{N}}$ is stationary and ergodic.

In the following, we provide the definition of an almost surely exponentially stable random fixed point used in this note and Zhu et al (2009).

Definition B.2 (Almost Sure Exponential Stability) *A random fixed point x_* of the random dynamical system φ is called almost surely exponentially stable with respect to a norm $\|\cdot\|$ on X if for any small $\bar{\varepsilon} > 0$, there exists $\bar{\delta} > 0$ such that for almost all ω ,*

$$\sup_{0 \leq n < \infty} \|\varphi(n, \omega, x_0) - x_*(\sigma^n\omega)\| < \bar{\varepsilon}$$

whenever $\|x_0 - x_\| < \bar{\delta}$, and if there exists $\gamma \in (0, 1)$ such that for almost all ω and any x_0 with $\|x_0 - x_*\| < \bar{\delta}$,*

$$\|\varphi(n, \omega, x_0) - x_*(\sigma^n\omega)\| < \gamma^n, \quad \forall n > N(x_0).$$

The stability of random fixed points can be characterized by Lyapunov exponents. The definition of Lyapunov exponents in this note follows Arnold (1998). For more detailed information, refer to Arnold (1998, pp.134, 172-175).

Definition B.3 (Lyapunov Exponents) *Let Φ be a linear cocycle over metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \{\sigma^n\})$. Assume that $\lim_{n \rightarrow \infty} (\Phi(n, \omega)^* \Phi(n, \omega))^{\frac{1}{2n}}$ exists. Let $\Psi(\omega) := \lim_{n \rightarrow \infty} (\Phi(n, \omega)^* \Phi(n, \omega))^{\frac{1}{2n}}$. If $\Lambda_i(\omega)$ are the eigenvalues of matrix $\Psi(\omega)$, then the Lyapunov exponents $\lambda_i(\omega)$ are defined by*

$$\lambda_i(\omega) = \log \Lambda_i(\omega).$$

If $(\Omega, \mathcal{F}, \mathbb{P}, \{\sigma^n\})$ is ergodic, then $\Lambda_i(\cdot), \lambda_i(\cdot)$ are constants. In particular, for random fixed point x_ , $\Phi(n, \omega) := D\varphi(n, \omega, x_*(\omega))$.*

Furthermore, we follow Duffie et al. (1994) to provide the definition of equilibrium states of the random dynamical systems in the usual sense.

Definition B.4 (Equilibrium) *Equilibrium is a time-homogeneous Markov process with an ergodic stationary measure.*

Hence if the perturbation corresponds to an i.i.d. process, the orbit of the almost surely exponentially stable random fixed point x_* will be an equilibrium.

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