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Existence and stability of stationary solutions of nonlinear difference equations under random perturbations

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Abstract

Existence and stability of stationary solutions of nonlinear random difference equations are studied in this note. Firstly, we give the weak conditions that guarantee the stability of Lyapunov exponents under small random perturbations. Secondly, we find out the conditions under which the ratio of the random norm and the standard Euclidean norm has deterministic bounds. Based on these new results, we provide easy-to-use conditions that guarantee the existence and almost sure stability of a stationary solution. In addition, we also prove that the stationary solution converges with probability one to the fixed point of the corresponding deterministic system as the noise intensity tends to zero.

Keywords: difference equation; random perturbation; stationary process; Lyapunov exponents; random norm

AMS Subject Classification: 39A11; 91B02; 37N35; 37N40

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1 Introduction

Random difference equations have been studied in many contexts, particularly in modeling physical and economic systems subject to exogenous random perturbations. For the random models, one of the basic problems is the long-term behavior of solutions, in particular the existence and stability of stationary solutions of the systems. Theoretical study in this direction is still far from maturity. Many complex practical models described by nonlinear random difference equations have to be analyzed by numerical simulations (see [4, 6, 9]).

Generally, random difference equations are treated by studying invariant measures or stationary measures. Araújo [1], Futia [8] and Zmarrou & Homburg [15] provided different conditions to guarantee the existence of invariant measures from different perspectives. Kifer [11] further studied the connections of properties of invariant measures in the random case with those in the deterministic case.

Furthermore, the existence of stationary solutions for random difference equations is also studied, which implies the existence of invariant measures. Arnold [2] gave the stationary processes of hyperbolic affine random dynamical systems. Tong [13] studied the existence of stationary solutions for a special class of random difference equations where the Jacobian matrices in the linear parts are deterministic. For more general nonlinear random difference equations, Arnold & Boxler [3] provided a theorem that additive noise turns a hyperbolic fixed point into a stationary solution. However, the conditions of the theorem are related to Lyapunov exponents and random norms, which make it difficult to apply directly theorem to practical models. Therefore, we try to develop a new mathematical tool to build up a bridge between the theoretical study and practical applications.

In this paper we provide easy-to-use conditions that guarantee the existence, stability and convergence of stationary solutions. Firstly, we give weaker conditions to generate the stability of Lyapunov exponents under small random perturbations by using some ideas of [14]. In fact, Young [14] studied a more general case. However, some practical models (see [16]) do not satisfy the conditions given by [14]. In order to deal with the special cases related to the practical models, we provide some weaker conditions than those given by [14]. Consistent with [14], in this paper we also concentrate on two-dimensional random difference equations. Secondly, to our knowledge, for the first time we give the conditions to guarantee that the ratio of the random norm and the standard Euclidean

norm has deterministic bounds, which plays a crucial role in obtaining the theorem and propositions of this paper. By using the relationship between the random norm and the standard Euclidean norm and some ideas of [7], we find out a deterministic neighborhood in which the Lipschitz constant in the case of random norm is deterministic and sufficiently small. Based on these new results and the idea of [3], we provide easy-to-use conditions that guarantee the existence of stationary solutions of nonlinear random difference equations. In addition, we further obtain that the stationary solutions are locally asymptotically stable and converge to the corresponding fixed points of the deterministic system as the noise intensity tends to zero both in the almost sure sense, which implies that the stationary measures also converge weakly to the Dirac measure supported on the corresponding deterministic fixed point as the intensity of noise tends to zero. These properties are significant for studies of practical models.

In order to obtain the above results, here we make two basic assumptions. On the one hand we assume that the perturbations are bounded. This assumption is motivated by some practical backgrounds, for example, dividends in financial markets vary randomly and in a bounded range. See also [15] for reasons of considering bounded noise. On the other hand we assume that the random map is C^2 not only with respect to state variables but also with respect to random perturbations. In fact, many practical models are all analytic with respect to all variables and parameters as well as random perturbations. Araujo [1] and Zmarrou & Homburg [15] have the similar assumptions.

The rest of this paper is set out as follows. Section 2 introduces the preliminaries and the main theorem. Section 3 provides three lemmas. Section 4 proves the main theorem. Section 5 presents several properties of stationary solutions. Section 6 gives a conclusion.

2 Random difference equation

Now consider nonlinear difference equation

$$Y_n = A(\epsilon \varepsilon_n) Y_{n-1} + H(\epsilon \varepsilon_n, Y_{n-1}) + h(\epsilon \varepsilon_n), \quad n \in \mathbb{Z}, \quad (2.1)$$

where $Y_n \in \mathbb{R}^2$, $A(\cdot) \in C^1([-b, b], GL(2, \mathbb{R}))$, $H(\cdot, \cdot) \in C^2([-b, b] \times \mathbb{R}^2, \mathbb{R}^2)$, $h(\cdot) \in C^2([-b, b], \mathbb{R}^2)$. Here b is a positive number. $\epsilon \varepsilon_n$ denotes the random perturbation, where $\epsilon \in [0, 1]$ is a constant revealing the corresponding perturbation intensity. As $\epsilon = 0$, the difference equation (2.1) is deterministic. Suppose that $\{\varepsilon_n\}$ is independent and identically distributed (i.i.d.) with noise ε_n independent of $\{Y_m, m < n\}$. Further, let ν denote the

probability measure of ε_n on $[-b, b]$. Let $\Omega := [-b, b]^{\mathbb{Z}}$ be the bi-infinite product of the interval $[-b, b]$ with itself. For each $\omega \in \Omega$, let its n 'th coordinate $(\omega)_n$ equal some value of ε_n . That is, $\omega = (\cdots \varepsilon_{-1}, \varepsilon_0, \varepsilon_1 \cdots)$. Let $\mathcal{F}$ be the Borel σ -algebra of subsets of Ω . Then $\nu^{\mathbb{Z}}$ is a probability measure on $\mathcal{F}$. On $(\Omega, \mathcal{F})$ define the shift operator $\sigma : \Omega \rightarrow \Omega$ by $(\sigma\omega)_n = (\omega)_{n+1}$. Thus based on the assumption that $\{\varepsilon_n\}$ is i.i.d., $(\Omega, \mathcal{F}, \nu^{\mathbb{Z}}, (\sigma^n)_{n \in \mathbb{N}})$ is a measure-preserving dynamical system ([2, 5]).

For convenience of expression, set

$$\begin{aligned} Y_n &= A(\varepsilon\varepsilon_n)Y_{n-1} + H(\varepsilon\varepsilon_n, Y_{n-1}) + h(\varepsilon\varepsilon_n) \\ &:= A_\varepsilon(\varepsilon_n)Y_{n-1} + H_\varepsilon(\varepsilon_n, Y_{n-1}) + h_\varepsilon(\varepsilon_n). \end{aligned} \quad (2.2)$$

Furthermore, we assume

$$A_\varepsilon(\varepsilon_n) = A(\varepsilon\varepsilon_n) = \begin{pmatrix} a^1(\varepsilon\varepsilon_n) & a^2(\varepsilon\varepsilon_n) \\ a^3(\varepsilon\varepsilon_n) & a^4(\varepsilon\varepsilon_n) \end{pmatrix} := \begin{pmatrix} a_\varepsilon^1(\varepsilon_n) & a_\varepsilon^2(\varepsilon_n) \\ a_\varepsilon^3(\varepsilon_n) & a_\varepsilon^4(\varepsilon_n) \end{pmatrix}.$$

Define the linear cocycle with one-sided time over the dynamical system $(\Omega, \mathcal{F}, \nu^{\mathbb{Z}}, (\sigma^n)_{n \in \mathbb{N}})$

$$\Phi_\varepsilon(n, \omega) := \begin{cases} A_\varepsilon(\varepsilon_n) \cdots A_\varepsilon(\varepsilon_2) A_\varepsilon(\varepsilon_1), & \text{for } n \geq 1, \\ \text{id}, & \text{for } n = 0, \end{cases} \quad (2.3)$$

where id denotes a 2×2 identity matrix. When $\varepsilon = 0$, $A_0(\varepsilon_n) = A(0)$ is a constant matrix independent of noise. Let $A_0 := \begin{pmatrix} a_0^1 & a_0^2 \\ a_0^3 & a_0^4 \end{pmatrix}$ denote $A_0(\varepsilon_n)$ and $\Phi_0(n)$ denote $\Phi_0(n, \omega) = A_0^n$.

Let $\bar{u} \in \mathbb{P}^1$ denote the direction represented by the unit vector $u \in \mathbb{R}^2$, where $\mathbb{P}^1$ is the projective space on $\mathbb{R}^2$. For sufficiently small ε and $t \in [-b, b]$, define a map $\overline{A_\varepsilon(t)}(\bar{u}) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ by $\overline{A_\varepsilon(t)}(\bar{u}) = \overline{A_\varepsilon(t)u}$. Then the map $\overline{A_\varepsilon(\varepsilon_n)}$ induces a Markov chain on $\mathbb{P}^1$. Further, let E be a Borel subset of $\mathbb{P}^1$. The transition probabilities are defined by

$$P^\varepsilon(\bar{u}, E) = \nu\{t \in [-b, b] : \overline{A_\varepsilon(t)}(\bar{u}) \in E\}.$$

In the following proof, we also need the definition and some properties of the random norm. First of all, we give the definition of the random norm. Here we use a definition similar to that in [3].

Assume that Lyapunov exponents $\{\lambda_i^\varepsilon\}$ of Φ_ε satisfy $\lambda_2^\varepsilon \leq \lambda_1^\varepsilon < 0$. Set $\lambda_s^\varepsilon := \lambda_1^\varepsilon$. Fix a $\kappa(\varepsilon) > 0$ such that $\lambda_s^\varepsilon + \kappa(\varepsilon) < 0$. For any $Y : \Omega \rightarrow \mathbb{R}^2$, let

$$\|Y(\omega)\|_\omega := \sum_{i=0}^{\infty} e^{-(\lambda_s^\varepsilon + \kappa(\varepsilon))i} \|\Phi_\varepsilon(i, \omega)Y(\omega)\| \quad (2.4)$$

denote the random norm of Y w.r.t the cocycle $\Phi_\epsilon(n, \omega)$, where $\|\cdot\|$ is the standard Euclidean norm in $\mathbb{R}^2$. The crucial advantage of this random norm is that for all $Y : \Omega \rightarrow \mathbb{R}^2$

$$\|\Phi_\epsilon(n, \omega)Y(\omega)\|_{\sigma^n \omega} \leq e^{(\lambda_s^\epsilon + \kappa)n} \|Y(\omega)\|_\omega, \quad \forall n = 0, 1, 2, \dots \quad (2.5)$$

This is because, for $n = 0, 1, 2, \dots$

$$\|\Phi_\epsilon(n, \omega)Y(\omega)\|_{\sigma^n \omega} \leq e^{(\lambda_s^\epsilon + \kappa)n} \sum_{i=-n}^{\infty} e^{-(\lambda_s^\epsilon + \kappa)(i+n)} \|\Phi_\epsilon(i+n, \omega)Y(\omega)\| = e^{(\lambda_s^\epsilon + \kappa)n} \|Y(\omega)\|_\omega.$$

Further, define the random norm for the linear operator $\Phi_\epsilon(n, \omega)$ by

$$\|\Phi_\epsilon(n, \omega)\|_{\omega, \sigma^n \omega} = \sup_{\|Y\|_\omega \neq 0} \frac{\|\Phi_\epsilon(n, \omega)Y(\omega)\|_{\sigma^n \omega}}{\|Y(\omega)\|_\omega}.$$

Hence based on (2.5),

$$\|\Phi_\epsilon(n, \omega)\|_{\omega, \sigma^n \omega} \leq e^{(\lambda_s^\epsilon + \kappa)n}.$$

Note that the above random norms for vectors and matrices as well as κ all depend on the noise intensity ϵ .

Set $\|Y\| := \text{ess sup } \|Y(\omega)\|_\omega$.

The main result of this paper is to provide a theorem that guarantees the existence of stationary solutions to the nonlinear random difference equation (2.1), i.e. the following Theorem 2.1. Its proof will be given in Section 4.

Theorem 2.1 *For random difference equation (2.1), assume*

- (i) *The two eigenvalues of $A(0)$ are both nonzero and less than one in modulus,*
- (ii) *For any given $\bar{u} \in \mathbb{P}^1$, the probability measure $P^\epsilon(\bar{u}, \cdot)$ induced by $\overline{A_\epsilon(\epsilon_1)}(\bar{u})$ is absolutely continuous with respect to Lebesgue measure $m(\cdot)$ on $\mathbb{P}^1$,*
- (iii) *$H(\epsilon\epsilon_n, 0) = 0$, $D_Y H(\epsilon\epsilon_n, 0) = 0$ for any $\epsilon \in [0, 1]$ and any $n \in \mathbb{Z}$,*
- (iv) *$h(0) = 0$.*

Then for sufficiently small ϵ , there exists exactly one stationary solution in the neighborhood of the origin whose initial value $\xi^\epsilon(\omega)$ satisfies

$$\|\xi^\epsilon\| \leq K\epsilon, \quad (2.6)$$

where $K > 0$ is a constant independent of ϵ .

Remark 2.1 Conditions (i), (iii) and (iv) imply that the origin 0 is a stable fixed point of the deterministic difference equation $Y_n = A(0)Y_{n-1} + H(0, Y_{n-1}) + h(0)$. In fact, any random difference equation $Y_n = F(\epsilon_n, Y_{n-1})$ can be transformed into equation (2.1), where $F(\cdot, \cdot) \in C^2([-b, b] \times \mathbb{R}^2, \mathbb{R}^2)$, if its corresponding deterministic difference equation $Y_n = F(0, Y_{n-1})$ has a fixed point.

Remark 2.2 If the measure ν is absolutely continuous with respect to Lebesgue measure on $[-b, b]$ and $\arctan \frac{a_\epsilon^3(t) \cos \theta + a_\epsilon^4(t) \sin \theta}{a_\epsilon^1(t) \cos \theta + a_\epsilon^2(t) \sin \theta}$ is monotone or piecewise monotone with respect to t for any given $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then the condition (ii) in Theorem 2.1 holds.

Remark 2.3 Since $\epsilon = 0$ is a degenerate case where 0 is a trivial solution, in the following we consider the case that $\epsilon > 0$ only.

3 Several lemmas

In order to prove Theorem 2.1, at first we provide several lemmas.

Lemma 3.1 *Let Φ_ϵ be a linear cocycle with one-sided time over the dynamical system $(\Omega, \mathcal{F}, \nu^{\mathbb{Z}}, (\sigma^n)_{n \in \mathbb{N}})$ as defined in (2.3), where $\{\epsilon_n\}$ is i.i.d.. Assume that the generator $A(\cdot) : [-b, b] \rightarrow Gl(2, \mathbb{R})$ satisfies that $A(\cdot)$ is C^1 and $A(0)$ is invertible. Further assume that the transition probability $P^\epsilon(\bar{u}, \cdot)$ is absolutely continuous with respect to Lebesgue measure $m(\cdot)$ on $\mathbb{P}^1$ for any given $\bar{u} \in \mathbb{P}^1$. Then the Lyapunov exponent λ_i^ϵ of Φ_ϵ satisfies that*

$$\lambda_i^\epsilon \rightarrow \lambda_i^0 \quad \text{as } \epsilon \rightarrow 0, \quad i = 1, 2,$$

where λ_i^0 is the corresponding Lyapunov exponent of Φ_0 .

Proof. According to Breiman [5], the i.i.d. stochastic process $\{\epsilon_n\}$ is stationary and ergodic. That is, every invariant set with respect to σ has probability zero or one. Hence the measure-preserving dynamical system $(\Omega, \mathcal{F}, \nu^{\mathbb{Z}}, (\sigma^n)_{n \in \mathbb{N}})$ is ergodic. Thus based on the multiplicative ergodic theorem, the Lyapunov exponents $\{\lambda_i^\epsilon\}$ of Φ_ϵ are deterministic, see for example [2]. In particular, for $\epsilon = 0$, the deterministic Lyapunov exponents $\{\lambda_i^0\}$ of Φ_0 are equal to the values of $\log |\bar{\lambda}_i|$, where $\bar{\lambda}_i$ is an eigenvalue of A_0 , see also [2]. For simplification of expression, we assume that $\lambda_1^0 \geq \lambda_2^0$.

In the case that $\lambda_1^0 > \lambda_2^0$, there are two unit eigenvectors $\{u_i\}$ corresponding to $\{\lambda_i^0\}$ respectively for the matrix $A_0 = A(0)$. Let θ_i denote the angle corresponding to the

vector u_i . Since $\mathbb{P}^1 \cong S^1/\{\theta, \theta + \pi\}$, then we can identify $\mathbb{P}^1$ with $S^1/\{\theta, \theta + \pi\}$ and use θ instead of $\bar{u}$. Further, let $\mathcal{U}(S^1/\{\theta, \theta + \pi\})$ denote the space of Borel probability measures on $S^1/\{\theta, \theta + \pi\}$. Define the operator $\mathcal{L}_\epsilon$ on $\mathcal{U}(S^1/\{\theta, \theta + \pi\})$ by

$$\mathcal{L}_\epsilon(\cdot)(E) = \int_{\mathbb{P}^1} P^\epsilon(\theta, E) d(\cdot).$$

Then a probability measure μ on $S^1/\{\theta, \theta + \pi\}$ is an invariant measure if and only if for any Borel subset E in $S^1/\{\theta, \theta + \pi\}$

$$\mathcal{L}_\epsilon(\mu)(E) = \mu(E).$$

Invariant measures on compact space $S^1/\{\theta, \theta + \pi\}$ always exist and are absolutely continuous with respect to Lebesgue measure, because we assume that $P^\epsilon(\theta, \cdot)$ is absolutely continuous with respect to Lebesgue measure for any θ .

Following the similar argument as in [14], the maximal Lypunov exponent λ_1^ϵ can be expressed by

$$\lambda_1^\epsilon = \int_{\mathbb{P}^1 \times [-b, b]} \log \|A(t')u\| d(\mu_\epsilon \times \nu_\epsilon), \quad (3.1)$$

where u is understood to be one of the unit vectors in the equivalent class $\bar{u}$, μ_ϵ is any invariant measure on $\mathbb{P}^1$ and ν_ϵ is the measure of $\epsilon \varepsilon_n$ induced by the linear map $t' = \epsilon t$ and the measure ν . Note that $\{\epsilon \varepsilon_n\}$ is also i.i.d. since $\{\varepsilon_n\}$ is i.i.d.. Hence the measure ν_ϵ is invariant. Thus in order to prove the lemma it is sufficient to prove that the invariant measure μ_ϵ converges weakly to δ_{u_1} as $\epsilon \rightarrow 0$.

If we define a sequence of probability measures from any probability measure μ by

$$\frac{1}{n} \sum_{k=0}^{n-1} (\mathcal{L}_\epsilon^k \mu), \forall \mu \in \mathcal{U}(S^1/\{\theta, \theta + \pi\}), n = 1, 2, 3, \dots,$$

then any (weak) limit of this sequence is an invariant measure, reversely any invariant measure can be constructed in this way. See for example, [2].

Thus to show the lemma, it suffices to show that there is a small neighborhood $\mathcal{N}$ of θ_2 in $S^1/\{\theta, \theta + \pi\}$ such that $\mu_\epsilon(\mathcal{N}) \rightarrow 0$ as $\epsilon \rightarrow 0$, i.e.,

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (\mathcal{L}_\epsilon^k \mu)(\mathcal{N}) = 0, \quad (3.2)$$

for some $\mu \in \mathcal{U}(S^1/\{\theta, \theta + \pi\})$.

Since A_0 is an invertible matrix, then $(a_0^1 \cos \theta + a_0^2 \sin \theta)^2 + (a_0^3 \cos \theta + a_0^4 \sin \theta)^2 \neq 0$ for any $\theta \in S^1$. Thus based on

$$A_0 \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} = \begin{pmatrix} a_0^1 \cos \theta + a_0^2 \sin \theta \\ a_0^3 \cos \theta + a_0^4 \sin \theta \end{pmatrix},$$

$$\overline{A_0}(\theta) = \arctan \frac{a_0^3 \cos \theta + a_0^4 \sin \theta}{a_0^1 \cos \theta + a_0^2 \sin \theta} \quad \text{or} \quad \frac{\pi}{2}, \quad \forall \theta \in S^1 / \{\theta, \theta + \pi\}.$$

Note that in the space $S^1 / \{\theta, \theta + \pi\}$, θ and $\pi + \theta$ are identical.

Hence

$$\overline{A_0}'(\theta) = \frac{a_0^1 a_0^4 - a_0^2 a_0^3}{(a_0^1 \cos \theta + a_0^2 \sin \theta)^2 + (a_0^3 \cos \theta + a_0^4 \sin \theta)^2}.$$

If $\theta = \theta_2$, then

$$a_0^1 \cos \theta_2 + a_0^2 \sin \theta_2 = \pm e^{\lambda_2^0} \cos \theta_2, \quad a_0^3 \cos \theta_2 + a_0^4 \sin \theta_2 = \pm e^{\lambda_2^0} \sin \theta_2.$$

Hence

$$\overline{A_0}'(\theta_2) = \frac{\pm e^{\lambda_1^0} e^{\lambda_2^0}}{e^{2\lambda_2^0} \cos^2 \theta_2 + e^{2\lambda_2^0} \sin^2 \theta_2} = \pm e^{\lambda_1^0 - \lambda_2^0}.$$

Thus we can choose a positive constant $\delta_1^* (< \frac{1}{4}|\theta_2 - \theta_1|)$ such that

$$|\overline{A_0}'(\theta)| \geq e^{\lambda_1^0 - \lambda_2^0 - \epsilon_0} > 1, \forall \theta \in [\theta_2 - \delta_1^*, \theta_2 + \delta_1^*], \quad (3.3)$$

where ϵ_0 is a positive number satisfying $\epsilon_0 < \frac{\lambda_1^0 - \lambda_2^0}{2}$. Take $\mathcal{N} = (\theta_2 - \delta_1^*, \theta_2 + \delta_1^*)$. Let $\rho_\epsilon(\theta)$ denote the density of $P^\epsilon(\theta, \cdot)$ for given θ . In the following we first show that $\text{supp } \rho_\epsilon(\theta_2 \pm \delta_1^*)$ is disjoint from the δ_1^* -neighborhood of θ_2 for sufficiently small ϵ . Namely the case 2 in the proof of Theorem 1 in [14] never occurs under our situation.

If the two eigenvalues of A_0 are both positive, then $\overline{A_0}'(\theta_2) = e^{\lambda_1^0 - \lambda_2^0} > 1$. Thus based on $\overline{A_0}(\theta_2) = \theta_2$, for the above sufficiently small positive constant δ_1^* ,

$$\overline{A_0}(\theta_2 - \delta_1^*) - \theta_2 < -\delta_1^*, \quad \overline{A_0}(\theta_2 + \delta_1^*) - \theta_2 > \delta_1^*.$$

That is,

$$\theta_-^* := \overline{A_0}(\theta_2 - \delta_1^*) < \theta_2 - \delta_1^* := \theta_-, \quad \theta_+^* := \overline{A_0}(\theta_2 + \delta_1^*) > \theta_2 + \delta_1^* := \theta_+.$$

Further, since $A(t')$ is continuous with respect to t' and A_0 is invertible, then $\exists \epsilon_1 \in (0, 1]$, such that as $\epsilon < \epsilon_1$,

$$a_\epsilon^1(t) a_\epsilon^4(t) - a_\epsilon^2(t) a_\epsilon^3(t) \neq 0, \quad \forall t \in [-b, b].$$

Thus $\overline{A(\epsilon t)}(\theta)$ is well defined. Moreover, for $\theta = \theta_-, \theta_+, \theta_1$, there exists $\epsilon_2 \in (0, \epsilon_1]$ such that for any $\epsilon < \epsilon_2$ and $t \in [-b, b]$,

$$\begin{aligned} |\overline{A(\epsilon t)}(\theta_-) - \overline{A_0}(\theta_-)| &< \frac{1}{4}(\theta_- - \theta_-^*), \\ |\overline{A(\epsilon t)}(\theta_+) - \overline{A_0}(\theta_+)| &< \frac{1}{4}(\theta_+^* - \theta_+), \\ |\overline{A(\epsilon t)}(\theta_1) - \overline{A_0}(\theta_1)| &< \delta_1^*. \end{aligned}$$

Note that $\overline{A_0}(\theta_1) = \theta_1$. Therefore, $\text{supp } \rho_\epsilon(\theta_2 \pm \delta_1^*)$ and $\text{supp } \rho_\epsilon(\theta_1)$ are disjoint from the δ_1^* -neighborhood of θ_2 as $\epsilon < \epsilon_2$, respectively. If the two eigenvalues of A_0 are both negative or have different signs, the same result still holds following the similar argument.

Let μ be any Borel probability measure on $S^1/\{\theta, \theta + \pi\}$ which is absolutely continuous with respect to Lebesgue measure. Let η_0 denote the density of μ on $S^1/\{\theta, \theta + \pi\}$, and let η_i denote the density of $\mathcal{L}_\epsilon^i \mu$ on $S^1/\{\theta, \theta + \pi\}$, $i = 1, 2, \dots$. Note that the densities η_0 and η_1 have the relation

$$\eta_1(\theta) = \int_{[-b, b]} \eta_0 \left(\left(\overline{A_\epsilon(t)} \right)^{-1}(\theta) \right) \cdot \left| \left(\left(\overline{A_\epsilon(t)} \right)^{-1} \right)'(\theta) \right| d\nu(t), \quad (3.4)$$

where $\left(\overline{A_\epsilon(t)} \right)^{-1}(\theta)$ is the inverse mapping of $\overline{A_\epsilon(t)}(\theta)$ with respect to θ for given t . In the case that $\epsilon < \epsilon_2$, since $\text{supp } \rho_\epsilon(\theta_2 \pm \delta_1^*)$ and $\text{supp } \rho_\epsilon(\theta_1)$ are disjoint from the δ_1^* -neighborhood of θ_2 and $\overline{A_\epsilon(t)}(\theta)$ is strictly monotone with respect to θ for any given t , then for any $\theta \notin [\theta_2 - \delta_1^*, \theta_2 + \delta_1^*]$, $\overline{A_\epsilon(t)}(\theta) \notin [\theta_2 - \delta_1^*, \theta_2 + \delta_1^*]$. That is, for $\theta \in [\theta_2 - \delta_1^*, \theta_2 + \delta_1^*]$,

$$\left(\overline{A_\epsilon(t)} \right)^{-1}(\theta) \in [\theta_2 - \delta_1^*, \theta_2 + \delta_1^*].$$

Furthermore, for any $\epsilon < \epsilon_2$ and $t \in [-b, b]$, based on $\frac{\partial \overline{A_\epsilon(t)}(\theta)}{\partial \theta} \neq 0$, $\left(\overline{A(t')} \right)^{-1}(\theta)$ is continuously differentiable with respect to $t' := \epsilon t$ and θ . Thus $\left(\left(\overline{A(t')} \right)^{-1} \right)'(\theta)$ is continuous with respect to t' and θ . Hence there exists $\epsilon_3 (0 < \epsilon_3 < \epsilon_2)$, such that as $0 < \epsilon < \epsilon_3$, for any $t \in [-b, b]$,

$$\max_{\theta \in \mathcal{N}} | \left(\left(\overline{A(\epsilon t)} \right)^{-1} \right)'(\theta) | \leq e^{\epsilon_0} \max_{\theta \in \mathcal{N}} | \left(\left(\overline{A_0} \right)^{-1} \right)'(\theta) | \leq e^{-\lambda_1^0 + \lambda_2^0 + 2\epsilon_0}, \quad (3.5)$$

where the second inequality is based on (3.3).

Let $\tilde{\eta}_i := \max\{\eta_i(\theta) : |\theta - \theta_2| \leq \delta_1^*\}$, $i = 0, 1, \dots$. Without loss of generality, we assume $\tilde{\eta}_0 < \infty$ and then $\tilde{\eta}_i$ is well defined for $i \geq 1$. Hence based on (3.4) and (3.5),

$$\tilde{\eta}_1 \leq \tilde{\eta}_0 \cdot e^{-\lambda_1^0 + \lambda_2^0 + 2\epsilon_0},$$

where $0 < e^{-\lambda_1^0 + \lambda_2^0 + 2\epsilon_0} < e^{-\lambda_1^0 + \lambda_2^0 + \lambda_1^0 - \lambda_2^0} = 1$.

Recursively,

$$\tilde{\eta}_k \leq \tilde{\eta}_0 \cdot e^{k(-\lambda_1^0 + \lambda_2^0 + 2\epsilon_0)}, \quad k = 1, 2, \dots.$$

Therefore,

$$\frac{1}{n} \sum_{k=0}^{n-1} (\mathcal{L}_\epsilon^k \mu)(\mathcal{N}) \leq \frac{2\delta_1^*}{n} \sum_{k=0}^{n-1} \tilde{\eta}_k \leq \frac{2\delta_1^*}{n} \tilde{\eta}_0 \frac{1}{1 - e^{-\lambda_1^0 + \lambda_2^0 + 2\epsilon_0}} \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

where the last convergence is uniform with respect to $\epsilon \in (0, \epsilon_3)$. That is to say that we have the equation (3.2). Hence this lemma holds in the case that $\lambda_1^0 > \lambda_2^0$.

The proof for the case that $\lambda_1^0 = \lambda_2^0$ is easy, and hence we omit it here. $\square$

Lemma 3.2 *If all the assumptions of Lemma 3.1 are satisfied and $\lambda_2^0 \leq \lambda_1^0 < 0$, then for sufficiently small ϵ , the random norm and the standard norm have the following relation*

$$\|Y(\omega)\| \leq \|Y(\omega)\|_\omega \leq B\|Y(\omega)\|, \quad \forall \omega \in \Omega, \quad (3.6)$$

where B is a positive constant independent of ϵ .

Proof. At first, according to the definition of the random norm (2.4),

$$\|Y(\omega)\| \leq \|Y(\omega)\|_\omega. \quad (3.7)$$

On the other hand, since the Lypunov exponents $\lambda_2^0 \leq \lambda_1^0 < 0$, then the spectral radius of the matrix A_0 denoted by $\underline{\rho}(A_0)$ satisfies $\underline{\rho}(A_0) < 1$. Hence there is a norm $\|\cdot\|_a$ in $\mathbb{R}^2$ and a corresponding operator norm $\|\cdot\|_a$ for matrices such that $\|A_0\|_a < 1$, see for example, [12]. Let $0 < \alpha < \beta$ be such that

$$\alpha\|Y\| \leq \|Y\|_a \leq \beta\|Y\|, \quad \forall Y \in \mathbb{R}^2.$$

Moreover, based on the assumptions, $\exists t'_\epsilon \in [-\epsilon b, \epsilon b]$ such that for any $t \in [-b, b]$

$$\|A_\epsilon(t)\|_a = \|A(\epsilon t)\|_a = \|A(t')\|_a \leq \max_{t' \in [-\epsilon b, \epsilon b]} \|A(t')\|_a = \|A(t'_\epsilon)\|_a = e^{M_\epsilon},$$

where $M_\epsilon = \log \|A(t'_\epsilon)\|_a$. In particular, for $\epsilon = 0$, $M_0 = \log \|A_0\|_a < 0$. Thus, for any $i = 1, 2, \dots$,

$$\|\Phi_\epsilon(i, \omega)\|_a \leq \|A_\epsilon(\varepsilon_i)\|_a \cdots \|A_\epsilon(\varepsilon_2)\|_a \|A_\epsilon(\varepsilon_1)\|_a \leq e^{iM_\epsilon}, \quad \forall \omega \in \Omega.$$

Hence

$$\begin{aligned}
\|Y(\omega)\|_\omega &\leq \sum_{i=0}^{\infty} e^{-i(\lambda_s^\epsilon + \kappa(\epsilon))} \|\Phi_\epsilon(i, \omega)Y(\omega)\| \\
&\leq \frac{1}{\alpha} \sum_{i=0}^{\infty} e^{-i(\lambda_s^\epsilon + \kappa(\epsilon))} \|\Phi_\epsilon(i, \omega)Y(\omega)\|_a \\
&\leq \frac{\beta}{\alpha} \sum_{i=0}^{\infty} e^{i(M_\epsilon - \lambda_s^\epsilon - \kappa(\epsilon))} \|Y(\omega)\|.
\end{aligned}$$

Furthermore, since $M_\epsilon \rightarrow M_0$ and $\lambda_s^\epsilon \rightarrow \lambda_s^0$ as $\epsilon \rightarrow 0$, then $\exists \epsilon_4 \in (0, \epsilon_3]$ such that for any $\epsilon < \epsilon_4$, we have

$$|\lambda_s^\epsilon - \lambda_s^0| < -\frac{1}{4}M_0, \quad |(M_\epsilon - \lambda_s^\epsilon) - (M_0 - \lambda_s^0)| < -\frac{1}{4}M_0.$$

Take $\kappa \equiv \kappa(\epsilon) = \frac{1}{2}M_0 - \lambda_s^0$ ($> M_0 - \lambda_s^0 \geq 0$). Then

$$\begin{aligned}
\lambda_s^\epsilon + \kappa(\epsilon) &< \lambda_s^0 - \frac{1}{4}M_0 + \frac{1}{2}M_0 - \lambda_s^0 = \frac{1}{4}M_0 < 0, \\
M_\epsilon - \lambda_s^\epsilon - \kappa(\epsilon) &< M_0 - \lambda_s^0 - \frac{1}{4}M_0 - \frac{1}{2}M_0 + \lambda_s^0 = \frac{1}{4}M_0 < 0.
\end{aligned}$$

Thus as $\epsilon < \epsilon_4$,

$$\|Y(\omega)\|_\omega \leq \frac{\beta}{\alpha} \sum_{i=0}^{\infty} e^{i(M_\epsilon - \lambda_s^\epsilon - \kappa(\epsilon))} \|Y(\omega)\| \leq \frac{\beta}{\alpha} \sum_{i=0}^{\infty} e^{\frac{iM_0}{4}} \|Y(\omega)\| = B\|Y(\omega)\|, \quad (3.8)$$

where $B = \frac{\beta}{\alpha} \sum_{i=0}^{\infty} e^{\frac{iM_0}{4}} = \frac{\beta}{\alpha} \cdot \frac{1}{1 - e^{\frac{M_0}{4}}}$.

Therefore, for sufficiently small ϵ , combination of (3.7) with (3.8) gives

$$\|Y(\omega)\| \leq \|Y(\omega)\|_\omega \leq B\|Y(\omega)\|.$$

□

Note that in the following $\kappa = \frac{1}{2}M_0 - \lambda_s^0$. Further, we set $\Lambda = 1 - e^{\frac{M_0}{4}} \in (0, 1)$.

Lemma 3.3 *Under the conditions of Lemma 3.2, let $H : [-b, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a C^2 function with $H_\epsilon(\varepsilon_1, 0) = H(\varepsilon_1, 0) = 0$ and $D_Y H_\epsilon(\varepsilon_1, 0) = D_Y H(\varepsilon_1, 0) = 0$ for any $\epsilon \in [0, 1]$. Then for sufficiently small ϵ , there exists a constant $r \in (0, 1]$ independent of ϵ and ω such that for any $\|Y_i(\omega)\| \leq r \leq 1$, $i = 1, 2$*

$$\|H_\epsilon(\varepsilon_1, Y_1(\omega)) - H_\epsilon(\varepsilon_1, Y_2(\omega))\|_{\sigma\omega} \leq L\|Y_1(\omega) - Y_2(\omega)\|_\omega, \quad \forall \omega \in \Omega,$$

where $L = \frac{\Lambda}{2} > 0$.

Proof. Since $H(\cdot, \cdot)$ is a C^2 function, for any $\|Y_1(\omega)\| \leq 1, \|Y_2(\omega)\| \leq 1$, we have

$$\|D_Y H(\epsilon \varepsilon_1, Y_1(\omega)) - D_Y H(\epsilon \varepsilon_1, Y_2(\omega))\| \leq J \|Y_1(\omega) - Y_2(\omega)\| < \infty, \forall \epsilon \in [0, 1], \varepsilon_1 \in [-b, b],$$

where J is a positive constant independent of ϵ and ω as well as $Y_i, i = 1, 2$. Take $r \in (0, 1]$. Since $D_Y H(\epsilon \varepsilon_1, 0) = 0$, then for any $\|Y(\omega)\| \leq r$,

$$\|D_Y H(\epsilon \varepsilon_1, Y(\omega))\| = \|D_Y H(\epsilon \varepsilon_1, Y(\omega)) - D_Y H(\epsilon \varepsilon_1, 0)\| \leq J \|Y(\omega)\| \leq Jr.$$

Hence for any $\|Y_i(\omega)\| \leq r \leq 1, i = 1, 2$,

$$\begin{aligned} \|H_\epsilon(\varepsilon_1, Y_1(\omega)) - H_\epsilon(\varepsilon_1, Y_2(\omega))\| &\leq \sup_{\|Y(\omega)\| \leq r} \|D_Y H(\epsilon \varepsilon_1, Y(\omega))\| \|Y_1(\omega) - Y_2(\omega)\| \\ &\leq Jr \|Y_1(\omega) - Y_2(\omega)\|. \end{aligned} \quad (3.9)$$

From Lemma 3.2, as $\epsilon < \epsilon_4$, then

$$\begin{aligned} \|H_\epsilon(\varepsilon_1, Y_1(\omega)) - H_\epsilon(\varepsilon_1, Y_2(\omega))\|_{\sigma\omega} &\leq B \|H_\epsilon(\varepsilon_1, Y_1(\omega)) - H_\epsilon(\varepsilon_1, Y_2(\omega))\| \\ &\leq BJr \|Y_1(\omega) - Y_2(\omega)\| \\ &\leq BJr \|Y_1(\omega) - Y_2(\omega)\|_\omega. \end{aligned}$$

Set

$$r = \min\left\{\frac{L}{BJ}, 1\right\} = \min\left\{\frac{\Lambda}{2BJ}, 1\right\}.$$

Therefore, for any $\|Y_i(\omega)\| \leq r, i = 1, 2$, as $\epsilon < \epsilon_4$, we have

$$\|H_\epsilon(\varepsilon_1, Y_1(\omega)) - H_\epsilon(\varepsilon_1, Y_2(\omega))\|_{\sigma\omega} \leq BJr \|Y_1(\omega) - Y_2(\omega)\|_\omega \leq L \|Y_1(\omega) - Y_2(\omega)\|_\omega.$$

□

4 Proof of Theorem 2.1

Proof. We first consider a special case and then provide the proof in a general case.

Step 1: If $H \equiv 0$, then (2.1) is an affine random difference equation. In this case, for sufficiently small ϵ , the variation of constants formula gives the solution with its initial value

$$\xi^\epsilon(\omega) := \sum_{i=-\infty}^0 \Phi_\epsilon(-i, \sigma^i \omega) h_\epsilon(\varepsilon_i). \quad (4.1)$$

The expression makes sense, for

$$\begin{aligned}
|||\xi^\epsilon||| &= \text{ess sup } \|\xi^\epsilon(\omega)\|_\omega \\
&\leq \text{ess sup } \sum_{i=-\infty}^0 \|\Phi_\epsilon(-i, \sigma^i \omega)\|_{\sigma^i \omega, \omega} \|h_\epsilon(\varepsilon_i)\|_{\sigma^i \omega} \\
&\leq \text{ess sup } \sum_{i=-\infty}^0 e^{-i(\lambda_s^\epsilon + \kappa)} \|h_\epsilon(\varepsilon_i)\|_{\sigma^i \omega} \\
&\leq \frac{1}{1 - e^{\lambda_s^\epsilon + \kappa}} |||h_\epsilon||| \\
&\leq \frac{1}{\Lambda} |||h_\epsilon|||,
\end{aligned} \tag{4.2}$$

where the last inequality holds because that $0 < 1 - e^{\frac{1}{4}M_0} = \Lambda < 1 - e^{\lambda_s^\epsilon + \kappa}$ as $\epsilon < \epsilon_4$. Note that all $\text{ess sup } \|h_\epsilon(\varepsilon_i)\|_{\sigma^i \omega}$, $i \in \mathbb{Z}$ are equal, we denote them by $|||h_\epsilon|||$.

Based on the assumptions that $h(\cdot) \in C^2$ and $h(0) = 0$, $\exists \epsilon_5 \in (0, \epsilon_4]$ such that as $\epsilon < \epsilon_5$,

$$\|h_\epsilon(\varepsilon_0)\| = \|h(\epsilon \varepsilon_0)\| \leq \frac{C_0}{b} |\epsilon \varepsilon_0| \leq C_0 \epsilon,$$

where C_0 is a positive constant independent of ϵ .

Furthermore,

$$\|h_\epsilon(\varepsilon_0)\|_\omega = \|h(\epsilon \varepsilon_0)\|_\omega \leq B \|h(\epsilon \varepsilon_0)\| \leq BC_0 \epsilon. \tag{4.3}$$

Hence

$$|||\xi^\epsilon||| \leq \frac{|||h_\epsilon|||}{\Lambda} \leq \frac{BC_0 \epsilon}{\Lambda} < \infty.$$

On the other hand, according to the cocycle property,

$$A_\epsilon(\varepsilon_1) \xi^\epsilon(\omega) + h_\epsilon(\varepsilon_1) = \sum_{i=-\infty}^1 \Phi_\epsilon(1-i, \sigma^i \omega) h_\epsilon(\varepsilon_i) = \xi^\epsilon(\sigma \omega).$$

Hence the stochastic process with initial value $\xi^\epsilon(\omega)$ is a stationary process. Namely, $\{\xi_n^\epsilon(\omega)\} := \{\xi^\epsilon(\sigma^n \omega)\}$ is a stationary solution to (2.1).

Furthermore, we will show that the stationary solution is unique in the neighborhood of the origin. If $\zeta_1^\epsilon, \zeta_2^\epsilon$ are both stationary solutions of (2.1), then $\zeta^\epsilon := \zeta_1^\epsilon - \zeta_2^\epsilon$ is also one stationary solution and satisfies $\zeta_{n+1}^\epsilon = A_\epsilon(\varepsilon_{n+1}) \zeta_n^\epsilon$. However, based on Lemma 3.1 and Lemma 3.2, the linearized system $\zeta_{n+1}^\epsilon = A_\epsilon(\varepsilon_{n+1}) \zeta_n^\epsilon$ is hyperbolic. Hence $\zeta^\epsilon \equiv 0$. That is, the stationary solution is unique.

Step 2: Consider now the full equation (2.1).

Just as [3], we now define a similar space having similar properties. Let

$$\mathcal{L}^\infty := \{Y : \Omega \rightarrow \mathbb{R}^2 \text{ measurable with } |||Y||| < \infty\}.$$

Then $\mathcal{L}^\infty$ is a Banach space. Thus for $r > 0$ in Lemma 3.3,

$$D_r := \{Y \in \mathcal{L}^\infty : |||Y||| \leq r\}.$$

is also complete. In the following we will prove the existence of a stationary solution using the Banach fixed point theorem. Hence we first define an operator and then test the conditions of the theorem.

(1) We now define a nonlinear operator $T_\epsilon : D_r \rightarrow \mathcal{L}^\infty$ by

$$T_\epsilon Y(\omega) = \sum_{i=-\infty}^0 \Phi_\epsilon(-i, \sigma^i \omega) \tilde{h}_\epsilon(\sigma^i \omega), \quad (4.4)$$

where

$$\tilde{h}_\epsilon(\omega) = H_\epsilon(\varepsilon_0, Y(\sigma^{-1}\omega)) + h_\epsilon(\varepsilon_0).$$

The definition makes sense since, according to Lemma 3.3 and (4.3),

$$\|\tilde{h}_\epsilon(\omega)\|_\omega \leq \|H_\epsilon(\varepsilon_0, Y(\sigma^{-1}\omega))\|_\omega + \|h_\epsilon(\varepsilon_0)\|_\omega \leq Lr + BC_0\epsilon.$$

That is,

$$|||\tilde{h}_\epsilon||| = \text{ess sup } \|\tilde{h}_\epsilon(\omega)\|_\omega \leq Lr + BC_0\epsilon. \quad (4.5)$$

Furthermore, the comparison of (4.1) with (4.4) shows that $T_\epsilon Y$ is just the initial value of the unique stationary solution of the affine equation $Y_n = A_\epsilon(\varepsilon_n)Y_{n-1} + \tilde{h}_\epsilon(\varepsilon_n)$. Hence from (4.2),

$$|||T_\epsilon Y||| \leq \frac{1}{1 - e^{\lambda_s^\epsilon + \kappa}} |||\tilde{h}_\epsilon||| \leq \frac{1}{\Lambda} |||\tilde{h}_\epsilon|||. \quad (4.6)$$

(2) T_ϵ maps D_r to D_r .

At first, $\exists \epsilon_6 = \frac{r\Lambda}{2BC_0} > 0$, such that as $0 < \epsilon < \epsilon_6$, $Lr + BC_0\epsilon \leq r\Lambda$. Furthermore, based on (4.5) and (4.6), as $0 < \epsilon < \epsilon_7 := \min\{\epsilon_5, \epsilon_6\}$, we have

$$|||T_\epsilon Y||| \leq \frac{Lr + BC_0\epsilon}{\Lambda} \leq \frac{r\Lambda}{\Lambda} = r.$$

(3) T_ϵ is contracting on D_r .

From Lemma 3.3 and the property of the random norm (2.5), for any $Y_1, Y_2 \in D_r$, we have

$$\begin{aligned}
\|T_\epsilon Y_1(\omega) - T_\epsilon Y_2(\omega)\|_\omega &= \left\| \sum_{i=-\infty}^0 \Phi_\epsilon(-i, \sigma^i \omega) (H_\epsilon(\varepsilon_i, Y_1(\sigma^{i-1} \omega)) - H_\epsilon(\varepsilon_i, Y_2(\sigma^{i-1} \omega))) \right\|_\omega \\
&\leq \sum_{i=-\infty}^0 \|\Phi_\epsilon(-i, \sigma^i \omega)\|_{\sigma^i \omega, \omega} \|H_\epsilon(\varepsilon_i, Y_1(\sigma^{i-1} \omega)) - H_\epsilon(\varepsilon_i, Y_2(\sigma^{i-1} \omega))\|_{\sigma^i \omega} \\
&\leq \sum_{i=-\infty}^0 e^{-i(\lambda_s^\epsilon + \kappa)} L \|Y_1(\sigma^{i-1} \omega) - Y_2(\sigma^{i-1} \omega)\|_{\sigma^{i-1} \omega} \\
&\leq \frac{1}{2} \|Y_1 - Y_2\|.
\end{aligned}$$

That is,

$$\|T_\epsilon Y_1 - T_\epsilon Y_2\| = \text{ess sup } \|T_\epsilon Y_1(\omega) - T_\epsilon Y_2(\omega)\|_\omega \leq \frac{1}{2} \|Y_1 - Y_2\|.$$

Therefore, by the Banach fixed point theorem, (2.1) has a stationary solution whose initial value $\xi^\epsilon(\omega)$ satisfies $\|\xi^\epsilon(\omega)\| \leq \|\xi^\epsilon(\omega)\|_\omega \leq \|\xi^\epsilon\| \leq r$ for almost all ω (and is unique with this property).

Furthermore, for the stationary solution with initial value $\xi^\epsilon(\omega)$, based on Lemma 3.3 and (4.6),

$$\begin{aligned}
\|\xi^\epsilon(\omega)\|_\omega &= \|T_\epsilon \xi^\epsilon(\omega)\|_\omega \\
&\leq \|T_\epsilon \xi^\epsilon\| \\
&\leq \frac{1}{\Lambda} \|H_\epsilon(\varepsilon_0, \xi^\epsilon(\sigma^{-1} \omega)) + h_\epsilon(\varepsilon_0)\| \\
&\leq \frac{1}{\Lambda} (\|H_\epsilon(\varepsilon_0, \xi^\epsilon(\sigma^{-1} \omega)) - H_\epsilon(\varepsilon_0, 0)\| + \|h_\epsilon\|) \\
&\leq \frac{1}{\Lambda} (L \|\xi^\epsilon\| + \|h_\epsilon\|).
\end{aligned}$$

That is,

$$\|\xi^\epsilon\| = \text{ess sup } \|\xi^\epsilon(\omega)\|_\omega \leq \frac{1}{\Lambda} (L \|\xi^\epsilon\| + \|h_\epsilon\|).$$

Hence for almost all ω ,

$$\|\xi^\epsilon(\omega)\| \leq \|\xi^\epsilon(\omega)\|_\omega \leq \|\xi^\epsilon\| \leq \frac{\|h_\epsilon\|}{\Lambda - L} = \frac{2\|h_\epsilon\|}{\Lambda}.$$

Moreover based on (4.3), for almost all ω ,

$$\|\xi^\epsilon(\omega)\| \leq \|\xi^\epsilon\| \leq \frac{2\|h_\epsilon\|}{\Lambda} \leq \frac{2BC_0\epsilon}{\Lambda}. \quad (4.7)$$

Take $K = \frac{2BC_0}{\Lambda}$. Then the proof of Theorem 2.1 is complete.

□

5 Properties

Proposition 5.1 *The stationary solution obtained in Theorem 2.1 is measurable.*

Proof. We start with $\xi^{\epsilon,0} \equiv 0$ (which is obviously measurable) and recursively define

$$\xi^{\epsilon,k+1}(\omega) := T_\epsilon \xi^{\epsilon,k}(\omega), \quad k = 0, 1, 2, \dots$$

Based on the assumptions of theorem 2.1, $A(\cdot)$ is C^1 , $H(\cdot, \cdot)$, $h(\cdot)$ are C^2 , and hence they are all measurable. Thus according to the definition of T_ϵ in (4.4), $\xi^{\epsilon,1}(\omega) := T_\epsilon \xi^{\epsilon,0}(\omega)$ is measurable. Hence recursively $\xi^{\epsilon,k}(\omega)$ is measurable. In addition, the proof of (2) of step 2 in section 4 also shows that $\xi^{\epsilon,k}(\omega) \in D_r$.

By the Banach fixed point theorem on D_r ,

$$\lim_{k \rightarrow \infty} \xi^{\epsilon,k} = \xi^\epsilon.$$

That is, for any $\tilde{\epsilon} > 0$, $\exists K(\tilde{\epsilon}) > 0$ such that $\forall k \geq K(\tilde{\epsilon})$,

$$\|\xi^{\epsilon,k} - \xi^\epsilon\| < \tilde{\epsilon}.$$

Furthermore, due to the relation of norms, as $k > K(\tilde{\epsilon})$,

$$\|\xi^{\epsilon,k}(\omega) - \xi^\epsilon(\omega)\| \leq \|\xi^{\epsilon,k}(\omega) - \xi^\epsilon(\omega)\|_\omega \leq \|\xi^{\epsilon,k} - \xi^\epsilon\| < \tilde{\epsilon}.$$

Hence for almost all ω ,

$$\lim_{k \rightarrow \infty} \xi^{\epsilon,k}(\omega) = \xi^\epsilon(\omega).$$

Given that the fixed point $\xi^\epsilon(\omega)$ is a pointwise limit of measurable functions, then it is measurable. $\square$

Note that the proof follows Lemma 7.3.4 in [2].

Proposition 5.2 *As $\epsilon \rightarrow 0$, the stationary solution obtained in Theorem 2.1 converges almost surely to zero, which is an isolated fixed point in the deterministic case.*

Proof. Since according to the assumptions of Theorem 2.1, $h_0 = h(0) = 0$ and $H_0(\varepsilon_n, 0) = H(0, 0) = 0$, then 0 is a fixed point for the deterministic difference equation, namely the equation (2.1) with $\epsilon = 0$. Based on (4.7), for almost all ω ,

$$\|\xi^\epsilon(\omega)\| \leq \|\xi^\epsilon(\omega)\|_\omega \leq \frac{2BC_0\epsilon}{\Lambda} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

That is, the random fixed point converges almost surely to the deterministic fixed point as the noise intensity tends to zero. $\square$

Note that in general study of stochastic stability, only weak convergence can be derived.

Proposition 5.3 *The stationary solution obtained in Theorem 2.1 is locally exponentially stable with probability one.*

Proof. If $\zeta_n(\omega)$ is any other solution of (2.1) on $\mathbb{Z}^+$ with initial value $|||\zeta_0||| \leq r$, then

$$\zeta_{n+1} = A_\epsilon(\varepsilon_{n+1})\zeta_n + H_\epsilon(\varepsilon_{n+1}, \zeta_n) + h_\epsilon(\varepsilon_{n+1}). \quad (5.1)$$

Since $\|\zeta_0(\omega)\|_\omega \leq r$, based on the above equation (5.1) and $1 - e^{\lambda_s^\epsilon + \kappa} = 1 - e^{\lambda_s^\epsilon + \frac{1}{2}M_0 - \lambda_s^0} > \Lambda$, as $\epsilon < \epsilon_7$,

$$\|\zeta_1\|_{\sigma\omega} \leq \|A_\epsilon(\varepsilon_1)\|_{\omega, \sigma^1\omega} \|\zeta_0\|_\omega + \|H_\epsilon(\varepsilon_1, \zeta_0)\|_{\sigma\omega} + \|h_\epsilon(\varepsilon_1)\|_{\sigma\omega} \leq re^{\lambda_s^\epsilon + \kappa} + Lr + BC_0\epsilon \leq r.$$

We recursively obtain $\|\zeta_n\|_{\sigma^n\omega} \leq r$. Hence

$$|||\zeta_n||| \leq r.$$

On the other hand, ξ_n^ϵ is the stationary solution of model (2.1) with initial value $\xi_0^\epsilon(\omega) = \xi^\epsilon(\omega)$. That is,

$$\xi^\epsilon(\sigma^{n+1}\omega) := \xi_{n+1}^\epsilon = A_\epsilon(\varepsilon_{n+1})\xi_n^\epsilon + H_\epsilon(\varepsilon_{n+1}, \xi_n^\epsilon) + h_\epsilon(\varepsilon_{n+1}). \quad (5.2)$$

Subtracting (5.2) from (5.1) yields the equation

$$\zeta_{n+1} - \xi_{n+1}^\epsilon = A_\epsilon(\varepsilon_{n+1})(\zeta_n - \xi_n^\epsilon) + H_\epsilon(\varepsilon_{n+1}, \zeta_n) - H_\epsilon(\varepsilon_{n+1}, \xi_n^\epsilon).$$

Set $\zeta_{n+1}^* := \zeta_{n+1} - \xi_{n+1}^\epsilon$, $n = 0, 1, 2, \dots$. Then ζ_{n+1}^* satisfies the following equation:

$$\zeta_{n+1}^* = A_\epsilon(\varepsilon_{n+1})\zeta_n^* + H_\epsilon(\varepsilon_{n+1}, \zeta_n^* + \xi_n^\epsilon) - H_\epsilon(\varepsilon_{n+1}, \xi_n^\epsilon). \quad (5.3)$$

Since $\|\zeta_n^* + \xi_n^\epsilon\|_{\sigma^n\omega} = \|\zeta_n\|_{\sigma^n\omega} \leq |||\zeta_n||| \leq r$ and $\|\xi_n^\epsilon\|_{\sigma^n\omega} \leq |||\xi^\epsilon||| \leq r$, then $\|H_\epsilon(\varepsilon_{n+1}, \zeta_n^* + \xi_n^\epsilon) - H_\epsilon(\varepsilon_{n+1}, \xi_n^\epsilon)\|_{\sigma^{n+1}\omega} \leq L\|\zeta_n^*\|_{\sigma^n\omega}$, based on Lemma 3.3.

Thus

$$\|\zeta_{n+1}^*\|_{\sigma^{n+1}\omega} \leq (e^{\lambda_s^\epsilon + \kappa} + L)\|\zeta_n^*\|_{\sigma^n\omega} \leq (e^{\lambda_s^\epsilon + \kappa} + L)^{n+1}\|\zeta_0^*\|_\omega = \gamma^{n+1}\|\zeta_0^*\|_\omega,$$

where $\gamma = e^{\lambda_s^\epsilon + \kappa} + L \leq 1 - \Lambda + \frac{\Lambda}{2} = 1 - \frac{\Lambda}{2} \in (0, 1)$.

For any $\tilde{\varepsilon} > 0$, without loss of generality, we assume $\tilde{\varepsilon} < r \leq 1$. Take $\tilde{\delta} = \frac{\tilde{\varepsilon}}{B}$. Since $B > 1$, $\tilde{\delta} < 1$. As $\|\zeta_0 - \xi_0^\epsilon\| < \tilde{\delta} < 1$, for almost all ω and any $n = 0, 1, 2, \dots$

$$\|\zeta_n - \xi_n^\epsilon\| \leq \|\zeta_n - \xi_n^\epsilon\|_{\sigma^n\omega} \leq \gamma^n \|\zeta_0 - \xi_0^\epsilon\|_\omega < B\tilde{\delta} = \tilde{\varepsilon}.$$

Hence the stationary solution with the initial value $\xi^\epsilon(\omega)$ is locally stable with probability one. Furthermore, for almost all ω

$$\|\zeta_n - \xi_n^\epsilon\| \leq \|\zeta_n - \xi_n^\epsilon\|_{\sigma^n\omega} < \gamma^n \|\zeta_0 - \xi_0^\epsilon\|_\omega < \gamma^n \tilde{\varepsilon} < \gamma^n, \text{ for any } n \in \mathbb{N}.$$

Therefore, the stationary solution with the initial value $\xi^\epsilon(\omega)$ is locally exponentially stable with probability one. $\square$

6 Conclusion and future directions

In this paper we provide some new easy-to-use conditions that guarantee the existence and stability of nonlinear random difference equations with a view of application. First, we obtain the stability of Lyapunov exponents for a special case under weaker conditions, namely Lemma 3.1. Second, we derive a new relationship between the random norm and the standard norm, i.e., the ratio of the random norm and the standard norm has deterministic bounds, namely Lemma 3.2. However, in general theory of random dynamical systems, the bound of the ratio of the random norm and the standard norm is random depending on sample paths (see [2]). This result plays a crucial role in obtaining a deterministic neighborhood in which the Lipschitz constant in the case of random norm is deterministic and sufficiently small, namely Lemma 3.3. But in the general theory of random dynamical systems, the neighborhood is random depending on sample paths (see [2, 7]). Eventually, by integrating the lemmas we provide a theorem that guarantees the existence of stationary solutions under easy-to-use conditions, as stated in Theorem 2.1. In addition, we also obtain several properties including measurability, local exponent stability with probability one and the relationship between the stationary solution and the corresponding deterministic fixed point. Under new conditions, the general result of local asymptotical stability in probability given by [2] is improved. Moreover, the result that the stationary solution converges to the deterministic fixed point with probability one as the noise intensity tends to zero also implies that the stationary measure converges to the Dirac measure supported on the deterministic fixed point with the noise intensity tending to zero.

That is, for nonlinear random difference equations, we not only establish the easy-to-use conditions to guarantee the existence of stationary solutions but also study stability and relation of the stationary solution and its corresponding deterministic fixed point. These properties are important for practical models. For example, in [10] the original random model is studied by using an approximate deterministic model which is given by setting the random dividend in the random model as a constant, its mean. With the theorem and propositions in this paper, it can be proved that there exists a unique stable stationary solution in a small neighborhood of the fixed point of the corresponding deterministic difference equation provided the perturbation intensity is sufficiently small (see [16]). Thus from the dynamic point of view, the dynamic properties are very close for the random model and the corresponding deterministic model near the fixed point

since random dividends in real markets are generally small. That is, the method they used in [10] is meaningful at least for the dynamics of the model near the fixed point. Hence in some sense this paper builds up a bridge between some theoretical researches of mathematics and their applications to practical models.

In future research, firstly, it is possible to provide new easy-to-use conditions that guarantee the existence and stability stationary solutions of high dimensional nonlinear random difference equations with a view of application. Secondly, it is significant to generalize the i.i.d. noise to more general cases. Finally, it is also interesting to study various kinds of bifurcations based on the obtained results from a dynamical point of view.

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