Hurwitz numbers, moduli of curves, topological recursion, Givental's theory and their relations

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–1– Introduction

In my thesis, I investigate four different concepts that play an important role in mathematical physics, and in particular I discuss some of the relations between these subjects. They are *Hurwitz numbers*, *moduli spaces of curves*, *Chekhov-Eynard-Orantin recursion theory* and *Givental's theory*.

The work presented here is based on the papers [112, 113, 114, 115, 116, 117]. Chapters 2 through 8 are basically a collection of these papers, though reorganized in a way I hope presents the material in a clearer way. The first chapter contains an introduction to provide an overview of the different topics and their relations.

In the section below, intended for non-experts, or even non-mathematicians, I try to give some intuitive idea of what all these things are, leaving out almost all formulas. In the rest of the introduction I then give more precise definitions and explain some of the basic properties, and I state and try to explain the results proved in the main part of the thesis.

1.1 Introduction for non-experts

One of the most basic objects occurring in this thesis is the concept of a *curve*. Curves play an important role in all four of the main subjects, so let me first explain what the word "curve" means for me.

The most important thing about curves is that they are one-dimensional and that they are allowed to be curved; that is, they are not necessarily straight, or as mathematicians would say, linear. Thus, one might think of a curve as the object represented by the picture in Figure 1.1.

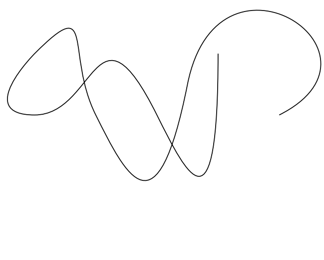

Figure 1.1: Cartoon of a curve.

However, Figure 1.1 is not entirely accurate, because in this case we want to work with *complex curves*. Since the complex numbers are two-dimensional over the real numbers, more accurate pictures are given in Figure 1.2. Mathematicians preferring to work over the real numbers would refer to these as *surfaces*.

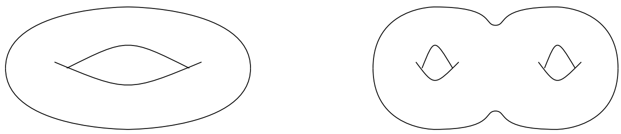

Figure 1.2: A curve of genus 1 and a curve of genus 2.

Looking at the picture in Figure 1.2, one could ask when two of those pictures actually represent the same curve. In different branches of mathematics, different answer would be given to this question. For instance in topology, a curve only depends on its number of 'holes', which is called the *genus*; for each genus, topologically there is precisely one curve of that genus.

Here, we consider curves with some additional structure, called a *complex structure*. Basically, that means that our curves locally resemble the plane of complex numbers. Then, we consider two curves to be the same if they have the same (or, mathematically; *isomorphic*) complex structures.

Once we have defined what we mean when we say that two curves are equivalent, we can:

- Count all the curves with certain properties. Of course, this will only be meaningful if we choose those properties in such a way that the number of curves having those properties is finite.

- Study the set of all curves with some given properties. Examples are moduli spaces of curves, and classes on those moduli spaces.

Now that we know something about curves, it is time to introduce the main subjects of this thesis.

Hurwitz numbers. Hurwitz numbers are an example of the first point above. They count how many *coverings* there are of the sphere by curves of a given genus, and with some specified *ramification profile*. Basically, that means that we look at pictures as in Figure 1.3; there is a curve C of genus g drawn above a sphere S (curve of genus 0). One should imagine that above every point in the sphere, there are some finite number d of points of C, and this number d is the same for all of them (this is what it means to be a covering), except for some special points where it can be smaller (this is specified by the ramification profile).

Moduli spaces of curves. The *moduli spaces of curves* that we are mostly interested in are those that parametrize curves of some specific genus g, where additionally, we have marked some specified number n of their points. When studying such moduli spaces, one of the most common things to do is to try to compute the *intersection* of some classes that are naturally defined on them. That is, we look at the set of all curves in the moduli space that have a certain natural property. In fact, we do this for all kinds of different properties. Then, we are interested in the intersection of those classes; that is, given two or more such properties, we try to find all the curves that have all of those properties, and we try to describe this new class once again by specifying some natural property that is obeyed precisely by the curves in this class. Sometimes, depending on the properties we started with, the number of curves in the intersection will be finite; in that case we are back to counting curves with specific properties, and the number is called an *intersection number*.

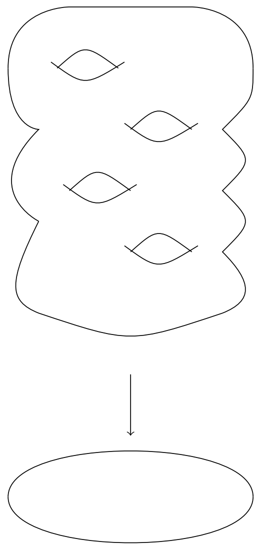

Figure 1.3: A Hurwitz cover.

Chekhov-Eynard-Orantin recursion theory. Whenever we are given a sequence of numbers indexed by some number g, as is the case with the Hurwitz numbers and some of the intersection numbers described above (here, g is always the genus of the curves in question), we could ask whether there is some way to compute the numbers corresponding to higher g directly from the ones corresponding to lower g. A formula describing this type of behaviour is called a *recursion formula*. In fact, in both cases described above there is also another index n, which is the number of points above some special point in S in the case of Hurwitz numbers, and the number of marked points in the case of intersection numbers, and it turns out that Hurwitz numbers as well as some types of intersection numbers obey a recursion of a specific type involving both g and n, called the *Chekhov-Eynard-Orantin recursion*.

One of the most important features of this theory is that the whole formula is encoded by a very simple set of data; basically, by just a curve. Thus, it allows to represent all the data that is encoded in for instance the infinite set of Hurwitz numbers by just giving the equation for one curve. Furthermore, general facts about the CEO recursion can then be used to relate such a set of numbers to other branches of mathematics, such as matrix models and cohomological field theories.

Givental's theory. To say something about Givental's theory, we have to introduce cohomological field theories, which could be described in many different ways. For now, let us think about them as a way to encompass the structure of *Gromov-Witten* theories, which means that they provide some formal way to study curves on a given space. That is, given some space, or *manifold* X (which is like a curve, but may be higher dimensional), we can study all the curves that could be 'drawn' on them. Note that 'drawn' is in parentheses because of the higher dimensional nature of these spaces. However, we can define a mathematical equivalent of drawing that still works in the higher dimensional case (using maps). Once again, we can count the number of curves with specific properties that can be 'drawn' on X, or we can intersect two or more classes of curves on X, and try to understand what class we end up with.

Computation like these are what constitute the Gromov-Witten theory of X, and cohomological field theories are then a way to formalise the properties of such Gromov-Witten theories.

Finally, Giventals theory is a powerful method to compute many things in cohomological field theories by relating them to the trivial cohomological field theory (which in fact is just the moduli space of curves) using operators which have been quantized in a certain way. The precise details of this construction are too involved for this part of the introduction, they are introduced more fully in Section 1.4.4 of this introduction, as well as in Chapter 5.

This concludes the informal introduction to the subjects of my thesis; in the next four sections I give their precise definitions and describe their basic properties.

1.2 Hurwitz numbers

In this section I introduce (various types of) Hurwitz numbers from two different points of view. The interplay between these viewpoints will be used later on, and is one of the main reasons for being interested in Hurwitz nubmers.

1.2.1 Hurwitz numbers

Hurwitz numbers enumerate branched covers of $\mathbb{P}^1$ by smooth curves of some specified genus and with specified ramification profile over some chosen points. I start by stating the most general definition, and then specialize it to the different cases that we encounter in this thesis.

Definition 1.1. Choose some number m, and let $p_1, \cdots, p_m$ be m chosen pair-wise different points on $\mathbb{P}^1$. Given some partitions $\mu^{(1)}, \cdots, \mu^{(m)}$ of a positive integer d and a non-negative integer g, the Hurwitz number $h_{g;\mu^{(1)},\cdots,\mu^{(m)}}$ is defined to be the weighted count of isomorphism classes of degree d covers $f\colon C \to \mathbb{P}^1$, where C is a (possibly disconnected) smooth algebraic curve of genus g, such that the ramification profile of f over p_i is given by $\mu^{(i)}$, and f is unramified on $\mathbb{P}^1 \backslash \{p_1, \cdots, p_m\}$. Two covers $f_1\colon C_1 \to \mathbb{P}^1$ and $f_2\colon C_2 \to \mathbb{P}^1$ are said to be isomorphic if there is an isomorphisms of curves $i\colon C_1 \to C_2$ such that $f_2 \circ i = f_1$, and covers are weighted by the reciprocal of the size of their automorphism group. Often, the inverse images of some of the p_i are considered to be labelled, and isomorphisms are required to respect the labelling. For each such branch point p_i, this has the effect of multiplying the Hurwitz number by $|\mathrm{Aut}(\mu^{(i)})|$.

Note that the Hurwitz number thus defined is finite, and does not depend on the chosen points $p_1, \ldots, p_m$, which justifies not including them in the notation.

Here, we do not study Hurwitz numbers in this general form; instead we limit the allowed ramification profiles in different ways. For instance, there are *double Hurwitz numbers* $h_{g;\mu,\nu}$, where the only arbitrary ramification (given by μ and ν) allowed is over two points (which we take to be 0 and ∞ for definiteness), and all other ramification is simple, which means that it has profile of the form $(2, 1, 1, \ldots, 1)$. For *single Hurwitz numbers* $h_{g;\mu}$, the only arbitrary ramification allowed is over infinity (it is given by the partition μ), there is no ramification over 0 and all other ramification is again simple.

For us, the most important types of Hurwitz numbers are variants of those called Hurwitz numbers *with completed* $(r+1)$*-cycles* and denoted $h^{(r)}_{g;\mu}$ and $h^{(r)}_{g;\mu,\nu}$. They are just the ordinary single and double Hurwitz numbers, but where any ramification over the points away from 0 and ∞ is not simple, but instead given by the *completed* $(r+1)$*-cycle*. That is, over those points, either $r+1$ sheets come together, corresponding to the partition $(r+1, 1, 1, \ldots, 1)$, or some completion effects take place which are more involved to define; their definition is given in Chapter 2.

The branch points where the ramification profile is specified by the partitions that appear in the notation are called *special* branch points (in the case of double Hurwitz numbers they are $0, \infty$; for single Hurwitz numbers it is ∞), whereas the other branch points are called *non-special*. The inverse images of the special branch points are considered to be labelled, the others are not.

It is important to note that in all four cases defined above, the number of non-special branch points is completely determined by the Riemann-Hurwitz formula; for ordinary double Hurwitz numbers it reads:

$$m = 2g - 2 + \ell(\mu) + \ell(\nu) \tag{1.1}$$

while for double Hurwitz numbers with completed $(r+1)$-cycles we have

$$m = \frac{2g - 2 + \ell(\mu) + \ell(\nu)}{r}. \tag{1.2}$$

Here $\ell(\mu)$ is the length of the partition μ (the number of entries). To get the formulas for single Hurwitz numbers, just insert the trivial partition $(1, 1, \ldots, 1)$ for ν, so that $\ell(\nu) = |\mu|$, where $|\mu|$ is the size of μ (the sum of all the entries). These formulas allow us to specify the number of non-special branch points instead of the genus; the notation then becomes $h^{(r),m}_{\mu,\nu}$ (and analogously for the other types).

1.2.2 Combinatorial definition

One of the main reasons that Hurwitz numbers are so interesting, is that they automatically combine geometry and combinatorics. As we have seen in the previous section, the definition of Hurwitz numbers, in terms of covers of curves, is geometric. Here, I will explain why this definition actually leads to combinatorial objects, by giving an equivalent definition in terms of factorizations in the symmetric groups.

For simplicity, we study the single Hurwitz number h^m_μ. We label the points of non-special ramification $p_1, \ldots, p_m$. Choose a base-point p away from $0, \infty, p_1, \ldots, p_m$. Since there is no ramification at p, there are $d = |\mu|$ points in the fibre above p, which we number $1, \ldots, d$ in some way. Now, moving along a simple loop γ_i based at p around one of the points p_i interchanges two of the points of the fibre, which can be described by a transposition σ_i in the symmetric group S_d. On the other hand, moving along a simple loop γ based at p around ∞ interchanges the points in the fibre above p in some way allowed by the ramification profile μ, which means that it can be described by an element σ of S_d of cycle-type μ. If we move around all the loops γ_i and γ consecutively, then the points of the fibre are interchanged according to the product $\sigma_1 \cdots \sigma_m \cdot \sigma$. However, since we are on the sphere, the concatenation of all those loops is equivalent to the trivial loop, so the interchange of the points in the fibre must also be trivial. Thus, any Hurwitz cover induces a factorization in the symmetric group

$$\sigma^{-1} = \sigma_1 \cdots \sigma_m. \tag{1.3}$$

In fact, it turns out that such factorizations are in one-to-one correspondence with Hurwitz covers (one can reconstruct the cover from the data of the factorization, as is shown in Chapter 2 for Hurwitz numbers with completed cycles). Therefore, the Hurwitz number h^m_μ is just the number of factorizations of elements of $S_{|\mu|}$ of cycle-type μ into transpositions.

Computing such numbers of factorizations is a purely combinatorial exercise that could be done on a computer for any individual Hurwitz number. Note that it is not obvious from the original definition of Hurwitz numbers that such a combinatorial way of computing these numbers should exist. In fact, in Chapter 2, I describe an algorithm to compute Hurwitz numbers using their connection to symmetric groups, which will also allow us to deduce more general properties of Hurwitz numbers, such as polynomial behaviour.

1.3 Moduli space of curves

In this section, I introduce the moduli space of curves, some standard classes on this moduli space, and a remarkable relation to Hurwitz numbers that has been used as a powerful tool in the proof of some famous theorems.

1.3.1 The moduli space of curves

A *nodal curve* is a compact connected algebraic curve with finitely many singularities, all of which are nodes. A node is a singularity locally isomorphic to $\{xy = 0\}$ in $\mathbb{C}^2$. Such a curve is called *stable* if it has finitely many automorphisms. For $g \geq 0$ and $n > 0$, we denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space parametrizing stable nodal curves of genus g with n marked points, labelled by $1, \ldots, n$. It turns out that a nodal curve is stable if and only if every irreducible component of genus 0 has at least three special points (nodes and marked points). Thus, $\overline{\mathcal{M}}_{g,n}$ is non-empty as long as $(g, n) \neq (0, 1),\ (0, 2)$. It naturally has the structure of a compact Deligne-Mumford stack of dimension $3g - 3 + n$. As an analytic space, it is an orbifold. An excellent introduction to the moduli space of curves is given in [104].

We will also need some natural maps between moduli space of curves that correspond to forgetting marked points and to gluing curves together at marked points. The *forgetful map* $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the map that forgets the last marked point. The *gluing maps* $\mathrm{gl}\colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ and $\mathrm{gl}\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ glue two curves together at their last marked points, or glue the last two marked points of a curve together, increasing the genus. Remark that we abuse notation and denote both maps by gl, and also that we do not include g and n in the notation.

1.3.2 Natural classes on $\overline{\mathcal{M}}_{g,n}$.

We introduce some natural classes on $\overline{\mathcal{M}}_{g,n}$. For the *boundary classes*, we associate to any curve in $\overline{\mathcal{M}}_{g,n}$ a graph in the following way;

1. replace each irreducible component by a vertex labelled by the genus of the component,

2. replace each node by an edge connecting the corresponding node(s),

3. attach a leaf to each vertex for each marked point on the corresponding component, labelled by the label of the marked point.

This graph is called the dual graph of the curve, and curves with the same dual graph are said to have the same topological type. Given any graph Γ that appears as the dual graph of some curve, the boundary class D_Γ is the class of all curves in $\overline{\mathcal{M}}_{g,n}$ of topological type Γ.

To introduce the *ψ-classes*, denote by $\mathcal{L}_i$ the line bundle on $\overline{\mathcal{M}}_{g,n}$ whose fibre at a point $(C, p_1, \ldots, p_n)$ is the cotangent line to the i^{th} marked point p_i, glued together in some natural way. Define $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n}; \mathbb{Q})$ to be the first chern class $c_1(\mathcal{L}_i)$ of this line bundle.

Using the forgetful map $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, we introduce *$\kappa$-classes* by the following formula:

$$\kappa_i = \pi_* \psi_{n+1}^{i+1} \tag{1.4}$$

Finally, we introduce so-called *λ-classes.* Let $\mathbb{E}$ be the rank g vector bundle whose fibre at each curve is the space of sections of the canonical line bundle, called the *Hodge bundle.* Then for any $i \in \{1, \ldots, g\}$, define λ_i to be the i^{th} chern class of that bundle; $\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$.

Now that we have defined some natural classes on the moduli space, one of the things to understand is their intersection theory. In particular, a natural question to ask is how many points lie in the intersection of a set of classes the sum of whose degrees is equal to the dimension on the space. Such a number of points is denoted by an integral. For instance

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \quad \text{for } d_1, \ldots d_n \in \mathbb{Z}_{\geq 0} \tag{1.5}$$

denotes the number of points in the intersection of d_1 copies of ψ_1 with d_2 copies of ψ_2, etc. Note that this intersection is zero-dimensional if and only if $d_1 + \cdots + d_n = 3g - 3 + n$; if this equality does not hold, the integral is defined to be zero. It turns out that those intersections of ψ classes are governed by the Korteweg-de-Vries hierarchy; this is the subject of the Witten-Kontsevich theorem.

1.3.3 The ELSV formula

Another example of natural occurring intersection numbers that are governed by an integrable hierarchy are the so-called *linear Hodge integrals*, which are of the form

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_i \psi_1^{d_1} \cdots \psi_n^{d_n} \quad \text{for } d_1, \ldots d_n \in \mathbb{Z}_{\geq 0}. \tag{1.6}$$

Again, the integral is defined to be zero when $i + d_1 + \cdots + d_n = 3g - 3 + n$ does not hold.

One of the reasons that these linear Hodge integrals are interesting for us is that they are related to Hurwitz numbers by a remarkable formula due to Ekedahl, Shapiro, Lando and Vainshtein [31];

$$h_{g,\mu} = \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \lambda_2 - \cdots \pm \lambda_g}{\prod_{i=1}^{\ell(\mu)} (1 - \mu_i \psi_i)} \ . \tag{1.7}$$

Here the right-hand side should be interpreted using the expansion

$$\frac{1}{1 - \mu_i \psi_i} = \sum_{k=0}^{\infty} (\mu_i \psi_i)^k \tag{1.8}$$

and all terms where the sum of the degrees of the classes does not match the dimension of the moduli space are disregarded. The formula is remarkable, because it allows us to compute some a priori geometrical quantities (intersection numbers) in terms of combinatorial ones (Hurwitz numbers). It turns out that this is a very powerful tool, which was for instance used in one of the proofs of the Witten-Kontsevich theorem ([63]).

1.4 Cohomological field theories

In this section, I give the definition of a cohomological field theory, and I try to explain why they are interesting to study. To do that, I introduce the notion of Gromov-Witten theory before continuing to more general cohomological field theories. This finally allows me to introduce Givental's formalism, which will later be related to the CEO recursion of the next section.

1.4.1 Gromov-Witten theory

In the previous section we studied intersection theory on the moduli space of curves. It turns out that it can be even more interesting to look not just at curves, but at curves inside some given space. One of the ways to describe such a situation is called Gromov-Witten theory; basically, given a space X, it studies the moduli space of curves in X, and in particular the intersections of naturally defined classes on that space. One of the reason for the interest in this type of theory comes from physics, where one is interested for instance in the world-sheet of a closed string in space-time, which can indeed be described mathematically as a curve in some manifold.

To be more precise, let $g \geq 0$ and $n \geq 1$ be integers, let X be a smooth projective manifold, and let $\beta \in H^2(X, \mathbb{Z})$ be a class in the second homology of X. Then a genus g, n-pointed stable map to (X, β)

$$f\colon (C, p_1, \ldots, p_n) \to X \tag{1.9}$$

is a map from an n-pointed nodal curve C of genus g to X such that f has only finitely many automorphisms (stability), and $f_*[C] = \beta$. Concretely, the stability means that C should have at least one stable irreducible component, and all irreducible components of C that are mapped to a point should be stable (both in the sense of stable curves). We denote by $\overline{\mathcal{M}}_{g,n}(X, \beta)$ the moduli space of such stable maps to (X, β). Note that when X is a point, this is just the moduli space of curves, so the Gromov-Witten theory of a point reduces to ordinary intersection theory on the moduli space of curves. Also note that there is a natural forgetful map to the moduli space of curves

$$\rho_\beta\colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n} \tag{1.10}$$

that forgets the map and the target, and returns the source curve after an appropriate stabilization.

Once again, we want to integrate some natural classes on the moduli space. However, it is quite singular (it is not even equidimensional), so one has to do some work to be able to do this. The solution turns out to be the construction of a *virtual fundamental class* $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{virt}}$, which behaves just like the fundamental class of a smooth space of dimension

$$\dim = (3 - \dim(X))\,(g-1) + c_1(TX) \cap \beta + n. \tag{1.11}$$

To define suitable classes to integrate, let $\mathcal{L}_i$ once again be the line bundle on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ with fibre given by the cotangent line to the i^{th} marked point, and let $\psi_i = c_1(\mathcal{L}_i)$. Furthermore, we define some classes associated to the space X. Let $\mathrm{ev}_i\colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$ be the map that sends a stable map to the image of the i^{th} marked point. Then, for any even cohomology class α on X, we get a class $\mathrm{ev}_i^*(\alpha)$ on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ by pull-back.

The most common integrals we are interested in are the intersection of these classes, called *correlators*, of the following form:

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \prod_{i=1}^{n} \psi_i^{d_i} ev_i^*(e_i), \tag{1.12}$$

where e_i are some classes in the even cohomology of X. The analogues of these correlators play an important role in the cohomological field theories introduced in the next subsection, where they are used to build the so-called *descendant potentials*.

1.4.2 Cohomological field theory

Cohomological field theories can be viewed as a way to axiomatize some of the properties of Gromov-Witten theory. The definition is the following.

Let V be a finite dimensional vector space with a scalar product $(\cdot, \cdot)$ and a distinguished vector $e_{\mathbf{1}}$. Denote by $H^*_{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ the even cohomology of the moduli space of curves. A cohomological field theory is a collection of linear homomorphisms $c_{g,n}\colon V^{\otimes n} \to H^*_{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ for all g and n that behaves nicely with respect to permutation of marked points and tensor factors, and also with respect to the standard morphism between moduli spaces of curves. To be precise, the following should hold:

- For any g and n, the morphism $c_{g,n}$ is $\mathbb{S}_n$ equivariant, where the action is given by permutation of tensor factors in $V^{\otimes n}$ and of marked points in $\overline{\mathcal{M}}_{g,n}$.
- For any $a, b \in V$ we have $(a,b) = c_{0,3}(e_{\mathbf{1}} \otimes a \otimes b) \in H^*(\overline{\mathcal{M}}_{0,3}) = \mathbb{C}$.

Remember that $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the morphism that forgets the last marked point, and gl denotes the gluing morphism, either gluing two marked points on one curve, increasing the genus, or gluing two curves together at marked points.

- For any $a_1, \ldots, a_n \in V$, we have
$$\pi^* c_{g,n}(a_1 \otimes \cdots \otimes a_n) = c_{g,n+1}(a_1 \otimes \cdots \otimes a_n \otimes \mathbf{1}). \tag{1.13}$$
- Let $\{e_i\}$ be a basis of V and denote $\eta^{ij} = (e_i, e_j)$. Then we have for all $a_1, \ldots, a_{n_1+n_2}$ in V:
$$\begin{aligned} \mathrm{gl}^* c_{g_1+g_2,n_1+n_2}(a_1 \otimes \cdots \otimes a_{n_1+n_2}) = \\ = c_{g_1,n_1+1}(a_1 \otimes \cdots \otimes a_{n_1} \otimes e_i) \cdot \\ \cdot c_{g_2,n_2+1}(a_{n_1+1} \otimes \cdots \otimes a_{n_2} \otimes e_j) \cdot \eta^{ij}. \end{aligned} \tag{1.14}$$
- Finally, for all $a_1, \ldots, a_n$, we have
$$\mathrm{gl}^* c_{g,n}(a_1 \otimes \cdots \otimes a_n) = c_{g-1,n+2}(a_1 \otimes \cdots \otimes a_n \otimes e_i \otimes e_j) \cdot \eta^{ij}. \tag{1.15}$$

Indeed, this definition generalizes Gromov-Witten theories in the following way. Let X be a smooth projective variety, and β a class in the second homology of X. Assume that X does not have odd cohomology. Let $V = H^*(X, \mathbb{C})$ with scalar product $(\cdot,\cdot)$ given by the Poincaré pairing and unit $\mathbf{1}$ by the unit in cohomology. Define

$$c_{g,n}(a_1 \otimes \cdots \otimes a_n) = (\rho_\beta)_* \left([\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}} \cap \prod_{i=1}^{n} \mathrm{ev}_i^*(a_i) \right). \tag{1.16}$$

Then $c_{g,n}$ obeys the axioms of a cohomological field theory, making Gromov-Witten theories into a large class of examples of cohomological field theories.

As in Gromov-Witten theory, it turns out that we are often interested in certain types of intersection numbers of cohomological field theories called correlators. They are denoted by $\langle \tau_{d_1}(a_1) \cdots \tau_{d_n}(a_n) \rangle_{g,n}$ and defined as

$$\langle \tau_{d_1}(a_1) \cdots \tau_{d_n}(a_n) \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}(a_1 \otimes \cdots \otimes a_n) \psi_1^{d_1} \cdots \psi_n^{d_n}. \tag{1.17}$$

It is convenient to collect correlators of a cohomological field theory in a generating function Z, called the *partition function* or *descendant potential*. It is defined as follows. Let $e_1, \ldots, e_N$ be some chosen basis for V. Then:

$$\begin{aligned} Z\left(\hbar; \{t^{d,i}\}\right) := \exp\left(\sum_g \hbar^{g-1} \mathcal{F}_g \right) = \\ = \exp\left(\sum_{g,n} \hbar^{g-1} \sum_{\substack{d_1,\ldots,d_n \geq 0 \\ 1 \leq i_1,\ldots,i_n \leq N}} \frac{\langle \tau_{d_1}(e_{i_1}) \cdots \tau_{d_n}(e_{i_n}) \rangle}{|\mathrm{Aut}((d_j, i_j)_{j=1}^n)|} t^{d_1,i_1} \cdots t^{d_n,i_n} \right). \end{aligned} \tag{1.18}$$

Here, $|\mathrm{Aut}((d_j, i_j)_{j=1}^n)|$ denotes the number of automorphism of the collection of multi-indices (d_j, i_j), and the sum over g, n contains all stable contributions. The variable $\hbar$ keeps track of the genus, and the variables $t^{d,i}$ keep track of the number of ψ-classes and the basis vectors in V.

1.4.3 Frobenius manifolds

It turns out that the genus zero part of a cohomological field theory has an interesting structure of its own; it is a Frobenius manifold. We describe it here.

Definition 1.2. A *Frobenius algebra* is a commutative associative algebra A with unit e together with a bilinear symmetric non-degenerate inner product

$$\langle\cdot,\cdot\rangle\colon A \times A \to \mathbb{C} \tag{1.19}$$

such that

$$\langle ab, c\rangle = \langle a, bc\rangle. \tag{1.20}$$

Definition 1.3. A *Frobenius manifold* is a manifold M together with a stucture of Frobenius algebra on the tangent space at each point, varying smoothly, such that the following hold.

1. The inner product $\langle\cdot,\cdot\rangle$ induces a flat metric η on M.
2. The unit vector field is covariantly constant with respect to the Levi-Cevitá connection ∇ for the metric η: $\nabla e = 0$.
3. Define a symmetric three-tensor by $c(u, v, w) = \langle u \cdot v, w\rangle$. The four-tensor $(\nabla_z c)(u, v, w)$ should be symmetric in the four vector fields z, u, v, w.

Such a Frobenius manifold is called *conformal*, if in addition, it is equiped with a vector field E, called the *Euler vector field*, satisfying

4. $\nabla(\nabla E) = 0$.

For any system of local flat coordinates $t^1, \ldots, t^N$ on a Frobenius manifold M, denote by $c^\alpha_{\beta\gamma}$ the structure constants of the algebra structure on the tangent space. That is

$$\frac{\partial}{\partial t^\beta} \cdot \frac{\partial}{\partial t^\gamma} = c^\alpha_{\alpha\beta} \frac{\partial}{\partial t^\alpha}. \tag{1.21}$$

For any N-dimensional Frobenius manifold M, locally, there exist flat coordinates $t^1, \ldots, t^N$ and a function $G(t)$ called the *potential* such that

$$e = \frac{\partial}{\partial t^1} \tag{1.22}$$

$$\eta_{\alpha\beta} = \frac{\partial^3 G}{\partial t^1 \partial t^\alpha \partial t^\beta} \tag{1.23}$$

$$c^\alpha_{\beta\gamma} = \eta^{\alpha\mu} \frac{\partial^3 G}{\partial t^\mu \partial t^\alpha \partial t^\beta}, \tag{1.24}$$

where the matrix $\eta_{\alpha\beta}$ is just the metric η in the coordinates $\{t^\alpha\}$ and $\eta^{\alpha\beta}$ is its inverse. It turns out that the definition of Frobenius manifolds implies that the potential G satisfies the WDVV equation:

$$\frac{\partial^3 G}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^3 G}{\partial t^\nu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 G}{\partial t^\beta \partial t^\delta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^3 G}{\partial t^\nu \partial t^\gamma \partial t^\alpha} \ . \tag{1.25}$$

Conversely, suppose a holomorphic function $G(t^1,\dots,t^N)$ on an open subset $U \subset \mathbb{C}^N$ and a constant matrix η satisfying equations (1.23) and (1.25) are given. Then equations (1.22) and (1.24) determine the structure of a Frobenius manifold on U.

Consider an N-dimensional cohomological field theory with vector space $V = \langle e_1,\dots,e_N\rangle$, unit e_1 and scalar product η. Let G be the genus zero part of the partition function with no descendants (ψ-classes):

$$G(t^{0,1},\dots,t^{0,N}) = \mathcal{F}_0(\{t^{d,i}\})\big|_{t^i_{\geq 1}=0} \tag{1.26}$$

Then by the definition of cohomological field theories, G and η fulfil the requirements of Equations (1.23) and (1.25). Since G does not necessarily converge to a function on some open subset of $\mathbb{C}^N$, we say that it defines a *formal Frobenius manifold* structure on V.

It turns out that the whole genus zero potential can be recovered from the series G using the so-called *topological recursion relation* in genus zero (note that the CEO recursion is also sometimes called topological recursion; to avoid confusion here we always call it CEO recursion). Thus, the information of a Frobenius manifold corresponding to a cohomological field theory is enough to recover the whole genus zero partition function.

In the next subsection we describe a construction, due to Givental and proved by Teleman, that allows to reconstruct the full partition function of a cohomological field theory from a local conformal Frobenius manifold structure.

1.4.4 Givental's formalism

Givental's formalism essentially consists of two parts. The first is a group action on the space of cohomological field theories, and the second is a way to reconstruct a group element that takes the trivial cohomological field theory to a given one.

Consider the space of partition functions of cohomological field theories on an N-dimensional vector space $V = \langle e_1,\dots,e_N\rangle$ with scalar product η:

$$Z = \exp\left(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}_g\right). \tag{1.27}$$

Let $r_l \in \mathrm{Hom}(V,V)$ for $l \geq 0$ be some operators such that the operators with odd indices are symmetric and those with even indices are skew-symmetric. We define their quantization $(r_l z^l)\hat{}$ by

$$(r_l z^l)\hat{} := -(r_l)^i_{\mathbf{1}}\frac{\partial}{\partial t^{l+1,i}} + \sum_{d=0}^{\infty} v^{d,i}(r_l)^j_i \frac{\partial}{\partial t^{d+l,j}} + \frac{\hbar}{2}\sum_{m=0}^{l-1}(-1)^{m+1}(r_l)^{i,j}\frac{\partial^2}{\partial t^{m,i}\partial t^{l-1-m,j}}. \tag{1.28}$$

Here the indices $i,j \in \{1,\dots,N\}$ on r_l correspond to the basis $\{e_1,\dots,e_N\}$ of V, and the index $\mathbf{1}$ corresponds to the unit vector $e_{\mathbf{1}}$. When we write r_l with two upper-indices we mean as usual that we raise one of the indices using the scalar product η.

Given such a sequence of operators r_l, we define an operator series $R(z)$ in the following way

$$R(z) = \sum_{l=0}^{\infty} R_l z^l := \exp\left(\sum_{l=1}^{\infty} r_l z^l\right). \tag{1.29}$$

The quantization $\hat{R}$ of this series is defined by

$$\hat{R} = \exp\left(\sum_{l=1}^{\infty}\left((-1)^l r_l z^l\right)\hat{}\right). \tag{1.30}$$

Givental observed that the action of such operators on tame formal power series (series for which the number of ψ-classes (given by the first index of $t^{d,i}$) at any monomial is not too high) is well defined. The infinitesimal form of the quantization (1.28) was found by Y.P. Lee [68, 69, 70].

It turns out that the natural action of such differential operators $\hat{R}$ on the space of partition functions of cohomological field theories with target space (V, η) is well-defined [44, 62, 103]. This is the first part of Givental's formalism.

For the second part, suppose that we start with some N-dimensional semi-simple conformal cohomological field theory partition function Z. Here semi-simplicity means that the associated Frobenius manifold has a semi-simple Frobenius algebra structure on the tangent space at a generic point. Then, from the data of just the genus zero part without descendants of Z, so from the associated Frobenius manifold, one can construct an operator series $R(z) = \sum_{k\geq 0} R_k z^k$ as above, such that the quantization of this operator takes the trivial N-dimensional cohomological field theory (properly rescaled) to the cohomological field theory we started with [45, 46, 47].

Thus, for semi-simple conformal cohomological field theories, the whole partition function can in principle be reconstructed from just the genus zero data without descendants. Since it is a bit involved, I leave the actual construction of the operator series R to Chapter 5.

For us, one of the most important applications of this theory is that the action of a series R can be represented using graphs, so the partition function of a given cohomological field theory is expressed as a sum of contributions over a specific set of labelled graphs, each of whose building blocks contribute in a specific way.

1.5 CEO recursion

In this section I describe the Chekhov-Eynard-Orantin recursion. It is a way to recursively define a series of invariants $\omega_{g,n}$ (where g is a non-negative integer, and n is a positive integer) from a simple set of data; a curve in $\mathbb{C}^2$ called the spectral curve, together with a meromorphic differential, often called the Bergman kernel. Note that in the literature, it often called topological recursion theory. Here we avoid that name to avoid any confusion with the topological recursion relation mentioned in the last section.

This recursion was first established to deal with expectation values in the theory of matrix models, but it turns out that many series of invariants that arise naturally in different parts of mathematics also fit into this framework. For instance, Hurwitz numbers, some intersection numbers on the moduli space of curves, and cohomological field theories can all be described in some way by CEO recursion.

Let $\mathcal{S}$ be a smooth complex curve in $\mathbb{C}^2$ with coordinates x, y. Let $a_1, \ldots, a_N$ be the critical points of the coordinate function x. Assume that x is a ramified covering with ramification of order 2 at each of the points $a_1, \ldots, a_N$, and no other ramification points. Thus, in the neighbourhood of each point a_i one can define a local coordinate $z^{(i)}$ such that

$$x(z^{(i)}) = (z^{(i)})^2 + x(a_i). \tag{1.31}$$

Furthermore, there is a natural involution σ_i that interchanges the sheets of the covering x; for simplicity, for any function or differential form α we often denote this involution by $\bar{\alpha} = \sigma_i^*(\alpha)$ when the ramification point a_i is clear. Now, let B be a meromorphic symmetric 2-differential on $\mathcal{S} \times \mathcal{S}$ such that near (a_i, a_j) it can be expressed as

$$B(z_1^{(i)}, z_2^{(j)}) = \delta_{ij} \frac{\mathrm{d}z_1^{(i)} \mathrm{d}z_2^{(j)}}{(z_1^{(i)} - z_2^{(j)})^2} + B_{\mathrm{reg}}^{(ij)}(z_1^{(i)}, z_2^{(j)}) \tag{1.32}$$

in the local coordinate $(z^{(i)}, z^{(j)})$, where $B_{\mathrm{reg}}^{(ij)}$ is a 2-differential holomorphic at (a_i, a_j).

To such a set of data $(\mathcal{S}, x, y, B)$ we recursively associate a tower of n-differentials $\omega_{g,n}$ on $\mathcal{S}^{\times n}$, for $g \geq 0$ and $n \geq 1$, in the following way:

$$\omega_{0,1}(z) = 0, \quad \omega_{0,2}(z_1, z_2) = B(z_1, z_2), \tag{1.33}$$

and for $2g - 2 + n > 0$,

$$\omega_{g,n}(z, z_2, \ldots, z_n) = -\sum_{i=1}^{N} \operatorname*{res}_{z'=a_i} \left(\frac{\tilde{\omega}_{g,n}(z', \bar{z}', z_2, \ldots, z_n)}{2(y(z')\mathrm{d}x - \bar{y}(z')\mathrm{d}x)} \int_{z'}^{\bar{z}'} B(z, \cdot) \right), \tag{1.34}$$

where

$$\tilde{\omega}_{g,n}(z', z'', z_2, \ldots, z_n) = \omega_{g-1,n+1}(z', z'', z_2, \ldots, z_n) + \\ + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\}}} \omega_{g_1,|I|+1}(z', z_I)\omega_{g_2,|J|+1}(z'', z_J). \tag{1.35}$$

Here $z_I := (z_{i_1}, \ldots, z_{i_{|I|}})$ for any subset $I \subset \{2, \ldots, N\}$. The $\omega_{g,n}$ thus defined are called the correlation functions associated to the spectral curve (C, x, y, B).

Furthermore, we also associate to this data a set of so-called symplectic invariants F_g, for $g \geq 2$, as follows:

$$F_g := \frac{1}{2-2g} \sum_{i=1}^{N} \operatorname*{res}_{z=a_i} \left(\omega_{g,1}(z)\Phi(z)\right), \tag{1.36}$$

where Φ is defined locally near a_i and obeys $\mathrm{d}\Phi = y\mathrm{d}x$.

Remark 1.4. In fact, it turns out that it is often interesting to study the just the local behaviour of the CEO recursion at the branch points of the spectral curve. Thus, one defines local CEO recursion, where the curve is replaced by a set of disks centered at the ramification points, and all differentials are replaced by their local germs. I will use this set-up when discussing the relation with Giventals formalism.

Example 1.5. Consider the spectral curve given by

$$x(z) = z^2 + a, \quad y(z) = z \quad \text{and} \quad B(z, z') = \frac{\mathrm{d}z \otimes \mathrm{d}z'}{(z - z')^2}, \tag{1.37}$$

which is called the Airy curve. It has just one ramification point at $(x, y) = (a, 0)$. The corresponding correlation functions $\omega_{g,n}$ encode the intersections of ψ-classes on the moduli spaces of curves:

$$\omega_{g,n}(z_1, \ldots, z_n) = \left(-\frac{1}{2}\right)^{2g-2+n} \sum_{d_1,\ldots,d_n \geq 0} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \prod_{i=1}^{n} \frac{(2d_i + 1)!!\mathrm{d}z_i}{z_i^{2d_i+2}}. \tag{1.38}$$

This example turns out to be very important, since it constitutes a basic building block for the correlation functions of any spectral curve. That is, for any spectral curve, the correlation functions can be written as a sum over a set of labelled graphs, where each part of a graphs is assigned some contribution. Then, the contribution of the vertices are precisely the correlation functions of the Airy curve described above (1.38).

1.6 Results

In this thesis, I describe and prove various results about the four areas described above, about their internal structure as well as the relations between them. Here I give a description of those results, as well as a plan of how they are presented.

1.6.1 Hurwitz numbers with completed cycles

In Chapter 2, I introduce Hurwitz numbers with completed $(r+1)$-cycles and prove various results about their structure.

After defining the Hurwitz numbers $h^{(r)}_{g,\mu,\nu}$ following [88], I show how to describe them as the vacuum expectation values of certain operators on the so-called semi-infinite wedge space. This allows us to define an algorithm to compute $h^{(r)}_{g,\mu,\nu}$ by commuting those operators, based on the algorithm described by Johnson in [59] for ordinary Hurwitz numbers. In fact, we prove the following theorem:

Theorem 1.6. *The Hurwitz number $h^{(r),m}_{\mu,\nu}$ is given by the following formula:*

$$h^{(r),m}_{\mu,\nu} = \frac{1}{\prod_{i=1}^{l(\mu)} \mu_i \prod_{j=1}^{l(\nu)} \nu_j} [z_1^{r+1} \cdots z_s^{r+1}] \left(\sum_{P \in \mathrm{CP}^s_{\mu,\nu}} \prod_{t \in T(P)} \frac{1}{\zeta(z_{S_t})} \prod_{l \in L(P)} \zeta \begin{pmatrix} P^l_I & P^l_J & P^l_K \\ P^l_M & P^l_N & P^l_R \end{pmatrix} \right) \tag{1.39}$$

Here $[z^k]Q(z)$ denotes the coefficient of z^k in the power series $Q(z)$, the set $\mathrm{CP}^s_{\mu,\nu}$ indexes the possible ways of commuting the operators in the vacuum expectation value describing $h^{(r),m}_{\mu,\nu}$, and the function ζ is given by

$$\zeta(z) = e^{z/2} - e^{-z/2}. \tag{1.40}$$

The matrix argument for ζ in equation (1.39) is just a notation for a certain comibination of the variables $z_1, \dots, z_s$ depending on the commutation pattern P.

The proof of Theorem 1.6 provides us with an algorithm (described in detail in Chapter 2) to compute any Hurwitz numbers $h^{(r)}_{g,\mu,\nu}$ individually. This algorithm can then be used to derive some of their general properties. For instance, it is used to prove the following theorem.

Choose two positive integers s and n and let V be the set of pairs of partitions of length s and n respectively, and of the same size. Define a function $h^{(r)}_g \colon V \to \mathbb{Q}$ by $h^{(r)}_g(\mu,\nu) = h^{(r)}_{g,\mu,\nu}$. For any subsets $I \subset \{1, \dots, s\}$ and $J \subset \{1, \dots, n\}$ let $W_{I,J}$ be the hyperplane

$$W_{I,J} := \left\{ (\mu,\nu) \in V \;\middle|\; \sum_{i \in I} x_i - \sum_{j \in J} y_J = 0 \right\} \subset V \tag{1.41}$$

Then the following theorem holds.

Theorem 1.7. *The function $h^{(r)}_g$ is piecewise polynomial on V, with walls given by the hyperplanes $W_{I,J}$ for any non-empty proper subsets I and J as above.*

In fact, the algorithm also allows to describe the structure of this piecewise polynomial (Theorem 2.47) and provide wall-crossing formulas for all the hyperplanes $W_{I,J}$ (Theorem 2.49). These theorems are direct generalizations of known theorems for ordinary Hurwitz numbers, which were first conjectured by Goulden, Jackson and Vakil in [51] and later partially proved by Cavalieri, Johnson and Markwig in [14] and fully proved by Johnson in [59]. In fact, the techniques used to prove the theorems described here are direct generalizations of the ones in [59].

The description of the completed Hurwitz number in terms of vacuum expectation values in the infinite wedge space can be used directly to prove two further theorems that are also direct analogues of known theorems for ordinary Hurwitz numbers.

For the first, define a generating series for completed Hurwitz numbers by

$$H_{r+1}(\beta, p_1, p_2, \dots, q_1, q_2, \dots) = \sum_{n,m,s} \sum_{\substack{\mu_1,\dots,\mu_s \\ \nu_1,\dots,\nu_n}} h^{(r+1),m}_{\mu,\nu} \frac{\beta^m}{m!} \frac{p_{\mu_1} \cdots p_{\mu_s}}{s!} \frac{q_{\nu_1} \cdots q_{\nu_n}}{n!}. \tag{1.42}$$

Then this generating series obeys a so-called cut-and-join equation.

Theorem 1.8. *The generating series H_{r+1} obeys the following equation:*

$$\frac{\partial H_{r+1}}{\partial \beta} = Q_{r+1} H_{r+1}, \tag{1.43}$$

where the operators Q_{r+1} are defined as the coefficients of the expansion of the following series

$$Q_1 z + Q_2 z^2 + \cdots = \frac{1}{\zeta(z)} \sum_{n\geq 1} \frac{1}{n!} \sum_{k_1+\cdots+k_n=0} \zeta(k_1 z)\cdots\zeta(k_n z) \frac{:a_{k_1}\cdots a_{k_n}:}{k_1\cdots k_n}. \tag{1.44}$$

Here, for $k \geq 1$ we define $a_{-k} = p_k$ and $a_k = k\frac{\partial}{\partial p_k}$, and the normal ordering $:\cdot:$ means that the differential operators go to the right.

For the second theorem define some 'combinatorial intersection numbers' by the formula

$$h^{(r)}_{g,\mu} = \frac{m!}{d} \int_{X^{(r)}_{g,n}} \frac{1 - \Lambda^{(r)}_2 + \Lambda^{(r)}_4 - \cdots + (-1)^g \Lambda^{(r)}_{2g}}{(1-\mu_1\Psi^{(r)}_1)\cdots(1-\mu_n\Psi^{(r)}_n)}. \tag{1.45}$$

Here, by combinatorial intersection numbers, we mean that the right-hand side of this formula is defined by the equality with the left-hand side, but one should hope to be able to define some spaces $X^{(r)}_{g,n}$ together with classes $\Lambda^{(r)}_{2l}$ and $\Psi^{(r)}_i$ on them such that the Equality (1.45) holds. Then, Equation (1.45) becomes an ELSV-type formula for one-part double Hurwitz numbers with completed cycles.

As in the case of the usual ELSV formula, it turns out that the generating series for these intersection numbers, after an appropriate change of variables, obeys a particular integrable hierarchy.

Theorem 1.9. *Let*

$$G(u) := \sum_{j,k_1,k_2,\ldots} (-1)^j \langle \Lambda_{2j} \tau_0^{k_0} \tau_1^{k_1} \ldots \rangle_g u^{2j} \frac{T_0^{k_0}}{k_0!} \frac{T_1^{k_1}}{k_1!} \cdots. \tag{1.46}$$

Then, after an appropriate triangular change of variables, we have that for any function $c(u)$ the series $c(u) + G(u, q_1, q_2, \ldots)$ is a solution of the Hirota equations in variables $q_1, q_2, \ldots$.

The first few equations of the change of variables are given by

$$\begin{aligned} T_0 &= q_1, \\ T_1 &= uq_1 + q_2, \\ T_2 &= u^2 q_1 + 3uq_2 + 2q_3, \end{aligned} \tag{1.47}$$

while the complete change is described in Chapter 2.

1.6.2 CEO recursion for Hurwitz numbers with completed cycles

In Chapter 3, I describe a spectral curve whose correlations functions conjecturally generate single Hurwitz numbers with completed cycles. This is a generalization of the Bouchard-Marino conjecture [10], which was proved in various ways [35, 8].

Let $H^{(r)}_{g,n}$ be the generating function for completed single Hurwitz numbers in the following sense:

$$H^{(r)}_{g,n}(x_1,\ldots,x_n) := \sum_{\mu,\ldots,\mu_n} \frac{h^{(r)}_{g,\mu}}{m!} \exp(\mu_1 x_1 + \cdots + \mu_n x_n), \tag{1.48}$$

where m denotes the Euler characteristic (Equation (1.2)) as always. Also, for any function $f(x_1, \ldots, x_n)$, let

$$Df(x_1, \ldots, x_n) = \frac{\partial^n f}{\partial x_1 \cdots \partial x_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{1.49}$$

Conjecture 1.10. *Let $\omega_{g,n}^{(r)}$ be the correlation functions associated to the spectral curve*

$$\begin{cases} x(z) &= -z^r + \log z \\ y(z) &= z \end{cases} \tag{1.50}$$

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}. \tag{1.51}$$

Then we have

$$DH_{g,n}^{(r)}(x_1, \ldots, x_n) = \omega_{g,n}(x_1, \ldots, x_n). \tag{1.52}$$

The evidence for this conjecture is four-fold. First, there is a proof of the Bouchard-Mariño conjecture by Borot, Eynard, Mulase and Safnuk ([8]) that uses matrix models, which can be fully generalized to the case of completed Hurwitz numbers. Unfortunately, this proof is not completely rigorous, so it should be regarded as strong evidence from mathematical physics for the conjecture.

Second is the general idea [30] that the spectral curve for an enumerative problem should be given by its $(0,1)$-geometry. In the case of completed Hurwitz numbers it is easy to see that the $(0,1)$-geometry leads to the spectral curve from Conjecture 1.10.

Third, there is a conjecture by Gukov and Sułkowski [53] which states that if an enumerative problem can be described using the CEO recursion, then there should be a way to 'quantize' the spectral curve to produce an operator that annihilates a certain specialization of the partition function for the enumerative problem. In the case of completed Hurwitz numbers, we prove that the spectral curve from Conjecture 1.10 can be quantized in such a way that the resulting operator annihilates this specialization of the partition function, simultaneously providing some evidence for the r-Bouchard-Mariño conjecture and the Gukov-Sułkowski conjecture.

Let $Z^{(r)}$ be the following generating function for single Hurwitz numbers with completed $(r+1)$-cycles:

$$Z^{(r)}(\lambda, p_1, p_2, \ldots) := \exp\left(\sum_{g,\mu} h_{g,\mu}^{(r)} \lambda^{2g-2+\ell(\mu)} p_{\mu_1} \cdots p_{\mu_{\ell(\mu)}} \right). \tag{1.53}$$

Remark 1.11. In the generating function above, the parameter λ keeps track of the Euler characteristic of the curve. We will also sometimes use parameters $\hbar$ or g_{s} for this purpose. In that case they will appear with exponents $\hbar^{g-1}$ and g_{s}^{2g-2}. We abuse notation and denote all three generating functions by $Z^{(r)}$.

Theorem 1.12. *Denote by*

$$Z^{(r)}(\lambda, x) = Z^{(r)}(\lambda, p_1, p_2, \ldots)|_{p_i \to x^i} \tag{1.54}$$

the principal specialization of $Z^{(r)}$. Then we have the following Schroedinger type equation:

$$\left(\lambda x \frac{\mathrm{d}}{\mathrm{d}x} - x^{\frac{3}{2}} \exp\left(\frac{\sum_{i=0}^{r} x^{-1} (\lambda x \frac{\mathrm{d}}{\mathrm{d}x})^i x (\lambda x \frac{\mathrm{d}}{\mathrm{d}x})^{r-i}}{r+1} \right) x^{-\frac{1}{2}} \right) Z^{(r)}(\lambda, x) = 0. \tag{1.55}$$

Furthermore, the operator in front of $Z^{(r)}$ in this equation is a quantization of the spectral curve (1.50) *in some appropriate sense.*

The precise sense in which the operator in Equation (1.55) is a quantization of the spectral curve (1.50) is explained in Chapter 3. One interesting fact is that is one of the first cases of the Gukov-Sułkowski conjecture where one has to use non-normal ordering for the operators in the quantization.

In fact, we prove a similar theorem for the so-called q-double Hurwitz numbers. In that case the spectral curve is known ([9, 26]), and we show that the quantization, once again using non-normal ordering for the operators, annihilates the principal specialization of their generating function.

The final piece of evidence for the r-Bouchard-Mariño conjecture is provided in Chapter 8, where we prove that it is equivalent to a generalization of the ELSV formula relating completed Hurwitz numbers with linear Hodge integrals on the moduli space of r-spin curves. Thus, the evidence for this conjecture (in the form of computer calculation, and proofs in small genus cases) also contributes to the evidence for the r-BM conjecture.

1.6.3 Integrals of ψ-classes over double ramifcation cycles.

In Chapter 4, I define so-called *double-ramification cycles*, which intuitively are classes on $\overline{\mathcal{M}}_{g,n}$ consisting of those curves on which a covering to $\mathbb{P}^1$ can be defined with some specified ramification over 0 and ∞. Thus, for a list of integers $a_1, \ldots, a_n$ satisfying $\sum a_i = 0$, the double ramification cycles $\mathrm{DR}_g(a_1, \ldots, a_n)$ is the class of curves in $\overline{\mathcal{M}}_{g,n}$ for which there exist a covering of $\mathbb{P}^1$ by that curve such that the ramification over 0 is of the type given by all the positive a_i and whose ramification over ∞ is of the type given by the absolute values of all the negative a_i.

Although it is known that these classes can be expressed in terms of the ψ, κ and boundary classes described above, there is no known explicit expression of this type. Here, we give a partial result towards this answer which is interesting in its own right; that is, we compute the intersection of any double-ramification cycles with any monomial in ψ-classes when the result is zero-dimensional.

Theorem 1.13. *Given a list of n integers $a_1, \ldots, a_n$, satisfying $\sum a_i = 0$ and a list of non-negative integers $d_1, \ldots, d_n$ satisfying $\sum d_i = 2g - 3 + n$, the integral*

$$\mathrm{DR}_g(a_1, \cdots a_n)\psi_1^{d_1} \cdots \psi_n^{d_n} \tag{1.56}$$

of a monomial in ψ-classes over a DR-cycle is equal to the coefficient of

$$z_1^{d_1} \cdots z_n^{d_n} \tag{1.57}$$

in the generating function

$$\frac{z_1 \cdots z_n}{\zeta(z_1 + \cdots + z_n)} \sum_{\substack{\sigma \in S_n \\ \sigma(1)=1}} \frac{\zeta\left(\begin{vmatrix} a_1' & a_2' \\ z_1' & z_2' \end{vmatrix}\right) \zeta\left(\begin{vmatrix} a_1' + a_2' & a_3' \\ z_1' + z_2' & z_3' \end{vmatrix}\right) \cdots \zeta\left(\begin{vmatrix} a_1' + \cdots + a_{n-1}' & a_n' \\ z_1' + \cdots + z_{n-1}' & z_n' \end{vmatrix}\right)}{z_1' \begin{vmatrix} a_1' & a_2' \\ z_1' & z_2' \end{vmatrix} \begin{vmatrix} a_2' & a_3' \\ z_2' & z_3' \end{vmatrix} \cdots \begin{vmatrix} a_{n-1}' & a_n' \\ z_{n-1}' & z_n' \end{vmatrix} z_n'}. \tag{1.58}$$

1.6.4 Correspondence between Givental and CEO theories

In Chapter 5, I describe a correspondence between Givental's action on cohomological field theories and the Chekhov-Eynard-Orantin recursion theory. That is, given a semi-simple conformal cohomological field theory, or equivalently, an operator series $R(z)$ as described in Section 1.4.4 of this introduction, together with a rescaling Δ of the trivial cohomological field theory (it will be properly introduced in Chapter 5), one can define a spectral curve such that the associated partition functions are equal.

Theorem 1.14. *Let $R(z)$ be some operators series on an N-dimensional vector space, as in Section 1.4.4, and let Δ be a rescaling of the trivial N-dimensional cohomological field theory. Let Z be the partition function of the corresponding semi-simple cohomological field theory.*

Define a local spectral curve by the following data

$$[z^{2p}w^{2q}]B^{i,j}(z,w) := \frac{1}{(2p-1)!!(2q-1)!!}[z^p w^q]\frac{\delta^{ij} - \sum_{s=1}^N R^i_s(-z)R(-w)^j_s}{z+w} \tag{1.59}$$

and

$$[z^{2k-1}]y^i(z) := \frac{1}{2(2k-1)!!}[z^{k-1}]\left(-R(-z))^i_{\mathbf{1}}\right) \quad (k>1) \tag{1.60}$$

$$[z^1]y^i(z) := -\frac{1}{2\sqrt{\Delta^i}}. \tag{1.61}$$

Then after an appropriate change of variables (that is introduced explicitly in Chapter 5) the partition function of the cohomological field theory and the CEO recursion invariants agree in the following sense:

$$Z = \exp\left(\sum_{g,n}\hbar\omega_{g,n}\right). \tag{1.62}$$

Remark 1.15. Note that in the theorem we only define the even coefficients of the the expansion of B, and only the odd coefficients of the expansion of y. It turns out to be a general property of the CEO invariants $\omega_{g,n}$ that they do not depend on the full data of the spectral curve and two-point differential, but only on those coefficients.

The proof of this theorem is combinatorial; in fact, we express both partition functions as a sum over the same set of decorated graphs, where each component of the a graph (vertex, edge, leaf) contributes in a standard way. Then we show that, using the identification of Theorem 1.14, the contribution of each individual component is equal in both partition functions.

1.6.5 CEO-recusrion for the Gromov-Witten theory of $\mathbb{P}^1$

The first application of Theorem 1.14 is a proof of the Norbury-Scott conjecture [84] in Chapter 6. That is, we describe a spectral curve whose CEO invariants completely describe the stationary Gromov-Witten invariants of $\mathbb{P}^1$. Here stationary means that we always pull-back the class of the point from $\mathbb{P}^1$.

Theorem 1.16. *Let $\omega_{g,n}$ be the CEO invariants associated to the spectral curve*

$$\begin{cases} x &= z + \frac{1}{z}; \\ y &= \log z, \end{cases} \tag{1.63}$$

$$B(z,w) = \frac{\mathrm{d}z \otimes \mathrm{d}w}{(z-w)^2} \tag{1.64}$$

Then, for $2g-2+n>0$ and $a_1,\dots,a_n \geq 0$, we have:

$$\prod_{j=1}^n \left(-\operatorname*{res}_{x_j=\infty}\frac{1}{(a_j+1)!}x_j^{a_j+1}\right)\omega_{g,n}(x_1,\dots,x_n) = \left\langle \prod_{j=1}^n \mathrm{ev}_j^*(\mathrm{pt})\psi_j^{a_j}\right\rangle_g, \tag{1.65}$$

where pt *is the class of the point on $\mathbb{P}^1$, and $\langle\cdot\rangle_g$ denotes the integral on the moduli space of stable genus g maps to $\mathbb{P}^1$.*

1.6.6 Equivalence of ELSV and Bouchard-Mariño; a new proof of the ELSV formula

The second application of Theorem 1.14 is a proof of the equivalence of the Bouchard-Mariño conjecture and the ELSV formula in Chapter 7. Together with a new proof of the Bouchard-Mariño conjecture that does not use the ELSV formula, this is also a new proof of the ELSV formula.

Theorem 1.17. *Let $\omega_{g,n}$ be the correlation functions associated to the spectral curve*

$$\begin{cases} x(z) &=& -z+\log z \\ y(z) &=& z \end{cases} \tag{1.66}$$

$$B(z_1,z_2)=\frac{\mathrm{d}z_1\otimes\mathrm{d}z_2}{(z_1-z_2)^2}. \tag{1.67}$$

Then we have

$$DH_{g,n}(x_1,\dots,x_n)=\omega(x_1,\dots,x_n) \tag{1.68}$$

where D is the operator defined in equation (1.49) *and $H_{g,n}$ is the generating function for single Hurwitz numbers:*

$$H_{g,n}(x_1,\dots,x_n):=\sum_{\mu,\ell(\mu)=n}\frac{h_{g,\mu}}{(2g-2+\ell(\mu)+|\mu|)!}\exp(\mu_1x_1+\cdots+\mu_nx_n). \tag{1.69}$$

Theorem 1.18. *The Bouchard Mariño conjecture (Theorem 1.17) is equivalent to the ELSV formula* (1.7).

In fact, as an intermediate step in the new proof of the Bouchard-Mariño conjecture, we prove that single Hurwitz numbers have a quasi-polynomial behaviour that is immediately obvious from the ELSV formula, but which had not been proved without the use of that formula.

Theorem 1.19. *Single Hurwitz numbers depend quasi-polynomially on the ramification data in the following sense:*

$$h^{\circ}_{g;\mu_1,\dots,\mu_n}=m!\left(\prod_{i=1}^{n}\frac{\mu_i^{\mu_i}}{\mu_i!}\right)P_{g,n}(\mu_1,\dots,\mu_n), \tag{1.70}$$

where $P_{g,n}(\mu_1,\dots,\mu_n)$ are some polynomials in $\mu_1,\dots,\mu_n$.

1.6.7 Equivalence of r-ELSV and r-BM conjectures

The final application of Theorem 1.14 in this thesis is a generalization of the previous one, which is the proof (mentioned earlier) of the equivalence of the r-spin variants of the Bouchard-Mariño conjecture and ELSV formula. This is the content of Chapter 8.

Theorem 1.20. *Introduce the following integrals on the space of r-spin structures:*

$$f^{(r)}_{g;k_1,\dots,k_n}=m!\,r^{m+n+2g-2}\prod_{i=1}^{n}\frac{\left(\frac{k_i}{r}\right)^{p_i}}{p_i!}\times\int_{\overline{\mathcal{M}}^{1/r}_{g,n;a_1,\dots,a_n}}\frac{\mathcal{S}}{\left(1-\frac{k_1}{r}\psi_1\right)\dots\left(1-\frac{k_n}{r}\psi_n\right)}. \tag{1.71}$$

and let $F^{(r)}_{g,n}$ be their generating function:

$$F^{(r)}_{g,n}(x_1,\ldots,x_n) := \sum_{\mu,\ell(\mu)=n} \frac{f^{(r)}_{g,\mu}}{m!} \exp(\mu_1 x_1 + \cdots + \mu_n x_n) \tag{1.72}$$

Let $\omega^{(r)}_{g,n}$ be the correlators of the spectral curve

$$\begin{cases} x(z) &= -z^r + \log z \\ y(z) &= z \end{cases} \tag{1.73}$$

$$B(z_1,z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1-z_2)^2} \tag{1.74}$$

that appears in Conjecture 1.10. Then we have

$$DF^{(r)}_{g,n}(x_1,\ldots,x_n) = \omega^{(r)}_{g,n}(x_1,\ldots,x_n). \tag{1.75}$$

Note that this theorem implies that Conjecture 1.10 is equivalent to the equation

$$f^{(r)}_{g;k_1,\ldots,k_n} = h^{(r)}_{g,k_1,\ldots,k_n} \tag{1.76}$$

which is called the r-ELSV formula. Indeed, in the case $r=1$ it reduces to the ordinary ELSV formula (1.7).