Hurwitz numbers, moduli of curves, topological recursion, Givental's theory and their relations

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–3– CEO recursion for Hurwitz numbers with completed cycles

3.1 Introduction

In this chapter we introduce the r-spin variant of the Bouchard-Mariño conjecture (the r-BM conjecture), stating that the spectral curve for Hurwitz numbers with completed $(r+1)$ cycles is the r-Lambert curve $x = \log(y) - y^r$, and we provide three pieces of evidence for this conjecture. A fourth piece of evidence is discussed in Chapter 8.

First is an attempt at a direct proof using matrix models, generalizing the direct proof of the Bouchard-Mariño conjecture in [8]. Unfortunately, the proof in [8] turned out not to be completely rigorous; thus, our generalization should be viewed as a strong indication in favour of the r-BM conjecture. More precisely, we prove Theorem 3.3, which states that Hurwitz numbers with completed $(r+1)$-cycles are given by a matrix model. Then we argue that the spectral curve corresponding to this matrix model should be the one described by the r-BM conjecture, and we show where there are still gaps in the proof of this second statement.

Second is the general idea that the spectral curve for an enumerative problem should be given by its $(0,1)$-geometry, as discussed in [30]. In the case of completed Hurwitz numbers, this leads us immediately to the r-BM conjecture.

For the third piece of evidence, we turn to a conjecture from physics [3, 23, 24, 25, 53]. That is, when we have some invariants $\omega_{g,n}$ coming from the CEO-recursion for a genus zero spectral curve, it is conjectured that the following holds.

- There exists a unique procedure to calculate the canonical primitive functions of the symmetric differential forms that are obtained by the CEO-recursion.

- The *partition function* of the theory, which is the exponential generating function of the *principal specialization* of these primitive functions, satisfies a holonomic system generated by a single stationary Schrödinger operator.

- Moreover, the total symbol of the holonomic system defines a Lagrangian subvariety immersed into the cotangent bundle of $\mathbb{C}^*$, which is exactly the same as the realization of the spectral curve as a plane curve.

- In other words, the spectral curve and its immersion as a Lagrangian into the cotangent bundle are recovered from the semi-classical limit of the Schrödinger equation.

In the physics literature cited above, this Schrödinger operator is called a *quantum curve*. It is the Weyl quantization of the defining equation of the spectral curve in the cotangent bungle. Mathematical proofs of this conjecture for a few simple cases have been established in [81].

In the case of simple Hurwitz numbers, whose spectral curve is the Lambert curve $x = \log(y) - y$ ([10, 8, 35, 33]), Zhou has shown [110] the existence of the quantum curve, quantizing in a proper way the equation of the Lambert curve (see also [81]).

Here, we show that a proper quantization of the r-Lambert curve indeed annihilates the principal specialization of the partition function for Hurwitz numbers with completed cycles, simultaneously providing evidence for the conjecture described above and the r-BM conjecture.

Furthermore, we show that the conjecture also holds for so-called q-double Hurwitz numbers (described below), as well as for the mixed case of q-double Hurwitz numbers with completed $(r+1)$-cycles.

Together, these already provide a good indication that the r-BM conjecture should be true. In Chapter 8, we provide a final piece of evidence, showing that the r-BM conjecture is equivalent to the r-ELSV conjecture, which has independent evidence of its own.

3.1.1 Plan of the chapter.

We start by stating once again the r-BM conjecture in Section 3.2. For the first part of the evidence, we state and prove Theorem 3.3 in Section 3.3, providing a matrix model for completed Hurwitz numbers. In Section 3.4, we indicate why it should be expected that the spectral curve for this matrix model is the one specified by the r-BM conjecture.

In Section 3.5, we collect the data of the spectral curves for the different types of Hurwitz numbers, as well as their quantizations.

For the second and third parts of the evidence, in each of the following three sections we (a) introduce a particular generalization of Hurwitz numbers; (b) derive the formula for the principal specialization of their partition function; (c) identify the formula for the spectral curve using the $(0,1)$-geometry; and (d) prove the existence of the quantum curve, or the stationary Schrödinger equation. The q-double Hurwitz numbers are studied in Section 3.6, the r-spin Hurwitz numbers in Section 3.7, and finally in Section 3.8 we prove the results for the mixed case.

3.2 The r-Bouchard-Mariño conjecture

We repeat the statement of the r-Bouchard-Mariño conjecture.

Let $H^{(r)}_{g,n}$ be n-point generating function of genus g for Hurwitz numbers with completed $(r+1)$-cycles as defined in Chapter 1:

$$H^{(r)}_{g,n}(x_1, \dots, x_n) = \sum_{k_1,\dots,k_n} \frac{h^{(r)}_{g;k_1,\dots,k_n}}{m!} \exp(k_1 x_1 + \dots + k_n x_n). \tag{3.1}$$

Remark 3.1. In this chapter, we use a different convention for the definition of the completed $(r+1)$-cycle from what we did in Chapter 2; the completed $(r+1)$-cycle here is $r!$ times the completed $(r+1)$ cycle from that chapter. To avoid too cluttered notation, we still denote Hurwitz numbers with completed cycles in the same way. See also Remark 2.5. If we would use the definition from Chapter 2, Equation (3.3) would have $x(z) = -\frac{z^r}{r!} + \log(z)$.

Also, for any function $f(x_1, \dots, x_n)$, let

$$Df = \frac{\partial^n f}{\partial x_1 \cdots \partial x_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{3.2}$$

Then we conjecture the following generalization of the Bouchard-Mariño conjecture.

Conjecture 3.2 (r-BM)**.** *Let $\omega_{g,n}$ be the n-point correlation forms associated by the CEO-recursion to the plane curve*

$$\begin{cases} x(z) &= -z^r + \log z \\ y(z) &= z \end{cases} \tag{3.3}$$

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}. \tag{3.4}$$

Then we have

$$DH_{g,r}(x_1, \ldots, x_n) = \omega_{g,n}. \tag{3.5}$$

3.3 Completed Hurwitz numbers as a matrix model

In this section we state and prove a theorem, representing the partition function for completed Hurwitz numbers as a matrix integral, that is shown in Section 3.4 to lead to strong evidence for the r-BM conjecture. Both the theorem presented in this section and the resulting evidence in the next section are based on a direct generalization of one of the proofs of the original Bouchard-Mariño conjecture in [8]. Unfortunately, it seems that this proof is not completely rigorous, meaning that we can only present the generalization to completed Hurwitz numbers as evidence for the r-BM conjecture.

3.3.1 The statement

Let Z be the partition function for Hurwitz numbers with completed $(r+1)$-cycles:

$$Z := \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2} \sum_{\mu} \frac{h^{(r)}_{g;\mu}}{m!} p_\mu \right). \tag{3.6}$$

The character formula for not necessarily connected Hurwitz numbers allows us to write

$$Z(\mathbf{p}, g_s; t) = \sum_{K=0}^{\infty} t^K \sum_{|\mu|=K} \sum_{m=0}^{\infty} \frac{g_s^{rm-K-l(\mu)}}{m!} p_\mu \times \sum_{|\lambda|=K} \left(\frac{\dim(\lambda)}{K!} \right)^2 \frac{|C_\mu| \chi_\lambda(\mu)}{\dim(\lambda)} \left(\frac{\mathbf{p}_{r+1}(\lambda)}{(r+1)} \right)^m. \tag{3.7}$$

Here $rm - K - |\mu|$ is equal to the Euler characteristic of the curve (by the Riemann-Hurwitz formula), λ and μ are partitions of K encoding an irreducible representation and a conjugacy class C_μ respectively, and $p_\mu = p_{\mu_1} \cdots p_{\mu_n}$ for $\mu = (\mu_1, \ldots, \mu_n)$. For every μ the coefficient of p_μ in this expression is a formal Laurent series in g_s with a finite number of negative degree terms.

Note that we have inserted an extra formal variable t to encode the degree of the covering. It is redundant, since the degree can also be recovered from the total degree in $\mathbf{p}$, but turns out to be convenient later on.

Fix a positive integer N. We use the following substitution for the variables p_k, $k = 1, 2, \ldots,$ as symmetric functions:

$$p_k = g_s \sum_{i=1}^{N} v_i^k. \tag{3.8}$$

Moreover, introduce the N-tuple of variables $\mathbf{v} := (v_1, \ldots, v_N)$ and the diagonal matrix of their formal logarithms $\mathbf{R} := \mathrm{diag}(\log v_1, \ldots \log v_N)$. We use Δ to denote the Vandermonde determinant:

$$\Delta(\mathbf{v}) := \prod_{1\le j<i\le N} (v_i - v_j), \tag{3.9}$$

and similarly for $\Delta(\mathbf{R})$, where we take the Vandermonde determinant of its diagonal entries.

Let B_k be the Bernoulli numbers, and introduce the following functions depending also on the parameter N:

$$\begin{aligned} A_{r+1}(x) = \sum_{k=0}^{r+1} \Bigg(& r! \frac{(-N+\frac{1}{2})^k}{k!} \frac{x^{r+1-k}}{(r+1-k)!} \\ & + (-1)^{r+1} r! \frac{(-1)^k B_k}{k!} \frac{(2^{k-1}-1)}{2^{k-1}} \frac{N^{r+1-k}}{(r+2-k)!} \Bigg); \end{aligned} \tag{3.10}$$

$$\begin{aligned} V(x) = & - g_s^{r+1} A_{r+1}(x/g_s) + g_s \log(g_s/t) A_1(x/g_s) \\ & + \mathrm{I}\pi x - g_s \log(\Gamma(-x/g_s)) + \mathrm{I}\pi g_s. \end{aligned} \tag{3.11}$$

The function A_{r+1} is just a function of x, whereas V is a function of x that also depends on the variables g_s and t that live on $\mathbb{C}\backslash(-\infty, 0)$. These functions originate from the combinatorics of Young diagrams, their meaning will be explained later in this section.

Let $\mathcal{C}_D$ be a fixed contour in the complex plane that goes around the integers h with $0 \le h \le D$. Let $\mathcal{H}_N(\mathcal{C}_D)$ be the space of N by N *normal matrices* M with eigenvalues in $\mathcal{C}_D$. In other words, $M \in \mathcal{H}_N(\mathcal{C}_D)$ if and only if M can be diagonalized by conjugation with a unitary matrix, and its eigenvalues belong to $\mathcal{C}_D$:

$$M = U^\dagger X U, \quad U \in U(N), \quad X = \mathrm{diag}(x_1, \dots, x_N), \quad x_i \in \mathcal{C}_D. \tag{3.12}$$

We use the following measure on $\mathcal{H}_N(\mathcal{C}_D)$:

$$\mathrm{d}M = \Delta(X)^2 \mathrm{d}X \mathrm{d}U, \tag{3.13}$$

where $\mathrm{d}U$ is the Haar measure on $U(N)$ and $\mathrm{d}X$ is the product of Lebesgue curvilinear measures along $\mathcal{C}_D$.

Now we formulate the main theorem of this section.

Theorem 3.3. *We have:*

$$Z(\mathbf{p}, g_s; t) \sim \frac{g_s^{-N^2}}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \int_{\mathcal{H}_N(\mathcal{C}_D)} \mathrm{d}M e^{-\frac{1}{g_s}\mathrm{Tr}(V(M) - M\mathbf{R})}. \tag{3.14}$$

Notice that the left-hand side is a formal Laurent series in $\mathbf{p}, g_s, t$, whereas the right-hand side is a meromorphic function of those variables, defined for t and g_s on the domain $\mathbb{C}\backslash(-\infty, 0)$, that also depends on two parameters N and D. The symbol $\sim$ means that for any K, the coefficient of t^K on the left-hand side is given by the coefficient of t^K in the series expansion around $t = 0$ of the function on the right-hand side for any choice of the parameters such that $N > K$ and $D > K + \frac{N-1}{2}$.

Remark 3.4. The form of Theorem 3.3 differs from that of the analogous statement in [8]; here the contour is around a finite set of integers, whereas in [8] it goes around all non-negative integers. According to our understanding, the contour should be finite in both cases, since the integral over the infinite contour does not converge to a meromorphic function, which makes it impossible to have an expansion for it in powers of t. See Remark 3.7 for a more precise discussion of the origin of this problem. Note that this is one of the reasons we are not able to convert the evidence in the next section into a formal theorem (see Section 3.4.1).

The proof of this theorem occupies the rest of this section.

3.3.2 Schur polynomials

We recollect some facts about the Schur polynomials $s_\lambda(\mathbf{v})$ that can be defined, for a sufficiently large N, by the following formula:

$$s_\lambda(\mathbf{v}) := \frac{\det(v_i^{\lambda_j - j + N})}{\Delta(\mathbf{v})} \ . \tag{3.15}$$

The Schur polynomials are related to representations of the symmetric group (and thus to Hurwitz numbers) by the Frobenius formula

$$s_\lambda(\mathbf{v}) = \frac{1}{n!} \sum_{|\mu|=n} |C_\mu| \chi_\lambda(C_\mu) \tilde{p}_\mu \quad (\text{where } \tilde{p}_m = \sum_{i=1}^{\ell(\mu)} v_i^m \ ) \ . \tag{3.16}$$

There is an expression for s_λ in terms of the Itzykson-Zuber integral (see [56, 8])

$$I(X,Y) := \int_{U(N)} \mathrm{d}U e^{\mathrm{Tr}(XUYU^\dagger)} = \frac{\det(e^{x_i y_j})}{\Delta(X)\Delta(Y)}, \tag{3.17}$$

where $\mathrm{d}U$ is the Haar measure on $U(N)$, normalized according to the second equality. Denote by $\mathbf{h}_\lambda$ the diagonal matrix $\mathrm{diag}(h_1 \dots h_N)$, where $h_i = \lambda_i - i + N$, and by $\Delta(\mathbf{h}_\lambda)$ the Vandermonde determinant of its diagonal entries. Then we have:

$$s_\lambda(\mathbf{v}) = \Delta(\mathbf{h}_\lambda) \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} I(\mathbf{h}_\lambda, \mathbf{R}). \tag{3.18}$$

3.3.3 Partition function in terms of the Itzykson-Zuber integral

First, we rearrange the partition function for completed Hurwitz numbers in the following way:

$$\begin{aligned} Z(\mathbf{p}, g_s; t) &= \sum_{K=0}^{\infty} t^K \sum_{m=0}^{\infty} \frac{g_s^{mr-K}}{m!} \sum_{\substack{|\lambda|=K \\ \ell(\lambda)\le N}} s_\lambda(\mathbf{v}) \frac{\dim(\lambda)}{|\lambda|!} \left(\frac{\mathbf{p}_{r+1}(\lambda)}{r+1} \right)^m \\ &= \sum_{\ell(\lambda)\le N} \left(\frac{t}{g_s} \right)^{|\lambda|} \frac{\dim(\lambda)}{|\lambda|!} s_\lambda(\mathbf{v}) e^{g_s^r \frac{\mathbf{p}_{r+1}(\lambda)}{r+1}} . \end{aligned} \tag{3.19}$$

Here we use the interpretation of p_k, $k = 1, 2, \dots,$ as symmetric functions in v_i, $1 \le i \le N$. Furthermore, the formula above should be interpreted order by order in powers of t; for any given power K of t, the formula is true for N larger than K. We should keep this interpretation in mind throughout the rest of the computations.

Suppose that we have found functions $A_{r+1}(x)$ such that

$$\sum_{i=1}^{N} A_{r+1}(h_i) = \frac{\mathbf{p}_{r+1}(\lambda)}{r+1}, \tag{3.20}$$

so that in particular

$$\sum_{i=1}^{N} A_1(h_i) = |\lambda|. \tag{3.21}$$

Then, applying Equation (3.18) and the equality

$$\frac{\dim(\lambda)}{|\lambda|!} = \frac{\Delta(\mathbf{h})}{\prod_{i=1}^{N} h_i!} \quad \text{for} \quad N \geq \ell(\lambda), \tag{3.22}$$

we get the following:

$$\begin{aligned} Z(\mathbf{p}, g_s; t) &= \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{\lambda} \left(\frac{t}{g_s}\right)^{|\lambda|} I(\mathbf{h}_\lambda, \mathbf{R}) \frac{(\Delta(\mathbf{h}_\lambda))^2}{\prod_{i=1}^{N} h_i!} \prod_{i=1}^{N} e^{g_s^r A_{r+1}(h_i)} \\ &= \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{h_1 > \cdots > h_N \geq 0} I(\mathbf{h}, \mathbf{R}) (\Delta(\mathbf{h}))^2 \prod_{i=1}^{N} \frac{e^{g_s^r A_{r+1}(h_i)} (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i + 1)} \\ &= \frac{1}{N!} \frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{h_1, \ldots, h_N \geq 0} I(\mathbf{h}, \mathbf{R}) (\Delta(\mathbf{h}))^2 \prod_{i=1}^{N} \frac{e^{g_s^r A_{r+1}(h_i)} (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i + 1)}. \end{aligned} \tag{3.23}$$

Remark 3.5. Note that we should be careful when writing $(g_s/t)^{A_1(h)}$, since this might introduce non-integer powers of the formal variables g_s and t. In fact, by Equation (3.27), we obtain half-integer powers. However, it is clear that in Equation (3.23) they will cancel in the full product over i.

3.3.4 Computation of A_{r+1}

In this section we compute explicitly the polynomials A_{r+1} using Equation (3.20) as a definition. The result will coincide with Equation (3.10).

We have

$$\begin{aligned} \mathbf{p}_{r+1}(\lambda) &= \sum_{i=1}^{N} \left((\lambda_i - i + \frac{1}{2})^{r+1} - (-i + \frac{1}{2})^{r+1} \right) \\ &= \sum_{i=1}^{N} \left((h_i - N + \frac{1}{2})^{r+1} - (-i + \frac{1}{2})^{r+1} \right) \\ &= \sum_{i=1}^{N} \sum_{k=0}^{r+1} \binom{r+1}{k} (-N + \frac{1}{2})^k h_i^{r+1-k} - \sum_{j=1}^{N} \left(\frac{-2j+1}{2} \right)^{r+1}. \end{aligned} \tag{3.24}$$

The second term can be represented in the following form:

$$\begin{aligned} \sum_{j=1}^{N} \left(\frac{-2j+1}{2} \right)^{r+1} &= \frac{1}{(-2)^{r+1}} \left(\sum_{j=1}^{2N-1} j^{r+1} - \sum_{k=1}^{N-1} (2k)^{r+1} \right) \\ &= \frac{(-1)^{r+1}}{r+2} \sum_{k=0}^{r+1} \binom{r+2}{k} (-1)^k B_k \left(\frac{1}{2^{r+1}} (2N)^{r+2-k} - N^{r+2-k} \right) \\ &= \frac{(-1)^r}{r+2} \sum_{k=0}^{r+1} \binom{r+2}{k} (-1)^k B_k \left(\frac{(2^{k-1} - 1) N^{r+2-k}}{2^{k-1}} \right) \end{aligned} \tag{3.25}$$

(here $B_k := B_k(0)$, $k = 0, 1, \ldots,$ are the Bernoulli numbers).

Thus we have the following formula for A_{r+1}:

$$A_{r+1}(x) = \sum_{k=0}^{r+1} \Bigg(\binom{r+1}{k} (-N + \frac{1}{2})^k \frac{x^{r+1-k}}{r+1} + \frac{(-1)^{r+1}}{(r+2)(r+1)} \binom{r+2}{k} (-1)^k B_k \frac{(2^{k-1}-1)N^{r+1-k}}{2^{k-1}} \Bigg)$$
$$= \sum_{k=0}^{r+1} \Bigg(r! \frac{(-N+\frac{1}{2})^k}{k!} \frac{x^{r+1-k}}{(r+1-k)!} + (-1)^{r+1} r! \frac{(-1)^k B_k}{k!} \frac{(2^{k-1}-1)}{2^{k-1}} \frac{N^{r+1-k}}{(r+2-k)!} \Bigg) \tag{3.26}$$

in agreement with (3.10). In particular, we have

$$A_1(x) = x - \frac{N-1}{2}. \tag{3.27}$$

3.3.5 Contour integral

We now replace the N sums in Equation (3.23) for the partition function by integrals over a contour $\mathcal{C}_D$ enclosing the non-negative integers less than or equal to D. For that we use a function which has simple poles with residue 1 at all integers:

$$f(\xi) := \frac{\pi e^{-\mathrm{I}\pi\xi}}{\sin(\pi\xi)} = -\Gamma(\xi+1)\Gamma(-\xi)e^{-\mathrm{I}\pi\xi}. \tag{3.28}$$

Note that for any K, only finitely many terms of the sum in (3.23) contribute to the coefficient of t^K. Thus, when we want to compute any such coefficient, we can replace the sum by a finite one:

$$[t^K]Z(\mathbf{p}, g_s; t) = [t^K]Z_D(\mathbf{p}, g_s; t) := [t^K]\frac{1}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{h_1,\dots,h_N=0}^{D} I(\mathbf{h},\mathbf{R})(\Delta(\mathbf{h}))^2 \prod_{i=1}^{N} \frac{e^{g_s^r A_{r+1}(h_i)}(g_s/t)^{-A_1(h_i)}}{\Gamma(h_i+1)},$$

which is true as long as $D \geq K + \frac{N-1}{2}$.

Remark 3.6. While Z is a Laurent series that does not converge to a function, the truncated series Z_D obviously does converge to a meromorphic function with domain $\mathbb{C}$ for all the variables, since it is a finite sum of such functions.

Using the function f defined in equation (3.28) we can rewrite the function Z_D in terms of residues, if we restrict the domain of g_s and t to $\mathbb{C}\backslash(-\infty, 0)$:

$$Z_D(\mathbf{p}, g_s; t) = \frac{1}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})} \sum_{h_1,\dots,h_N=0}^{D} \operatorname*{res}_{z_1\to h_1} \cdots \operatorname*{res}_{z_N\to h_N} I(\mathbf{z},\mathbf{R})(\Delta(\mathbf{z}))^2 \prod_{i=1}^{N} \frac{f(z_i)e^{g_s^r A_{r+1}(z_i)}(g_s/t)^{-A_1(z_i)}}{\Gamma(z_i+1)} \tag{3.29}$$

On the right-hand side, $(g_s/t)^{-A_1(z)}$ is defined as $\exp(-A_1(x)\log(g_s/t))$, which requires a choice of branch of the logarithm (one can see that the end result does not depend on this choice), and explains the change in domain. Here, it is important that g_s and t are no longer just formal variables, but the arguments of a function.

Finally, the sum over residues can be replaced by a contour integral:

$$Z_D(\mathbf{p}, g_s; t) = \frac{1}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\oint_{\mathcal{C}_D^N} \mathrm{d}h_1 \cdots \mathrm{d}h_N (\Delta(\mathbf{h}))^2 I(\mathbf{h}, \mathbf{R}) \prod_{i=1}^{N} \frac{f(h_i) e^{g_s^r A_{r+1}(h_i)} (g_s/t)^{-A_1(h_i)}}{\Gamma(h_i+1)}. \tag{3.30}$$

Remark 3.7. Equation (3.30) is an equality of functions, and the function on the left-hand side is defined as a (converging) series in t, implying that the function on the right has the same series expansion at $t = 0$. Note that we have to work with Z_D because it is not possible to write a formula like (3.29) for Z, since it is only a formal series, and does not converge to any function. Another way to see this is that the integral around all non-negative integers does not converge, so that it is meaningless to take the coefficient of t^K in that integral.

Rescaling the integration variables $h_i \to h_i/g_s$, we get:

$$Z_D(\mathbf{p}, g_s; t) = \frac{g_s^{-N^2}}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\oint_{\mathcal{C}_D^N} \mathrm{d}h_1 \cdots \mathrm{d}h_N (\Delta(\mathbf{h}))^2 \times I\left(\frac{\mathbf{h}}{g_s}, \mathbf{R}\right) \prod_{i=1}^{N} -\Gamma\left(-\frac{h_i}{g_s}\right) e^{g_s^r A_{r+1}(\frac{h_i}{g_s}) - \frac{I\pi h_i}{g_s}} \left(\frac{g_s}{t}\right)^{-A_1(\frac{h_i}{g_s})}. \tag{3.31}$$

3.3.6 Normal matrices and final formula

As it is done in [8], we now replace the integration along the N copies of the contour $\mathcal{C}_D$ by integration over the space $\mathcal{H}_N(\mathcal{C}_D)$ of N by N normal matrices with eigenvalues in $\mathcal{C}_D$. We get

$$Z_D(\mathbf{p}, g_s; t)) = \lim_{N\to\infty} \frac{1}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\int_{\mathcal{H}_N(\mathcal{C}_\mathcal{D})} \mathrm{d}M e^{-\mathrm{Tr}V(M) + \mathrm{Tr}(M\mathbf{R})}, \tag{3.32}$$

where V is as in formula(3.11):

$$V(\xi) = -g_s^r A_{r+1}(\frac{\xi}{g_s}) + g_s \log(\frac{g_s}{t}) A_1(\frac{\xi}{g_s}) + \mathrm{I}\pi\xi - g_s \log(\Gamma(-\frac{\xi}{g_s})) + \mathrm{I}\pi g_s.$$

In particular, it implies

$$Z(\mathbf{p}, g_s; t) \sim \frac{g_s^{-N^2}}{N!}\frac{\Delta(\mathbf{R})}{\Delta(\mathbf{v})}\int_{\mathcal{H}_N(\mathcal{C}_\mathcal{D})} \mathrm{d}M e^{-\frac{1}{g_s}\mathrm{Tr}(V(M) - M\mathbf{R})}, \tag{3.33}$$

concluding the proof of Theorem 3.3.

3.4 Spectral curve for the r-spin Hurwitz matrix model

3.4.1 Spectral curve associated to a matrix model

In [8], a “physics proof” of the Bouchard-Mariño conjecture is given by first representing the generating function for Hurwitz numbers as a matrix model, and then showing that the spectral curve for this matrix model is equal to the one predicted by the Bouchard-Mariño conjecture.

We generalized the first part of this proof in the previous section, where we showed that the generating function for completed Hurwitz numbers is given by the matrix model (3.14).

Unfortunately, it seems that the reasoning in [8] does not constitute a precise mathematical proof, nor can it easily be made into one. On the other hand, all the reasoning in [8] generalizes directly to the case of completed Hurwitz numbers. Thus, if a way could be found to transform this into a rigorous mathematical proof, it would immediately prove the r-BM conjecture.

Remark 3.8. There is another proof of the Bouchard-Mariño conjecture in [35], but it is based on the ELSV formula, so it is not useful for our purposes.

In the rest of this section, we briefly describe the steps taken in [8] and how they generalize to completed Hurwitz numbers, and we note the places were we believe the reasoning is not mathematically rigorous. Since all the steps in [8] generalize directly to our case, we do not repeat the detailed steps of that paper, and just give an overview of the reasoning.

3.4.2 Loop equations and topological expansion

It is a general theme in the theory of matrix models that they can be related to a spectral curve by way of so-called *loop equations*. That is, to any matrix model one can associate a free energy $F = \log Z$ and a tower of n-point correlation functions $W_n(x_1, \ldots, x_n)$, $n \geq 1$. Then, one can ask whether there exists a curve such that those invariants coincide with the symplectic invariants and n-point correlation forms associated to this curve by CEO-recursion.

In general, the answer to this question is given by varying the integration variable in the matrix integral in a specific way. The resulting equations are called the loop equations, and in good situations they imply that the free energy and correlation functions of the matrix model are given by a specific spectral curve.

To derive such loop equations, we need F and W_n to have a Laurent series expansion in powers of g_s with finite tail (this is called the topological expansion property of the matrix model). In particular, this allows us to define invariants F_g and $W_{g,n}$ as the coefficients of powers of g_s in the expansions of F and W_n, which in turn makes it possible to compare them to the symplectic invariants and correlation forms of a spectral curve. In [8], it is shown that the matrix model discussed there has this topological expansion property, and the proof goes through in exactly the same way for our matrix model.

Remark 3.9. In [8], the correlators $W_{g,n}$ are themselves power series in g_s. Because of the triangular nature of the relation between $W_{g,n}$ in that paper and the coefficients of powers of g_s in W_n, it is immediate that those coefficients of powers of g_s are well-defined and they are non-zero only for finitely many negative powers of g_s. This does mean that the powers of g_s in the expansion of W_n do not necessarily increase in steps of 2, but that does not present any problems in the rest of the reasoning.

Loop equations

Given that our matrix model has the topological expansion property, the loop equations are derived as the invariance of the integral under a certain change of variables. In [8], the change

$$M \to M + \epsilon \frac{1}{x - M} \frac{1}{y - R} \tag{3.34}$$

for ϵ small is used, but this does not preserve the property of being a normal matrix, so we prefer the change

$$M \to M + \frac{\epsilon}{2} \frac{1}{x - M} \frac{1}{y - R} + \frac{\epsilon}{2} \frac{1}{y - R} \frac{1}{x - M} \tag{3.35}$$

which does preserve that property. In Appendix D of [36] the spectral curve equation and the CEO-recursion for the correlators are derived from such a change of variables for a matrix model of the form (3.14). However, their derivation depends on the potential $V(x)$ being a rational function of x, independent of g_s, neither of which holds for the matrix model in [8] or the generalization described here. This does not affect their reasoning when deriving the spectral curve, but they really use those properties of V to show that the $W_{g,n}$ indeed obey the CEO-recursive relations associated to that spectral curve.

Furthermore, the invariance of the integral under the change of variables depends on the fact that the domain of integration does not change under this change of variables. When using the infinite contour of [8], this in fact holds, but for the finite contour $\mathcal{C}_D$ (see Remark 3.4), it is not the case. That is, one easily sees that the change (3.35) does not affect the property of a matrix being Hermitian (normal matrix with real eigenvalues), but it sends the space of normal matrices with eigenvalues in $\mathcal{C}_D$ to the space of normal matrices with eigenvalues on some different contour $\widetilde{\mathcal{C}}_D$.

Remark 3.10. If we were integrating over the space of diagonal matrices with eigenvalues in $\mathcal{C}_D$ instead of those that are diagonalizable using unitary matrices, the space would also effectively be invariant under the change of variables, since the integral would only depend on the homotopy type of the contour with respect to the non-negative integers. However, the unitary matrices spoil this symmetry.

Spectral curve for completed Hurwitz numbers

Suppose that we would overcome the problems described above in some way. Then, the loop equations would lead to a spectral curve (depending on g_s) and corresponding topological recursion for the $W_{g,n}$. The proof of Conjecture 3.2 could then be completed as in [8], using the relation between the n-point genus g correlation functions for Hurwitz numbers and the free energy of the matrix model

$$\frac{\partial^n H^{(r)}_{g,n}(R_1,\dots,R_n)}{\partial R_1\cdots\partial R_n} = \frac{1}{g_s^n}\frac{\partial^n F_g}{\partial R_1\cdots\partial R_n}\bigg|_{g_s=0} \tag{3.36}$$

and some properties of the topological recursion theory and its relation to matrix models. Together, those show that the spectral curve for completed Hurwitz numbers is given by $\mathcal{S}^{(r)}_{\mathrm{s}}: x = -y^r + \log y$, concluding the evidence for the spectral curve of the matrix model (3.14).

3.5 Spectral curve and quantization

In the remainder of this chapter we consider two different generalizations of the usual simple Hurwitz numbers. The first are the Hurwitz numbers with completed $(r+1)$-cycles discussed in the previous sections, and one of the main subjects in this thesis. In the remainder of this chapter we denote them by $h^{r,1}_{g,\mu}$; the reason for the extra index 1 in the notation will become clear when we introduce the mixed case.

The other type of Hurwitz numbers we study here are a certain kind of double Hurwitz numbers. They count ramified coverings of $\mathbb{P}^1$ with two special fibers as follows. One of the special fibers has an arbitrary fixed cyclic type of monodromy $\mu = (\mu_1, \dots, \mu_\ell)$, and the other has the cyclic type of monodromy equal to $(q, q, \dots, q)$. All other critical points are assumed to be simple. This type of Hurwitz numbers we call *q-double Hurwitz numbers* and denote by $h^{1,q}_{g,\mu}$. There is a closed formula for these numbers in terms of the so-called Hurwitz-Hodge integrals, see [60]. For $q = 1$ we recover the usual simple Hurwitz numbers.

Finally we also consider the mixed case of the above two generalizations. Geometrically, this is the case of two special fibers, where one has an arbitrary fixed monodromy, the other has the cyclic type of $(q, q, \ldots, q)$, and all other ramifications are the completed $(r+1)$-cycles. We call these numbers *q-double r-spin Hurwitz numbers*, and denote them by $h^{r,q}_{g,\mu}$.

3.5.1 Spectral curves

If we have a partition function Z that is the exponential generating function of the *free energies*, i.e., if Z has an expansion of the form

$$Z = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} \sum_{\ell=1}^{\infty} F_{g,\ell}\right), \tag{3.37}$$

then a natural question is whether we can produce a spectral curve and the other input data of the CEO-recursion procedure so that the ℓ-point differential forms $\omega_{g,\ell}$ determined by the recursion would coincide with the exterior derivatives $\mathrm{d}_1 \cdots \mathrm{d}_\ell F_{g,\ell}$ of the free energies.

We do not have a general answer to this question. If we can find the holonomic system satisfied by Z, then its semi-classical limit gives a spectral curve as a holomorphic Lagrangian subvariety. Another mechanism was proposed in [30]. The idea is that the spectral curve can be obtained via the analysis of the $(0,1)$-geometry, that is, the spectral curve is the Riemann surface (the maximal domain of holomorphy) of the one variable function $F_{0,1}$. This mechanim works for many examples, including simple Hurwitz numbers [30].

Note that in both cases, it is not à priori clear that the ℓ-point differential forms produced from the resulting spectral curve will coincide with the exterior derivatives of the free energies; it appears to be the case in many known examples, but has to be proved in each individual case.

We first examine the latter idea in the case of various generalizations of simple Hurwitz numbers described above. This way we obtain the spectral curves in Table 3.1.

q-Double Hurwitz Numbers	$x = y^{1/q} e^{-y}$
r-Spin Hurwitz Numbers	$x = y e^{-y^r}$
Mixed q-Double r-Spin Hurwitz Numbers	$x = y^{1/q} e^{-y^r}$

Table 3.1: Spectral Curves.

Remark 3.11. In the table above, a change of coordinates $x \to \exp(x)$ occurred compared to the previous section. This does not change the spectral curve itself, but it does change the functions x and y defined on it, so it also changes the correlators $\omega_{g,n}$. Thus, when computing correlators, one should always use the parametrization of the previous section. In this section we use the new coordinates since they make for slightly more attractive formulas. Note that one could also do all the computations in this section in the old coordinates, as long as one uses the quantization $\hat{y} = \lambda \mathrm{d}/\mathrm{d}x$ instead of $\hat{y} = \lambda x \mathrm{d}/\mathrm{d}x$. See also Remark 3.13 and Equation (3.59).

We note that the spectral curve for the case of q-double Hurwitz numbers was recently proved in [9, 26]. The formula for the spectral curve for r-spin Hurwitz numbers is the subject of the r-BM conjecture. The mixed case is so far still conjectural.

3.5.2 Schrödinger equations

The formulas for the spectral curves, even still conjectural for the most general case, give enough input to test the conjecture of the existence of the quantum curves, or the Schrödinger equation for the principal specialization of the partition function. We prove it in all three cases mentioned above, generalizing in this way the result of [110] for simple Hurwitz numbers.

It is worth mentioning that when we apply Weyl quantization, we need to find the correct ordering of the operators. Our guiding principle is the straightforward application of the semi-infinite wedge product formalism of the various Hurwitz numbers, as described in Chapter 2.

The main result of the quantum curves we establish are summarized in the following table.

q-Double Hurwitz Numbers	$\hat{y} - \left(e^{\frac{q-1}{2}\hat{y}}\hat{x}e^{-\frac{q-1}{2}\hat{y}}\right)^q e^{q\hat{y}}$
r-Spin Hurwitz Numbers	$\hat{y} - \hat{x}^{\frac{3}{2}} \exp\left(\frac{\sum_{i=0}^r \hat{x}^{-1}\hat{y}^i\hat{x}\hat{y}^{r-i}}{r+1}\right)\hat{x}^{-\frac{1}{2}}$
Mixed Hurwitz Numbers	$\hat{y} - \hat{x}^{q+1/2}e^{\frac{q}{r+1}\sum_{i=0}^r \hat{x}^{-q}\hat{y}^i\hat{x}^q\hat{y}^{r-1}}\hat{x}^{-1/2}$

Table 3.2: Quantum Curves.

Here the canonical quantization of the coordinate functions x and y are defined by

$$\begin{cases} \hat{x} = x \\ \hat{y} = \lambda x \frac{d}{dx}, \end{cases} \tag{3.38}$$

reflecting the nature of the cotangent bundle $T^*(\mathbb{C}^*)$ and the holomorphic tautological 1-form $y\ \mathrm{d}(\log x)$ on it.

3.6 q-Double Hurwitz numbers

In this section, we study *q-double Hurwitz numbers*. Their geometric definition, mentioned in the previous section, is equivalent ([85, 59]) to the following one in terms of connected (see Definition 2.34) vacuum expectation values in the infinite wedge space.

Definition 3.12. We define the (connected) q-double Hurwitz numbers as

$$h_{g;\mu}^{1,q} := [w_1^2 \cdots w_m^2] \left\langle \prod_{i=1}^{\ell(\mu)} \frac{\alpha_{\mu_i}}{\mu_i} \cdot \prod_{j=1}^{m} \tilde{\mathcal{E}}_0(w_j) \cdot \frac{(\alpha_{-q})^s}{q^s \cdot s!} \right\rangle^\circ , \tag{3.39}$$

where $[w_1^{d_1} \cdots w_n^{d_n}]$ denotes the coefficient of the monomial $w_1^{d_1} \cdots w_n^{d_n}$ in the power series that follows it. Note that $s = |\mu|/q$ is an integer since $|\mu|$ is the degree of the covering, and m is the number of simple ramification points away from 0 and ∞; it is given by the Riemann-Hurwitz formula:

$$m = 2g - 2 + \ell(\mu) + s . \tag{3.40}$$

Note that the Hurwitz numbers defined here differ slightly from those in [59] in that we do not remember the ordering of the branch points over ∞, reflected in the factor $1/s!$. Write

$$F_{g,\ell}^{1,q}(p_1, p_2, \ldots) := \sum_{\mu:\ \ell(\mu)=\ell} \frac{h_{g;\mu}^{1,q}}{m!}\, p_{\mu_1} \cdots p_{\mu_n} \tag{3.41}$$

for the generating series of genus g, q-double Hurwitz numbers whose partition μ has ℓ parts.

The full generating series is given by

$$\log Z^{1,q}(p_1,p_2,\ldots;\lambda) := \sum_{g,\ell} F^{1,q}_{g,\ell}(p_1,p_2,\ldots)\lambda^{2g-2+\ell} \tag{3.42}$$

$$= \sum_{g,\mu} \frac{h^{1,q}_{g;\mu}}{m!}\,\lambda^{2g-2+\ell(\mu)} p_{\mu_1}\cdots p_{\mu_{\ell(\mu)}}$$

$$= \left\langle \exp\left(\sum_{i=1}^{\infty}\frac{\alpha_i p_i}{i\lambda^{i/q}}\right)\exp\left([w^2]\tilde{\mathcal{E}}_0(w)\lambda\right)\exp\left(\frac{\alpha_{-q}}{q}\right)\right\rangle^{\circ}.$$

3.6.1 Spectral curve from $(0,1)$ geometry

To find an equation for the spectral curve, we compute the $(g,n)=(0,1)$ part of the generating function

$$F^{1,q}_{0,1}(\mathbf{p}) = [w_1^2\cdots w_{n-1}^2]\sum_{n=1}^{\infty} p_{nq}\left\langle \frac{\alpha_{nq}}{nq}\cdot\prod_{i=1}^{n-1}\frac{\tilde{\mathcal{E}}_0(w_i)}{(n-1)!}\cdot\frac{\alpha_{-q}^n}{q^n n!}\right\rangle^{\circ}. \tag{3.43}$$

Using the commutation relation (2.20) to commute the operator α_{nq} to the right, we obtain:

$$F^{1,q}_{0,1}(\mathbf{p}) = \sum_{n=1}^{\infty}\frac{(nq)^{n-2}}{n!}p_{nq}. \tag{3.44}$$

We will abuse notation and write $F^{1,q}_{0,1}(x) = F^{1,q}_{0,1}(\mathbf{p})|_{p_i\mapsto x^i}$ for the principal specialization of $F^{1,q}_{0,1}$.

Remark 3.13. Suppose the generating function for these Hurwitz numbers comes from a spectral curve in $\mathbb{C}^2$. Denote by x and y the coordinates on the two copies of $\mathbb{C}$. Then by the topological recursion theory, the one-form $\omega_{0,1}(x) = \mathrm{d}F^{1,q}_{0,1}(x)$ should be equal to $y(x)\mathrm{d}x$. Sometimes, it will be more natural to think of the spectral curve as living in $\mathbb{C}^*\times\mathbb{C}$ or in $(\mathbb{C}^*)^2$. In that case $\omega_{0,1}(x)$ should be equal to $y(x)\frac{\mathrm{d}x}{x}$ or $\log(y)\frac{\mathrm{d}x}{x}$ respectively.

We define an auxiliary function. Let W be the main branch of the Lambert function [21]. It has a power-series expansion around zero with radius of convergence of $1/e$ given by

$$W(z) = -\sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}(-z)^n. \tag{3.45}$$

and has the property that

$$W(z)e^{W(z)} = z. \tag{3.46}$$

Using this definition, we have

$$\omega_{0,1}(x) = \mathrm{d}F^{1,q}_{0,1}(x) = \frac{1}{q}\sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}(qx^q)^n\frac{\mathrm{d}x}{x} = -\frac{1}{q}W(-qx^q)\frac{\mathrm{d}x}{x}, \tag{3.47}$$

where the last equality is true as long as $|x|\le (qe)^{-1/q}$.

Therefore, Remark 3.13 leads us to think of the spectral curve $S^{1,q}$ as living in $\mathbb{C}^*\times\mathbb{C}$, given by the equation

$$S^{1,q}:\ y = -\frac{1}{q}W(-qx^q). \tag{3.48}$$

which can be rewritten to get

$$-qx^q = -qye^{-qy} \quad\Leftrightarrow\quad x = y^{1/q}e^{-y}. \tag{3.49}$$

3.6.2 Principal specialization

Here we once again abuse notation and write

$$Z^{1,q}(x;\lambda) := Z^{1,q}(\mathbf{p};\lambda)|_{p_i\mapsto x^i} \tag{3.50}$$

for the principal specialization of $Z^{1,q}$.

Let $s_\sigma(\mathbf{p})$ be the Schur function corresponding to a partition σ, which is given as the following vacuum expectation value in the infinite wedge space

$$s_\sigma(\mathbf{p}) := \left\langle 0 \middle| \exp\left(\sum_{i=0}^{\infty} \frac{\alpha_i p_i}{i}\right) \middle| v_\sigma \right\rangle. \tag{3.51}$$

It is a standard fact in the theory of Schur functions that its principal specialization is given by

$$s_\sigma(\mathbf{p})|_{p_i\mapsto x^i} = \begin{cases} x^l & \text{if } \sigma = (l, 0, \ldots) \text{ for some } l \\ 0 & \text{otherwise .} \end{cases} \tag{3.52}$$

Using this it is easy to see that the principal specialization of $Z^{1,q}$ is given by

$$Z^{1,q}(x;\lambda) = \sum_{i=0}^{\infty} \frac{x^{iq}}{i!(\lambda q)^i} \exp\left(\lambda \frac{(iq-\frac{1}{2})^2 - (-\frac{1}{2})^2}{2}\right). \tag{3.53}$$

To find an operator that annihilates this power-series, we proceed as follows. Denote the i^{th} summand in $Z^{1,q}(x;\lambda)$ by a_i:

$$a_i := \frac{x^{iq}}{i!(\lambda q)^i} \exp\left(\lambda \frac{(iq-\frac{1}{2})^2 - (-\frac{1}{2})^2}{2}\right). \tag{3.54}$$

Then

$$\frac{a_{i+1}}{a_i} = \frac{x^q}{(i+1)\lambda q} e^{\lambda\left(iq^2 + \frac{q(q-1)}{2}\right)}, \tag{3.55}$$

which implies that the coefficients of $Z^{1,q}(x;\lambda)$ are related by

$$\lambda q(i+1)a_{i+1} = \left(x e^{\lambda\frac{q-1}{2}}\right)^q e^{\lambda i q^2} a_i. \tag{3.56}$$

In terms of operators, this can be rewritten as

$$\lambda x \frac{\mathrm{d}}{\mathrm{d}x} a_{i+1} - \left(x e^{\lambda\frac{q-1}{2}}\right)^q e^{q\lambda x \frac{\mathrm{d}}{\mathrm{d}x}} a_i = 0, \tag{3.57}$$

which implies that the operator

$$\lambda x \frac{\mathrm{d}}{\mathrm{d}x} - \left(x e^{\lambda\frac{q-1}{2}}\right)^q e^{q\lambda x \frac{\mathrm{d}}{\mathrm{d}x}} \tag{3.58}$$

annihilates $Z^{1,q}(x;\lambda)$.

3.6.3 Quantization

We show that the operator that annihilates the principal specialization of $Z^{1,q}$ can be obtained as a quantization of the equation of the spectral curve $S^{1,q}$.

The spectral curve $S^{1,q}$ is defined in $\mathbb{C}^* \times \mathbb{C}$, where the symplectic form is $\lambda \mathrm{d}(\log(x)) \wedge \mathrm{d}y$, so we have the following rules of quantization:

$$\begin{cases} \hat{x} = x\cdot \\ \hat{y} = \lambda \frac{\mathrm{d}}{\mathrm{d}(\log(x))} = \lambda x \frac{\mathrm{d}}{\mathrm{d}x} \end{cases}. \tag{3.59}$$

In order to have the right ordering, we rewrite the equation for $S^{1,q}$ as follows:

$$S^{1,q}\colon \; y - \left(e^{\frac{q-1}{2}y} x e^{-\frac{q-1}{2}y}\right)^q e^{qy} = 0. \tag{3.60}$$

Theorem 3.14. *Quantization of the equation of $S^{1,q}$ in this form annihilates $Z^{1,q}(x,\lambda)$.*

Proof. Indeed, direct computation implies that

$$\hat{y} - \left(e^{\frac{q-1}{2}\hat{y}} \hat{x} e^{-\frac{q-1}{2}\hat{y}}\right)^q e^{q\hat{y}} = \lambda x \frac{\mathrm{d}}{\mathrm{d}x} - \left(x e^{\lambda\frac{q-1}{2}}\right)^q e^{q\lambda x \frac{\mathrm{d}}{\mathrm{d}x}}, \tag{3.61}$$

and we have seen in the previous section that this operator annihilates $Z^{1,q}(x,\lambda)$. □

We see that q-double Hurwitz numbers are an example of a theory obeying a Schrödinger-like equation with respect to the quantization of the spectral curve as expected by [53], but contrary to the previous known cases [81, 109, 110] we have to take a non-trivial ordering of the operators to obtain this result.

3.7 r-Spin Hurwitz numbers

In this section, we look at the r-spin single Hurwitz numbers. They were described as vacuum expectation values in the infinite wedge space in Chapter 2; we repeat that description here as a definition.

Definition 3.15. We define the (connected) r-spin Hurwitz numbers as

$$h^{r,1}_{g,\mu} := \left\langle \prod_{i=1}^{\ell(\mu)} \frac{\alpha_{\mu_i}}{\mu_i} \cdot \left(r![w^{r+1}]\tilde{\mathcal{E}}_0(w)\right)^m \cdot \frac{(\alpha_{-1})^{|\mu|}}{|\mu|!} \right\rangle^\circ, \tag{3.62}$$

where m is the number of ramification points other than 0, which is given by the Riemann-Hurwitz formula:

$$m = \frac{2g-2+\ell(\mu)+|\mu|}{r}. \tag{3.63}$$

Note the extra factor of $\frac{1}{|\mu|!}$ compared to Proposition 2.23; it appears because for single Hurwitz numbers, we only label the the inverse images of the special ramification point.

For $r = 1$, this definition reduces to the definition of ordinary single Hurwitz numbers. We remind the reader that there are different conventions on the coefficient of $[w^{r+1}]\tilde{\mathcal{E}}_0(w)$ in different sources; in particular, a different convention is used in [88, 112].

Similar to to previous section, we denote by $F^{r,1}_{g,n}(\mathbf{p})$ the generating function for genus g, r-spin Hurwitz numbers $h^{r,1}_{g,\mu}$ whose partition μ has ℓ parts. That is,

$$F^{r,1}_{g,\ell}(\mathbf{p}) := \sum_{\mu\colon\, \ell(\mu)=\ell} h^{r,1}_{g;\mu} p_{\mu_1} \cdots p_{\mu_\ell}. \tag{3.64}$$

For the full generating function $Z^{r,1}$ we then have

$$\log Z^{r,1}(\mathbf{p},\lambda) := \sum_{g,\ell} F^{r,1}_{g,\ell}(\mathbf{p})\lambda^{2g-2+\ell} \tag{3.65}$$
$$= \left\langle \exp\left(\sum_{i=1}^{\infty}\frac{\alpha_i p_i}{i\lambda^i}\right)\exp\left(r![w^{r+1}]\tilde{\mathcal{E}}_0(w)\lambda^r\right)\exp(\alpha_{-1})\right\rangle^{\circ}.$$

3.7.1 Spectral curve from $(0,1)$-geometry

To find an equation for the spectral curve, we compute the $(g,n)=(0,1)$ part of the generating function. Commuting the operator α_d responsible for the total ramification over 0 in (3.65) to the right, we obtain

$$F^{r,1}_{0,1}(\mathbf{p}) = \sum_{n=0}^{\infty}\frac{(rn+1)^{n-2}}{n!}p_{rn+1}. \tag{3.66}$$

Applying the principal specialization, this means that

$$F^{r,1}_{0,1}(x) = \sum_{n=0}^{\infty}\frac{(rn+1)^{n-2}}{n!}x^{rn+1}, \tag{3.67}$$

which leads to

$$\omega_{0,1}(x) = \mathrm{d}F^{r,1}_{0,1} = \sum_{n=0}^{\infty}\frac{(rn+1)^{n-1}}{n!}x^{rn+1}\frac{\mathrm{d}x}{x}. \tag{3.68}$$

We use the following formula from [21]:

$$\left(\frac{W(x)}{x}\right)^{\alpha} = \sum_{n=0}^{\infty}\frac{\alpha(n+\alpha)^{n-1}}{n!}(-x)^n \tag{3.69}$$

to express the right hand side of Equation (3.68) in a more convenient way. That is

$$\sum_{n=0}^{\infty}\frac{(rn+1)^{n-1}}{n!}x^{rn+1} = x\sum_{n=0}^{\infty}\frac{\frac{1}{r}(n+\frac{1}{r})^{n-1}}{n!}(rx^r)^n \tag{3.70}$$
$$= x\left(\frac{W(-rx^r)}{-rx^r}\right)^{1/r} = \frac{W(-rx^r)^{1/r}}{(-r)^{1/r}}$$

Thus, by Remark 3.13 we arrive at the following equation for the spectral curve $S^{r,1}$ in $\mathbb{C}^*\times\mathbb{C}$:

$$S^{r,1}\colon\; y = \left(\frac{W(-rx^r)}{-r}\right)^{1/r} \quad\Leftrightarrow\quad x = ye^{-y^r}. \tag{3.71}$$

3.7.2 Principal specialization

Once again we look at the principal specialization of the full generating function

$$Z^{r,1}(x;\lambda) = Z^{r,1}(x;\lambda)|_{p_i\mapsto x^i} = \sum_{d=0}\frac{x^d}{\lambda^d d!}\exp\left(\lambda^r\frac{(d-\frac{1}{2})^{r+1}-(-\frac{1}{2})^{r+1}}{r+1}\right). \tag{3.72}$$

We define a_d to be the d^{th} summand this expression. The quotient of a_{d+1} and a_d is given by

$$\frac{a_{d+1}}{a_d} = \frac{x}{\lambda(d+1)}\exp\left(\lambda^r\frac{(d+\frac{1}{2})^{r+1}-(d-\frac{1}{2})^{r+1}}{r+1}\right), \tag{3.73}$$

which is equivalent to

$$(d+1)\lambda a_{d+1} = x \exp\left(\lambda^r \frac{(d+\frac{1}{2})^{r+1} - (d-\frac{1}{2})^{r+1}}{r+1}\right) a_d. \tag{3.74}$$

To get this into a more convenient form to compare later on with quantization, we define an operator

$$\mathcal{A} := x^{\frac{3}{2}} \exp\left(\frac{x^{-1}\sum_{i=0}^r \left(\lambda x \frac{\mathrm{d}}{\mathrm{d}x}\right)^i x \left(\lambda x \frac{\mathrm{d}}{\mathrm{d}x}\right)^{r-i}}{r+1}\right) x^{-\frac{1}{2}}. \tag{3.75}$$

Observe that

$$\begin{aligned} \mathcal{A}x^n &= \exp\left(\frac{\lambda^r}{r+1}\sum_{i=0}^r (n+\frac{1}{2})^i (n-\frac{1}{2})^{r-i}\right) x^{n+1} \\ &= \exp\left(\frac{\lambda^r}{r+1}\left((n+\frac{1}{2})^{r+1} - (n-\frac{1}{2})^{r+1}\right)\right) x^{n+1}. \end{aligned} \tag{3.76}$$

Thus, equation (3.74) implies that

$$(\lambda x \frac{\mathrm{d}}{\mathrm{d}x} - \mathcal{A}) Z^{r,1}(x;\lambda) = 0. \tag{3.77}$$

3.7.3 Quantization

We show that the operator that annihilates the principal specialization of $Z^{r,1}$ can be obtained as a quantization of the equation of the spectral curve $S^{r,1}$.

We can rewrite the equation of the spectral curve (3.71) as

$$S^{r,1}:\; y - x^{\frac{3}{2}} \exp\left(\frac{\sum_{i=0}^r x^{-1} y^i x y^{r-i}}{r+1}\right) x^{-\frac{1}{2}} = 0. \tag{3.78}$$

Theorem 3.16. *Quantization of the equation of $S^{r,1}$ in this form annihilates $Z^{r,1}(x,\lambda)$.*

Proof. Indeed, apllying the standard quantization (3.59) to Equation (3.78) we obtain the operator $\lambda x \frac{\mathrm{d}}{\mathrm{d}x} - \mathcal{A}$. □

3.8 Mixed case

In this section we provide a slight generalization of the previous two sections, were we look at (connected) r-spin q-double Hurwitz numbers $h^{r,q}_{g,\mu}$. Since the computations are basically the same as in the previous two sections, we just give the main formulas. Note that for $r = 1$ this reduces to the computations of Section 3.6, and for $q = 1$ this reduces to those of Section 3.7.

These Hurwitz numbers are given as vacuum expectation values by:

$$h^{r,q}_{g,\mu} = \left\langle \prod_{i=1}^{\ell} \frac{\alpha_{\mu_i}}{\mu_i} \cdot \left(r![z^{r+1}]\tilde{\mathcal{E}}_0(z)\right)^m \cdot \frac{(\alpha_{-q})^s}{q^s s!} \right\rangle^{\circ}. \tag{3.79}$$

Here the degree of the covering is given by $d = \sum_{i=1}^{\ell(\mu)} \mu_i = qs$, and the Riemann-Hurwitz formula reads $2g - 2 + \ell(\mu) = mr - s$.

The full generating function is given by

$$\log Z^{r,q}(\mathbf{p};\lambda) = \sum_{g,\mu} \frac{h^{r,q}_{g;\mu}}{m!} \lambda^{2g-2+\ell(\mu)} p_{\mu_1} \cdots p_{\mu_{\ell(\mu)}} \tag{3.80}$$
$$= \left\langle \exp\left(\sum_{i=1}^{n} \frac{\alpha_i p_i}{i\lambda^{i/q}} \right) \exp\left(r![w^{r+1}]\tilde{\mathcal{E}}_0(w)\lambda^r \right) \exp\left(\frac{\alpha_{-q}}{q} \right) \right\rangle^{\circ},$$

and the $(0,1)$-function is given by

$$F_{0,1}(x) = q \sum_{n=0}^{\infty} \frac{((nr+1)q)^{n-2}}{n!} x^{(nr+1)q} \tag{3.81}$$

This leads to the following spectral curve:

$$S\colon\ x = y^{1/q} e^{-y^r}, \tag{3.82}$$

which means that

$$y = -\left(\frac{1}{rq}\right)^{\frac{1}{r}} W(-rqx^{rq})^{\frac{1}{r}}, \tag{3.83}$$

where W is the standard Lambert function.

The principal specialization ($p_i \mapsto x^i$) of $Z^{r,q}$ is given by

$$Z^{r,q}(x,\lambda) = \sum_{n=0}^{\infty} \frac{x^{qn}}{\lambda^n q^n n!} e^{\frac{\lambda^r}{r+1}\left((qn-\frac{1}{2})^{r+1} - (-\frac{1}{2})^{r+1}\right)}, \tag{3.84}$$

which is annihilated by the operator

$$\lambda x \frac{d}{dx} - x^{q+1/2} e^{\frac{q}{r+1}\sum_{i=0}^{r} x^{-q}\left(\lambda x \frac{d}{dx}\right)^i x^q \left(\lambda x \frac{d}{dx}\right)^{r-i}} x^{-1/2} \tag{3.85}$$

This operator dequantizes to $y - x^q \exp(qy^r)$, which is equivalent to the equation (3.82) of the spectral curve $S^{r,q}$ computed from the $(0,1)$-geometry.

Furthermore, one sees immediately that under the specializations $(r,q) = (1,q)$ and $(r,1)$ we recover all the formulas we had in Sections 3.6 and 3.7.