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Hurwitz numbers, moduli of curves, topological recursion, Givental's theory and their relations

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–5– *Givental’s action and CEO-recursion*

5.1 Introduction

5.1.1 Givental theory

Givental theory [46, 45, 47] is one of the most important tools in the study of Gromov-Witten invariants of target varieties and general cohomological field theories that allows, in particular, to obtain explicit relations between the partition functions of different theories, to reconstruct higher genera correlators from the genus 0 data, and to establish general properties of semi-simple theories.

The core of the theory is Givental’s formula that gives a formal Gromov-Witten potential associated to a calibrated semi-simple Frobenius structure. Teleman proves [103] that the formal Gromov-Witten potential associated to the calibrated Frobenius structure of a target variety with semi-simple quantum cohomology coincides with the actual Gromov-Witten potential in all genera.

Roughly speaking, to a calibrated Frobenius structure of dimension N with a chosen semi-simple point t one can associate two $N \times N$ matrix series, $S_t(\zeta^{-1})$ and $R_t(\zeta)$, and $N \times N$ matrices Ψ_t and Δ_t (the latter one is diagonal), such that for a certain quantization of these matrices we have the following formula for the corresponding Gromov-Witten potential

$$\hat{S}_t^{-1} \hat{\Psi}_t \hat{R}_t \hat{\Delta}_t Z_{\text{KdV}}^{\otimes r}, \quad (5.1)$$

where by Z_{KdV} we denote the Kontsevich-Witten tau-function of the KdV hierarchy; that is, the function parametrizing the intersection indices of ψ -classes on the moduli space of curves.

5.1.2 CEO-recursion theory

The Chekhov-Eynard-Orantin-recursion, or CEO-recursion (see [36, 38]), is a procedure that takes the following objects as input. First, a particular Riemann surface, which is usually called the *spectral curve*. Second, two functions x and y on this surface, and third, a choice of a bi-differential on this surface, which we will call the *two-point function* (it has often been referred to as the Bergman kernel, but since this term has other uses as well, we refrain from using it here). Occasionally, a particular extra choice of a coordinate on an open part of the Riemann surface is also made. The output of the CEO-recursion is a set of n -forms $\omega_{g,n}$, whose expansion in this additional coordinate generates interesting numbers.

In some cases these numbers are correlators of a matrix model (that was the original motivation for introducing the CEO-recursion; it is a natural generalization of the reconstruction procedure for the correlators of a certain class of matrix models, see, e.g. [5]), in some other cases they appear to be related to Gromov-Witten theory and to various intersection numbers on the moduli space of curves.

One of the ways to think about the input data of the CEO-recursion theory is to say that the $(g, n) = (0, 1)$ part of a partition function in some geometrically motivated theory

determines the spectral curve; the $(g, n) = (0, 2)$ part of a partition function determines the two-point function, and the rest of the correlators can be reconstructed from these two via CEO-recursion, in terms of a proper expansion of $\omega_{g,n}$ (see [30]).

The CEO-recursion theory is often used to reproduce known partition functions, to extract from them some higher genus correlators which were until then unreachable and to give new non-trivial relations for the correlators, see e. g. [39].

5.1.3 Identification

As we see, there is a lot of similarities in both theories (which was first noted by Alexandrov, Mironov and Morozov in [4, 5, 6]). In both cases we have to start with a small amount of data fixed in genus zero, and in both cases the intersection indices of ψ -classes on the moduli space of curves are some kind of structure constants of the reconstruction procedure (in the case of Givental this is just a part of Givental's formula for the formal Gromov-Witten potential, and in the case of CEO-recursion it is recovered locally in an expansion near a simple critical point of the spectral curve, see [33]).

Moreover, in both cases we have an expansion of the correlators in terms of Feynman graphs, see [118] on the Givental side and [34, 39, 67] on the spectral curve side. So, the natural question is whether we can precisely identify both theories in some setup.

On the Givental side we restrict ourselves to a part of the Givental formula, namely, $\hat{R}\hat{\Delta}Z_{\text{KdV}}^{\otimes r}$ (this expression gives the so-called *total ancestor potential*, written in the normalized canonical basis). In some sense, it is the most important part of the Givental formula since it determines the underlying Frobenius structure, while the rest of the formula is a linear change of variables (action of the matrix $\hat{\Psi}$) and a change of calibration rather than of the Frobenius structure itself (action of the matrix series \hat{S}^{-1}). Note that for a cohomological field theory which does not have quadratic terms in the potential, the S -action becomes trivial when one takes the origin as the chosen point on the Frobenius manifold. For Gromov-Witten applications, where quadratic terms do appear, the S -action is nontrivial, but, together with Ψ -action, it amounts to a linear change of variables. This still allows for the correspondence below to be established, as long as one makes a specific choice of coordinates on the CEO-recursion side. We describe this in detail in the case of the particular example of \mathbb{P}^1 , see below.

On the CEO-recursion side we consider a collection of local germs of a spectral curve at a finite number of points, with fixed expansions of the coordinate functions x and y and the two-point function near these points. The result of the CEO-recursion are local germs of n -forms $\omega_{g,n}$ defined on the products of the given germs of the curve, which we expand in a particular basis of forms that also depends on the expansions of the two-point function.

The resulting systems of correlators coincide for consistent choices of the input data in both theories. We prove this fact, essentially using the graphical interpretation of the formulas given in [118, 34], and provide a dictionary to translate Givental data into local spectral curve data and vice versa.

Thus, we solve the problem about the mysterious relation between CEO-recursion and enumerative geometry.

5.1.4 Organization of the chapter

The chapter assumes some pre-knowledge of both Givental and CEO-recursion theory; we refer to [38, 71, 97] as possible sources. In Section 5.2 we recall the Givental theory, and we present the Givental formula as a sum over graphs. In Section 5.3 we do the same for CEO-recursion theory. In Section 5.4 we prove the theorem on identification of the two theories and provide a corresponding dictionary.

5.2 Givental group action as a sum over graphs

In this section we review the Givental group action and we remind the reader how it can be used to write the partition function of an N -dimensional semi-simple cohomological field theory as an operator acting on the product of N KdV τ -functions. Using this, we write the partition function for such a cohomological field theory as a sum over decorated graphs. This is essentially the same as what was done in [118]; in the present chapter the contributions are distributed in a slightly different way over the components of the graph to make the comparison with the CEO-recursion.

5.2.1 Givental group action

We remind the reader of the original formulation, due to Y.-P. Lee, of the infinitesimal Givental group action in terms of differential operators [68, 69, 70].

Consider the space of partition functions for N -dimensional cohomological field theories

$$Z = \exp \left(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}_g \right) \quad (5.2)$$

in variables $v^{d,i}$, $d \geq 0$, $i = 1, \dots, N$. There is a fixed scalar product $\eta_{ij} = \delta_{ij}$ on the vector space $V := \langle e_1, \dots, e_N \rangle$ of primary fields corresponding to the indices $i = 1, \dots, N$. Furthermore, we will denote by e_1 the vector in V that plays the role of the unit.

Later on we will also use the so-called *correlators*

$$\langle \tau_{d_1}(e_{i_1}) \tau_{d_2}(e_{i_2}) \cdots \tau_{d_k}(e_{i_k}) \rangle_g \quad (5.3)$$

which correspond to the coefficients of formal power series \mathcal{F}_g in the following way:

$$\mathcal{F}_g = \sum \frac{\langle \tau_{d_1}(e_{i_1}) \tau_{d_2}(e_{i_2}) \cdots \tau_{d_k}(e_{i_k}) \rangle_g}{|\text{Aut}((i_m, d_m)_{m=1}^k)|} v^{d_1, i_1} \cdots v^{d_k, i_k}, \quad (5.4)$$

where $|\text{Aut}((i_m, d_m)_{m=1}^k)|$ denotes the number of automorphisms of the collection of multi-indices (i_m, d_m) and where the sum is such that it includes each monomial $v^{d_1, i_1} \cdots v^{d_k, i_k}$ exactly once. Note that in the special case of a Gromov-Witten theory for some manifold X , these correlators carry the following meaning:

$$\langle \tau_{d_1}(e_{i_1}) \tau_{d_2}(e_{i_2}) \cdots \tau_{d_k}(e_{i_k}) \rangle_g = \sum_{\text{deg}} \int_{[X_{g,k,\text{deg}}]} ev_1^*(e_{i_1}) \psi_1^{d_1} ev_2^*(e_{i_2}) \psi_2^{d_2} \cdots ev_k^*(e_{i_k}) \psi_k^{d_k}, \quad (5.5)$$

where $[X_{g,k,\text{deg}}]$ is the moduli space of degree deg stable maps to X of genus- g curves with k marked points, ev_i is the evaluation map at the i^{th} point and the ψ correspond to ψ -classes.

Consider a sequence of operators $r_l \in \text{Hom}(V, V)$ for $l \geq 1$, such that the operators with odd (resp., even) indices are symmetric (resp., skew-symmetric). Then we denote by $(r_l z^l)^\wedge$ the following differential operator:

$$\begin{aligned} (r_l z^l)^\wedge := & - (r_l)_1^i \frac{\partial}{\partial v^{l+1, i}} + \sum_{d=0}^{\infty} v^{d, i} (r_l)_i^j \frac{\partial}{\partial v^{d+l, j}} \\ & + \frac{\hbar}{2} \sum_{m=0}^{l-1} (-1)^{m+1} (r_l)^{i, j} \frac{\partial^2}{\partial v^{m, i} \partial v^{l-1-m, j}}. \end{aligned} \quad (5.6)$$

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Here the indices $i, j \in \{1, \dots, N\}$ on r_l correspond to the basis $\{e_1, \dots, e_N\}$ of V , and the index $\mathbf{1}$ corresponds to the unit vector e_1 . When we write r_l with two upper-indices we mean as usual that we raise one of the indices using the scalar product η .

Given such a sequence of operators r_l , we define an operator series $R(z)$ in the following way

$$R(z) = \sum_{l=0}^{\infty} R_l z^l := \exp \left(\sum_{l=1}^{\infty} r_l z^l \right). \quad (5.7)$$

The quantization \hat{R} of this series is defined by

$$\hat{R} = \exp \left(\sum_{l=1}^{\infty} ((-1)^l r_l z^l)^\wedge \right). \quad (5.8)$$

Givental observed that the action of such operators \hat{R} on formal power series Z for which the number of ψ -classes (given by the first index of $v^{d,\mu}$) at any monomial of degree n is no more than $3g - 3 + n$, is well-defined. The main theorem of [44] states that this action preserves the property that Z is a generating function of the correlators of a cohomological field theory with target space (V, η) (see also [62, 103]).

Remark 5.1. Note that this definition of \hat{R} differs from the one in [118] by the sign $(-1)^l$. It is needed here to agree with Givental's notation in Proposition 5.3, cf. [46, Proposition 7.3]. For the same reason, in order to agree with the conventions of Givental, we label in a matrix by the upper index the column and by the lower index the row.

5.2.2 Givental operator for a Frobenius manifold

Let $Z(\{t^{d,\mu}\})$ be the partition function of some N -dimensional semi-simple conformal cohomological field theory. We recall the construction (due to Givental [46, 45, 47], see also Dubrovin [28]) of an operator series $R(z)$ as in the previous section whose quantization takes the product of N KdV τ -functions to Z .

Let F be the restriction of $\log(Z)$ to the genus zero part without descendants. Denote $t^\mu := t^{0,\mu}$. Then F can be interpreted as a formal Frobenius manifold with metric

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \quad (5.9)$$

and Frobenius algebra structure $c_{\alpha\beta}^\gamma$

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}. \quad (5.10)$$

We can assume that $\eta_{\alpha\beta} = \delta_{\alpha+\beta, n+1}$ and $e_1 = e_1$. According to [27] it is always possible by an appropriate choice of these flat coordinates t^μ .

Canonical coordinates

Another set of coordinates is given by the *canonical coordinates* $\{u^i\}$ which can be found as solutions to Equation (3.54) from [27], and have the property that $\{\partial_i := \partial/\partial u^i\}$ forms a basis of canonical idempotents of the Frobenius algebra product. In these coordinates the metric is diagonal and the unit vector field is given by $e_1 = \partial_1 + \dots + \partial_N$.

Define $\Delta_i := 1/(\partial_i, \partial_i)$ to be the inverse of the square of the length of the i^{th} canonical basis element, and call $\{\partial/\partial v^i := \Delta_i^{1/2} \partial/\partial u^i\}$ the normalized canonical basis in the tangent

space. We denote the coordinates corresponding to this basis by v^i , and the formal variables corresponding to these coordinates by $v^{d,i}$. They are precisely the formal variables $v^{d,i}$ appearing in the previous section.

Let U be the matrix of canonical coordinates $U = \text{diag}(u^1, \dots, u^N)$ and denote by Ψ the transition matrix from the flat to the normalized canonical bases. That is, denoting $dt = (dt^1, \dots, dt^N)^T$ and $du = (du^1, \dots, du^N)^T$, one has

$$\Delta^{-1/2} du = \Psi dt, \quad (5.11)$$

where $\Delta = \text{diag}(\Delta_1, \dots, \Delta_N)$.

Remark 5.2. Note that Ψ obtained by the definition above depends on the point p of the Frobenius manifold.

Recursion

Construct an operator series $R(z) = \sum_{k \geq 0} R_k z^k$ as in the previous section in the following way.

Recursively define the off-diagonal entries of R_k in normalized canonical coordinates by solving the equation

$$\Psi^{-1} d(\Psi R_{k-1}) = [dU, R_k]. \quad (5.12)$$

using $R_0 = \mathbf{I}$ as a base case. Construct the diagonal entries of R_k by integrating the next equation

$$\Psi^{-1} d(\Psi R_k) = [dU, R_{k+1}] \quad (5.13)$$

using the fact that the diagonal entries of $[dU, R_{k+1}]$ are equal to zero. To fix the integration constant, use the Euler equation

$$R_k = -(i_E d R_k)/k, \quad (5.14)$$

where $E = \sum u^i \partial_i$ is the Euler field (here we use the fact that we started with a conformal cohomological field theory).

This procedure recursively defines R_k for all k . The following proposition is essentially proved in Givental's papers [46, 45].

Proposition 5.3. *Let F be a local N -dimensional Frobenius manifold structure, semisimple at the origin, and let (R_k) be the series of operators constructed from this F by the recursive procedure described above, at the origin. Let Ψ and Δ be as above, taken at the origin as well. Then we have the following formula:*

$$\mathcal{F}_0 = \text{Res}_{\hbar=0} d\hbar \cdot \log \hat{\Psi} \hat{R} \hat{\Delta} \mathcal{T}. \quad (5.15)$$

Here $\mathcal{F}_0 = \mathcal{F}_0(\{t^{d,\mu}\})$ is the genus 0 descendant potential of cohomological field theory associated to F ; \mathcal{T} is the product of N KdV tau-functions,

$$\mathcal{T} := Z_{\text{KdV}}(\{u^{d,1}\}) \cdots Z_{\text{KdV}}(\{u^{d,N}\});$$

$\hat{\Delta}$ replaces the variables of i^{th} KdV τ -function according to $u^{d,i} = \Delta_i^{1/2} v^{d,i}$ and replaces \hbar with $\Delta_i \hbar$, while $\hat{\Psi}$ is the change of variables $v^{d,i} = \Psi_v^i t^{d,\nu}$. The unit for the R -action is given by $(\Psi_1^1, \dots, \Psi_1^N)$.

Remark 5.4. In fact, using Teleman's result in [103], one has a refined version of Equation (5.15):

$$Z = \hat{\Psi} \hat{R} \hat{\Delta} \mathcal{T}. \quad (5.16)$$

Note that it holds for cohomological field theories. In the Gromov-Witten case, when quadratic terms in the potential cannot be neglected, there appears an additional complication, see the next remark below.

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Remark 5.5. Givental's formula [46] for a Gromov-Witten total descendant potential (without the $(g = 1, n = 0)$ -term),

$$Z = \hat{S}^{-1} \hat{\Psi} \hat{R} \hat{\Delta} \mathcal{T}, \quad (5.17)$$

also includes the operator \hat{S} , given by

$$\hat{S} = \exp \left(\sum_{l=1}^{\infty} (s_l z^{-l})^\wedge \right), \quad (5.18)$$

where the operators $(s_l z^{-l})^\wedge$ are defined in the following way (see, e. g., [44, Section 4.2]):

$$\begin{aligned} \sum_{l=1}^{\infty} (s_l z^{-l})^\wedge &= - (s_1)_1^\mu \frac{\partial}{\partial t^{0,\mu}} + \frac{1}{\hbar} \sum_{d=0}^{\infty} (s_{d+2})_{1,\mu} t^{d,\mu} \\ &+ \sum_{\substack{d=0 \\ l=1}}^{\infty} (s_l)_\nu^\mu t^{d+l,\nu} \frac{\partial}{\partial t^{d,\mu}} + \frac{1}{2\hbar} \sum_{\substack{d_1, d_2 \\ \mu_2, \mu_2}} (-1)^{d_1} (s_{d_1+d_2+1})_{\mu_1, \mu_2} t^{d_1, \mu_1} t^{d_2, \mu_2}. \end{aligned} \quad (5.19)$$

Note that formula (5.18) for the quantization of S differs from the analogous formula (5.8) for R by a factor of $(-1)^l$ in the exponent, which agrees with the definition in Givental's papers [45, 46].

The matrices s_k are defined through the following relation:

$$S(z) = \sum_{k=0}^{\infty} S_k z^{-k} = \exp \left(\sum_{l=0}^{\infty} s_l z^{-l} \right), \quad (5.20)$$

where for $S(z)$, taken at a point p of the Frobenius manifold, we have (see [46]), for any points a and b of the Frobenius manifold,

$$(a, b S_p) := (a, b) + \sum_{k=0}^{\infty} \langle \tau_0(a) \exp(\tau_0(p)) \tau_k(b) \rangle_0 z^{-1-k}. \quad (5.21)$$

Here on the left hand side the brackets stand for the scalar product on the tangent space to the Frobenius manifold at p , and we used an identification of the tangent space with the whole Frobenius manifold, since in this case the Frobenius manifold is itself a vector space. If p is the origin, we have just

$$(a, b S) := (a, b) + \sum_{k=0}^{\infty} \langle \tau_0(a) \tau_k(b) \rangle z^{-1-k}. \quad (5.22)$$

Note that this S action is defined in the general case when the total descendant genus 0 potential is known. For the case when only a Frobenius potential is specified, the choice of S is then called a *calibration* of the Frobenius manifold, see [45, 28] for related details. In the case of cohomological field theory when we disregard quadratic terms, the S action is trivial if p is taken to be the origin.

It turns out that in most of the relevant cases, e.g. for the Gromov-Witten theory of $\mathbb{C}P^1$ (see section 6.1 below), the only relevant term in equation (5.19) is

$$\sum_{\substack{d=0 \\ l=1}}^{\infty} (s_l)_\nu^\mu t^{d+l,\nu} \frac{\partial}{\partial t^{d,\mu}}, \quad (5.23)$$

since $(s_1)_1^\mu$ vanishes and all other terms just change the unstable terms in the potential.

This means, that in these cases \hat{S}^{-1} just performs a linear change of formal variables $t^{d,\mu}$ in the following way:

$$t^{d,\mu} \mapsto \sum_{m=d}^{\infty} (S_{m-d})_\nu^\mu t^{m,\nu}. \quad (5.24)$$

5.2.3 Expressions in terms of graphs

In [118] the action of an operator series as in equation (5.8) is written as a sum over graphs. By Remark 5.4, this allows us to construct the potential of any semi-simple conformal cohomological field theory as a sum over graphs. Here we repeat the construction of [118] in a slightly different way that will be more convenient for the comparison with the CEO-recursion formalism. Furthermore, we also include the action of $\hat{\Delta}$. It is easy to see that the construction is equivalent to that of [118].

Notation 5.6. Let γ be any graph. By a half-edge we mean either a leaf or an edge together with a choice of one of the two vertices it is attached to. By $V(\gamma)$, $E(\gamma)$, $H(\gamma)$ and $L(\gamma)$ we denote the sets of vertices, edges, half-edges and leaves of γ . For any vertex v of γ , denote by $H(v)$ the set of half-edges connected to v .

Let $\tilde{\Gamma}$ be the set of all connected graphs γ together with a choice of disjoint splitting $L(\gamma) = L^*(\gamma) \amalg L^\bullet(\gamma)$, a labelling of the vertices by pairs $(g, i) \in \mathbb{Z}_{\geq 0} \times \{1, \dots, N\}$ and a labelling of the elements of $H(\gamma)$ by non-negative integers, such that the label of a leaf in L^\bullet is always greater than one. The elements of $L^*(\gamma)$ are called *ordinary leaves*, the elements of L^\bullet are called *dilaton leaves*. We denote by Γ the subset of all graphs in $\tilde{\Gamma}$ that are *stable*; that is, any vertex labelled $(0, i)$ for some i is of valence at least three.

For any graph γ denote by $\mathbf{g}: V(\gamma) \rightarrow \mathbb{Z}_{\geq 0}$ and $\mathbf{i}: V(\gamma) \rightarrow \{1, \dots, N\}$ the maps that associate to any vertex its first and second label respectively, and by $\mathbf{e}: H(\gamma) \rightarrow \mathbb{Z}_{\geq 0}$ the map that associates to any half-edge its label. Denote by $\mathbf{v}: L(\gamma) \rightarrow V(\gamma)$ the map that associates to each leaf the corresponding vertex, and by $\mathbf{v}_1, \mathbf{v}_2: E(\gamma) \rightarrow V(\gamma)$ and by $\mathbf{h}_1, \mathbf{h}_2: E(\gamma) \rightarrow H(\gamma)$ the maps that associate to an edge the first and second vertex, and the corresponding half-edges respectively.

Remark 5.7. The labels introduced above are used to keep track of different data for the trivial cohomological field theory; g is for the genus, i for the primary field in canonical coordinates and the labelling of the marked half-edges is for the power of ψ -class.

Remark 5.8. As in [118], edges of a graph in Γ are considered to be oriented (this allows to define the maps \mathbf{v}_1 and \mathbf{v}_2 unambiguously); the final result does not depend on the orientation.

Let $R(z)_j^i$ be the components of the operator series $R(z)$ in normalized canonical basis as computed in Section 5.2.2. To each part of a graph $\gamma \in \Gamma$ we assign some polynomial in formal variables \hbar and $v^{d,i}$. Here \hbar is used to keep track of the genus, while the first index of $v^{d,i}$ keeps track of the number of ψ -classes and the second index keeps track of the normalized canonical coordinate.

Leaves

To each ordinary leaf $l \in L^*$ marked by k attached to a vertex marked by the pair (g, i) , we assign

$$(\mathcal{L}^*)_k^i(l) := [z^k] \left(\sum_{d \geq 0} ((R(-z))_j^i)^d v^{d,j} z^d \right), \quad (5.25)$$

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which corresponds to the second term in (5.6).

To a dilaton leaf $\lambda \in L^\bullet(\gamma)$ marked by k attached to a vertex marked by (g, i) we assign

$$(\mathcal{L}^\bullet)_k^i(\lambda) := [z^{k-1}] (-R(-z))_1^i, \quad (5.26)$$

which corresponds to the first term in (5.6), which is called the *dilaton shift*.

Edges

To an edge e connecting a vertex v_1 marked by (g_1, i_1) to a vertex v_2 marked by (g_2, i_2) and with markings k_1 and k_2 at the corresponding half-edges, we assign

$$\mathcal{E}_{k_1, k_2}^{i_1, i_2}(e) := [z^{k_1} w^{k_2}] \left(\hbar \cdot \frac{\delta^{i_1 i_2} - \sum_s (R(-z))_s^{i_1} (R(-w))_s^{i_2}}{z + w} \right). \quad (5.27)$$

Note that this does not depend on the choice of ordering of the vertices and that it follows from the fact that $R(z)$ can be written as $R(z) = \exp(\sum r_l z^l)$ that the numerator on the right-hand side is equal to the product of $(z + w)$ with some power series in z and w , so this definition makes sense.

Vertices

Let v be a vertex marked by (g, i) with n half-edges attached to it (this includes all ordinary and dilaton leaves and also half-edges that are parts of internal edges) labelled by k_1, \dots, k_n . Then we assign to v the following expression:

$$\mathcal{V}_{\{k_1, \dots, k_n\}}^{(g, i)}(v) := \hbar^{g-1} (\Delta_i)^{\frac{1}{2}(2g-2+n)} \int_{\overline{\mathcal{M}}_{g, n}} \psi_1^{k_1} \dots \psi_n^{k_n}. \quad (5.28)$$

Z as a sum over graphs

It is easy to see that the sum over all graphs in Γ of the product of the contributions described above, weighted by the inverse order of the automorphism group of the graph, is equal to the graph-sum described in [118] (the only difference is that now we have specialized to the action on the trivial cohomological field theory, leading to ψ -class integrals (5.28) as vertex contributions). Thus, we recover the partition function Z of the cohomological field theory we started with as a sum over Γ :

$$(\hat{R}\hat{\Delta}\mathcal{T})(\{v^{d, j}\}) = \sum_{\gamma \in \Gamma} \frac{1}{|\text{Aut}(\gamma)|} \prod_{v \in V(\Gamma)} \hbar^{g(v)-1} (\Delta_{i(v)})^{\frac{1}{2}(2g(v)-2+\text{val}(v))} \left\langle \prod_{h \in H(v)} \tau_{\mathfrak{k}(h)} \right\rangle_{\mathfrak{g}} \\ \prod_{e \in E(\gamma)} \mathcal{E}_{\mathfrak{k}(b_1(e)), \mathfrak{k}(b_2(e))}^{i(v_1(e)), i(v_2(e))}(e) \prod_{l \in L^*(\gamma)} (\mathcal{L}^*)_{\mathfrak{k}(l)}^{i(v(l))}(l) \prod_{\lambda \in L^\bullet(\gamma)} (\mathcal{L}^\bullet)_{\mathfrak{k}(l)}^{i(v(l))}(\lambda). \quad (5.29)$$

5.3 CEO-recursion as a sum over graphs

In this section, we define a local version of the CEO-recursion and write the corresponding invariants as a sum over graphs, which allows us to compare it to the Givental action in the next section.

5.3.1 Local CEO-recursion

We define a local version of the CEO-recursion in the following way. The term local refers to the fact that the data are all defined locally around the canonical coordinates without any reference to the possible existence of a global manifold where these functions can be defined.

Definition 5.9. For $N \in \mathbb{N}^*$, we call times a set of N families of complex numbers $\{h_k^i\}_{k \in \mathbb{N}}$ for $i = 1, \dots, N$ and jumps another set of $N \times N$ infinite families of complex numbers $\{B_{k,l}^{i,j}\}_{(k,l) \in \mathbb{N}^2}$ for $i, j = 1, \dots, N$. We finally define a set of canonical coordinates $\{a_i\}_{i=1}^N \in \mathbb{C}^N$ subject to $a_i \neq a_j$ for $i \neq j$.

For all $i, j \in \{1, \dots, N\}$, we define the following set of analytic functions and differential forms in a neighborhood of $0 \in \mathbb{C}$:

$$x^i(z) := z^2 + a_i, \quad y^i(z) := \sum_{k=0}^{\infty} h_k^i z^k \quad (5.30)$$

and

$$B^{i,j}(z, z') = \delta_{i,j} \frac{dz \otimes dz'}{(z - z')^2} + \sum_{k,l=0}^{\infty} B_{k,l}^{i,j} z^k z'^l dz \otimes dz'. \quad (5.31)$$

For $2g - 2 + n > 0$, we define the genus g , n -point correlation functions $\omega_{g,n}^{i_1, \dots, i_n}(z_1, \dots, z_n)$ recursively by

$$\omega_{g,n+1}^{i_0, i_1, \dots, i_n}(z_0, z_1, \dots, z_n) := \sum_{j=1}^N \operatorname{res}_{z \rightarrow 0} \frac{\int_{-z}^z B^{i_0, j}(z_0, \cdot)}{2(y^j(z) - y^j(-z)) dx^j(z)} \left(\omega_{g-1, n+2}^{j, i_1, \dots, i_n}(z, -z, z_1, \dots, z_n) + \sum_{A \cup B = \{1, \dots, n\}} \sum_{h=0}^g \omega_{h, |A|+1}^{j, \mathbf{i}_A}(z, \mathbf{z}_A) \omega_{g-h, |B|+1}^{j, \mathbf{i}_B}(-z, \mathbf{z}_B) \right), \quad (5.32)$$

where for any set A , we denote by \mathbf{z}_A (resp., \mathbf{i}_A) the set $\{z_k\}_{k \in A}$ (resp., $\{i_k\}_{k \in A}$), and where the base of the recursion is given by

$$\omega_{0,1}^i(z) := 0; \quad \omega_{0,2}^{i,j}(z, z') := B^{i,j}(z, z'). \quad (5.33)$$

For convenience, in the sequel we denote

$$K^{i,j}(z, z') = \frac{\int_{-z}^z B^{i,j}(z', \cdot)}{2(y^j(z) - y^j(-z)) dx^j(z)} \quad (5.34)$$

and

$$\omega_{g,n}(\vec{z}) = \sum_{\vec{i}} \omega_{g,n}^{\vec{i}}(\vec{z}), \quad (5.35)$$

where the length of \vec{z} and \vec{i} is n .

5.3.2 Correlation functions and intersection numbers

The correlation functions built by this CEO-recursion can actually be written in terms of intersections of ψ classes on the moduli space of Riemann surfaces. This result is a slight generalization of [33, 34] to the local CEO-recursion.

5.3.3 One-branch point case

The link between the CEO-recursion formalism and intersection numbers on the moduli space of Riemann surfaces comes from the application of this formalism to the Airy curve. This case corresponds to $N = 1$ and:

$$x(z) = z^2 + a, \quad y(z) = z \quad \text{and} \quad B(z, z') = \frac{dz \otimes dz'}{(z - z')^2}. \quad (5.36)$$

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Remark 5.10. Since there is only one branch point in this case, i.e. $N = 1$, we omit the superscript indicating which branch point we consider in the notations of this section.

For further convenience, we introduce two additional parameters by considering the curve

$$x(z) = z^2 + a, \quad y(z) = \alpha z \quad \text{and} \quad B(z, z') = \beta \frac{dz \otimes dz'}{(z - z')^2}, \quad (5.37)$$

the usual Airy curve being $\alpha = \beta = 1$. In this case, the CEO-recursion reads

$$\omega_{g,n+1}(z_0, z_1, \dots, z_n) := \operatorname{res}_{z \rightarrow 0} \frac{\beta \, dz_0}{2\alpha \, 2z \, dz} \frac{1}{(z_0^2 - z^2)} \left(\omega_{g-1,n+2}(z, -z, z_1, \dots, z_n) + \sum_{A \cup B = \{1, \dots, n\}} \sum_{h=0}^g \omega_{h,|A|+1}(z, \mathbf{z}_A) \omega_{g-h,|B|+1}(-z, \mathbf{z}_B) \right) \quad (5.38)$$

and one has

Lemma 5.11. *The correlation functions of the Airy curve can be expressed in terms of intersection numbers:*

$$\omega_{g,n}(z_1, \dots, z_n) = \left(-\frac{\beta}{2\alpha} \right)^{2g+n-2} \beta^{g+n-1} \sum_{\alpha_1, \dots, \alpha_n \geq 0} \langle \tau_{\alpha_1} \dots \tau_{\alpha_n} \rangle_{g,n} \prod_{i=1}^n \frac{(2\alpha_i + 1)!! \, dz_i}{z_i^{2\alpha_i+2}}. \quad (5.39)$$

This lemma was proved many times by direct computation [32, 33, 38, 109], matching the CEO-recursion with the recursive definition of the intersection numbers.

As a side note, the first few correlation functions are

$$\omega_{0,3}(z_1, z_2, z_3) = -\frac{\beta^3}{2\alpha} \prod_{i=1}^3 \frac{dz_i}{z_i^2}, \quad (5.40)$$

$$\omega_{0,4}(z_1, z_2, z_3, z_4) = \frac{\beta^5}{4\alpha^2} \prod_{i=1}^4 \frac{dz_i}{z_i^2} \sum_{i=1}^4 \frac{3}{z_i^2}, \quad (5.41)$$

$$\omega_{1,1}(z) = \frac{-\beta^2 \, dz}{2\alpha \, 8z^4} \quad (5.42)$$

and

$$\omega_{1,2}(z_1, z_2) = \frac{\beta^4 \, dz_1 \, dz_2}{4\alpha^2 \, 8z_1^2 z_2^2} \left(\frac{5}{z_1^4} + \frac{5}{z_2^4} + \frac{3}{z_1^2 z_2^2} \right). \quad (5.43)$$

Remark 5.12. It is important to remark that there exist different conventions in the literature for defining the CEO-recursion, mainly differing by a change of sign of the recursion kernel. The latter can be recovered by a change of sign $\alpha \rightarrow -\alpha$.

Let us now consider a deformation of the Airy curve which we will refer to as the KdV curve in the following. It has only one branch point, $N = 1$, and reads

$$\begin{cases} x(z) = z^2 + a_i \\ y(z) = \alpha \sum_{k=1}^{\infty} h_k z^k \\ B(z, z') = \beta B_{\text{KdV}}(z, z') = \beta \frac{dz \otimes dz'}{(z - z')^2} \end{cases}. \quad (5.44)$$

The corresponding correlation functions can also be expressed in terms intersection numbers as follows:

Lemma 5.13. *The correlation functions of the KdV curve read:*

$$\omega_{g,n}(z_1, \dots, z_n) = \left(-\frac{\beta}{2\alpha h_1} \right)^{2g+n-2} \beta^{g+n-1} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\vec{\alpha} \in \mathbb{N}^{*m}} \prod_{k=1}^m (2\alpha_k + 1)!! \frac{h_{2\alpha_k+1}}{h_1} \prod_{i=1}^n \frac{(2d_i + 1)!! dz_i}{z_i^{2d_i+2}} \left\langle \prod_{j=1}^n \tau_{d_j} \prod_{k=1}^m \tau_{\alpha_k+1} \right\rangle_{g,n+m}. \quad (5.45)$$

Proof. Once again the proof can be found in the literature [32, 37, 33]. However, let us study a graphical interpretation of this result when considering an arbitrary convention for the CEO-recursion. For $f(z)$ an analytic function around $z \rightarrow 0$ and $\{T_k\}_{k \in \mathbb{Z}}$ a set of parameters, one can compute

$$\operatorname{res}_{Z_1 \rightarrow 0} \operatorname{res}_{Z_2 \rightarrow 0} K(Z_1, z) \left\{ \left(\sum_{k \geq 1} T_k Z_1^k \right) dZ_1 K(Z_2, -Z_1) f(Z_2) [dZ_2]^2 - \left(\sum_{k \geq 1} T_k (-Z_1)^k \right) dZ_1 K(Z_2, Z_1) f(Z_2) [dZ_2]^2 \right\} \quad (5.46)$$

where the recursion kernel is the one of the Airy curve, i.e. the one for which $h_k = 0$ for $k \geq 2$:

$$K(z, z_0) = \frac{\beta}{2\alpha h_1} \frac{dz_0}{2z dz} \frac{1}{(z_0^2 - z^2)}. \quad (5.47)$$

One can move the integration contours to get

$$\operatorname{res}_{Z_1 \rightarrow 0} \operatorname{res}_{Z_2 \rightarrow 0} = \operatorname{res}_{Z_2 \rightarrow 0} \operatorname{res}_{Z_1 \rightarrow 0} + \operatorname{res}_{Z_2 \rightarrow 0} \operatorname{res}_{Z_1 \rightarrow Z_2} + \operatorname{res}_{Z_2 \rightarrow 0} \operatorname{res}_{Z_1 \rightarrow -Z_2}. \quad (5.48)$$

The first term of the right hand side vanishes since the integrand does not have any pole at $Z_1 \rightarrow 0$. Let us now compute one of the other two terms:

$$\begin{aligned} & \operatorname{res}_{Z_2 \rightarrow 0} \operatorname{res}_{Z_1 \rightarrow Z_2} K(Z_1, z) \left(\sum_{k \geq 1} T_k Z_1^k dZ_1 \right) K(Z_2, -Z_1) f(Z_2) [dZ_2]^2 \\ &= - \operatorname{res}_{Z_2 \rightarrow 0} \frac{\beta}{2\alpha h_1} \frac{dZ_2}{2Z_2} f(Z_2) \operatorname{res}_{Z_1 \rightarrow Z_2} \frac{dz}{2Z_1} \frac{1}{(z^2 - Z_1^2)} \frac{\beta}{2\alpha h_1} \frac{1}{(Z_1^2 - Z_2^2)} \left(\sum_{k \geq 1} T_k Z_1^k dZ_1 \right) \\ &= - \operatorname{res}_{Z_2 \rightarrow 0} \frac{\beta}{2\alpha h_1} \frac{dZ_2 dz}{2Z_2} f(Z_2) \frac{1}{(z^2 - Z_2^2)} \frac{\beta}{2\alpha h_1} \sum_{k \geq 1} \frac{T_k}{4} Z_2^{k-2}. \end{aligned} \quad (5.49)$$

In the same way,

$$\begin{aligned} & \operatorname{res}_{Z_2 \rightarrow 0} \operatorname{res}_{Z_1 \rightarrow -Z_2} K(Z_1, z) \left(\sum_{k \geq 1} T_k Z_1^k dZ_1 \right) K(Z_2, -Z_1) f(Z_2) [dZ_2]^2 = \\ &= - \operatorname{res}_{Z_2 \rightarrow 0} \frac{\beta}{2\alpha h_1} \frac{dZ_2 dz}{2Z_2} f(Z_2) \frac{1}{(z^2 - Z_2^2)} \frac{\beta}{2\alpha h_1} \sum_{k \geq 1} \frac{T_k}{4} (-Z_2)^{k-2}. \end{aligned} \quad (5.50)$$

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The sum of these two terms reads

$$\begin{aligned} \operatorname{res}_{Z_2 \rightarrow 0} \operatorname{res}_{Z_1 \rightarrow \pm Z_2} K(Z_1, z) \left(\sum_{k \geq 1} T_k Z_1^k dZ_1 \right) K(Z_2, -Z_1) f(Z_2) [dZ_2]^2 = \\ = - \operatorname{res}_{Z_2 \rightarrow 0} \frac{\beta}{2\alpha h_1} \frac{dZ_2 dz}{2Z_2} f(Z_2) \frac{1}{(z^2 - Z_2^2)} \frac{\beta}{2\alpha h_1} \sum_{k \geq 1} \frac{T_{2k}}{2} (Z_2)^{2k-2} \end{aligned} \quad (5.51)$$

and finally:

$$\begin{aligned} \operatorname{res}_{Z_1 \rightarrow 0} \operatorname{res}_{Z_2 \rightarrow 0} K(Z_1, z) \left\{ \left(\sum_{k \geq 1} T_k Z_1^k \right) dZ_1 K(Z_2, -Z_1) f(Z_2) [dZ_2]^2 \right. \\ \left. - \left(\sum_{k \geq 1} T_k (-Z_1)^k \right) dZ_1 K(Z_2, Z_1) f(Z_2) [dZ_2]^2 \right\} = \\ = \operatorname{res}_{Z_2 \rightarrow 0} \frac{\beta}{2\alpha h_1} \frac{dZ_2 dz}{2Z_2} f(Z_2) \frac{1}{(z^2 - Z_2^2)} \left(-\frac{\beta}{2\alpha h_1} \right) \sum_{k \geq 1} T_{2k} (Z_2)^{2k-2}. \end{aligned} \quad (5.52)$$

On the other hand, plugging in the times h_k amounts to computing similar quantities:

$$\begin{aligned} \operatorname{res}_{z \rightarrow 0} \frac{\beta}{2\alpha h_1} \frac{dz_0}{2z dz} \frac{1}{(z_0^2 - z^2)} \frac{1}{\left(1 + \sum_{k=1}^{\infty} \frac{h_{2k+1}}{h_1} z^{2k} \right)} f(z) [dz]^2 = \\ = \operatorname{res}_{z \rightarrow 0} \frac{\beta}{2\alpha h_1} \frac{dz_0}{2z dz} \frac{1}{(z_0^2 - z^2)} f(z) [dz]^2 \\ \cdot \left(1 - \sum_{k=1}^{\infty} \frac{h_{2k+1}}{h_1} z^{2k} + \left[\sum_{k=1}^{\infty} \frac{h_{2k+1}}{h_1} z^{2k} \right]^2 + \dots \right) \end{aligned} \quad (5.53)$$

The first term of this sum is the Airy recursion kernel. The second one is of the shape of the preceding one with $T_{2k+2} = \frac{2\alpha h_{2k+1}}{h_1}$ for $k \geq 1$ so that:

$$\begin{aligned} - \operatorname{res}_{z \rightarrow 0} \frac{\beta}{2\alpha} \frac{dz_0}{2z dz} \frac{1}{(z_0^2 - z^2)} f(z) [dz]^2 \sum_{k=1}^{\infty} \frac{h_{2k+1}}{h_1} z^{2k} = \\ = \operatorname{res}_{Z_1 \rightarrow 0} \operatorname{res}_{Z_2 \rightarrow 0} K(Z_1, z_0) \left\{ g(Z_1) dZ_1 K(Z_2, -Z_1) f(Z_2) [dZ_2]^2 \right. \\ \left. - g(-Z_1) dZ_1 K(Z_2, Z_1) f(Z_2) [dZ_2]^2 \right\} \end{aligned} \quad (5.54)$$

where

$$g(z) := \sum_{k \geq 1} \frac{2\alpha h_{2k+1}}{\beta h_1} z^{2k+2}. \quad (5.55)$$

This same procedure can be applied to the other terms of the sum. The k^{th} order term can be written as a sequence of $k + 1$ residues computed with the Airy recursion kernel with $g(z)dz$ on one of the outgoing legs. This computation shows that introducing non-vanishing times amounts to introducing a non-vanishing $\omega_{0,1}(z) := g(z)dz$ in the CEO-recursion.

It is often useful to represent the CEO-recursion in a graphical form by representing the interaction kernel $K(z, z_0)$ by an edge oriented from z_0 towards a trivalent vertex labelled by z

and the function $\omega_{0,2}(z_1, z_2)$ by a non-oriented edge (see [36] for more details about this set of graphs). In this form, $\omega_{g,n}(z_1, \dots, z_n)$ is a sum over trivalent graphs of genus g with n leaves labelled by the arguments z_1, \dots, z_n . The preceding computation shows that the correlation functions of the KdV curve can be obtained from the correlation functions of the Airy curve by introducing a set of new leaves, called dilation leaves, in the definition of the graphs used. A dilation leaf decorated by a label k is weighted by

$$(2d-1)!! \operatorname{res}_{z \rightarrow 0} g(z) \frac{dz}{z^{2d+1}} = (2d-1)!! \frac{2\alpha h_{2d-1}}{\beta}. \quad (5.56)$$

Plugging this expression into the formula for the Airy correlation functions proves the result. \square

General case

In this section we give a formula for the correlation function of the local CEO-recursion.

Definition 5.14. Let $\Gamma_{g,n}$ be the subset of Γ (see Notation 5.6) consisting of graphs of genus g' such that $g' + \sum_{v \in V(\Gamma)} \mathfrak{g}(v) = g$ and with n ordinary leaves. Let us also introduce orderings on the ordinary leaves and denote by $\check{\Gamma}_{g,n}$ the set of all graphs from $\Gamma_{g,n}$ with all possible orderings on the ordinary leaves. For a given graph with a fixed ordering $\check{\gamma} \in \check{\Gamma}_{g,n}$ and for an ordinary leaf of that graph $l \in L^*(\check{\gamma})$ we denote by $\mathfrak{m}(l)$ the index of this particular leaf (then $\mathfrak{m}(l)$ is an integer from 1 to n such that different leaves have different values $\mathfrak{m}(l)$ assigned to them).

Theorem 5.15. *The correlation functions can be written as a sum over decorated graphs whose vertices are weighted by intersection of ψ -classes on $\overline{\mathcal{M}}_{g,n}$, edges by the jumps, ordinary leaves by primitives of B and dilaton leaves by the times.*

For $2 - 2g - n < 0$, one has

$$\begin{aligned} \omega_{g,n}(\vec{z}) = \frac{1}{n!} \sum_{\check{\gamma} \in \check{\Gamma}_{g,n}} \prod_{v \in V(\check{\gamma})} \left(-2h_1^{i(v)} \right)^{2-2\mathfrak{g}(v)-\operatorname{val}(v)} \left\langle \prod_{h \in H(v)} \tau_{\mathfrak{t}(h)} \right\rangle_{\mathfrak{g}(v), \operatorname{val}(v)} \\ \prod_{e \in E(\check{\gamma})} \check{B}_{\mathfrak{t}(h_1(e)), \mathfrak{t}(h_2(e))}^{\mathfrak{i}(v_1(e)), \mathfrak{i}(v_2(e))} \prod_{l \in L^*(\check{\gamma})} \sum_{j=1}^N d\xi_{\mathfrak{t}(l)}^{\mathfrak{i}(v(l))}(z_{\mathfrak{m}(l)}, j) \prod_{\lambda \in L^*(\check{\gamma})} \check{h}_{\mathfrak{t}(\lambda)}^{\mathfrak{i}(v(\lambda))} \end{aligned} \quad (5.57)$$

with

$$\check{h}_k^i := 2(2k-1)!! h_{2k-1}^i, \quad (5.58)$$

$$d\xi_d^i(z_\alpha, j) := \operatorname{res}_{z \rightarrow 0} \frac{(2d+1)!! dz}{z^{2d+2}} \int^z B^{i,j}(z, z_\alpha), \quad (5.59)$$

$$\check{B}_{d_1, d_2}^{i,j} := B_{2d_1, 2d_2}^{i,j} (2d_1-1)!! (2d_2-1)!! \quad (5.60)$$

and

$$\left\langle \prod_{i=1}^n \tau_{k_i} \right\rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \psi_2^{k_2} \dots \psi_n^{k_n}. \quad (5.61)$$

Proof. The proof is very similar to the one presented in [34, 67]. However, we prefer to present a completely graphical proof so that the link with the next sections becomes clear.

We follow the proof of [34]. From the definition, one can write the correlation functions as a sum over graphs with oriented and non-oriented arrows linking trivalent vertices resulting in the following expression:

$$\omega_{g,n}^{\vec{\gamma}}(\vec{z}) = \sum_{G \in \check{\mathcal{G}}_{g,n}} \omega(G) \quad (5.62)$$

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with $\widehat{G}_{g,n}$ the set of genus g trivalent graphs with one root and $n - 1$ leaves labelled by the arguments z_i and a skeleton tree of oriented edges pointing from the root towards the leaves weighted by

$$\omega(G) = \prod_{v \in V(G)} \operatorname{res}_{Z_v \rightarrow 0} \prod_{e \in E_{\text{oriented}}(G)} K^{i(v_1(e)), i(v_2(e))}(Z_{v_1(e)}, Z_{v_2(e)}) \prod_{e \in E_{\text{unoriented}}(G)} B^{i(v_1(e)), i(v_2(e))}(Z_{v_1(e)}, Z_{v_2(e)}) \quad (5.63)$$

where each leaf is considered as a one-valent vertex v and one denotes Z_v the variable z_i associated to this leaf in the correlation function, $E_{\text{oriented}}(G)$ is the set of oriented leaves of G and $E_{\text{unoriented}}(G)$ is the set of unoriented leaves of G (see [36] for further details). The product of residues $\prod_{v \in V(G)} \operatorname{res}_{Z_v \rightarrow 0}$ is oriented following the arrows, i.e. one first computes the residue corresponding to the end of an arrow before the one associated to its root.

It is useful to remark that, for any edge, oriented or not, one has two types of contributions. Indeed, the functions $B^{i,j}(z, z')$ have a singular part

$$B_{\text{KdV}}^{i,j}(z, z') := \delta_{i,j} \frac{dz \otimes dz'}{(z - z')^2} \quad (5.64)$$

and a regular part

$$B_{\text{reg}}^{i,j}(z, z') := \sum_{k,l=0}^{\infty} B_{k,l}^{i,j} z^k z'^l dz \otimes dz' \quad (5.65)$$

when $z \rightarrow z'$:

$$B^{i,j}(z, z') = B_{\text{KdV}}^{i,j}(z, z') + B_{\text{reg}}^{i,j}(z, z'). \quad (5.66)$$

In the same way, one has

$$K^{i,j}(z, z') = K_{\text{KdV}}^{i,j}(z, z') + K_{\text{reg}}^{i,j}(z, z'). \quad (5.67)$$

One can translate this by representing $B_{\text{KdV}}^{i,j}(z, z')$ and $B_{\text{reg}}^{i,j}(z, z')$ by dashed and dotted unoriented edges respectively, while representing $K_{\text{KdV}}^{i,j}(z, z')$ and $K_{\text{reg}}^{i,j}(z, z')$ by dashed and dotted oriented edges from z to z' . The preceding sum is thus transformed into a sum over graphs where the edges are dotted or dashed and weighted accordingly.

The dashed edges can be expressed in a slightly different way. Indeed, one has

$$B_{\text{reg}}^{i,j}(z, z') = \operatorname{res}_{z_1 \rightarrow z} \operatorname{res}_{z_2 \rightarrow z'} B_{\text{KdV}}^{i,i}(z, z_1) \left[\int^{z_1} \int^{z_2} B_{\text{reg}}^{i,j}(z_1, z_2) \right] B_{\text{KdV}}^{j,j}(z_2, z') \quad (5.68)$$

and

$$K_{\text{reg}}^{i,j}(z, z') = \operatorname{res}_{z_1 \rightarrow z} \operatorname{res}_{z_2 \rightarrow \pm z'} B_{\text{KdV}}^{i,i}(z, z_1) \left[\int^{z_1} \int^{z_2} B_{\text{reg}}^{i,j}(z_1, z_2) \right] K_{\text{KdV}}^{j,j}(z_2, z') \quad (5.69)$$

by a simple application of the Cauchy formula.

Remember that such an edge comes with integration of its boundary variables, thus, one typically has to compute

$$\operatorname{res}_{z \rightarrow 0} \operatorname{res}_{z' \rightarrow 0} g(z) K_{\text{reg}}^{i,j}(z, z') f(z') \quad (5.70)$$

which reads

$$\operatorname{res}_{z \rightarrow 0} \operatorname{res}_{z_1 \rightarrow z} \operatorname{res}_{z' \rightarrow 0} \operatorname{res}_{z_2 \rightarrow \pm z'} \left(g(z) B_{\text{KdV}}^{i,i}(z, z_1) \left[\int^{z_1} \int^{z_2} B_{\text{reg}}^{i,j}(z_1, z_2) \right] K_{\text{KdV}}^{j,j}(z_2, z') f(z') \right). \quad (5.71)$$

One can move the integration contours around 0 thanks to:

$$\operatorname{res}_{z \rightarrow 0} \operatorname{res}_{z_1 \rightarrow z} = \operatorname{res}_{z_1 \rightarrow 0} \operatorname{res}_{z \rightarrow 0} - \operatorname{res}_{z \rightarrow 0} \operatorname{res}_{z_1 \rightarrow 0} \quad (5.72)$$

and

$$\operatorname{res}_{z' \rightarrow 0} \operatorname{res}_{z_2 \rightarrow \pm z'} = \operatorname{res}_{z_2 \rightarrow 0} \operatorname{res}_{z' \rightarrow 0} - \operatorname{res}_{z' \rightarrow 0} \operatorname{res}_{z_2 \rightarrow 0}. \quad (5.73)$$

Since, the integrand does not have any pole as $z_1 \rightarrow 0$ nor $z_2 \rightarrow 0$, this shows that (5.70) is equal to

$$\operatorname{res}_{z_1 \rightarrow 0} \operatorname{res}_{z \rightarrow 0} \operatorname{res}_{z_2 \rightarrow 0} \left(g(z) B_{\text{KdV}}^{i,i}(z, z_1) \left[\int^{z_1} \int^{z_2} B_{\text{reg}}^{i,j}(z_1, z_2) \right] \operatorname{res}_{z' \rightarrow 0} K_{\text{KdV}}^{j,j}(z_2, z') f(z') \right). \quad (5.74)$$

In the same way, one gets that

$$\operatorname{res}_{z \rightarrow 0} \operatorname{res}_{z' \rightarrow 0} g(z) B_{\text{reg}}^{i,j}(z, z') f(z') \quad (5.75)$$

is equal to

$$\operatorname{res}_{z_1 \rightarrow 0} \operatorname{res}_{z \rightarrow 0} \operatorname{res}_{z_2 \rightarrow 0} \left(g(z) B_{\text{KdV}}^{i,i}(z, z_1) \left[\int^{z_1} \int^{z_2} B_{\text{reg}}^{i,j}(z_1, z_2) \right] \operatorname{res}_{z' \rightarrow 0} B_{\text{KdV}}^{j,j}(z_2, z') f(z') \right). \quad (5.76)$$

One can finally proceed in a similar way for re-expressing the weights of the root and the leaves by writing

$$\operatorname{res}_{z' \rightarrow 0} K^{i,j}(z, z') f(z') = \operatorname{res}_{z_2 \rightarrow 0} \left[\int^{z_2} B^{i,j}(z, z_2) \right] \operatorname{res}_{z' \rightarrow 0} K_{\text{KdV}}^{j,j}(z_2, z') f(z') \quad (5.77)$$

and

$$\operatorname{res}_{z \rightarrow 0} g(z) B^{i,j}(z, z') = \operatorname{res}_{z_1 \rightarrow 0} \operatorname{res}_{z \rightarrow 0} g(z) B_{\text{KdV}}^{i,i}(z, z_1) \left[\int^{z_1} B^{i,j}(z_1, z') \right]. \quad (5.78)$$

Remark that, for the roots and leaves, in opposition to the inner edges, the functions are the full ones, not just the regular part.

As a result, by applying this transformation to each dotted line, any graph is composed of a set of dotted subgraphs whose vertices have the same label separated by dashed lines. Since each subgraph with label i also includes a root and leaves, it is a contribution to the correlation functions obtained for the case $N = 1$, times h_k^i and vanishing jumps $B_{k,l}^{i,i} = 0$. In the sum over graphs, one can thus replace every sum over such sub-graphs by vertices of corresponding genus weighted by the correlation function for $N = 1$, which reads

$$\omega_{g,n}^{\vec{i}}(\vec{z}) = \sum_{\gamma \in \Gamma_{g,n}} \Omega(\gamma) \quad (5.79)$$

where

$$\begin{aligned} \Omega(\gamma) = & \prod_{v \in V(\gamma)} \prod_{h \in H(v)} \operatorname{res}_{Z_h \rightarrow 0} \omega_{g(v), \text{val}(v)}^{\text{KdV}, i(v)} \left(\{Z_h\}_{h \in H(v)} \right) \\ & \prod_{e \in E(\gamma)} \int^{Z_{h_1(e)}} \int^{Z_{h_2(e)}} B_{\text{reg}}^{i(v_1(h)), i(v_2(h))}(Z_{h_1(e)}, Z_{h_2(e)}) \prod_{h \in L^*(\gamma)} \int^{Z_h} B^{i,j}(Z_h, z_h) \end{aligned} \quad (5.80)$$

5.3. CEO-RECURSION AS A SUM OVER GRAPHS

where $\omega_{g,n}^{\text{KdV},i}(z_1, \dots, z_n)$ is the genus g , n -pointed correlation function obtained from the CEO-recursion in the case $N = 1$ and the initial data:

$$\begin{cases} x(z) = z^2 + a_i \\ y(z) = \sum_{k=1}^{\infty} h_k^i z^k \\ B(z, z') = B_{\text{KdV}}(z, z') = \frac{dz \otimes dz'}{(z-z')^2} \end{cases} . \quad (5.81)$$

As explained in the preceding section, it can be expressed in terms of intersection numbers:

$$\begin{aligned} \omega_{g,n}^{\text{KdV},i}(z_1, \dots, z_n) &= (-2h_1^i)^{2-2g-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \\ &\sum_{\vec{\alpha} \in \mathbb{N}^{*m}} \prod_{k=1}^m (2\alpha_k + 1)!! \frac{h_{2\alpha_k+1}^i}{h_1^i} \prod_{i=1}^n \frac{(2d_i + 1)!! dz_i}{z_i^{2d_i+2}} \left\langle \prod_{j=1}^n \tau_{d_j} \prod_{k=1}^m \tau_{\alpha_k+1} \right\rangle_{g,n+m} \end{aligned} \quad (5.82)$$

which can be made more symmetric under the exchange of the ordinary and dilation leaves by writing

$$\begin{aligned} \omega_{g,n}^{\text{KdV},i}(z_1, \dots, z_n) &= \sum_{m=0}^{\infty} (-2h_1)^{2-2g-n-m} \frac{1}{m!} \\ &\sum_{\vec{\alpha} \in \mathbb{N}^{*m}} \prod_{k=1}^m (2\alpha_k - 1)!! 2h_{2\alpha_k-1}^i \prod_{i=1}^n \frac{(2d_i + 1)!! dz_i}{z_i^{2d_i+2}} \left\langle \prod_{j=1}^n \tau_{d_j} \prod_{k=1}^m \tau_{\alpha_k} \right\rangle_{g,n+m} . \end{aligned} \quad (5.83)$$

Absorbing the factors of the form $\frac{(2d+1)!! dz}{z^{2d+2}}$ into the corresponding half-edge contribution, the weight of an inner edge becomes

$$\text{res}_{z_1 \rightarrow 0} \text{res}_{z_2 \rightarrow 0} \int^{z_1} \int^{z_2} B_{\text{reg}}^{i,j}(z_1, z_2) \frac{(2d_1 + 1)!! dz_1}{z_1^{2d_1+2}} \frac{(2d_2 + 1)!! dz_2}{z_2^{2d_2+2}} \quad (5.84)$$

which is equal to

$$\check{B}_{d_1, d_2}^{i,j} := B_{2d_1, 2d_2}^{i,j} (2d_1 - 1)!! (2d_2 - 1)!! , \quad (5.85)$$

while the weight of the ordinary leaves becomes

$$d\xi_d^i(z_\alpha, j) := \text{res}_{z \rightarrow 0} \frac{(2d + 1)!! dz}{z^{2d+2}} \int^z B^{i,j}(z, z_\alpha), \quad (5.86)$$

where one has to consider both the singular and non-singular part of $B^{i,j}(z, z_\alpha)$. Collecting these contributions together proves the theorem. \square

5.3.4 Change of scales

An important property of the correlation functions built in this way is their homogeneity property which reads

$$\forall \lambda \in \mathbb{C}, \omega_{g,n}(\vec{z}_N | x, \lambda y, B) = \lambda^{2-2g-n} \omega_{g,n}(\vec{z}_N | x, y, B) \quad (5.87)$$

One can thus get an additional factor λ^i by replacing $h_k^i \rightarrow \lambda^i h_k^i$ resulting in a rescaling of the weight of the vertices by $(\lambda^{i(v)})^{2-2g(v)-\text{val}(v)}$.

5.3.5 Weights, Laplace transform and recursive definition

It is interesting to note that the weights of the edges are the coefficients of the Laplace transform of B ;

$$\check{B}^{i,j}(u, v) := \sum_{(k,l) \in \mathbb{N}^2} \check{B}_{k,l}^{i,j} u^{-k} v^{-l} \quad (5.88)$$

is equal to

$$\check{B}^{i,j}(u, v) = \delta_{i,j} \frac{uv}{u+v} + \frac{\sqrt{uv} e^{ua_i+va_j}}{2\pi} \int_{x(z)-a_i \in \mathbb{R}^+} \int_{x(z')-a_j \in \mathbb{R}^+} B^{i,j}(z, z') e^{-ux(z)-vx(z')}. \quad (5.89)$$

In [34], it was proved that, if dx is a meromorphic form defined on a Riemann surface, $\check{B}^{i,j}(u, v)$ can be factorized and expressed in terms of some basic functions. Here, we will consider the converse and build $B_{k,l}^{i,j}$ by induction in such a way that there exist a set of functions $\{f_{i,j}(u)\}_{i,j=1}^N$ such that

$$\check{B}^{i,j}(u, v) = \frac{uv}{u+v} \left(\delta_{i,j} - \sum_{k=1}^N f_{i,k}(u) f_{k,j}(v) \right). \quad (5.90)$$

Let us define the coefficients $B_{k,l}^{i,j}$ recursively in terms of the initial data $B_{k,0}^{i,j}$ by imposing that

$$\xi_{d+1}^i(z, j) := -2 \frac{d\xi_d^i(z, j)}{dx^{[l]}(z)} - \sum_{k=1}^b \check{B}_{d,0}^{i,k} \xi_0^k(z, j), \quad (5.91)$$

or, in terms of the Laplace transform

$$\begin{aligned} f_d^i(u, j) &:= \frac{\sqrt{u}}{2\sqrt{\pi}} \int_{x(z)-a_j \in \mathbb{R}^+} e^{-u(x(z)-a_j)} dx^j(z) \xi_d^i(z) \\ &= \delta_{i,j} (-1)^d u^d - \sum_{d'} \check{B}_{d,d'}^{i,j} u^{-d'-1} \end{aligned} \quad (5.92)$$

this reads

$$f_{d+1}^i(u, j) := -2u f_d^i(u, j) - \sum_{k=1}^N \check{B}_{d,0}^{i,k} f_0^k(z, j). \quad (5.93)$$

With this definition, one has

$$\check{B}^{i,j}(u, v) = \frac{uv}{u+v} \left(\delta_{i,j} - \sum_{k=1}^b f_0^k(u, i) f_0^k(v, j) \right). \quad (5.94)$$

5.4 Identification of the two theories

In this section we show how to find a local spectral curve corresponding to any semi-simple conformal Frobenius manifold.

Suppose some local spectral curve is given. For any $i \in \{1, \dots, N\}$ and $k \in \mathbb{Z}_{\geq 0}$ define

$$W_k^i(z) := \sum_{j=1}^N d \left(\left(-\frac{1}{z} \frac{d}{dz} \right)^k \xi_0^i(z, j) \right). \quad (5.95)$$

5.4. IDENTIFICATION OF THE TWO THEORIES

Theorem 5.16. *Let R be some series of operators on an N -dimensional vector space V as in Section 5.2. Let $Z = \hat{R}\hat{\Delta}\mathcal{T}$, where \mathcal{T} is a product of N KdV τ -functions, be the partition function of the corresponding semi-simple cohomological field theory.*

Define a local spectral curve by the following data

$$\check{B}_{p,q}^{i,j} := [z^p w^q] \frac{\delta^{ij} - \sum_{s=1}^N R_s^i(-z) R(-w)_s^j}{z+w} \quad (5.96)$$

and

$$\check{h}_k^i := [z^{k-1}] (-R(-z))_1^i \quad (5.97)$$

$$h_1^i := -\frac{1}{2\sqrt{\Delta^i}}. \quad (5.98)$$

Let $\omega_{g,n}$ be the genus g , n -pointed CEO-recursion invariant of this spectral curve and denote by

$$\Omega(\{v^{d,i}\}) = \left(\sum_{g,d} \omega_{g,d}(z_1, \dots, z_d) \Big|_{W_d^i(z_m)=v^{d,i}} \hbar^{g-1} \right) \quad (5.99)$$

their sum after a change of variables $W_k^i(z_m) \leftrightarrow v^{d,i}$ for all m . Then the partition function of the cohomological field theory and the CEO-recursion invariants agree in the following sense:

$$Z(\{v^{d,i}\}) = \exp(\Omega(\{v^{d,i}\})). \quad (5.100)$$

Proof. In Sections 5.2 and 5.3 we have given representations of Z and $\omega_{g,n}$ as sums over the set Γ (in fact, in the case of $\omega_{g,n}$ this set is $\tilde{\Gamma}$ rather than Γ , but after changing the variables $W_k^i(z_m) \leftrightarrow v^{d,i}$ we can take the sum over orderings and arrive at the sum over Γ acquiring an additional factor of $n!$, which cancels with the corresponding factor in (5.57)). We prove the theorem by showing that the contribution of each individual graph to Z is equal to the contribution to Ω .

Let $\gamma \in \Gamma$ be some graph. Note that on both sides we assign the same weight to the vertices of γ , namely to a vertex labelled (g, i) with n half-edges attached to it labelled d_1, \dots, d_n we associate

$$(-2h_1^i)^{2-2g-n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}. \quad (5.101)$$

Furthermore, by equation (5.96), any edge in γ contributes the same to Z and Ω .

Let l be an ordinary leaf of γ labelled by k attached to a vertex labelled by (g, i) . We use induction on k to show that the contribution to Z is the same as the contribution to Ω .

The contribution of l to Z is given by

$$\mathcal{L}_k^i(l) = [z^k] \left(\sum_d (R(-z))_j^i v^{d,j} z^d \right) = \sum_{d=0}^k (-1)^{k-d} (R_{k-d})_j^i v^{d,j}. \quad (5.102)$$

When $k=0$, the contribution of l to Ω is given by

$$\sum_j d\xi_0^i(z_j, j) = W_0^i. \quad (5.103)$$

Since $(R_0)_j^i = \delta_j^i$, the contributions to Z and Ω agree when $k=0$.

Now suppose that they agree for some $k \in \mathbb{Z}_{\geq 0}$. That is, suppose that

$$\sum_j d\xi_k^i(z^j, j) = \sum_{l=0}^k (-1)^{k-l} (R_{k-l})_s^i W_l^s. \quad (5.104)$$

Then, using Equation (5.91), the contribution of the leaf to Ω for the index $k+1$ is given by

$$\begin{aligned} \sum_j d\xi_{k+1}^i(z^j, j) &= \sum_j d \left(-2 \frac{\partial \xi_k^i(z^j, j)}{\partial x^j} - \sum_{t=1}^N \check{B}_{k,0}^{i,t} \xi_t^i(z^j, j) \right) \\ &= \sum_j d \left(-\frac{1}{z^j} \frac{\partial}{\partial z^j} \xi_k^i(z^j, j) - \sum_{t=1}^n -(-1)^{k+1} (R_{k+1})_t^i \xi_t^i(z^j, j) \right) \\ &= \sum_{l=0}^k (-1)^l (R_l)_t^i W_{k+1-l}^t + (-1)^{k+1} (R_{k+1})_t^i W_0^t = \sum_{l=0}^{k+1} (-1)^l (R_l)_t^i W_{k+1-l}^t, \end{aligned} \quad (5.105)$$

where we used equation (5.96) to write

$$\check{B}_{k,0}^{i,t} = -(-1)^{k+1} (R_{k+1})_t^i. \quad (5.106)$$

This completes the induction, and since it is clear that the dilaton leaves contribute the same in both cases, it also completes the proof of the theorem. \square

Remark 5.17. The theorem above deals with the potential of a cohomological field theory written in terms of formal variables $v^{d,i}$ corresponding to normalized canonical basis. In order to pass to flat coordinates one can change the variables in the following way:

$$v^{d,i} = \Psi_\mu^i t^{d,\mu}. \quad (5.107)$$

On the spectral curve side it will correspond to changing the variables W_k^i in the following way:

$$W_k^i = \Psi_\mu^i V_k^\mu. \quad (5.108)$$

Thus, the theorem holds in the same form for the potential of cohomological field theory written in terms of formal variables $t^{d,\mu}$, only one should identify $t^{d,\mu}$ with V_d^μ .

Remark 5.18. Above we established the correspondence between cohomological field theories and symplectic invariants of spectral curves. However, as noted in Remark 5.5, in the case of Gromov-Witten theories we cannot disregard quadratic terms. So, in the formula for the total descendent potential an additional operator \hat{S} appears. In some cases, again see Remark 5.5, it performs only a linear change of formal variables $t^{d,\mu}$ on which the potential depends. Thus, to establish the correspondence in this case, one has to change the variables W_k^i in precisely the same way, and then identify the resulting variables with $t^{d,\mu}$, similar to the case of previous remark. Occasionally, the changes of variables performed by $\hat{\Psi}$ and \hat{S}^{-1} can be a re-expansion of $\omega_{g,n}$ in a new coordinate on the spectral curve. In Chapter 6 we explain this procedure in detail for the case of \mathbb{P}^1 .

Remark 5.19. The system of equations obtained via a Laplace transform from the equations of Givental for the R -matrix (that is, the so-called equations of deformed flat connection) is studied in detail in [28, Section 5]. This gives, in particular, a recipe to reconstruct the two-point function directly from the Frobenius structure bypassing the reconstruction of the R -matrix. This also explains why we call the critical values a_1, \dots, a_N of x the canonical coordinates.