Hurwitz numbers, moduli of curves, topological recursion, Givental's theory and their relations

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–6– The Norbury-Scott conjecture

As a first application of Theorem 5.16, in this chapter we recall and prove the Norbury-Scott conjecture on the stationary sector of the Gromov-Witten theory of $\mathbb{P}^1$.

6.1 Gromov-Witten theory of $\mathbb{P}^1$

The Gromov-Witten theory of $\mathbb{P}^1$ is discussed from the geometric point of view in many sources, see e. g. [86]. Givental proved in [46] that his formula for the formal Gromov-Witten potential coincides with the geometric Gromov-Witten potential of $\mathbb{P}^1$, so we discuss it here only from the Givental point of view, ignoring the geometric background. One can find the same computations in [101, 102].

The underlying structure of Frobenius manifold is determined by the following solution of the WDVV equation

$$\frac{1}{2}(t^1)^2t^2 + e^{t^2}, \tag{6.1}$$

and the scalar product given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.2}$$

All ingredients of the Givental formula depend on a particular choice of the point on the Frobenius manifold, and in this case we choose the point $(0,0)$ in the coordinates (t_1, t_2).

We perform a direct computation following the recipe of Givental in [45], see also Section 5.2.2. As a possible choice of the canonical coordinates, we use

$$u^1 = t^1 + 2\exp(t^2/2); \tag{6.3}$$

$$u^2 = t^1 - 2\exp(t^2/2). \tag{6.4}$$

In particular, for $t^1 = t^2 = 0$ we have $u^1 = -u^2 = 2$. Then,

$$\Delta_1^{-1} = \left\langle \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1} \right\rangle = \frac{\exp(-t^2/2)}{2}; \tag{6.5}$$

$$\Delta_2^{-1} = \left\langle \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2} \right\rangle = \frac{-\exp(-t^2/2)}{2}, \tag{6.6}$$

so we can choose the square roots as

$$\Delta_1^{-1/2} = \frac{\exp(-t^2/4)}{\sqrt{2}}; \tag{6.7}$$

$$\Delta_2^{-1/2} = \frac{-i\exp(-t^2/4)}{\sqrt{2}}. \tag{6.8}$$

For this choice we have the following matrix of transition from the basis given by $(\partial/\partial t^1, \partial/\partial t^2)$ to the normalized canonical basis:

$$\Psi = \begin{pmatrix} \frac{\exp(-t^2/4)}{\sqrt{2}} & \frac{-i\exp(-t^2/4)}{\sqrt{2}} \\ \frac{\exp(t^2/4)}{\sqrt{2}} & \frac{i\exp(t^2/4)}{\sqrt{2}} \end{pmatrix}. \tag{6.9}$$

It is the matrix $\Psi = \Psi^i_\alpha$, where α labels the rows and corresponds to the flat basis, while i labels the columns and corresponds to the normalized canonical basis.

The recipe of reconstruction of the matrix R from [45] gives at the origin the matrix $R(\zeta) = \sum_{k=0}^{\infty} R_k\zeta^k$, where

$$R_k = \frac{(2k-1)!!(2k-3)!!}{2^{4k}k!} \cdot \begin{pmatrix} -1 & (-1)^{k+1}2ki \\ 2ki & (-1)^{k+1} \end{pmatrix} \tag{6.10}$$

The S matrix is given by the derivatives of the deformed flat coordinates, computed in [29, Example 3.7.9] At the origin we have:

$$\begin{aligned} S(\zeta^{-1}) = \mathbf{I} + \zeta^{-1} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ + \sum_{k=1}^{\infty} \frac{\zeta^{-2k}}{(k!)^2} \begin{pmatrix} 1 - 2k\left(\frac{1}{1} + \cdots \frac{1}{k}\right) & 0 \\ 0 & 1 \end{pmatrix} \\ + \sum_{k=1}^{\infty} \frac{\zeta^{-2k-1}}{(k!)^2} \begin{pmatrix} 0 & -2\left(\frac{1}{1} + \cdots \frac{1}{k}\right) \\ \frac{1}{k+1} & 0 \end{pmatrix}. \end{aligned} \tag{6.11}$$

(Note once again that we are using the convention that the matrices are acting on vector rows, opposite to the standard one).

The unit vector at the origin in the normalized canonical basis is equal to

$$e = (1,0) \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \left(\frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}\right). \tag{6.12}$$

Therefore, the dilaton leaves (cf. Equation (5.26)) in the Givental formula for $\mathbb{P}^1$ at the origin are

$$(\mathcal{L}^\bullet)^1_{k+1} = \frac{1}{\sqrt{2}} \cdot \frac{(-1)^{k+1}\left((2k-1)!!\right)^2}{k!2^{4k}}; \tag{6.13}$$

$$(\mathcal{L}^\bullet)^2_{k+1} = \frac{i}{\sqrt{2}} \cdot \frac{\left((2k-1)!!\right)^2}{k!2^{4k}} \tag{6.14}$$

for $k \geq 0$.

Proposition 6.1. *The Gromov-Witten potential of* $\mathbb{P}^1$*,*

$$Z_{\mathbb{P}^1}(\hbar, \{t^{\ell,1}, t^{\ell,2}\}_{\ell=0}^{\infty}), \tag{6.15}$$

is obtained from $\hat{R}\hat{\Delta}Z_{\mathrm{KdV}}^{\otimes 2}$ *(understood as a sum over graphs in the sense of Section 5.2.3 and written down in the normalized canonical basis, that is, in the variables* $v^{d,i}$*,* $d \geq 0$*,* $i = 1, 2$*) via a linear change of variables given by*

$$\sum_{m \geq k}^{\infty} \left(t^{m,1}, t^{m,2}\right) S_k\zeta^{m-k} = \sum_{\ell=0}^{\infty} \left(v^{\ell,1}, v^{\ell,2}\right)\zeta^\ell \cdot \Psi^{-1}, \tag{6.16}$$

and a correction of the unstable terms (that is, (g,n)*-correlators with* $2g-2+n \leq 0$*).*

Proof. In order to get the Gromov-Witten potential of $\mathbb{P}^1$ as given by the Givental formula, we have to apply the $\hat{\Psi}$- and $\hat{S}^{-1}$-action to the expression in terms of graphs discussed in Section 5.2.3 that corresponds to $\hat{R}\hat{\Delta}Z_{\mathrm{KdV}}^{\otimes 2}$. The $\hat{\Psi}$-action is just a linear change of variable by definition. The general S-action is discussed in [44, Section 4.2]. It is a combination of a shift of variables that vanishes in our case (indeed, $(1,0)S_1 = (0,0)$), the linear change of variables that we have in the statement of Proposition, and a correction of unstable terms that is not essential for us. □

6.2 The Norbury-Scott conjecture

Norbury and Scott [84] propose the following construction. They consider a spectral curve given by

$$\begin{cases} x &= z + \frac{1}{z}; \\ y &= \log z, \end{cases} \tag{6.17}$$

and the standard two-point function

$$B(z,z') = \frac{\mathrm{d}z \otimes \mathrm{d}z'}{(z-z')^2}. \tag{6.18}$$

Via CEO-recursion they obtain the n-forms $\omega_{g,n}$ that they consider in the global variable x, and they conjecture the following theorem (they prove it for $g = 0,1$):

Theorem 6.2. *For $2g-2+n>0$, we have:*

$$\prod_{j=1}^{n} \left(- \operatorname*{res}_{x_j=\infty} \frac{1}{(a_j+1)!} x_1^{a_j+1} \right) \omega_{g,n}(x_1,\dots,x_n) = \langle \prod_{j=1}^{n} \tau_{2,a_j} \rangle_g, \tag{6.19}$$

where $\langle \prod_{j=1}^{n} \tau_{2,a_j} \rangle_g$ is the corresponding correlator in $Z_{\mathbb{P}^1}$, that is, the coefficient of

$$\hbar^{g-1} \prod_{j=1}^{n} t_{2,a_j} / |Aut(a_1,\dots,a_n)| \tag{6.20}$$

in $\log Z_{\mathbb{P}^1}$.

In the rest of this section we prove this theorem, identifying all ingredients of the CEO-recursion with the corresponding parts of the Givental formula.

6.3 Proof of the Norbury-Scott conjecture

6.3.1 Local coordinates near the branch points

We denote the local coordinates by $z_1 = \sqrt{x-2}$ and $z_2 = \sqrt{x+2}$. Then we have:

$$x = z_1^2 + 2 \text{ near } x = 2,\ z = 1,\ z_1 = 0; \tag{6.21}$$
$$x = z_2^2 - 2 \text{ near } x = -2,\ z = -1,\ z_2 = 0. \tag{6.22}$$

Therefore,

$$z = 1 + \frac{z_1^2}{2} \pm z_1 \sqrt{1 + \frac{z_1^2}{4}}; \tag{6.23}$$

$$z = -1 + \frac{z_2^2}{2} \pm i z_2 \sqrt{1 - \frac{z_2^2}{4}}. \tag{6.24}$$

In both cases we choose $+$ for $\pm$.

6.3.2 Expansion of y

Recall that $y = \log z$. A direct computation shows:

$$y = \int \frac{\mathrm{d}z_1}{\sqrt{1+\frac{z_1^2}{4}}}; \tag{6.25}$$

$$y = \int \frac{-i\,\mathrm{d}z_2}{\sqrt{1-\frac{z_2^2}{4}}}; \tag{6.26}$$

Note that

$$\frac{1}{\sqrt{1+\frac{z_1^2}{4}}} = 1 + \sum_{k=1}^{\infty} z_1^{2k} \cdot \frac{(-1)^k (2k-1)!!}{k! 2^{3k}}; \tag{6.27}$$

$$\frac{-i}{\sqrt{1-\frac{z_2^2}{4}}} = -i + \sum_{k=1}^{\infty} z_2^{2k} \cdot \frac{(-i)\cdot(2k-1)!!}{k! 2^{3k}}. \tag{6.28}$$

Therefore

$$y = z_1 + \sum_{k=1}^{\infty} z_1^{2k+1} \cdot \frac{(-1)^k (2k-1)!!}{k! 2^{3k}(2k+1)}; \tag{6.29}$$

$$y = -iz_2 + \sum_{k=1}^{\infty} z_2^{2k+1} \cdot \frac{(-i)\cdot(2k-1)!!}{k! 2^{3k}(2k+1)}. \tag{6.30}$$

Thus the coefficients $\check{h}^i_{k+1}$, $k \geq 0$, are given by the following formulas:

$$\check{h}^1_{k+1} = 2 \cdot \frac{(-1)^k \left((2k-1)!!\right)^2}{k! 2^{3k}}; \tag{6.31}$$

$$\check{h}^2_{k+1} = 2 \cdot \frac{(-i)\cdot \left((2k-1)!!\right)^2}{k! 2^{3k}}. \tag{6.32}$$

6.3.3 The matrix $f_{i,j}(w)$

We use the following definition of the matrix $f_{ij}(w)$ (cf. Equation (5.90)):

$$f_{ij}(w) = \delta_{ij} - w\check{B}^{[ij]}(0, w^{-1}), \tag{6.33}$$

where $w = v^{-1}$. We use $\tilde{B}^{ij}_{0,l} = (B^{ij}_{\mathrm{reg}})_{0,2l}(2l-1)!!$, and the following expressions:

$$B^{11}_{\mathrm{reg}}(0, z_1) = \left[\frac{\mathrm{d}z(z_1')\otimes \mathrm{d}z(z_1)}{(z(z_1')-z(z_1))^2} - \frac{\mathrm{d}z_1'\otimes \mathrm{d}z_1}{(z_1'-z_1)^2}\right]_{z_1'=0} \tag{6.34}$$

$$B^{12}_{\mathrm{reg}}(0, z_2) = \left[\frac{\mathrm{d}z(z_1')\otimes \mathrm{d}z(z_2)}{(z(z_1')-z(z_2))^2}\right]_{z_1'=0} \tag{6.35}$$

$$B^{21}_{\mathrm{reg}}(0, z_1) = \left[\frac{\mathrm{d}z(z_2')\otimes \mathrm{d}z(z_1)}{(z(z_2')-z(z_1))^2}\right]_{z_2'=0} \tag{6.36}$$

$$B^{22}_{\mathrm{reg}}(0, z_2) = \left[\frac{\mathrm{d}z(z_2')\otimes \mathrm{d}z(z_2)}{(z(z_2')-z(z_2))^2} - \frac{\mathrm{d}z_2'\otimes \mathrm{d}z_2}{(z_2'-z_2)^2}\right]_{z_2'=0} \tag{6.37}$$

Therefore,

$$B^{11}_{\text{reg}}(0, z_1) = \frac{1}{z_1^2}\left(\frac{1}{\sqrt{1+\frac{z_1^2}{4}}} - 1\right) \tag{6.38}$$

$$B^{12}_{\text{reg}}(0, z_2) = \frac{i}{4(1-\frac{z_2^2}{4})^{3/2}} \tag{6.39}$$

$$B^{21}_{\text{reg}}(0, z_1) = \frac{i}{4(1+\frac{z_1^2}{4})^{3/2}} \tag{6.40}$$

$$B^{22}_{\text{reg}}(0, z_2) = \frac{1}{z_2^2}\left(\frac{1}{\sqrt{1-\frac{z_2^2}{4}}} - 1\right) \tag{6.41}$$

So, we have the following expansions:

$$B^{11}_{\text{reg}}(0, z_1) = \sum_{k=0}^{\infty} z_1^{2k} \cdot \frac{(-1)^{k+1}(2k+1)!!}{(k+1)!2^{3(k+1)}} \tag{6.42}$$

$$B^{12}_{\text{reg}}(0, z_2) = \sum_{k=0}^{\infty} z_2^{2k} \cdot \frac{i(2k+1)!!}{(k)!2^{3k+2}} \tag{6.43}$$

$$B^{21}_{\text{reg}}(0, z_1) = \sum_{k=0}^{\infty} z_2^{2k} \cdot \frac{i(-1)^k(2k+1)!!}{(k)!2^{3k+2}} \tag{6.44}$$

$$B^{22}_{\text{reg}}(0, z_2) = \sum_{k=0}^{\infty} z_2^{2k} \cdot \frac{(2k+1)!!}{(k+1)!2^{3(k+1)}}. \tag{6.45}$$

The formulas for $f_{ij}(w)$ are then

$$f_{11}(w) = 1 + \sum_{k=1}^{\infty} w^k \cdot \frac{(-1)^{k+1}(2k-1)!!(2k-3)!!}{k!2^{3k}} \tag{6.46}$$

$$f_{12}(w) = \sum_{k=1}^{\infty} w^k \cdot \frac{-i(2k-1)!!(2k-3)!!}{(k-1)!2^{3k-1}} \tag{6.47}$$

$$f_{21}(w) = \sum_{k=1}^{\infty} w^k \cdot \frac{(-1)^k i(2k-1)!!(2k-3)!!}{(k-1)!2^{3k-1}} \tag{6.48}$$

$$f_{22}(w) = 1 + \sum_{k=1}^{\infty} w^k \cdot \frac{-(2k-1)!!(2k-3)!!}{k!2^{3k}} \tag{6.49}$$

This coincides with the formula for the $\sum_{k=0}^{\infty} R_k 2^k(-w)^k$ at the point $(0,0)$.

6.3.4 Comparison of the coefficient of (g, n, m)-vertex

In this section we consider a vertex of genus g with n attached half-edges or ordinary leaves, and m dilaton leaves, with an associated intersection number $\langle \prod_{i=1}^{n} \tau_{d_i} \prod_{i=1}^{m} \tau_{a_i+1} \rangle_{g,n+m}$. There are vertices of type 1 and type 2, depending on the canonical coordinate that we associate to the vertex. We compare the coefficients that we associate to these vertices in the Givental case, using the data from Section 6.1 in Formula (5.29), and in the case of local CEO-recursion, using the data from Sections 6.3.1-6.3.3 in Formula (5.57).

The coefficients that we have in Formula (5.57) (at the vertex of the type 1 and 2 resp.) are:

$$(-2)^{2-2g-n-m} \qquad \text{and} \qquad (2i)^{2-2g-n-m}. \tag{6.50}$$

Let us compute how these coefficients change if we take into account all the differences between R-matrix and the dilaton leaves. For convenience, from now on we rescale the differential forms on the leaves, $W^i_a \to 2^{-a}W^i_a$, $i = 1, 2$, $a = 0, 1, 2, \dots$. Observe that this rescaling, the extra factor of 2^k in R_k and, in addition, an extra factor of $\sqrt{2}$ that we have to put by hand on each ordinary leave give us together the extra factors of

$$2^{\sum_{i=1}^n d_i} 2^{n/2} \qquad \text{and} \qquad 2^{\sum_{i=1}^n d_i} 2^{n/2}. \tag{6.51}$$

Then the quotient of the contributions of the dilaton leaves gives us extra factors of

$$2^{\sum_{i=1}^m (a_i+1)} 2^{m/2}(-1)^m \qquad \text{and} \qquad 2^{\sum_{i=1}^m (a_i+1)} 2^{m/2}(-1)^m. \tag{6.52}$$

Let us assign by hand an extra factor of $(-1)^{2g-2+n}$ to each (g, n, m)-vertex. This way we get the following coefficients:

$$2^{g-1+n/2+m/2} \qquad \text{and} \qquad 2^{g-1+n/2+m/2} i^{2g-2+n+m}. \tag{6.53}$$

These coefficients are precisely

$$\left((\Delta_1)^{1/2}\right)^{2g-2+n+m} \qquad \text{and} \qquad \left((\Delta_2)^{1/2}\right)^{2g-2+n+m}. \tag{6.54}$$

Therefore, the coefficient of $\prod_{k=1}^{\hat{n}} 2^{-d_k} W^{i_k}_{d_k}$ in a graph of global genus $\hat{g}$ with $\hat{n}$ marked leaves in the Formula (5.57) for the set up of Norbury-Scott, multiplied by

$$2^{\hat{n}/2}(-1)^{2\hat{g}-2+\hat{n}} = (-\sqrt{2})^{\hat{n}}, \tag{6.55}$$

is equal to the coefficient of $\prod_{k=1}^{\hat{n}} t^{d_k, i_k}$ in the same graph in Formula (5.29). This extra factor will be taken into account via a rescaling of the variables by $-\sqrt{2}$.

6.3.5 The Ψ-action

Let us apply the Ψ-operator to the leaves. After comparing the R-action with the graph expansion given by formulas (5.29) and (5.57), and taking into account the extra factor of $-\sqrt{2}$, we have the following identification of the marking on the leaves:

$$\sum_{a-b=c} (t^{a,1}, t^{a,2}) S_b = \left(2^{-c}W^1_c, 2^{-c}W^2_c\right) \Psi^{-1}/(-\sqrt{2}). \tag{6.56}$$

Here

$$W^1_0 = \left.\frac{\mathrm{d}z}{(1-z)^2}\right|_{z=z(z_1)} + \left.\frac{\mathrm{d}z}{(1-z)^2}\right|_{z=z(z_2)} \tag{6.57}$$

$$W^2_0 = \left.\frac{i\,\mathrm{d}z}{(1+z)^2}\right|_{z=z(z_1)} + \left.\frac{i\,\mathrm{d}z}{(1+z)^2}\right|_{z=z(z_2)}, \tag{6.58}$$

and

$$2^{-c}W^i_c = \mathrm{d}\left(\left(-\frac{d}{dx}\right)^c \int W^i_0\right), \tag{6.59}$$

so we can work in the global coordinate z rather than in the local coordinates z_1, z_2.

Since

$$\Psi^{-1}/(-\sqrt{2}) = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-i}{2} & \frac{i}{2} \end{pmatrix}, \tag{6.60}$$

we have:

$$\sum_{a-b=c} (t^{a,1}, t^{a,2}) S_b = \left(U_c^1, U_c^2\right), \tag{6.61}$$

where

$$U_0^1 = \frac{1}{2}\left(-\frac{dz}{(1-z)^2} + \frac{dz}{(1+z)^2}\right) \tag{6.62}$$

$$U_0^2 = \frac{-1}{2}\left(\frac{dz}{(1-z)^2} + \frac{dz}{(1+z)^2}\right) \tag{6.63}$$

and

$$U_c^i = \mathrm{d}\left(\left(-\frac{\partial}{\partial x}\right)^c \int U_0^i\right), \qquad i = 1, 2; c = 0, 1, 2, \ldots. \tag{6.64}$$

6.3.6 The S-action

The S-action is just a linear change of variables prescribed by Equation (6.61). This means that we replace each U_c^i with a linear combination of times $t^{a,j}$, $a \geq c$, where the coefficient of $t^{a,2}$ (this is the series of variables corresponding to the stationary sector) is equal to

$$\begin{cases} 0, & \text{if } a-c \text{ is even;} \\ \frac{1}{(k+1)\cdot(k!)^2}, & \text{if } a-c = 2k+1. \end{cases} \tag{6.65}$$

for $i = 1$, and

$$\begin{cases} \frac{1}{(k!)^2}, & \text{if } a-c = 2k; \\ 0, & \text{if } a-c \text{ is odd.} \end{cases} \tag{6.66}$$

for $i = 2$.

Norbury and Scott make the same kind of a linear change of variables, with the coefficient of $t^{a,2}$ in U_c^j, $j = 1, 2$, given by

$$-\operatorname*{res}_{x=\infty} \frac{1}{(a+1)!} x^{a+1} U_c^j = \frac{1}{(a+1)!} \operatorname*{res}_{z=0} \left(z + \frac{1}{z}\right)^{a+1} U_c^j. \tag{6.67}$$

In order to complete the proof of Theorem 6.2, we have to check two things: (1) that the Norbury-Scott formula for the contribution depends only on the difference $a-c$; (2) that for $c = 0$ Equation (6.67) gives exactly the same coefficients as we have in Equations (6.65) and (6.66).

The first thing follows directly from the formula. Indeed,

$$\begin{aligned} -\oint \frac{x^{a+1}}{(a+1)!} \mathrm{d}\left(\left(-\frac{\partial}{\partial x}\right)^c \int U_0^j\right) &= \oint \frac{x^a}{(a)!} \left(\left(-\frac{\partial}{\partial x}\right)^c \int U_0^j\right) \mathrm{d}x \\ &= \oint \frac{x^{a-c}}{(a-c)!} \left(\int U_0^j\right) \mathrm{d}x \\ &= -\oint \frac{x^{a+1-c}}{(a+1-c)!} \mathrm{d}\left(\int U_0^j\right). \end{aligned} \tag{6.68}$$

In particular, we see that the coefficient is equal to 0 if $a < c$.

Then, a direct computation shows that

$$\begin{aligned}
&\frac{1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1}U_0^1 \\
&=\frac{1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1}\frac{1}{2}\left(-\frac{\mathrm{d}z}{(1-z)^2}+\frac{\mathrm{d}z}{(1+z)^2}\right) \\
&=\frac{1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1}\frac{-2z\,\mathrm{d}z}{(1-z^2)^2} \\
&=\begin{cases} 0, & \text{if } a \text{ is even;} \\ \frac{-2}{(2k+2)!}\left(\binom{2k+2}{0}(k+1)+\binom{2k+2}{1}k+\cdots+\binom{2k+2}{k}1\right) & \text{if } a=2k+1. \end{cases} \\
&=\begin{cases} 0, & \text{if } a \text{ is even;} \\ \frac{-1}{(k+1)(k!)^2} & \text{if } a=2k+1. \end{cases}
\end{aligned} \tag{6.69}$$

and

$$\begin{aligned}
&\frac{1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1}U_0^2 \\
&=\frac{1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1}\frac{-1}{2}\left(\frac{\mathrm{d}z}{(1-z)^2}+\frac{\mathrm{d}z}{(1+z)^2}\right) \\
&=\frac{-1}{(a+1)!}\operatorname*{res}_{z=0}\left(z+\frac{1}{z}\right)^{a+2}\frac{z\,\mathrm{d}z}{(1-z^2)^2} \\
&=\begin{cases} \frac{-1}{(2k+1)!}\left(\binom{2k+2}{0}\cdot(k+1)+\binom{2k+2}{1}\cdot k+\cdots\binom{2k+2}{k}\cdot 1\right) & \text{if } a=2k; \\ 0, & \text{if } a \text{ is odd} \end{cases} \\
&=\begin{cases} \frac{-1}{(k!)^2} & \text{if } a=2k; \\ 0, & \text{if } a \text{ is odd.} \end{cases}
\end{aligned} \tag{6.70}$$

We see that there is an extra factor of (-1) in all coefficients. This means that the (g,n)-correlation functions of Norbury-Scott differ from the stationary Gromov-Witten invariants of $\mathbb{P}^1$ by a factor of $(-1)^n$. But this factor is exactly the difference we must have because Norbury and Scott are using a different convention of the sign in the CEO-recursion, cf. Remark 5.12. This completes the proof of Theorem 6.2.