Hurwitz numbers, moduli of curves, topological recursion, Givental's theory and their relations
Spitz, L.

Citation for published version (APA):
Spitz, L. (2014). Hurwitz numbers, moduli of curves, topological recursion, Givental's theory and their relations
8.1 Introduction
In this chapter we discuss two conjectures related to Hurwitz numbers with completed cycles that generalize two important theorems in the theory of ordinary Hurwitz numbers. The first is a generalization of the celebrated ELSV formula relating Hurwitz numbers and Hodge integrals [31] that we call the r-ELSV conjecture. The second is a generalization of the (now proved) Bouchard-Marino conjecture [10, 35, 8] relating Hurwitz numbers to the CEO-recursion procedure of Chekhov, Eynard, and Orantin [15, 36], that we call the r-BM conjecture. We prove that these two conjectures are equivalent. This completes our evidence for the r-BM conjecture started in Chapter 3, while simultaneously providing evidence for the r-ELSV conjecture.

8.1.1 The space of r-spin structures
The r-spin structures on smooth curves
Fix an integer \( r \geq 1 \). Let \( C \) be a (smooth compact connected) complex curve of genus \( g \) with \( n \) marked (pairwise distinct numbered) points \( x_1, \ldots, x_n \). Denote by \( K \) the canonical (cotangent) line bundle over \( C \). Choose \( n \) integers \( a_1, \ldots, a_n \in \{0, \ldots, r-1\} \) such that \( 2g-2-\sum a_i \) is divisible by \( r \).

**Definition 8.1.** An r-spin structure on \( C \) is a line bundle \( L \) together with an identification \( L^\otimes r \cong K(\sum a_i x_i) \). (8.1)

The moduli space \( \mathcal{M}_{g,n,a_1,\ldots,a_n}^{1/r} \) is the space of all r-spin structures on all smooth curves.

We will denote this space by \( \mathcal{M}_{g,n}^{1/r} \) omitting the other indices if this does not lead to ambiguity.

The element \( K(-\sum a_i x_i) \in \text{Pic}(C) \) can be divided by \( r \) in \( r^{2g} \) different ways. Hence there are exactly \( r^{2g} \) different r-spin structures on every smooth curve \( C \). Thus the natural morphism \( \pi : \mathcal{M}_{g,n}^{1/r} \to \mathcal{M}_{g,n} \) to the moduli space of smooth curves is actually a nonramified \( r^{2g} \)-sheeted covering.

The space \( \mathcal{M}_{g,n}^{1/r} \) has a structure of orbifold or smooth Deligne-Mumford stack. The stabilizer of a generic r-spin structure is \( \mathbb{Z}/r\mathbb{Z} \) (except for several cases with small \( g \) and \( n \) where it can be bigger), because every line bundle \( L \) has \( r \) automorphisms given by the multiplication by \( r \)-th roots of unity. In particular, the degree of \( \pi \) is not \( r^{2g} \) but \( r^{2g-1} \).

The compactification
A natural compactification \( \overline{\mathcal{M}}_{g,n,a_1,\ldots,a_n}^{1/r} \) of \( \mathcal{M}_{g,n,a_1,\ldots,a_n}^{1/r} \) was constructed in [58], [2], [12], [16]. It has the structure of an orbifold or a smooth Deligne-Mumford stack. There are three different
8.1. Introduction

Constructions that involve different versions of the universal curve \( \mathcal{T}^{(1/r)}_{g,n,a_1,\ldots,a_n} \) but the moduli space \( \overline{M}_{g,n,a_1,\ldots,a_n}^{(1/r)} \) is the same.

In the geometrically most natural construction the universal curve over the compactification has orbifold fibers and \( \mathcal{L} \) is extended into a line bundle in the orbifold sense. The second construction is a fiberwise coarsification of the first one. After the coarsification the universal curve becomes a singular orbifold with \( A_{-1} \) singularities and \( \mathcal{L} \) becomes a rank 1 torsion-free sheaf rather than a line bundle. Finally, the third construction is obtained from the second one after resolving its singularities by a sequence of blow-ups. In the third construction the universal curve is smooth and \( \mathcal{L} \) is again a line bundle, but the morphism \( \mathcal{L}^{(1/r)} \to K(\sum a_i \omega_i) \) has zeros on the exceptional divisors. The third construction is most convenient for intersection theoretic computations.

On a singular stable curve with an \( r \)-spin structure, every branch of every node carries an index \( a \in \{0, \ldots, r-1\} \) such that the sum of indices at each node equals \(-2\) modulo \( r \). The divisibility condition \( 2g - 2 - \sum a_i = 0 \) mod \( r \) is satisfied on each irreducible component of the curve. Thus for a separating node the indices of the branches are determined uniquely, while for a nonseparating node there are \( r \) choices.

From now on we write simply \( \overline{M}^{(1/r)} \) and \( \mathcal{T}^{(1/r)} \) for the compactified space of \( r \)-spin structures and its universal curve, if this does not lead to ambiguity.

Remark 8.2. The interest in \( \overline{M}^{(1/r)} \) was initially motivated by Witten’s conjecture asserting that some natural intersection numbers on this space can be arranged into a generating series that satisfies the \( r \)-KdV (or \( r \)th higher Gelfand-Dikii) hierarchy of partial differential equations [108]. The conjecture is now proved [44] and other beautiful results on the intersection theory of \( \overline{M}^{(1/r)} \) have been obtained [17, 100].

One of the main ingredients in Witten’s conjecture is the so-called “Witten top Chern class” whose definition uses the sheaves \( R^i p_* \mathcal{L} \) and \( R^i p_* \mathcal{L} \) on \( \overline{M}^{(1/r)} \) in a rather involved way (see [93] or [18]). The \( r \)-ELSV conjecture, on the other hand, uses the more straightforward total Chern class \( c(R^i p_* \mathcal{L})/c(R^0 p_* \mathcal{L}) \). Therefore it is not at all clear if the \( r \)-ELSV formula can be related to Witten’s conjecture. However, it would provide a link between the intersection theory on \( \overline{M}^{(1/r)} \) and integrable hierarchies via Hurwitz numbers, see Theorem 8.12.

The map \( \pi : \mathcal{M}^{(1/r)} \to \mathcal{M}_{g,n} \) extends to the compactifications \( \overline{\pi} : \overline{\mathcal{M}}^{(1/r)} \to \overline{\mathcal{M}_{g,n}} \). Thus the classes \( \psi_1, \ldots, \psi_v \in H^2(\overline{\mathcal{M}_{g,n}}/\mathbb{Q}) \) that one usually defines on \( \overline{\mathcal{M}_{g,n}} \) (see, for instance, [104], Section 3.5) can be lifted to \( \overline{\mathcal{M}}^{(1/r)} \). By abuse of notation we will denote the lifted classes by \( \overline{\psi}_i \) instead of \( \overline{\pi}^* (\psi_i) \).

The projection \( p : \mathcal{T}^{(1/r)} \to \overline{\mathcal{T}}^{(1/r)} \) induces a push-forward of \( \mathcal{L} \) in the sense of derived functors:

\[
R^i p_* \mathcal{L} = R^i p_* \mathcal{L} - R^{i-1} p_* \mathcal{L}.
\]

(8.2)

Notation 8.3. We denote by \( S \) the total Chern class

\[
S = c(-R^0 p_* \mathcal{L}) = c(R^1 p_* \mathcal{L})/c(R^0 p_* \mathcal{L}).
\]

(8.3)

8.1.2 The \( r \)-ELSV Conjecture

Let \( \mathcal{M}^{(1)}_{g,n,k_1,\ldots,k_n}(\mathbb{P}^1) \) be the space of stable genus \( g \) maps to \( \mathbb{P}^1 \) relative to \( \infty \in \mathbb{P}^1 \) with profile \((k_1,\ldots,k_n)\) and with \( m \) marked points in the source curve. It is a compactification of the space of meromorphic functions on genus \( g \) curves with \( n \) numbered poles of orders \( k_1,\ldots,k_n \) and \( m \) more marked points. See [104], Section 5 for a precise definition and main properties. Let \( \omega \in H^2(\mathbb{P}^1) \) be the Poincaré dual class of a point.
8. The \( r \)-ELSV Formula and the \( r \)-BM Conjecture

We write every \( k_i \) in the form \( k_i = r p_i + (r - 1 - a_i) \), where \( p_i \) is the quotient and \( r - 1 - a_i \) the remainder of the division of \( k_i \) by \( r \). Let \( m = (\sum k_i + n + 2g - 2)/r \).

Denote by \( h_{g,r,k_1,\ldots,k_n} \) the following Gromov-Witten invariants

\[
h_{g,r,k_1,\ldots,k_n} := (\text{ev}_1^*)^m \int \psi_1^* \cdots \psi_n^* \omega^r.
\]

\((8.4)\)

Okounkov and Pandharipande \([88]\) studied Gromov-Witten invariants as above and proved that they are equal to the Hurwitz numbers with completed cycles \( h^{(r)}_{g,k_1,\ldots,k_n} \) that were introduced in Chapter 2.

Introduce the following integrals over the space of \( r \)-spin structures:

\[
f_{g,r,k_1,\ldots,k_n} = m! \int \prod_{i=1}^n (\frac{k_i}{p_i})^{p_i} \times \int_{\mathcal{M}_{g,n+1,-m}} \mathcal{S} \frac{1}{(1 - \frac{r}{p_1} \psi_1) \cdots (1 - \frac{r}{p_n} \psi_n)}
\]

\((8.5)\)

Chiado \([17]\) studied the class \( \mathcal{S} \) expressing it in terms of standard cohomology classes on the moduli space, the coefficients being equal to values of Bernoulli polynomials at rational points with denominator \( r \). His work allows one to compute the numbers \( f_{g,r,k_1,\ldots,k_n} \).

Conjecture 8.4 (r-ELSV). We have

\[
h_{g,r,k_1,\ldots,k_n} = f_{g,r,k_1,\ldots,k_n}.
\]

\((8.6)\)

This conjecture suggests a hidden equality between virtual fundamental classes. More precisely, if we consider the space of relative stable maps \( f \) such that \( df \) has an \( r \)-th root, the conjecture seems to imply that its virtual fundamental class is obtained from the virtual fundamental class of the space of all stable maps after multiplication by \( r \)-th powers of \( \psi \)-classes. However this is not elucidated so far and our evidence for this conjecture is not geometric.

The \( r \)-ELSV formula was first conjectured in \([111]\). Since both sides are computable, the conjecture can be tested on a computer, and there is virtually no doubt that it is correct. The conjecture is proved in genus 0, and also for \((g,n) = (1,1)\) and arbitrary \( r \) (unpublished).

8.1.3 The \( r \)-Bouchard-Marino Conjecture

In \([15, 36]\) Chekhov, Eynard, and Orantin assigned to every plane analytic complex curve \( S \) a series of invariants \( \omega_{g,n}(S) \) called correlation \( n \)-forms. Each \( \omega_{g,n} \) is a meromorphic \( n \)-form on \( S^n \). The construction is described in detail in Chapter 1.

Definition 8.5. The \( n \)-point functions of genus \( g \) for \( r \)-Hurwitz numbers and for \( r \)-spin integrals are defined as

\[
H^{(r)}_{g,n}(x_1, \ldots, x_n) = \sum_{k_1,\ldots,k_n} \frac{h^{(r)}_{g,k_1,\ldots,k_n}}{m!} \exp(k_1x_1 + \cdots + k_nx_n).
\]

\((8.7)\)

\[
F_{g,r}(x_1, \ldots, x_n) = \sum_{k_1,\ldots,k_n} \frac{f_{g,r,k_1,\ldots,k_n}}{m!} \exp(k_1x_1 + \cdots + k_nx_n).
\]

\((8.8)\)

We also let, for any function \( f(x_1, \ldots, x_n) \),

\[
Df = \frac{\partial f}{\partial x_1} dx_1 \cdots \frac{\partial f}{\partial x_n} dx_n.
\]

Conjecture 8.6 (r-BM). Let \( \omega_{g,n} \) be the \( n \)-point correlation forms of the plane curve \( x = -y^r + log y \). Then we have

\[
DH^{(r)}_{g,n}(x_1, \ldots, x_n) = \omega_{g,n}.
\]

\((8.10)\)

149
8.1.4 Equivalence of the two conjectures
The goal of this chapter is to prove the following theorem.

**Theorem 8.7.** Let \( \omega_{g,n} \) be the \( n \)-point correlation forms of the plane curve \( x = -y^r + \log y \).
Then we have
\[
DF_{g,r}(x_1, \ldots, x_n) = \omega_{g,n}.
\]  
(8.11)

It shows that the \( r \)-ELSV conjecture is equivalent to \( r \)-BM conjecture. Since there is independent evidence for both conjectures, this theorem transfers that evidence between them. It is also interesting in its own right, providing a relation between two a priori very different aspects of the theory of Hurwitz numbers.

8.1.5 Plan of the chapter
In Section 8.2 we describe once again the completed cycles and completed Hurwitz numbers involved in the \( r \)-ELSV conjecture.

In Section 8.3 we review the work of Chiodo showing that the cohomology classes \( S \) form a cohomological field theory. We determine that cohomological field theory and the Givental \( R \)-matrix assigned to \( S \).

Section 8.4 contains the proof of the main theorem. It is based on the identification [113] of the Givental-Teleman theory of semi-simple cohomological field theories with the Chekhov-Eynard-Orantin topological recursion theory. We check that the \( R \)-matrix for the class \( S \) coincides with the \( R \)-matrix assigned to the curve \( x = -y^r + \log y \).

8.1.6 Notation
The following notation is used consistently throughout this chapter.

- \( r \geq 1 \) is an integer; we work with the space of \( r \)-spin structures.
- \( I \) is the imaginary unit, \( I^2 = -1 \).
- \( J \) is the primitive \( r \)th root of unity, \( J^r = 1 \).
- \( g \geq 0 \) is the (arithmetic) genus of the curves \( C \) under consideration.
- \( n \geq 1 \) is the number of marked points on \( C \); the points themselves are denoted by \( x_1, \ldots, x_n \).
- \( k_1, \ldots, k_n \) is an \( n \)-tuple of positive integers such that \( m = (\sum k_i + n + 2g - 2)/r \) is also an integer. We consider meromorphic functions on \( C \) with \( n \) poles of orders \( k_1, \ldots, k_n \) at the marked points \( x_1, \ldots, x_n \). Such a function represents a ramified covering of \( \mathbb{P}^1 \) of degree \( \sum k_i \) with a ramification point of type \( (k_1, \ldots, k_n) \) at \( \infty \). We also require the covering to have exactly \( m \) additional ramification points of multiplicity \( r \).
- Each \( k_i \) determines a unique couple of integers \( p_i, a_i \) such that \( k_i = rp_i + (r - 1 - a_i) \) and \( a_i \in \{0, \ldots, r - 1\} \). Thus \( p_i \) is the quotient and \( a_i \) the “reversed remainder” of the division of \( k_i \) by \( r \). The \( r \)-spin structures we consider will be \( r \)th roots of the line bundle \( K(\sum a_ix_i) \).

The integers \( g, n, r, k_1, \ldots, k_n \) determine the integers \( m, p_i, \) and \( a_i \) uniquely.

150
8. The \( r \)-ELSV Formula and the \( r \)-BM Conjecture

8.2 Completed cycles and completed Hurwitz numbers

Okounkov and Pandharipande proved in [88] that the left-hand side of Conjecture 8.4 is equal to Hurwitz numbers with completed cycles. To make this chapter more independent from the rest of the text, we introduce them once more in a way that is slightly different from what we did in Chapter 2. Note that in this chapter, as well as in Chapter 3, we use a different normalization from what we did in Chapter 2 (see Remark 2.5).

8.2.1 Completed cycles

A partition \( \lambda \) of an integer \( q \) is a non-increasing finite sequence \( \lambda_1 \geq \cdots \geq \lambda_r \) such that \( \sum \lambda_i = q \).

It is known that the irreducible representations \( \rho_r \) of the symmetric group \( S_r \) are in a natural one-to-one correspondence with the partitions \( \lambda \) of \( q \). On the other hand, to a partition \( \lambda \) of \( q \) one can assign a central element \( C_{p,\lambda} \) of the group algebra \( C S_p \) for any positive integer \( p \).

The coefficient of a given permutation \( \sigma \in S_r \) in \( C_{p,\lambda} \) is defined as the number of ways to choose and number \( l \) cycles of \( \sigma \) so that their lengths are \( \lambda_1, \ldots, \lambda_l \), and the remaining \( p - q \) elements are fixed points of \( \sigma \). Thus the coefficient of \( \sigma \) vanishes unless its cycle lengths are \( \lambda_1, \ldots, \lambda_l, 1, \ldots, 1 \). In particular, \( C_{p,\lambda} = 0 \) if \( p < q \). Thus \( C_{p,\lambda} \) is the sum of permutations with \( l \) numbered cycles of lengths \( \lambda_i, \ldots, \lambda_l \) and any number of non-numbered fixed points.

The collection of elements \( C_{p,\lambda} \) for \( p = 1, 2, \ldots \) is called a stable center element \( C_{\lambda} \). For example, the stable element \( C_{\lambda} \) is the sum of all transpositions in \( C S_p \), which is well-defined for each \( p \), (in particular, equal to zero for \( p = 1 \)).

Let \( \lambda \) be a partition of \( q \) and \( \mu \) a partition of \( p \). Since \( C_{p,\lambda} \) lies in the center of \( C S_p \), it is represented by a scalar (multiplication by a constant) in the representation \( \rho_p \) of \( S_p \). Denote this scalar by \( f_\lambda(\mu) \). Thus to a stable center element \( C_{\lambda} \) we have assigned a function \( f_\lambda \) defined on the set of all partitions. We are interested in the vector space spanned by the functions \( f_\lambda \).

To study this space, one defines some new functions on the set of partitions as follows\(^2\):

\[
p_{r+1}(\mu) = \frac{1}{r+1} \sum_{i \geq 1} \left( u_i - i + \frac{1}{2} \right)^{r+1} + i + \frac{1}{2} \right)^{r+1} \quad (r \geq 0).
\]

**Theorem 8.8** (Kerov, Olshansky [65]). The vector space spanned by the functions \( f_\lambda \) coincides with the algebra generated by the functions \( p_1, p_2, \ldots \).

As a corollary, to each stable center element \( C_\lambda \) we can assign a polynomial in \( p_1, p_2, \ldots \) and, conversely, each \( p_{r+1} \) corresponds to a linear combination of stable center elements \( C_\lambda \).

**Definition 8.9.** The linear combination of stable center elements corresponding to \( p_{r+1} \) is called the completed \((r+1)\)-cycle and denoted by \( \overline{C}_{r+1} \).

The first completed cycles are:

\[
\begin{align*}
\overline{C}_1 &= C_1, \\
\overline{C}_2 &= C_2, \\
\overline{C}_3 &= C_3 + C_{1,1} + \frac{1}{2}C_1, \\
\overline{C}_4 &= C_4 + 2C_{2,1} + \frac{1}{3}C_2, \\
\overline{C}_5 &= C_5 + 3C_{3,1} + 4C_{2,2} + \frac{1}{4}C_3 + 4C_{1,1,1} + \frac{1}{5}C_{1,1} + \frac{1}{2}C_1.
\end{align*}
\]

For reasons that will become clear later, we say that a stable center element \( C_\lambda \) involved in the completed cycle \( \overline{C}_{r+1} \) has genus defect \( \left[ r + 2 - \sum(\lambda_i + 1) \right]/2 \).

---

\(^1\)This has nothing to do with stable curves.

\(^2\)The standard definition involves certain additive constants that we have dropped to simplify the expression, since these constants play no role here.
8.2.2 Completed Hurwitz numbers

Let \( K = \sum k_i \). Recall that the completed \((r + 1)\)-cycle can be considered as a central element of the group algebra \( \mathbb{C}S_K \). An \( r \)-factorization of type \((k_1, \ldots, k_n)\) in the symmetric group \( S_K \) is a factorization

\[
\sigma_1 \ldots \sigma_m = \sigma \tag{8.14}
\]

such that (i) the cycle lengths of \( \sigma \) equal \( k_1, \ldots, k_n \) and (ii) each permutation \( \sigma_i \) enters the completed \((r + 1)\)-cycle with a nonzero coefficient. The product of these coefficients for \( i \) going from 1 to \( m \) is called the weight of the \( r \)-factorization.

Choose \( m \) points \( y_1, \ldots, y_m \in \mathbb{C} \) and a system of \( m \) loops \( s_i \in \pi_1(\mathbb{C} \setminus \{y_1, \ldots, y_m\}) \), \( s_i \) going around \( y_i \). Then to an \( r \)-factorization one can assign a family of stable maps from nodal curves to \( \mathbb{P}^1 \). This is done in the following way: (i) Consider the covering of \( \mathbb{P}^1 \) ramified over \( y_1, \ldots, y_m \), and \( \infty \) with monodromies given by \( \sigma_1, \ldots, \sigma_m \) and \( \sigma^{-1} \) (relative to the chosen loops). (ii) If \( \sigma_i \) has \( l_i \) distinguished cycles and genus defect \( g_i \), glue a curve of genus \( g_i \) with \( l_i \) marked points to the \( l_i \) preimages of the \( i \)th ramification point that correspond to the distinguished cycles. The covering mapping is extended on this new component by saying that it is entirely projected to the \( i \)th ramification point. (iii) Among the newly added components, contract those that are unstable.

One can easily check that the arithmetic genus of the curve \( C \) constructed in this way is equal to \( g \). The complex structure on the newly added components of \( C \) can be chosen arbitrarily, which implies that in general we obtain not a unique stable map, but a family of stable maps.

An \( r \)-factorization is called transitive if the curve \( C \) assigned to the factorization is connected, in other words if one can go from every element of \( \{1, \ldots, K\} \) to any other by applying the permutations \( \sigma_i \) and jumping from one distinguished cycle of \( \sigma_i \) to another one.

**Definition 8.10.** The completed Hurwitz number \( h_{g,r,k_1,\ldots,k_n} \) is the sum of weights of the transitive \( r \)-factorizations of type \((k_1, \ldots, k_n)\).

**Theorem 8.11** (Okounkov, Pandharipande [88]). The relative Gromov-Witten invariant

\[
(r!)^m \int \prod_{i=1}^{r} \psi_i^\ast(\omega) \prod_{i=1}^{m} \psi_i^\ast(\omega) \ldots \prod_{i=1}^{r} \psi_i^\ast(\omega) \prod_{i=1}^{m} \psi_i^\ast(\omega) \tag{8.15}
\]

is equal to the corresponding completed Hurwitz number.

Thus there is no clash of notation if we denote both by \( h_{g,r,k_1,\ldots,k_n} \).

8.2.3 A digression on Kadomtsev-Petviashvili

The Hurwitz numbers can be arranged into a generating series

\[
G_r(\beta; p_1, p_2, \ldots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{k_1, \ldots, k_n} h_{g,r,k_1,\ldots,k_n} \frac{\beta^m}{m!} p_1 \ldots p_n. \tag{8.16}
\]

(One can prove that \( h_{g,r,k_1,\ldots,k_n} = 0 \) whenever \( m = (\sum k_i + n + 2g - 2)/r \) is not an integer.)

**Theorem 8.12.** The series \( G_r(\beta) \) is a solution of the Kadomtsev-Petviashvili (KP) hierarchy in variables \( p_i, \beta \) being a parameter.

The proof of this theorem is a straightforward generalization of [63] and [85].
8.3 The class $S$

8.3.1 Chiodo’s formula

Chiodo computed the Chern characters

$$h^e_k(R^ep_*L) = h^e_k(R^1p_*L) - h^e_k(R^0p_*L) \quad (8.17)$$

and obtained the following expression.

**Theorem 8.13** (Chiodo [17], Theorem 1.1.1). We have

$$h^e_k(R^ep_*L) = \frac{B_{k+1}(\hat{x})}{(k+1)!} - \sum_{i=1}^{r} \frac{B_{k+1}(\frac{\hat{x}+i}{r})}{(k+1)!} \psi_i^e$$

$$+ \sum_{a=0}^{r-1} \frac{B_{k+1}(\frac{\hat{x}+a}{r})(ja)_a}{(k+1)!} \prod_{i=1}^{r} (\psi_i^e)^{a_i}$$

Here $ja_a$ is the push-forward from the boundary components with a chosen node and a chosen branch of index $a$.

Chiodo’s formula makes it possible to compute any class of the form

$$\exp \left( \sum_{k \geq 1} s_k h^e_k(R^ep_*L) \right) . \quad (8.19)$$

Specifically, we will need

$$c(-R^ep_*L) = \exp \left( - \sum_{k \geq 1} (k-1)! h^e_k(R^ep_*L) \right) , \quad (8.20)$$

with $s_k = -(k-1)!$.

8.3.2 Topological field theory and $R$-matrix

In this section we use the notion of a cohomological field theory, topological field theory and $R$-matrix action. See [92] or [97] for an introduction.

Let $\Omega_{p,n}(a_1, \ldots, a_n) = \pi_*(\mathcal{S})$ be the push-forward of the class $\mathcal{S} = c(R^ep_*L)/c(R^pL)$ from the space of $r$-spin structures to $\overline{M}_{p,n}$. Further, denote by $\omega_{p,n}$ the degree 0 part of $\Omega_{p,n}$.

**The topological field theory**

The degree 0 class $\omega_{p,n}$ is equal to the push-forward of the cohomology class 1 from the space of $r$-spin structures to the space of stable curves. Thus it is easy to compute:

$$\omega_{p,n}(a_1, \ldots, a_n) = r^{2g-2-\sum a_i \mod r} , \quad (8.21)$$

because the degree of $\pi: \overline{M}_{p,n}^{(r)} \to \overline{M}_{p,n}$ equals $r^{2g-2}$. The classes $\omega_{p,n}$ form a topological field theory with unit on the vector space $\langle e_0, \ldots, e_{r-1} \rangle$ with quadratic form

$$\eta_{ab} = \frac{1}{r} \delta_{a+b \equiv 2 \mod r} \quad (8.22)$$

and unit $e_0$. 

153
8.4. Equivalence of the $r$-ELSV and $r$-BM conjectures

The 3-point correlators in genus 0 are equal to
\[ \langle e_a e_b e_c \rangle = \frac{1}{r} b_{a+b+c+2} \text{ mod } r. \] (8.23)

Therefore the quantum product is given by
\[ e_a \bullet e_b = e_{a+b} \text{ mod } r. \] (8.24)

The idempotents of the quantum product are
\[ \frac{1}{r} \sum_{a=0}^{r-1} J^{2a} e_a, \] (8.25)
where $J$ is the primitive $r$th root of unity and $0 \leq i \leq r - 1$.

The $R$-matrix and the correlators

It follows from Chiodo’s formula that the family of cohomology classes $\Omega_{g,n}$ can be obtained from $\omega_{g,n}$ by Givental’s $R$-matrix action, where
\[ R(z) = \exp \left[ - \sum_{k \geq 1} \text{diag}^{k-1}_{k+1} \frac{B_{k+1}}{k(k+1)} z^{k} \right] \] (8.26)

in basis $(e_a)$.

This fact has been observed in [20] in the presence of Witten’s class, then in [19] in the setting we use here. It follows that the classes $\Omega_{g,n}$ form a semi-simple cohomological field theory.

We define the correlators $\langle \tau^{a_1} \cdots \tau^{a_n} \rangle^{\text{coh}}_{g,d_1,\ldots,d_n}$ of this cohomological field theory:
\[ \langle \tau^{a_1} \cdots \tau^{a_n} \rangle^{\text{coh}}_{g,d_1,\ldots,d_n} = \int_{\mathcal{M}_{g,n}} \Omega_{g,n}(a_1, \ldots, a_n) \psi^{d_1} \cdots \psi^{d_n}, \] (8.27)
which are just the coefficients of the $n$-point function
\[ F_{g,n}(x_1, \ldots, x_n) = \sum_{0 \leq d_1, \ldots, d_n \leq r-1}^{d_1 + 2n - 2 \geq \sum_{i=1}^{n} d_i} \frac{(-1)^{\sum_{i=1}^{n} d_i}}{\prod_{i=1}^{n} \psi^{d_i} + r - a_i - 1} \prod_{i=1}^{n} (r \psi^{d_i} + r - a_i - 1)^{x_i}. \] (8.28)

8.4 Equivalence of the $r$-ELSV and $r$-BM conjectures

In this section we derive a formula for the $n$-point differentials $\omega_{g,n}$ of the curve $S(r)$: $x = -y + \log y$ and prove Theorem 8.7. In particular, this implies that the $r$-ELSV conjecture and the $r$-BM conjecture are equivalent.

To do that, we perform a number of local computations with the spectral curve, following the recipe of [113] in order to present the cohomological field theory corresponding to the spectral curve as a particular $R$-matrix action in the sense of Givental, applied to a topological field theory equal to a direct sum of $r$ one-dimensional topological field theories, properly rescaled.

We prove that the cohomological field theory that we obtain this way is a multiple of \{\Omega_{g,n}\} defined in Section 8.3.2 rewritten in the basis of normalized idempotents.
8. The r-ELSV formula and the r-BM conjecture

Proof of Theorem 8.7. Our goal is to compare the coefficients of the n-differentials $\omega_{g,n}$ with the correlators $\langle \tau_{a_1}^{d_1} \cdots \tau_{a_n}^{d_n} \rangle_{g}^{\text{coh}}$ of the cohomological field theory described in Section 8.3.2 in terms of the Givental group action. In [113] the coefficients of the differentials $\omega_{g,n}$ are, under some conditions and modulo some extra factors, expressed in terms of correlators of a cohomological field theory. We use the result of [113] and compare the cohomological field theories and the extra factors that appear in the statement of the theorem and in the identification formula in [113].

Let us outline this comparison. We have the following formula for $\omega_{g,n}$ given in [113]:

$$\omega_{g,n} = \sum_{i_1, \ldots, i_n} \langle \tau_{i_1}^{d_1} \cdots \tau_{i_n}^{d_n} \rangle_{g} D \left( \prod_{j=1}^{n} \left( -2 \frac{\partial}{\partial x_j} \right) d_i \right) \xi_i(x_j),$$

where $\xi_i$ are some auxiliary functions of the coordinate $x$ on the curve, we recall them below.

The correlators $\langle \tau_{a_1}^{d_1} \cdots \tau_{a_n}^{d_n} \rangle_{g}^{\text{coh}}$ are represented as a sum over Givental graphs, whose structure is described in [118, 113].

Remark 8.14. The local computations that we do in this section in order to specify all ingredients of Equation (8.29) are a direct generalization of the computations of Eynard in [34, Section 8.2] for the curve $x = -y + \log y$ that shows that the usual ELSV formula is equivalent to the Bouchard-Marino conjecture for the usual Hurwitz numbers.

Note that the indices $i_1, \ldots, i_n$ that we have in Equation (8.29) refer to a summation in the basis of the normalized idempotents (cf. Section 8.3.2), while the indices $a_1, \ldots, a_n$ in Theorem 8.7 refer to a summation in the natural flat basis of the cohomological field theory.

Notation 8.15. Throughout this text, when we write an object with a tilde we mean refer to this object expressed in the natural flat basis, while when we write the same object without a tilde it should be interpreted as expressed in the basis of canonical idempotents. Furthermore, $\langle \cdot \rangle^{\text{coh}}$ refers to the correlators of the cohomological field theory associated to the spectral curve $S^{(r)}$ by the methods of [113], whereas $\langle \cdot \rangle^g$ refers to the correlators of the cohomological field theory $\Omega_{g,n}$.

Changing the coordinates, we can rewrite Equation (8.29) in the following form:

$$\omega_{g,n} = \sum_{a_1, \ldots, a_n} \langle \tau_{a_1}^{d_1} \cdots \tau_{a_n}^{d_n} \rangle_{g}^{\text{coh}} D \left( \prod_{j=1}^{n} \left( -2 \frac{\partial}{\partial x_j} \right) d_i \right) \tilde{\xi}_a(x_j),$$

where

$$\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \rangle_{g}^{\text{coh}} := \sum_{i_1, \ldots, i_n} \langle \tau_{i_1}^{d_1} \cdots \tau_{i_n}^{d_n} \rangle_{g} \prod_{j=1}^{n} \frac{g^{a_j+1}}{J^j i_j},$$

and

$$\tilde{\xi}_a := \sum_{i=0}^{r-1} p^{-1} J^{-(a+1)} \xi_i.$$

Below, in Lemma 8.18, we prove that

$$\tilde{\xi}_a = i \sqrt{2} r^2 \sum_{n=0}^{\infty} \frac{(rn+r-a-1)^n}{n!} e^{(rn+r-a-1)x}.$$
8.4. Equivalence of the $r$-ELSV and $r$-BM conjectures

Further, in Lemma 8.19 we prove that
\[ \langle \tau a_1 d_1 \cdots \tau a_n d_n \rangle \tilde{r} g_n \prod_{j=1}^{n} \left( \sqrt{2} r \cdot \tau a_j + \log \tau a_j \right) \]
\[ = \langle \tau a_1 d_1 \cdots \tau \tilde{a} \tilde{d} \rangle \tilde{r} g_2 g_2 + 2 \ell - 2 + \sum_{i=1}^{\ell} a_i r - \sum_{i=1}^{\ell} d_i, \]
(8.34)

Substituting these two expression in Equation (8.30), using equation (8.28), we obtain the equality of Theorem 8.7, which proves the theorem.

The rest of the section consists of a step-by-step computation of the expansions of various local objects on the curve $x = -g' + \log y$ that are needed to formulate and prove Lemmas 8.18 and 8.19. We follow the scheme of computations proposed in [113] and use the same notation as there.

\[ y_i := r^{-i/r} J_i, \quad i = 0, \ldots, r - 1. \]
(8.35)

The critical values of the function $x$ at these points are
\[ x_i := -\frac{1}{r} + \frac{12\pi i}{r} - \log r, \quad i = 0, \ldots, r - 1. \]
(8.36)

We denote by $z_i$ the local coordinates near the critical points, that is, $z_i^2 + x_i = x$, $i = 0, 1, \ldots, r - 1$. Let us make a choice of the expansion of function $y$ in $z_i$. One of the possible choices, that fixes $y = y(z_i)$ unambiguously is
\[ y(z_i) = y_i + y_{i1} z_i + O(z_i^3), \]
(8.37)

where
\[ y_{i1} := \sqrt{2} r^{-1/2} J_i, \quad i = 0, \ldots, r - 1. \]
(8.38)

8.4.1 Reciprocal Gamma function

We are using the following expansion of the reciprocal Gamma function:
\[ \frac{1}{\Gamma(w+v)} = \frac{1}{2\pi} \int_{C_n} (-t)^{-w-v} e^{-t} dt \]
(8.39)
\[ \sim \frac{w^{-w-v+1/2} e^w}{\sqrt{2\pi}} \exp \left( \sum_{j=1}^{\infty} \frac{B_{j+1}(v)}{j(j+1)} (-w)^j \right), \]
where $C_n$ is the Hankel contour [76] that goes around the positive real numbers, and $B_j(v)$ are the Bernoulli polynomials defined by
\[ \frac{w e^{w+x}}{e^w - 1} = \sum_{j=0}^{\infty} B_j(v) \frac{w^j}{j!}. \]
(8.40)

Note that $B_j(1 - v) = (-1)^j B_j(v)$, $j = 0, 1, \ldots$.
8.4.2 Local expansions of $y$
Let us fix $0 \leq i \leq r - 1$. We are interested in the expansion of the odd part of $y = y(z_i)$ in $z_i$, that is, we consider $(y(z_i) - y(-z_i))/2$. We consider also the coordinate $t := ry'$. It is easy to see that
\begin{align}
y &= t^i r^{-i} y'; \\
-x_i - y' + \log y &= \frac{1}{\rho} - \frac{t}{\rho} + \log t, \tag{8.42}

dy &= t^{i+1} e^{-1-i} t^i dt. \tag{8.43}
\end{align}

Note also that $t$ runs along the negative Hankel contour when $z_i$ runs from $-\infty$ to $+\infty$.

**Lemma 8.16.** Denote the coefficients of the odd part of $y(z_i)$ by:
\[ y(z_i) - y(-z_i) = \frac{1}{2} \sqrt{2\pi} r^{-\frac{1}{2} + i} \sum_{j=0}^{\infty} V_j (2j)! z_i^{2j}. \tag{8.44} \]
Then we have
\[ \sum_{j=0}^{\infty} V_j z^j = \exp \left( -\sum_{j=1}^{\infty} B_{j+1}(1/2) \cdot z^j \right). \tag{8.45} \]

**Proof.** The Laplace method gives the following asymptotic expansion:
\[ \int_{-\infty}^{\infty} y'(z_i) e^{-r w z_i^2} dz_i = \frac{1}{2} \sqrt{2\pi} r^{-1+i} y' e^{-\frac{1}{2} + i} \sum_{j=0}^{\infty} V_j w^{-j}. \tag{8.46} \]
On the other hand,
\begin{align}
\int_{-\infty}^{\infty} y'(z_i) e^{-r w z_i^2} dz_i &= \int_{C_-} e^{w(-x_i - y' + \log y)} dy \\
&= e^{-\frac{i}{2} + i} \int_{C_-} e^{w(1+i+\log \rho) - \frac{i}{2} + i} dt \\
&= e^{-\frac{i}{2} + i} \int_{C_-} e^{w(1+i+\log \rho) - \frac{i}{2} + i} dt \\
&= -e^{-\frac{i}{2} + i} \int_{C_-} e^{w(1+i+\log \rho) - \frac{i}{2} + i} dt.
\end{align}

(we used the substitution $p = -\rho t$ that transforms the negative Hankel contour $C_-$ to the Hankel contour $C_+$. Thus we see that
\begin{align}
\int_{-\infty}^{\infty} y'(z_i) e^{-r w z_i^2} dz_i &= \frac{2\pi i e^{-\frac{i}{2} + i} \int_{C_-} e^{w(1+i+\log \rho) - \frac{i}{2} + i} dt}{1(w+1+i)} \\
&\sim \frac{2\pi i e^{-\frac{i}{2} + i} \int_{C_-} e^{w(1+i+\log \rho) - \frac{i}{2} + i} dt}{\sqrt{2\pi}} \sum_{j=1}^{\infty} B_{j+1}(1-\frac{1}{2}) (-w)^{-j} \\
&= \sqrt{2\pi} e^{-\frac{i}{2} + i} \int_{C_-} e^{w(1+i+\log \rho) - \frac{i}{2} + i} dt \sum_{j=1}^{\infty} B_{j+1}(1-\frac{1}{2}) (-w)^{-j} \\
&= \sqrt{2\pi} e^{-\frac{i}{2} + i} \int_{C_-} e^{w(1+i+\log \rho) - \frac{i}{2} + i} dt \sum_{j=1}^{\infty} B_{j+1}(1-\frac{1}{2}) (-w)^{-j},
\end{align}
which proves
\[ \sum_{j=0}^{\infty} V_j w^{-j} = \exp \left( -\sum_{j=1}^{\infty} B_{j+1}(1/2) \cdot w^{-j} \right). \tag{8.49} \]
8.4. Equivalence of the \( r \)-ELSV and \( r \)-BM conjectures

8.4.3 Two-point function

Now we consider the two-point function. According to [34], since \( d \) is a meromorphic function in \( y \), we know that the Laplace transform of the two-point function is represented as a Givental-type edge contribution. So, we have to compute only the even coefficients of the local expansion of half of the two-point function in order to specify the Givental operator imposed by the topological recursion, namely, we are interested in the coefficients of the function

\[
Y_{i_1i_2} = \frac{y_{i_1}y_{i_2}(z_{i_2})}{(y_{i_1} - y(z_{i_2}))^2} = \frac{\delta_{i_1i_2}}{2} + O(1). \tag{8.50}
\]

Lemma 8.17. We have:

\[
\frac{Y_{i_1i_2}(z_{i_2}) + Y_{i_1i_2}(-z_{i_2})}{2} = \sum_{k=0}^{\infty} \frac{(U_k)_{i_1i_2}(2r)^{z_{i_2}^2-2}}{(2k-3)!}, \tag{8.51}
\]

where \((U_k)_{i_1i_2}\) is given by

\[
\sum_{k=0}^{\infty} (U_k)_{i_1i_2} z^k = \frac{1}{r} \sum_{c=0}^{r-1} \exp \left( -\sum_{k=1}^{r} \frac{B_{k+1}(z)}{k(k+1)} \right) \exp^{-ci}. \tag{8.52}
\]

Proof. Observe that

\[
\int_{-\infty}^{\infty} \left( \frac{y_{i_1}y_{i_2}(z_{i_2})}{(y_{i_1} - y(z_{i_2}))^2} \right) e^{-rwy} dz_{i_2} \sim -2\sqrt{\pi r^2} w^{\frac{1}{2}} \sum_{k=0}^{\infty} (U_k)_{i_1i_2} w^{-k}. \tag{8.53}
\]

On the other hand,

\[
\int_{-\infty}^{\infty} \left( \frac{y_{i_1}y_{i_2}(z_{i_2})}{(y_{i_1} - y(z_{i_2}))^2} \right) e^{-rwy} dz_{i_2}
= rw_{i_1} \int_{-\infty}^{\infty} \frac{1}{y_{i_1} - y(z_{i_2})} e^{-rwy} 2wz_{i_2} dz_{i_2}
= rw_{i_1} \int_{-\infty}^{\infty} \frac{y(z_{i_2}) - y_{i_1}}{y(z_{i_2}) - y_{i_1}} e^{-rwy} \exp(1-i\log t) dt.
\]

Here we can use that

\[
rw_{i_1} \frac{y(z_{i_2}) - y_{i_1}}{y(z_{i_2}) - y_{i_1}} = 1 \sqrt{2\pi r} \frac{1}{2}\sum_{c=0}^{r-1} \exp^{-ci}. \tag{8.55}
\]

Therefore,

\[
\int_{-\infty}^{\infty} \left( \frac{y_{i_1}y_{i_2}(z_{i_2})}{(y_{i_1} - y(z_{i_2}))^2} \right) e^{-rwy} dz_{i_2}
= 1 \sqrt{2\pi r} \frac{1}{2}\sum_{c=0}^{r-1} \exp^{-ci} \int_{C_+} e^{rwy} \exp^{-1+i} dt
= -1 \sqrt{2\pi r} \frac{1}{2}\sum_{c=0}^{r-1} \exp^{-ci} \int_{C_+} e^{-p} \exp^{-1+i} dp
\sim -2\sqrt{\pi r^2} w^{\frac{1}{2}} \sum_{c=0}^{r-1} \exp^{-ci}. \tag{8.56}
\]

\[158\]
Thus we see that
\[ \sum_{k=0}^{\infty} (U_k)_{i_1i_2} z^k = \frac{1}{r} \sum_{c=0}^{r-1} \exp \left( - \sum_{k=1}^{\infty} B_{0,k}(\xi) z^k \right) J^{a_2-c_2} . \] (8.57)

### 8.4.4 Functions on the leaves

According to [113], the auxiliary function \( \xi_i(x) \) that we put on the leaves in the graph expression for the correlation forms of the spectral curve are given by the following formula:

\[ \xi_i := \sqrt{2} r^{-1} \sum_{i=0}^{r-1} \left( - \frac{1}{2} \right)^{r-1} J^{-(a+1)i} \xi_i \] (8.58)

Here the index \( i \) corresponds to the basis of normalized idempotents, so in the standard flat basis we have to consider the functions \( \tilde{\xi}_a \) given by Equation (8.32):

\[ \tilde{\xi}_a := \sum_{i=0}^{r-1} r^{-1} J^{-(a+1)i} \xi_i \] (8.59)

**Lemma 8.18.** We have:

\[ \tilde{\xi}_a = 1 \sqrt{2} r^{-1} + \frac{1}{r} \sum_{n=0}^{\infty} \frac{r n + r - a - 1}{n!} e^{(m+r-a-1)x} . \] (8.60)

**Proof.** First, observe that

\[ \tilde{\xi}_a = \sum_{i=0}^{r-1} r^{-1} J^{-(a+1)i} \xi_i = 1 \sqrt{2} r^{-1} + \sum_{i=0}^{r-1} \frac{r^{-1} (a+1)i}{1 - i + J^{-i} y} \] (8.61)

Following [21], we define the Lambert function

\[ W(z) := \sum_{n=0}^{\infty} \frac{n^{n-1}}{n!} (-z)^n , \] (8.62)

and use its property

\[ \left( \frac{W(z)}{z} \right)^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha(n+\alpha)^{n-1}}{n!} (-z)^n . \] (8.63)

We have the following equation: \( e^x = ye^{-y} \). This equation implies (cf. [114])

\[ y = \left( \frac{W(-re^{r^2})}{-r} \right)^{\frac{1}{r}} \] (8.64)

and

\[ \frac{dy^{r^{r-1}}}{dx} = \frac{(r-a-1)y^{r-1}}{1 - ry} . \] (8.65)

159
8.4. Equivalence of the $r$-ELSV and $r$-BM conjectures

Therefore,

$$(r-a-1)g^{r-a-1} = \frac{d}{dx} \left(W(-re^{rx})\right)_{-r} = (-r)^{-r+1} \frac{d}{dx}(-re^{rx})_{-r} \sum_{n=0}^{\infty} \frac{(-r-a-1)^{n-1}}{n!} (re^{rx})^n = (r-a-1) \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(rn+r-a-1)^{n-1}}{n!} e^{(rn+r-a-1)x} = (r-a-1) \sum_{n=0}^{\infty} \frac{(rn+r-a-1)^{n}}{n!} e^{(rn+r-a-1)x}$$

So, we see that

$$\xi_x = 1 \sqrt{Z_r} + \sum_{n=0}^{\infty} \frac{(rn+r-a-1)^{n}}{n!} e^{(rn+r-a-1)x} \quad (8.67)$$

8.4.5 Comparison of the correlators

Consider the $n$-point correlators $\langle \phi^{a_1}_{d_1} \cdots \phi^{a_n}_{d_n} \rangle^r$ introduced in Equation (8.30). They are obtained via a linear change of the indices from the correlators $\langle \phi^{a_1}_{d_1} \cdots \phi^{a_n}_{d_n} \rangle^r$, where the latter ones are defined in [113] as the sum over Givental graphs. The structure constants of these graphs, that is, the parameters that we put on vertices, edges, leaves, and dilaton leaves, are defined in terms of local data of the curve $x = -y^r + \log y$ at the ramification points. More precisely, they are defined via the coefficients of the expansions of the function $g(z_i)$ in the local coordinate $z_i$, and the components of the Bergman kernel $Y_{i,j}$ in the local coordinate $z_j$.

In this Section we prove that the correlators $\langle \phi^{a_1}_{d_1} \cdots \phi^{a_n}_{d_n} \rangle^r_g$ are equal, up to some multiplicative factors, to the correlators $\langle \phi^{a_1}_{d_1} \cdots \phi^{a_n}_{d_n} \rangle^{coh}$ of the cohomological field theory described in Section (8.3.2). Namely, we prove the following lemma.

Lemma 8.19.

$$\sum_{\alpha_1 \cdots \alpha_{n-1}} \langle \phi^{a_1}_{d_1} \cdots \phi^{a_n}_{d_n} \rangle^r \prod_{k=1}^{n} \frac{j!}{j!(a_{k+1})!} \prod_{k=1}^{n} \sqrt{-2\pi r} (-2)^{a_k} \quad (8.68)$$

$$= \langle \phi^{a_1}_{d_1} \cdots \phi^{a_n}_{d_n} \rangle^{coh} \sqrt{2\pi} \cdot \frac{2^{2n+2}}{\sum_{k=1}^{n} a_k} - \sum_{k=1}^{n+2} d_k$$

Proof. The proof follows from the comparison of the ingredients of the Givental graph expressions on both sides, following the identification theorem in [113]. Let us rewrite both sides of the equality in the basis of normalized idempotents:

$$\langle \phi^{a_1}_{d_1} \cdots \phi^{a_n}_{d_n} \rangle^r \prod_{k=1}^{n} (-2r)^{d_k} = \langle \phi^{a_1}_{d_1} \cdots \phi^{a_n}_{d_n} \rangle^{coh} \sqrt{2\pi} \cdot \frac{2^{2n+2}}{\sum_{k=1}^{n} a_k} - \sum_{k=1}^{n+2} d_k$$

where

$$\langle \phi^{a_1}_{d_1} \cdots \phi^{a_n}_{d_n} \rangle^{coh} := \sum_{\alpha_1 \cdots \alpha_{n-1}} \langle \phi^{a_1}_{d_1} \cdots \phi^{a_n}_{d_n} \rangle^{coh} \prod_{k=1}^{n} j!^{-(a_{k+1})!} \quad (8.69)$$

160
The r-ELSV formula and the r-BM conjecture

The result of Chiodo (see Theorem 8.13) implies that the generating function of the correlators \( \langle \tau_{d_1} \cdots \tau_{d_r} \rangle \) is obtained from the r properly normalized copies of the Kontsevich-Witten tau function by application of the quantization of the operator

\[
R_i^j (\zeta) := \exp \left( - \sum_{k=1}^{\infty} \frac{1}{r} \sum_{a=0}^{r-1} \frac{y^{a-aj} B_{k+1} \left( \frac{z}{r} \right)}{k(k+1)} \right).
\]

In particular (we use it below), we have:

\[
R_i^j (\zeta) := \sum_{j=0}^{r-1} \sum_{a=0}^{r-1} \frac{y^{a-aj} \exp \left( - \sum_{k=1}^{\infty} \frac{\zeta B_{k+1} \left( \frac{z}{r} \right)}{k(k+1)} \right)}{r}.
\]

The weight of the correlators \( \prod_{j=1}^{p} \tau_{d_j} \), of the i-th copy of the Kontsevich-Witten tau function (or, in other words, the weight of the vertex labelled by \( i \) in the graphical representation of the Givental formula, as in [113, Theorem 4.1]) is equal to

\[
r^{-2q-1} \sum_{a_1, \ldots, a_p} \prod_{j=1}^{p} \prod_{j=1}^{R_i^j (\zeta)} = r^{-2q-2p} \frac{R_i^j (\zeta)}{r}.
\]

Let us compare that with the formula we get from the topological recursion, following the lines of [113]. We compare the coefficients of the expansion of \( y \) and the two-point function in the coordinate \( z_i, i = 0, \ldots, r-1 \) (that determine the ingredients of the graphs in the formula for \( \langle \tau_{d_1} \cdots \tau_{d_r} \rangle \) with the corresponding formulas in terms of the operator \( R_i^j (\zeta) \) that are used in the Givental graphical formula for \( \langle \tau_{d_1} \cdots \tau_{d_r} \rangle \), in the same way as it is done in [113, Theorem 4.1].

Lemma 8.16 implies that, in the notation of [113],

\[
B_{k+1} = 12r \sqrt{2} r^{-1/4} \frac{1}{2} \sum_{i=0}^{\infty} \sum_{a=0}^{r-1} \frac{B_{k+1} (\frac{z}{r})}{(k+1)^{a+1}} \frac{y^{a-aj}}{r}.
\]

Lemma 8.17 implies that, also in notation of [113],

\[
B_{k}^j = (2r)^{j+1} \frac{1}{2} \sum_{i=0}^{\infty} \frac{B_{k+1} (\frac{z}{r})}{(k+1)^{a+1}} \frac{y^{a-aj}}{r}.
\]

The vertex labelled by \( \prod_{j=1}^{p} \tau_{d_j} \) and an extra index \( i \) is also multiplied by (again in the notation of [113])

\[
(-2B_i)^{2-2p} = (-2r)^{j+1} \frac{1}{2} \sum_{i=0}^{\infty} \frac{B_{k+1} (\frac{z}{r})}{(k+1)^{a+1}} \frac{y^{a-aj}}{r}.
\]

This all together (including the factors \( -2r \sqrt{2} r^{-1/4} \frac{1}{2} \sum_{i=0}^{\infty} \sum_{a=0}^{r-1} \frac{B_{k+1} (\frac{z}{r})}{(k+1)^{a+1}} \frac{y^{a-aj}}{r} \)) gives the following extra factor for the vertex labelled by \( \prod_{j=1}^{p} \tau_{d_j} \), with \( p_d \) attached dilaton leaves and \( p_o \) ordinary leaves and/or half-edges \( (p = p_d + p_o) \), and an extra index \( i \):

\[
\frac{(-2r)^{j+1} \frac{1}{2} \sum_{i=0}^{\infty} \frac{B_{k+1} (\frac{z}{r})}{(k+1)^{a+1}} \frac{y^{a-aj}}{r}}{(-2r)^{j+1} \frac{1}{2} \sum_{i=0}^{\infty} \frac{B_{k+1} (\frac{z}{r})}{(k+1)^{a+1}} \frac{y^{a-aj}}{r}} = r^{(1+4^j)(2p+2p-2)} r^{2q+2p-2}.
\]
8.4. Equivalence of the $r$-ELSV and $r$-BM conjectures

(we used that $a_1 + \cdots + a_p = 3q - 3 + p$). Note that the factor $r^{(2q+p-2)j(2q+p-2)i}$ coincides with the weight that we have in the Givental’s presentation of Chiodo’s formula, cf. Equation (8.73). Meanwhile, the sum of the exponents $2q + p_0 - 2$ over all vertices in a graph of genus $g$ with $n$ global leaves is equal to $2g + n - 2$. Therefore, the product of the factors $r^{(1+i)r(2q+p_0-2)}$ over all vertices of a graph is exactly equal to the extra factor $r^{3g+n-2+2g+n-2}$ that we have on the right-hand side of Equation (8.69).