Hurwitz numbers, moduli of curves, topological recursion, Givental's theory and their relations

Spitz, L.

Publication date
2014

–8– The r-ELSV formula and the r-BM conjecture

8.1 Introduction

In this chapter we discuss two conjectures related to Hurwitz numbers with completed cycles that generalize two important theorems in the theory of ordinary Hurwitz numbers. The first is a generalization of the celebrated ELSV formula relating Hurwitz numbers and Hodge integrals [31] that we call the r-ELSV conjecture. The second is a generalization of the (now proved) Bouchard-Mariño conjecture [10, 35, 8] relating Hurwitz numbers to the CEO-recusion procedure of Chekhov, Eynard, and Orantin [15, 36], that we call the r-BM conjecture. We prove that these two conjectures are equivalent. This completes our evidence for the r-BM conjecture started in Chapter 3, while simultaneously providing evidence for the r-ELSV conjecture.

8.1.1 The space of r-spin structures

The r-spin structures on smooth curves

Fix an integer $r \geq 1$. Let C be a (smooth compact connected) complex curve of genus g with n marked (pairwise distinct numbered) points $x_1, \ldots, x_n$. Denote by K the canonical (cotangent) line bundle over C. Choose n integers $a_1, \ldots, a_n \in \{0, \ldots, r-1\}$ such that $2g - 2 - \sum a_i$ is divisible by r.

Definition 8.1. An *r-spin structure* on C is a line bundle $\mathcal{L}$ together with an identification

$$\mathcal{L}^{\otimes r} \xrightarrow{\sim} K\left(-\sum a_i x_i\right). \tag{8.1}$$

The *moduli space* $\mathcal{M}^{1/r}_{g,n;a_1,\ldots,a_n}$ is the space of all r-spin structures on all smooth curves.

We will denote this space by $\mathcal{M}^{1/r}$ omitting the other indices if this does not lead to ambiguity.

The element $K(-\sum a_i x_i) \in \mathrm{Pic}(C)$ can be divided by r in r^{2g} different ways. Hence there are exactly r^{2g} different r-spin structures on every smooth curve C. Thus the natural morphism $\pi : \mathcal{M}^{1/r} \to \mathcal{M}_{g,n}$ to the moduli space of smooth curves is actually a nonramified r^{2g}-sheeted covering.

The space $\mathcal{M}^{1/r}$ has a structure of orbifold or smooth Deligne-Mumford stack. The stabilizer of a generic r-spin structure is $\mathbb{Z}/r\mathbb{Z}$ (except for several cases with small g and n where it can be bigger), because every line bundle $\mathcal{L}$ has r automorphisms given by the multiplication by rth roots of unity. In particular, the degree of π is not r^{2g} but r^{2g-1}.

The compactification

A natural compactification $\overline{\mathcal{M}}^{1/r}_{g,n;a_1,\ldots,a_n}$ of $\mathcal{M}^{1/r}_{g,n;a_1,\ldots,a_n}$ was constructed in [58], [2], [12], [16]. It has the structure of an orbifold or a smooth Deligne-Mumford stack. There are three different constructions that involve different versions of the universal curve $\overline{C}^{1/r}_{g,n;a_1,\dots,a_n}$, but the moduli space $\overline{\mathcal{M}}^{1/r}_{g,n;a_1,\dots,a_n}$ is the same.

In the geometrically most natural construction the universal curve over the compactification has orbifold fibers and $\mathcal{L}$ is extended into a line bundle in the orbifold sense. The second construction is a fiberwise coarsification of the first one. After the coarsification the universal curve becomes a singular orbifold with A_{r-1} singularities and $\mathcal{L}$ becomes a rank 1 torsion-free sheaf rather than a line bundle. Finally, the third construction is obtained from the second one after resolving its singularities by a sequence of blow-ups. In the third construction the universal curve is smooth and $\mathcal{L}$ is again a line bundle, but the morphism $\mathcal{L}^{\otimes r} \to K(-\sum a_i x_i)$ has zeros on the exceptional divisors. The third construction is most convenient for intersection theoretic computations.

On a singular stable curve with an r-spin structure, every branch of every node carries an index $a \in \{0,\dots,r-1\}$ such that the sum of indices at each node equals -2 modulo r. The divisibility condition $2g-2-\sum a_i = 0 \mod r$ is satisfied on each irreducible component of the curve. Thus for a separating node the indices of the branches are determined uniquely, while for a nonseparating node there are r choices.

From now on we write simply $\overline{\mathcal{M}}^{1/r}$, and $\overline{C}^{1/r}$ for the compactified space of r-spin structures and its universal curve, if this does not lead to ambiguity.

Remark 8.2. The interest in $\overline{\mathcal{M}}^{1/r}$ was initially motivated by Witten's conjecture asserting that some natural intersection numbers on this space can be arranged into a generating series that satisfies the r-KdV (or rth higher Gelfand-Dikii) hierarchy of partial differential equations [108]. The conjecture is now proved [44] and other beautiful results on the intersection theory of $\overline{\mathcal{M}}^{1/r}$ have been obtained [17, 100].

One of the main ingredients in Witten's conjecture is the so-called "Witten top Chern class" whose definition uses the sheaves $R^0p_*\mathcal{L}$ and $R^1p_*\mathcal{L}$ on $\overline{\mathcal{M}}^{1/r}$ in a rather involved way (see [93] or [18]). The r-ELSV conjecture, on the other hand, uses the more straight-forward total Chern class $c(R^1p_*\mathcal{L})/c(R^0p_*\mathcal{L})$. Therefore it is not at all clear if the r-ELSV formula can be related to Witten's conjecture. However, it would provide a link between the intersection theory on $\overline{\mathcal{M}}^{1/r}$ and integrable hierarchies via Hurwitz numbers, see Theorem 8.12.

The map $\pi\colon \mathcal{M}^{1/r} \to \mathcal{M}_{g,n}$ extends to the compactifications; $\pi\colon \overline{\mathcal{M}}^{1/r} \to \overline{\mathcal{M}}_{g,n}$. Thus the classes $\psi_1,\dots,\psi_n \in H^2(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$ that one usually defines on $\overline{\mathcal{M}}_{g,n}$ (see, for instance, [104], Section 3.5) can be lifted to $\overline{\mathcal{M}}^{1/r}$. By abuse of notation we will denote the lifted classes by ψ_i instead of $\pi^*(\psi_i)$.

The projection $p : \overline{C}^{1/r} \to \overline{\mathcal{M}}^{1/r}$ induces a push-forward of $\mathcal{L}$ in the sense of derived functors:

$$R^\bullet p_*\mathcal{L} = R^0p_*\mathcal{L} - R^1p_*\mathcal{L}. \tag{8.2}$$

Notation 8.3. We denote by $\mathcal{S}$ the total Chern class

$$\mathcal{S} = c(-R^\bullet p_*\mathcal{L}) = c(R^1p_*\mathcal{L})/c(R^0p_*\mathcal{L}). \tag{8.3}$$

8.1.2 The r-ELSV conjecture

Let $\overline{\mathcal{M}}^{\text{rel}} = \overline{\mathcal{M}}_{g,m;k_1,\dots,k_n}(\mathbb{P}^1)$ be the space of stable genus g maps to $\mathbb{P}^1$ relative to $\infty \in \mathbb{P}^1$ with profile $(k_1,\dots,k_n)$ and with m marked points in the source curve. It is a compactification of the space of meromorphic functions on genus g curves with n numbered poles of orders $k_1,\dots,k_n$ and m more marked points. See [104], Section 5 for a precise definition and main properties. Let $\omega \in H^2(\mathbb{P}^1)$ be the Poincaré dual class of a point.

We write every k_i in the form $k_i = rp_i + (r-1-a_i)$, where p_i is the quotient and $r-1-a_i$ the remainder of the division of k_i by r. Let $m = (\sum k_i + n + 2g - 2)/r$.

Denote by $h_{g,r;k_1,\dots,k_n}$ the following Gromov-Witten invariants

$$h_{g,r;k_1,\dots,k_n} := (r!)^m \int_{\overline{\mathcal{M}}^{\mathrm{rel}}} \mathrm{ev}_1^*(\omega)\psi_1^r \cdots \mathrm{ev}_n^*(\omega)\psi_m^r. \tag{8.4}$$

Okounkov and Pandharipande [88] studied Gromov-Witten invariants as above and proved that they are equal to the Hurwitz numbers with completed cycles $h^{(r)}_{g,k_1,\dots,k_n}$ that where introduced in Chapter 2.

Introduce the following integrals over the space of r-spin structures:

$$f_{g,r;k_1,\dots,k_n} = m!\, r^{m+n+2g-2} \prod_{i=1}^{n} \frac{\left(\frac{k_i}{r}\right)^{p_i}}{p_i!} \times \int_{\overline{\mathcal{M}}^{1/r}_{g,n;a_1,\dots,a_n}} \frac{\mathcal{S}}{\left(1-\frac{k_1}{r}\psi_1\right)\dots\left(1-\frac{k_n}{r}\psi_n\right)}. \tag{8.5}$$

Chiodo [17] studied the class $\mathcal{S}$ expressing it in terms of standard cohomology classes on the moduli space, the coefficients being equal to values of Bernoulli polynomials at rational points with denominator r. His work allows one to compute the numbers $f_{g,r;k_1,\dots,k_n}$.

Conjecture 8.4 (r-ELSV)**.** *We have*

$$h_{g,r;k_1,\dots,k_n} = f_{g,r;k_1,\dots,k_n}. \tag{8.6}$$

This conjecture suggests a hidden equality between virtual fundamental classes. More precisely, if we consider the space of relative stable maps f such that $\mathrm{d}f$ has an r-th root, the conjecture seems to imply that its virtual fundamental class is obtained from the virtual fundamental class of the space of all stable maps after multiplication by rth powers of ψ-classes. However this is not elucidated so far and our evidence for this conjecture is not geometric.

The r-ELSV formula was first conjectured in [111]. Since both sides are computable, the conjecture can be tested on a computer, and there is virtually no doubt that it is correct. The conjecture is proved in genus 0, and also for $(g,n) = (1,1)$ and arbitrary r (unpublished).

8.1.3 The r-Bouchard-Mariño conjecture

In [15, 36] Chekhov, Eynard, and Orantin assigned to every plane analytic complex curve $\mathcal{S}$ a series of invariants $\omega_{g,n}(\mathcal{S})$ called *correlation n-forms*. Each $\omega_{g,n}$ is a meromorphic n-form on $\mathcal{S}^n$. The construction is described in detail in Chapter 1.

Definition 8.5. The *n-point functions of genus g* for r-Hurwitz numbers and for r-spin integrals are defined as

$$H^{(r)}_{g,n}(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n} \frac{h^{(r)}_{g;k_1,\dots,k_n}}{m!} \exp(k_1x_1 + \dots + k_nx_n). \tag{8.7}$$

$$F_{g,r}(x_1,\dots,x_n) = \sum_{k_1,\dots,k_n} \frac{f_{g,r;k_1,\dots,k_n}}{m!} \exp(k_1x_1 + \dots + k_nx_n). \tag{8.8}$$

We also let, for any function $f(x_1,\dots,x_n)$,

$$Df = \frac{\partial^n f}{\partial x_1 \cdots \partial x_n} \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{8.9}$$

Conjecture 8.6 (r-BM)**.** *Let $\omega_{g,n}$ be the n-point correlation forms of the plane curve $x = -y^r + \log y$. Then we have*

$$DH^{(r)}_{g,n}(x_1,\dots,x_n) = \omega_{g,n}. \tag{8.10}$$

8.1.4 Equivalence of the two conjectures

The goal of this chapter is to prove the following theorem.

Theorem 8.7. *Let $\omega_{g,n}$ be the n-point correlation forms of the plane curve $x = -y^r + \log y$. Then we have*

$$DF_{g,r}(x_1, \dots, x_n) = \omega_{g,n}. \tag{8.11}$$

It shows that the r-ELSV conjecture is equivalent to r-BM conjecture. Since there is independent evidence for both conjectures, this theorem transfers that evidence between them. It is also interesting in it's own right, providing a relation between to a priori very different aspects of the theory of Hurwitz numbers.

8.1.5 Plan of the chapter

In Section 8.2 we describe once again the completed cycles and completed Hurwitz numbers involved in the r-ELSV conjecture.

In Section 8.3 we review the work of Chiodo showing that the cohomology classes $\mathcal{S}$ form a cohomological field theory. We determine that cohomological field theory and the Givental R-matrix assigned to $\mathcal{S}$.

Section 8.4 contains the proof of the main theorem. It is based on the identification [113] of the Givental-Teleman theory of semi-simple cohomological field theories with the Chekhov-Eynard-Orantin topological recursion theory. We check that the R-matrix for the class $\mathcal{S}$ coincides with the R-matrix assigned to the curve $x = -y^r + \log y$.

8.1.6 Notation

The following notation is used consistently throughout this chapter.

- $r \geq 1$ is an integer; we work with the space of r-spin structures.
- I is the imaginary unit, $\mathrm{I}^2 = -1$.
- J is the primitive rth root of unity, $\mathrm{J}^r = 1$.
- $g \geq 0$ is the (arithmetic) genus of the curves C under consideration.
- $n \geq 1$ is the number of marked points on C; the points themselves are denoted by $x_1, \dots, x_n$.
- $k_1, \dots, k_n$ is an n-tuple of positive integers such that $m = (\sum k_i + n + 2g - 2)/r$ is also an integer. We consider meromorphic functions on C with n poles of orders $k_1, \dots, k_n$ at the marked points $x_1, \dots, x_n$. Such a function represents a ramified covering of $\mathbb{P}^1$ of degree $\sum k_i$ with a ramification point of type $(k_1, \dots, k_n)$ at ∞. We also require the covering to have exactly m additional ramification points of multiplicity r.
- Each k_i determines a unique couple of integers p_i, a_i such that $k_i = rp_i + (r - 1 - a_i)$ and $a_i \in \{0, \dots, r-1\}$. Thus p_i is the quotient and a_i the "reversed remainder" of the division of k_i by r. The r-spin structures we consider will be rth roots of the line bundle $K(-\sum a_i x_i)$.

The integers g, n, r, $k_1, \dots, k_n$ determine the integers m, p_i, and a_i uniquely.

8.2 Completed cycles and completed Hurwitz numbers

Okounkov and Panharipande proved in [88] that the left-hand side of Conjecture 8.4 is equal to Hurwitz numbers with completed cycles. To make this chapter more independent from the rest of the text, we introduce them once more in a way that is slightly different from what we did in Chapter 2. Note that in this chapter, as well as in Chapter 3, we use a different normalization from what we did in Chapter 2 (see Remark 2.5).

8.2.1 Completed cycles

A *partition* λ of an integer q is a non-increasing finite sequence $\lambda_1 \geq \cdots \geq \lambda_l$ such that $\sum \lambda_i = q$.

It is known that the irreducible representations ρ_λ of the symmetric group S_q are in a natural one-to-one correspondence with the partitions λ of q. On the other hand, to a partition λ of q one can assign a central element $C_{p,\lambda}$ of the group algebra $\mathbb{C}S_p$ for any positive integer p. The coefficient of a given permutation $\sigma \in S_p$ in $C_{p,\lambda}$ is defined as the number of ways to choose and number l cycles of σ so that their lengths are $\lambda_1, \dots, \lambda_l$, and the remaining $p-q$ elements are fixed points of σ. Thus the coefficient of σ vanishes unless its cycle lengths are $\lambda_1, \dots, \lambda_l, 1, \dots, 1$. In particular, $C_{p,\lambda} = 0$ if $p < q$. Thus $C_{p,\lambda}$ is *the sum of permutations with l numbered cycles of lengths $\lambda_1, \dots, \lambda_l$ and any number of non-numbered fixed points.*

The collection of elements $C_{p,\lambda}$ for $p = 1, 2, \dots$ is called a *stable center element*[1] C_λ. For example, the stable element C_2 is the sum of all transpositions in $\mathbb{C}S_p$, which is well-defined for each p, (in particular, equal to zero for $p = 1$).

Let λ be a partition of q and μ a partition of p. Since $C_{p,\lambda}$ lies in the center of $\mathbb{C}S_p$, it is represented by a scalar (multiplication by a constant) in the representation ρ_μ of S_p. Denote this scalar by $f_\lambda(\mu)$. Thus to a stable center element C_λ we have assigned a function f_λ defined on the set of all partitions. We are interested in the vector space spanned by the functions f_λ.

To study this space, one defines some new functions on the set of partitions as follows[2]:

$$\mathbf{p}_{r+1}(\mu) = \frac{1}{r+1} \sum_{i \geq 1} \left[(\mu_i - i + \frac{1}{2})^{r+1} - (-i + \frac{1}{2})^{r+1} \right] \qquad (r \geq 0). \tag{8.12}$$

Theorem 8.8 (Kerov, Olshansky [65]). *The vector space spanned by the functions f_λ coincides with the algebra generated by the functions $\mathbf{p}_1, \mathbf{p}_2, \dots$.*

As a corollary, to each stable center element C_λ we can assign a polynomial in $\mathbf{p}_1, \mathbf{p}_2, \dots$ and, conversely, each $\mathbf{p}_{r+1}$ corresponds to a linear combination of stable center elements C_λ.

Definition 8.9. The linear combination of stable center elements corresponding to $\mathbf{p}_{r+1}$ is called the *completed* $(r+1)$*-cycle* and denoted by $\overline{C}_{r+1}$.

The first completed cycles are:

$$\begin{array}{rcl} \overline{C}_1 &=& C_1, \\ \overline{C}_2 &=& C_2, \\ \overline{C}_3 &=& C_3 + C_{1,1} + \frac{1}{12} C_1, \\ \overline{C}_4 &=& C_4 + 2C_{2,1} + \frac{5}{4} C_2, \\ \overline{C}_5 &=& C_5 + 3C_{3,1} + 4C_{2,2} + \frac{11}{3} C_3 + 4C_{1,1,1} + \frac{3}{2} C_{1,1} + \frac{1}{80} C_1. \end{array} \tag{8.13}$$

For reasons that will become clear later, we say that a stable center element C_λ involved in the completed cycle $\overline{C}_{r+1}$ has *genus defect* $[r + 2 - \sum(\lambda_i + 1)]/2$.

---

[1]This has nothing to do with stable curves.

[2]The standard definition involves certain additive constants that we have dropped to simplify the expression, since these constants play no role here.

8.2.2 Completed Hurwitz numbers

Let $K = \sum k_i$. Recall that the completed $(r+1)$-cycle can be considered as a central element of the group algebra $\mathbb{C}S_K$. An *r-factorization of type* $(k_1, \dots, k_n)$ in the symmetric group S_K is a factorization

$$\sigma_1 \dots \sigma_m = \sigma \tag{8.14}$$

such that (i) the cycle lengths of σ equal $k_1, \dots, k_n$ and (ii) each permutation σ_i enters the completed $(r+1)$-cycle with a nonzero coefficient. The product of these coefficients for i going from 1 to m is called the *weight* of the r-factorization.

Choose m points $y_1, \dots, y_m \in \mathbb{C}$ and a system of m loops $s_i \in \pi_1(\mathbb{C} \setminus \{y_1, \dots, y_m\})$, s_i going around y_i. Then to an r-factorization one can assign a family of stable maps from nodal curves to $\mathbb{P}^1$. This is done in the following way. (i) Consider the covering of $\mathbb{P}^1$ ramified over $y_1, \dots, y_m$, and ∞ with monodromies given by $\sigma_1, \dots, \sigma_m$ and σ^{-1} (relative to the chosen loops). (ii) If σ_i has l_i distinguished cycles and genus defect g_i, glue a curve of genus g_i with l_i marked points to the l_i preimages of the ith ramification point that correspond to the distinguished cycles. The covering mapping is extended on this new component by saying that it is entirely projected to the ith ramification point. (iii) Among the newly added components, contract those that are unstable.

One can easily check that the arithmetic genus of the curve C constructed in this way is equal to g. The complex structure on the newly added components of C can be chosen arbitrarily, which implies that in general we obtain not a unique stable map, but a family of stable maps.

An r-factorization is called *transitive* if the curve C assigned to the factorization is connected, in other words if one can go from every element of $\{1, \dots, K\}$ to any other by applying the permutations σ_i and jumping from one distinguished cycle of σ_i to another one.

Definition 8.10. The *completed Hurwitz number* $h_{g,r;k_1,\dots,k_n}$ is the sum of weights of the transitive r-factorizations of type $(k_1, \dots, k_n)$.

Theorem 8.11 (Okounkov, Pandharipande [88]). *The relative Gromov-Witten invariant*

$$(r!)^m \int_{\overline{\mathcal{M}}^{\mathrm{rel}}} \mathrm{ev}_1^*(\omega)\psi_1^r \cdots \mathrm{ev}_n^*(\omega)\psi_m^r \tag{8.15}$$

is equal to the corresponding completed Hurwitz number.

Thus there is no clash of notation if we denote both by $h_{g,r;k_1,\dots,k_n}$.

8.2.3 A digression on Kadomtsev-Petviashvili

The Hurwitz numbers can be arranged into a generating series

$$G_r(\beta; p_1, p_2, \dots) = \sum_{g,n} \frac{1}{n!} \sum_{k_1,\dots,k_n} h_{g,r;k_1,\dots,k_n} \frac{\beta^m}{m!} p_{k_1} \dots p_{k_n}. \tag{8.16}$$

(One can prove that $h_{g,r;k_1,\dots,k_n} = 0$ whenever $m = (\sum k_i + n + 2g - 2)/r$ is not an integer.)

Theorem 8.12. *The series G_r is a solution of the Kadomtsev-Petviashvili (KP) hierarchy in variables p_i, β being a parameter.*

The proof of this theorem is a straightforward generalization of [63] and [85].

8.3 The class $\mathcal{S}$

8.3.1 Chiodo's formula

Chiodo computed the Chern characters

$$h_k^\circ(R^\bullet p_*\mathcal{L}) = h_k^\circ(R^0 p_*\mathcal{L}) - h_k^\circ(R^1 p_*\mathcal{L}) \tag{8.17}$$

and obtained the following expression.

Theorem 8.13 (Chiodo [17], Theorem 1.1.1)**.** *We have*

$$\begin{aligned} h_k^\circ(R^\bullet p_*\mathcal{L}) =& \frac{B_{k+1}(\frac{1}{r})}{(k+1)!}\kappa_k - \sum_{i=1}^{n} \frac{B_{k+1}(\frac{a_i+1}{r})}{(k+1)!}\psi_i^k \\ &+ \frac{r}{2}\sum_{a=0}^{r-1} \frac{B_{k+1}(\frac{a_i+1}{r})}{(k+1)!}(j_a)_* \frac{(\psi')^k + (\psi'')^k}{\psi' + \psi''}. \end{aligned} \tag{8.18}$$

Here j_a is the push-forward from the boundary components with a chosen node and a chosen branch of index a.

Chiodo's formula makes it possible to compute any class of the form

$$\exp\left(\sum_{k\geq 1} s_k h_k^\circ(R^\bullet p_*\mathcal{L})\right). \tag{8.19}$$

Specifically, we will need

$$c(-R^\bullet p_*\mathcal{L}) = \exp\left(-\sum_{k\geq 1}(k-1)! h_k^\circ(R^\bullet p_*\mathcal{L})\right), \tag{8.20}$$

with $s_k = -(k-1)!$.

8.3.2 Topological field theory and R-matrix

In this section we use the notion of a *cohomological field theory*, *topological field theory* and R-matrix action. See [92] or [97] for an introduction.

Let $\Omega_{g,n}(a_1,\ldots,a_n) = \pi_*(\mathcal{S})$ be the push-forward of the class $\mathcal{S} = c(R^1 p_*\mathcal{L})/c(R^0 p_*\mathcal{L})$ from the space of r-spin structures to $\overline{\mathcal{M}}_{g,n}$. Further, denote by $\omega_{g,n}$ the degree 0 part of $\Omega_{g,n}$.

The topological field theory

The degree 0 class $\omega_{g,n}$ is equal to the push-forward of the cohomology class 1 from the space of r-spin structures to the space of stable curves. Thus it is east to compute:

$$\omega_{g,n}(a_1,\ldots,a_n) = r^{2g-1}\delta_{2g-2-\sum a_i \bmod r}, \tag{8.21}$$

because the degree of $\pi : \overline{\mathcal{M}}^{1/r} \to \overline{\mathcal{M}}_{g,n}$ equals r^{2g-1}. The classes $\omega_{g,n}$ form a topological field theory with unit on the vector space $\langle e_0,\ldots,e_{r-1}\rangle$ with quadratic form

$$\eta_{ab} = \frac{1}{r}\delta_{a+b+2 \bmod r} \tag{8.22}$$

and unit e_0.

The 3-point correlators in genus 0 are equal to

$$\langle e_a e_b e_c \rangle = \frac{1}{r} \delta_{a+b+c+2 \mod r}. \tag{8.23}$$

Therefore the quantum product is given by

$$e_a \bullet e_b = e_{a+b \bmod r}. \tag{8.24}$$

The idempotents of the quantum product are

$$\frac{1}{r} \sum_{a=0}^{r-1} \mathrm{J}^{ai} e_a, \tag{8.25}$$

where J is the primitive rth root of unity and $0 \leq i \leq r-1$.

The R-matrix and the correlators

It follows from Chiodo's formula that the family of cohomology classes $\Omega_{g,n}$ can be obtained from $\omega_{g,n}$ by Givental's R-matrix action, where

$$R(z) = \exp \left[- \sum_{k \geq 1} \frac{\mathrm{diag}_{a=0}^{r-1} B_{k+1}\left(\frac{a+1}{r}\right)}{k(k+1)} z^k \right] \tag{8.26}$$

in basis (e_a).

This fact has been observed in [20] in the presence of Witten's class, then in [19] in the setting we use here. It follows that the classes $\Omega_{g,n}$ form a semi-simple cohomological field theory.

We define the *correlators* $\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\widetilde{\mathrm{coh}}}$ of this cohomological field theory:

$$\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\widetilde{\mathrm{coh}}} := \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(a_1, \ldots, a_n)\, \psi_1^{d_1} \cdots \psi_n^{d_n} = \int_{\overline{\mathcal{M}}_{g,n;a_1,\ldots,a_n}^{1/r}} \mathcal{S}\, \psi_1^{d_1} \cdots \psi_n^{d_n}, \tag{8.27}$$

which are just the coefficients of the n-point function

$$F_{g,r}(x_1, \ldots, x_n) = \sum_{\substack{d_1,\ldots,d_n \geq 0 \\ 0 \leq a_1,\ldots,a_n \leq r-1}} \left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\widetilde{\mathrm{coh}}} r^{2g+2n-2+\frac{2g-2-\sum_{i=1}^n a_i}{r} - \sum_{i=1}^n d_i}$$
$$\prod_{i=1}^n \sum_{p_i=0}^{\infty} \frac{(rp_i + r - a_i - 1)^{p_i+d_i}}{p_i!} e^{(rp_i+r-a_i-1)x_i}. \tag{8.28}$$

8.4 Equivalence of the r-ELSV and r-BM conjectures

In this section we derive a formula for the n-point differentials $\omega_{g,n}$ of the curve $\mathcal{S}^{(r)}\colon x = -y^r + \log y$ and prove Theorem 8.7. In particular, this implies that the r-ELSV conjecture and the r-BM conjecture are equivalent.

To do that, we perform a number of local computations with the spectral curve, following the recipe of [113] in order to present the cohomological field theory corresponding to the spectral curve as a particular R-matrix action in the sense of Givental, applied to a topological field theory equal to a direct sum of r one-dimentional topological field theories, properly rescaled. We prove that the cohomological field theory that we obtain this way is a multiple of $\{\Omega_{g,n}\}$ defined in Section 8.3.2 rewritten in the basis of normalized idempotents.

Proof of Theorem 8.7. Our goal is to compare the coefficients of the n-differentials $\omega_{g,n}$ with the correlators $\widetilde{\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\text{coh}}}$ of the cohomological field theory described in Section 8.3.2 in terms of the Givental group action. In [113] the coefficients of the differentials $\omega_{g,n}$ are, under some conditions and modulo some extra factors, expressed in terms of correlators of a cohomological field theory. We use the result of [113] and compare the cohomological field theories and the extra factors that appear in the statement of the theorem and in the identification formula in [113].

Let us outline this comparison. We have the following formula for $\omega_{g,n}$ given in [113]:

$$\omega_{g,n} = \sum_{\substack{i_1,\ldots,i_n \\ d_1,\ldots,d_n}} \left\langle \tau_{d_1}^{i_1} \cdots \tau_{d_n}^{i_n} \right\rangle_g^{\text{t.r.}} D \left(\prod_{j=1}^{n} \left(-2 \frac{\partial}{\partial x_j} \right)^{d_i} \xi_{i_j}(x_j) \right), \tag{8.29}$$

where ξ_i are some auxilliary functions of the coordinate x on the curve, we recall them below. The correlators $\left\langle \tau_{d_1}^{i_1} \cdots \tau_{d_n}^{i_n} \right\rangle_g^{\text{t.r.}}$ are represented as a sum over Givental graphs, whose structure is described in [118, 113].

Remark 8.14. The local computations that we do in this section in order to specify all ingredients of Equation (8.29) are a direct generalization of the computations of Eynard in [34, Section 8.2] for the curve $x = -y + \log y$ that shows that the usual ELSV formula is equivalent to the Bouchard-Mariño conjecture for the usual Hurwitz numbers.

Note that the indices $i_1, \ldots, i_n$ that we have in Equation (8.29) refer to a summation in the basis of the normalized idempotents (cf. Section 8.3.2), while the indices $a_1, \ldots, a_n$ in Theorem 8.7 refer to a summation in the natural flat basis of the cohomological field theory.

Notation 8.15. Throughout this text, when we write an object with a tilde we mean refer to this object expressed in the natural flat basis, while when we write the same object without a tilde it should be interpreted as expressed in the basis of canonical idempotents. Furthermore, $\langle \cdot \rangle^{\text{t.r.}}$ refers to the correlators of the cohomological field theory associated to the the spectral curve $\mathcal{S}^{(r)}$ by the methods of [113], whereas $\langle \cdot \rangle^{\text{coh}}$ refers to the correlators of the cohomological field theory $\Omega_{g,n}$.

Changing the coordinates, we can rewrite Equation (8.29) in the following form:

$$\omega_{g,n} = \sum_{\substack{a_1,\ldots,a_n \\ d_1,\ldots,d_n}} \widetilde{\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\text{t.r.}}} D \left(\prod_{j=1}^{n} \left(-2 \frac{\partial}{\partial x_j} \right)^{d_i} \tilde{\xi}_{a_j}(x_j) \right), \tag{8.30}$$

where

$$\widetilde{\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\text{t.r.}}} := \sum_{i_1,\ldots,i_n} \left\langle \tau_{d_1}^{i_1} \cdots \tau_{d_n}^{i_n} \right\rangle_g^{\text{t.r.}} \prod_{j=1}^{n} \mathrm{J}^{(a_j+1)i_j}, \tag{8.31}$$

and

$$\tilde{\xi}_a := \sum_{i=0}^{r-1} r^{-1} \mathrm{J}^{-(a+1)i} \xi_i . \tag{8.32}$$

Below, in Lemma 8.18, we prove that

$$\tilde{\xi}_a = \mathrm{I}\sqrt{2} r^{\frac{1}{2} - \frac{a+1}{r}} \sum_{n=0}^{\infty} \frac{(rn + r - a - 1)^n}{n!} e^{(rn+r-a-1)x}. \tag{8.33}$$

Further, in Lemma 8.19 we prove that

$$\widetilde{\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_n}^{a_n} \right\rangle_g^{\text{t.r.}}} \prod_{j=1}^{n} \mathrm{I}\sqrt{2r} \cdot r^{\frac{-(a_j+1)}{r}} (-2)^{d_n} \tag{8.34}$$
$$= \widetilde{\left\langle \tau_{d_1}^{a_1} \cdots \tau_{d_\ell}^{a_\ell} \right\rangle_g^{\text{coh}}} r^{2g+2\ell-2+\frac{2g-2-\sum_{i=1}^{\ell} a_i}{r} - \sum_{i=1}^{\ell} d_i}$$

Substituting these two expression in Equation (8.30), using equation (8.28), we obtain the equality of Theorem 8.7, which proves the theorem. □

The rest of the section consists of a step-by-step computation of the expansions of various local objects on the curve $x = -y^r + \log y$ that are needed to formulate and prove Lemmas 8.18 and 8.19. We follow the scheme of computations proposed in [113] and use the same notation as there.

$$y_i := r^{-1/r} \mathrm{J}^i, \qquad i = 0, \ldots, r-1. \tag{8.35}$$

The critical values of the function x at these points are

$$x_i := -\frac{1}{r} + \frac{\mathrm{I}2\pi i}{r} - \frac{\log r}{r}, \qquad i = 0, \ldots, r-1. \tag{8.36}$$

We denote by z_i the local coordinates near the critical points, that is, $z_i^2 + x_i = x$, $i = 0, 1, \ldots, r-1$. Let us make a choice of the expansion of function y in z_i. One of the possible choices, that fixes $y = y(z_i)$ unambiguously is

$$y(z_i) = y_i + y_{i1} z_i + O(z_i^3), \tag{8.37}$$

where

$$y_{i1} := \mathrm{I}\sqrt{2} r^{-\frac{1}{2}-\frac{1}{r}} \mathrm{J}^i, \qquad i = 0, \ldots, r-1. \tag{8.38}$$

8.4.1 Reciprocal Gamma function

We are using the following expansion of the reciprocal Gamma function:

$$\frac{1}{\Gamma(w+v)} = \frac{\mathrm{I}}{2\pi} \int_{C_+} (-t)^{-w-v} e^{-t} \mathrm{d}t \tag{8.39}$$
$$\sim \frac{w^{-w-v+\frac{1}{2}} e^w}{\sqrt{2\pi}} \exp\left(\sum_{j=1}^{\infty} \frac{B_{j+1}(v)}{j(j+1)} (-w)^j \right),$$

where C_+ is the Hankel contour [76] that goes around the positive real numbers, and $B_j(v)$ are the Bernoulli polynomials defined by

$$\frac{w e^{wv}}{e^w - 1} = \sum_{j=0}^{\infty} B_j(v) \frac{w^j}{j!} . \tag{8.40}$$

Note that $B_j(1-v) = (-1)^j B_j(v)$, $j = 0, 1, \ldots$.

8.4.2 Local expansions of y

Let us fix $0 \leq i \leq r-1$. We are interested in the expansion of the odd part of $y = y(z_i)$ in z_i, that is, we consider $(y(z_i) - y(-z_i))/2$. We consider also the coordinate $t := ry^r$. It is easy to see that

$$y = t^{\frac{1}{r}} r^{-\frac{1}{r}} \mathrm{J}^i; \tag{8.41}$$

$$-x_i - y^r + \log y = \frac{1}{r} - \frac{t}{r} + \frac{\log t}{r}; \tag{8.42}$$

$$\mathrm{d}y = t^{\frac{1-r}{r}} r^{-1-\frac{1}{r}} \mathrm{J}^i \mathrm{d}t. \tag{8.43}$$

Note also that t runs along the negative Hankel contour when z_i runs from $-\infty$ to $+\infty$.

Lemma 8.16. *Denote the coefficients of the odd part of $y(z_i)$ by:*

$$\frac{y(z_i) - y(-z_i)}{2} = \mathrm{I}\sqrt{2} r^{-\frac{1}{2}-\frac{1}{r}} \mathrm{J}^i z_i \sum_{j=0}^{\infty} \frac{V_j (2r)^j z_i^{2j}}{(2j+1)!!} . \tag{8.44}$$

Then we have

$$\sum_{j=0}^{\infty} V_j \zeta^j = \exp\left(- \sum_{i=1}^{\infty} \frac{B_{j+1}(\frac{1}{r})}{j(j+1)} \zeta^j \right) . \tag{8.45}$$

Proof. The Laplace method gives the following asymptotic expansion:

$$\int_{-\infty}^{\infty} y'(z_i) \exp(-rwz_i^2) \mathrm{d}z_i = \mathrm{I}\sqrt{2\pi} r^{-1-\frac{1}{r}} \mathrm{J}^i w^{-\frac{1}{2}} \sum_{j=0}^{\infty} V_j w^{-j} . \tag{8.46}$$

On the other hand,

$$\begin{aligned} \int_{-\infty}^{\infty} y'(z_i) e^{-rwz_i^2} \mathrm{d}z_i &= \int_{\tilde{C}_-} e^{-rw(-x_i - y^r + \log(y))} \mathrm{d}y \\ &= r^{-\frac{1+r}{r}} \mathrm{J}^i \int_{C_-} e^{-w(1-t+\log t)} t^{\frac{1-r}{r}} \mathrm{d}t \\ &= r^{-\frac{1+r}{r}} \mathrm{J}^i e^{-w} \int_{C_-} e^{wt} t^{-w+\frac{1-r}{r}} \mathrm{d}t \\ &= -r^{-\frac{1+r}{r}} \mathrm{J}^i e^{-w} w^{w-\frac{1}{r}} \int_{C_+} e^{-p} (-p)^{-w+\frac{1-r}{r}} \mathrm{d}p \end{aligned} \tag{8.47}$$

(we used the substitution $p = -wt$ that transforms the negative Hankel contour C_- to the Hankel contour C_+). Thus we see that

$$\begin{aligned} \int_{-\infty}^{\infty} y'(z_i) e^{-rwz_i^2} \mathrm{d}z_i &= \frac{2\pi \mathrm{I} r^{-\frac{1+r}{r}} \mathrm{J}^i e^{-w} w^{w-\frac{1}{r}}}{\Gamma(w+1-\frac{1}{r})} \\ \sim \frac{2\pi \mathrm{I} r^{-\frac{1+r}{r}} \mathrm{J}^i e^{-w} w^{w-\frac{1}{r}} w^{-w+\frac{1}{r}-\frac{1}{2}} e^w}{\sqrt{2\pi}} &\exp\left(\sum_{j=1}^{\infty} \frac{B_{j+1}(1-\frac{1}{r})}{j(j+1)} (-w)^{-j} \right) \\ = \sqrt{2\pi} \mathrm{I} r^{-\frac{1+r}{r}} \mathrm{J}^i w^{-\frac{1}{2}} &\exp\left(- \sum_{j=1}^{\infty} \frac{B_{j+1}(\frac{1}{r})}{j(j+1)} w^{-j} \right) , \end{aligned} \tag{8.48}$$

which proves

$$\sum_{j=0}^{\infty} V_j w^{-j} = \exp\left(- \sum_{j=1}^{\infty} \frac{B_{j+1}(\frac{1}{r})}{j(j+1)} w^{-j} \right) . \tag{8.49}$$

□

8.4.3 Two-point function

Now we consider the two-point function. According to [34], since $\mathrm{d}x$ is a meromorphic 1-form in y, we know that the Laplace transform of the two-point function is represented as a Givental-type edge contribution. So, we have to compute only the even coefficients of the local expansion of half of the two-point function in order to specify the Givental operator imposed by the topological recursion, namely, we are interested in the coefficients of the function

$$\mathrm{Y}_{i_1 i_2} := \frac{y_{i_1 1} y'(z_{i_b})}{(y_{i_1} - y(z_{i_2}))^2} = \frac{\delta_{i_1 i_2}}{z^2} + O(1). \tag{8.50}$$

Lemma 8.17. *We have:*

$$\frac{\mathrm{Y}_{i_1 i_2}(z_{i_2}) + \mathrm{Y}_{i_1 i_2}(-z_{i_2})}{2} = -\sum_{k=0}^{\infty} \frac{(U_k)_{i_1 i_2} (2r)^i z_{i_2}^{2k-2}}{(2k-3)!!}\ , \tag{8.51}$$

where $(U_k)_{i_1 i_2}$ is given by

$$\sum_{k=0}^{\infty} (U_k)_{i_1 i_2} z^k = \frac{1}{r} \sum_{c=0}^{r-1} \exp\left(- \sum_{k=1}^{\infty} \frac{B_{k+1}(\frac{c}{r}) z^k}{k(k+1)} \right) \mathrm{J}^{c i_2 - c i_1}\ . \tag{8.52}$$

Proof. Observe that

$$\int_{-\infty}^{\infty} \left(\frac{y_{i_1 1} y'(z_{i_2})}{(y_{i_1} - y(z_{i_2}))^2} \right) e^{-rwz_{i_2}^2} \mathrm{d}z_{i_2} \sim -2\sqrt{\pi} r^{\frac{1}{2}} w^{\frac{1}{2}} \sum_{k=0}^{\infty} (U_k)_{i_1 i_2} w^{-k}. \tag{8.53}$$

On the other hand,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{y_{i_1 1} y'(z_{i_2})}{(y_{i_1} - y(z_{i_2}))^2} e^{-rwz_{i_2}^2} \mathrm{d}z_{i_2} \\ &= rwy_{i_1 1} \int_{-\infty}^{\infty} \frac{1}{y_{i_1} - y(z_{i_2})} e^{-rwz_{i_2}^2} 2z_{i_2} \mathrm{d}z_{i_2} \\ &= rwy_{i_1 1} \int_{C_-} \frac{y(z_{i_2})^r - \frac{1}{r}}{y(z_{i_2}) - y_{i_1}} e^{-w(1-t+\log t)} \frac{\mathrm{d}t}{t}\ . \end{aligned} \tag{8.54}$$

Here we can use that

$$\begin{aligned} ry_{i_1 1} \frac{y(z_{i_2})^r - \frac{1}{r}}{y(z_{i_2}) - y_{i_1}} &= r\mathrm{I}\sqrt{2} r^{-\frac{1}{2} - \frac{1}{r}} \mathrm{J}^{i_1} \frac{(t^{\frac{1}{r}} r^{-\frac{1}{r}} \mathrm{J}^{i_2})^r - r^{-1}}{t^{\frac{1}{r}} r^{-\frac{1}{r}} \mathrm{J}^{i_2} - r^{-\frac{1}{r}} \mathrm{J}^{i_1}} \\ &= \mathrm{I}\sqrt{2} r^{-\frac{1}{2}} \sum_{c=0}^{r-1} \mathrm{J}^{c i_2 - c i_1} t^{\frac{c}{r}}\ . \end{aligned} \tag{8.55}$$

Therefore,

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{y_{i_1 1} y'(z_{i_2})}{(y_{i_1} - y(z_{i_2}))^2} e^{-rwz_{i_2}^2} \mathrm{d}z_{i_2} \\ &= \mathrm{I}\sqrt{2} r^{-\frac{1}{2}} w e^{-w} \sum_{c=0}^{r-1} \mathrm{J}^{c i_2 - c i_1} \int_{C_-} e^{wt} t^{-w-1+\frac{c}{r}} \mathrm{d}t \\ &= -\mathrm{I}\sqrt{2} r^{-\frac{1}{2}} w^{w+1} e^{-w} \sum_{c=0}^{r-1} \mathrm{J}^{c i_2 - c i_1} w^{-\frac{c}{r}} \int_{C_+} e^{-p} (-p)^{-w-1+\frac{c}{r}} \mathrm{d}p \\ &\sim -2\sqrt{\pi} r^{-\frac{1}{2}} w^{\frac{1}{2}} \sum_{c=0}^{r-1} \mathrm{J}^{c i_2 - c i_1} \exp\left(- \sum_{n=1}^{\infty} \frac{B_{n+1}(\frac{c}{r})}{n(n+1)} w^{-c} \right)\ . \end{aligned} \tag{8.56}$$

Thus we see that

$$\sum_{k=0}^{\infty}(U_k)_{i_1 i_2} z^k = \frac{1}{r}\sum_{c=0}^{r-1}\exp\left(-\sum_{k=1}^{\infty}\frac{B_{k+1}(\frac{c}{r})z^k}{k(k+1)}\right)\mathrm{J}^{ci_2-ci_1} \ . \tag{8.57}$$

□

8.4.4 Functions on the leaves

According to [113], the auxilliary function $\xi_i(x)$ that we put on the leaves in the graph expression for the correlation forms of the spectral curve are given by the following formula:

$$\xi_i := \frac{\mathrm{I}\sqrt{2}r^{-\frac{1}{2}-\frac{1}{r}}\mathrm{J}^i}{r^{-\frac{1}{r}}\mathrm{J}^i - y}. \tag{8.58}$$

Here the index i corresponds to the basis of normalized idempotents, so in the standard flat basis we have to consider the functions $\tilde{\xi}_a$ given by Equation (8.32):

$$\tilde{\xi}_a := \sum_{i=0}^{r-1} r^{-1}\mathrm{J}^{-(a+1)i}\xi_i \ . \tag{8.59}$$

Lemma 8.18. *We have:*

$$\tilde{\xi}_a = \mathrm{I}\sqrt{2}r^{\frac{1}{2}-\frac{a+1}{r}}\sum_{n=0}^{\infty}\frac{(rn+r-a-1)^n}{n!}e^{(rn+r-a-1)x} \ . \tag{8.60}$$

Proof. First, observe that

$$\begin{aligned}\tilde{\xi}_a &= \sum_{i=0}^{r-1} r^{-1}\mathrm{J}^{-(a+1)i}\xi_i = \mathrm{I}\sqrt{2}r^{-1-\frac{1}{2}}\sum_{i=0}^{r-1}\frac{\mathrm{J}^{-(a+1)i}}{1-r^{\frac{1}{r}}\mathrm{J}^{-i}y} \\ &= \mathrm{I}\sqrt{2}r^{-\frac{1}{2}}\left(\frac{r^{\frac{r-a-1}{r}}y^{r-a-1}}{1-ry^r}\right).\end{aligned} \tag{8.61}$$

Following [21], we define the Lambert function

$$W(z) := -\sum_{n=1}^{\infty}\frac{n^{n-1}}{n!}(-z)^n \ , \tag{8.62}$$

and use its property

$$\left(\frac{W(z)}{z}\right)^{\alpha} = \sum_{n=0}^{\infty}\frac{\alpha(n+\alpha)^{n-1}}{n!}(-z)^n \ . \tag{8.63}$$

We have the following equation: $e^x = ye^{-y^r}$. This equation implies (cf. [114])

$$y = \left(\frac{W(-re^{rx})}{-r}\right)^{\frac{1}{r}} \tag{8.64}$$

and

$$\frac{\mathrm{d}y^{r-a-1}}{\mathrm{d}x} = \frac{(r-a-1)y^{r-a-1}}{1-ry^r} \ . \tag{8.65}$$

Therefore,

$$\frac{(r-a-1)y^{r-a-1}}{1-ry^r} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{W(-re^{rx})}{-r}\right)^{\frac{r-a-1}{r}} \tag{8.66}$$
$$= (-r)^{-\frac{r-a-1}{r}}\frac{\mathrm{d}}{\mathrm{d}x}(-re^{rx})^{\frac{r-a-1}{r}}\sum_{n=0}^{\infty}\frac{\frac{r-a-1}{r}(n+\frac{r-a-1}{r})^{n-1}}{n!}(re^{rx})^n$$
$$= (r-a-1)\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}\frac{(rn+r-a-1)^{n-1}}{n!}e^{(rn+r-a-1)x}$$
$$= (r-a-1)\sum_{n=0}^{\infty}\frac{(rn+r-a-1)^{n}}{n!}e^{(rn+r-a-1)x}$$

So, we see that

$$\tilde{\xi}_a = \mathrm{I}\sqrt{2}r^{\frac{1}{2}-\frac{a+1}{r}}\sum_{n=0}^{\infty}\frac{(rn+r-a-1)^n}{n!}e^{(rn+r-a-1)x} \tag{8.67}$$

□

8.4.5 Comparison of the correlators

Consider the n-point correlators $\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle}_g^{\text{t.r.}}$ introduced in Equation (8.30). They are obtained via a linear change of the indices from the correlators $\left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{t.r.}}$, where the latter ones are defined in [113] as the sum over Givental graphs. The structure constants of these graphs, that is, the parameters that we put on vertices, edges, leaves, and dilaton leaves, are defined in terms of local data of the curve $x=-y^r+\log y$ at the ramification points. More precisely, they are defined via the coefficients of the expansions of the function $y(z_i)$ in the local coordinate z_i and the components of the Bergman kernel $Y_{i_1i_2}$ in the local coordinate z_{i_2}.

In this Section we prove that the correlators $\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tilde{\tau}_{d_n}^{a_n}\right\rangle}_g^{\text{t.r.}}$ are equal, up to some multiplicative factors, to the correlators $\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle}_g^{\text{coh}}$ of the cohomological field theory described in Section (8.3.2). Namely, we prove the following lemma.

Lemma 8.19.

$$\sum_{i_1,\dots,i_\ell}\left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{t.r.}}\prod_{k=1}^{n}r^{\frac{1}{2}}\mathrm{J}^{(a_k+1)i_k}\prod_{k=1}^{n}\mathrm{I}\sqrt{2}r^{\frac{-(a_k+1)}{r}}(-2)^{d_k} \tag{8.68}$$
$$=\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle}_g^{\text{coh}} r^{2g+2n-2+\frac{2g-2-\sum_{k=1}^n a_k}{r}-\sum_{k=1}^n d_i}$$

Proof. The proof follows from the comparison of the ingredients of the Givental graph expressions on both sides, following the identification theorem in [113]. Let us rewrite both sides of the equality in the basis of normalized idempotents:

$$\left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{t.r.}}\prod_{k=1}^{n}(-2r)^{d_k+\frac{1}{2}} = \left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{coh}} r^{2g+n-2+\frac{2g+n-2}{r}}, \tag{8.69}$$

where

$$\left\langle\tau_{d_1}^{i_1}\cdots\tau_{d_n}^{i_n}\right\rangle_g^{\text{coh}} := \sum_{a_1,\dots,a_n}\widetilde{\left\langle\tau_{d_1}^{a_1}\cdots\tau_{d_n}^{a_n}\right\rangle}_g^{\text{coh}}\prod_{j=1}^{n}\mathrm{J}^{-(a_j+1)i_j} \tag{8.70}$$

The result of Chiodo (see Theorem 8.13) implies that the generating function of the correlators $\left\langle \tau_{d_1}^{i_1} \cdots \tau_{d_n}^{i_n} \right\rangle_g^{\mathrm{coh}}$ is obtained from the r properly normalized copies of the Kontsevich-Witten tau function by application of the quantization of the operator

$$R_i^j(\zeta) := \exp\left(- \sum_{k=1}^{\infty} \frac{\zeta^k}{r} \sum_{a=0}^{r-1} \mathrm{J}^{ai-aj} \frac{B_{k+1}\left(\frac{a}{r}\right)}{k(k+1)} \right) . \tag{8.71}$$

In particular (we use it below), we have:

$$\begin{aligned} R_i^{\mathbf{1}}(\zeta) &:= \sum_{j=0}^{r-1} \sum_{a=0}^{r-1} \frac{\mathrm{J}^j}{r} \frac{\mathrm{J}^{ai-aj}}{r} \exp\left(- \sum_{k=1}^{\infty} \zeta^k \frac{B_{k+1}\left(\frac{a}{r}\right)}{k(k+1)} \right) \\ &= \frac{\mathrm{J}^i}{r} \exp\left(- \sum_{k=1}^{\infty} \zeta^k \frac{B_{k+1}\left(\frac{1}{r}\right)}{k(k+1)} \right) . \end{aligned} \tag{8.72}$$

The weight of the correlators $\langle \prod_{i=1}^p \tau_{a_i} \rangle_q$ of the i-th copy of the Kontsevich-Witten tau function (or, in other words, the weight of the vertex labelled by i in the graphical representation of the Givental formula, as in [113]) is equal to

$$r^{2q-1} \sum_{\substack{a_1,\dots,a_p: \\ r | 2g-2-a_1+\dots+a_p}} \prod_{j=1}^{p} \mathrm{J}^{-(a_j+1)i} = r^{2q+p-2} \mathrm{J}^{-(2q+p-2)i} . \tag{8.73}$$

Let us compare that with the formula we get from the topological recursion, following the lines of [113]. We compare the coefficients of the expansion of y and the two-point function in the coordinate z_i, $i = 0, \dots, r-1$ (that determine the ingredients of the graphs in the formula for $\left\langle \tau_{d_1}^{i_1} \cdots \tau_{d_n}^{i_n} \right\rangle_g^{\mathrm{t.r.}}$) with the corresponding formulas in terms of the operator $R_i^j(\zeta)$ that are used in the Givental graphical formula for $\left\langle \tau_{d_1}^{i_1} \cdots \tau_{d_n}^{i_n} \right\rangle_g^{\mathrm{coh}}$, in the same way as it is done in [113, Theorem 4.1].

Lemma 8.16 implies that, in the notation of [113],

$$\begin{aligned} \check{h}_{k+1}^i &= \mathrm{I} 2\sqrt{2} r^{-\frac{1}{2}-\frac{1}{r}} \mathrm{J}^i (2r)^k [\zeta^k] \exp\left(- \sum_{i=1}^{\infty} \frac{B_{i+1}(\frac{1}{r})}{i(i+1)} \zeta^i \right) \\ &= \mathrm{I}\sqrt{2r} r^{-1-\frac{1}{r}} (-2r)^{k+1} [\zeta^k] \left(-R_i^{\mathbf{1}}(-\zeta) \right) . \end{aligned} \tag{8.74}$$

Lemma 8.17 implies that, also in notation of [113],

$$\check{B}_{0,k}^{ji} = -(2r)^{k+1} [\zeta^{k+1}] R_i^j(\zeta) = (-2r)^{k+1} [\zeta^k] \left(\frac{1 - R(-\zeta)}{\zeta} \right) . \tag{8.75}$$

The vertex labelled by $\langle \prod_{i=1}^p \tau_{a_i} \rangle_q$ and an extra index i is also multiplied by (again in the notation of [113])

$$(-2h_1^i)^{2-2q-p} = (-\mathrm{I}2\sqrt{2} r^{-\frac{1}{2}-\frac{1}{r}} \mathrm{J}^i)^{2-2q-p}. \tag{8.76}$$

This all together (including the factors $(-2r)^{d_n+\frac{1}{2}}$ that we have on the global leaves in Equation (8.69)) gives the following extra factor for the vertex labelled by $\langle \prod_{i=1}^p \tau_{a_i} \rangle_q$, with p_d attached dilaton leaves and p_o ordinary leaves and/or half-edges ($p = p_d + p_o$), and an extra index i:

$$\frac{(-2r)^{a_1+\dots+a_p} (\mathrm{I}\sqrt{2r})^p r^{-p_d - \frac{p_d}{r}}}{(-\mathrm{I}2\sqrt{2} r^{-\frac{1}{2}-\frac{1}{r}} \mathrm{J}^i)^{2q+p-2}} = r^{(1+\frac{1}{r})(2q+p_o-2)} r^{(2q+p-2)} \mathrm{J}^{(2q+p-2)i} \tag{8.77}$$

(we used that $a_1 + \cdots + a_p = 3q - 3 + p$). Note that the factor $r^{(2q+p-2)} \mathrm{J}^{(2q+p-2)i}$ coinsides with the weight that we have in the Givental's presentation of Chiodo's formula, cf. Equation (8.73). Meanwhile, the sum of the exponents $2q + p_o - 2$ over all vertices in a graph of genus g with n global leaves is equal to $2g + n - 2$. Therefore, the product of the factors $r^{(1+\frac{1}{r})(2q+p_o-2)}$ over all vertices of a graph is exactly equal to the extra factor $r^{2g+n-2+\frac{2g+n-2}{r}}$ that we have on the right-hand side of Equation (8.69). □