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Next-to-leading power resummed rapidity distributions near threshold for Drell-Yan and diphoton production

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Abstract: We consider Drell-Yan production and QCD-induced diphoton production and compute their rapidity distributions up to next-to-leading power (NLP) in the threshold variable. We give results for rapidity distributions of the Drell-Yan process up to NNLO accuracy and show that a factorised structure occurs for the leading logarithms (LL) at NLP, generalising the result at leading power. For diphoton production, we generalise methods based on kinematical shifts to find the NLO cross section up to NLP for rapidity distributions. From the results for these two processes, we derive resummed cross sections at NLP LL accuracy that are double differential in the threshold variable and the rapidity variable, which generalise results for single differential resummed cross sections.

Keywords: Resummation, Higher-Order Perturbative Calculations

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1 Introduction

The increasing precision of experimental measurements from the Large Hadron Collider requires similar improvements in the accuracy of Standard Model theoretical predictions, especially for perturbative QCD. Progress can be made by including ever higher order contributions in the strong coupling constant, and by resumming certain kinematically enhanced contributions to all orders in perturbation theory, e.g. the production of particles near their kinematic threshold. In such processes, one can define a (partonic) threshold

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variable ξ, such that $\xi \to 0$ when approaching threshold. In this regime partonic cross sections take the generic form

$$\frac{d\hat{\sigma}}{d\xi} = \sigma_0 \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1} \left[c_{nm}^{(-1)} \left(\frac{\log^m \xi}{\xi}\right)_+ + c_n^{(\delta)} \delta(\xi) + c_{nm}^{(0)} \log^m \xi + \mathcal{O}(\xi) \right], \tag{1.1}$$

where σ_0 represents the Born-level cross section. The summation of logarithms related to the first contribution in the square brackets on the r.h.s. of eq. (1.1) has been much studied, see e.g. refs. [1–14], and also the summation of the second contribution proportional to $\delta(\xi)$ is known for processes with electroweak final states (see, for example, refs. [15, 16]). These two terms define the *leading power* (LP) (in ξ) contribution to the partonic cross section. Next-to-leading power (NLP) contributions are given in the third term, and their summation has seen much recent study. While they are not as divergent as their LP counterparts, they are still singular in the limit $\xi \to 0$, and can be numerically sizeable in this region (see e.g. [17]). Mapping the structure of large NLP logarithms is more challenging than at LP, where large logarithms can be related to the emission of soft and collinear gluons [18–20], and the factorisation of the cross section into universal jet and soft functions, together with a process-dependent hard function. At NLP radiation becomes sensitive to additional features of the underlying hard scattering process, such as the spin of the emitting particles, and it starts to resolve the structure of the hard scattering kernel, as well as the structure of clusters of virtual particles collinear to the directions of the incoming/outgoing hard particles. These features have been investigated both by means of direct-QCD (or diagrammatic) [20–40] and effective field theory methods, the latter based on soft-collinear effective theory (SCET) [41–53].

Currently, resummation of the large threshold logarithms at NLP is achieved at LL accuracy for scattering processes with a colour-singlet final state. This is done both in the direct-QCD approach [17, 40, 54–59] and in the SCET approach [60–64]. However, these discussions are limited to only a few observables, such as an inclusive cross section or invariant mass distribution. It is also important to consider other observables, in particular more differential distributions. We note that the Drell-Yan rapidity distribution near threshold has been recently investigated at NLP from a somewhat different perspective in [36, 57], based on previous works at LP, see e.g. [65–67].

In this paper we look specifically at rapidity distributions and cross sections that are double differential in the threshold variable (or invariant mass), and in the rapidity of (one of) the final state particle(s). It is well known that at leading power the partonic cross section for rapidity distributions takes on a factorised form [68–72], which enables the result for the invariant mass distribution to be conveniently extended to cross sections that are differential in the rapidity of one of the outgoing particles as well. In this paper we investigate whether, or to what extent, the results obtained in refs. [25, 54], which are both next-to-leading power results, can be extended to include rapidity distributions for scattering processes with colour-singlet final states. One of the main results of the latter references is a certain universality of the NLO partonic cross section up to NLP accuracy for colour-singlet final states, in that they be written as a universal factor multiplied by the Born-level partonic cross section with shifted Born kinematics. We investigate here

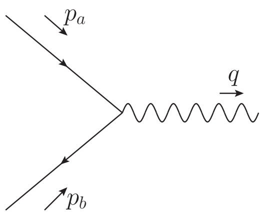

Figure 1. Tree-level contribution to the Drell-Yan process.

whether such universality also exists in the case of rapidity distributions, which would moreover aid the resummation of the leading logarithms at next-to-leading power for rapidity distributions.

The structure of our paper is as follows. In section 2, we investigate the perturbative structure of the Drell-Yan cross section differential in the invariant mass and the rapidity of the virtual photon, in particular whether a factorised form occurs. In section 3, we analyse the QCD-induced production of two photons, where we stay differential in the rapidity of one of the photons. We compute the NLP corrections at NLO both directly and by exploiting the method of ref. [25]. In section 4 we perform the resummation of the leading threshold logarithms at NLP for both diphoton production and the Drell-Yan process rapidity distributions. Section 5 contains our conclusions, while appendices contain useful expressions for phase space measures.

2 Rapidity distribution for fixed-order Drell-Yan process

In this section we consider the Drell-Yan process

$$A(p_A) + B(p_B) \to \gamma^*(q) + X(p_X), \tag{2.1}$$

where $A(p_A)$ and $B(p_B)$ represent the incoming hadrons with respectively momenta p_A and p_B, with $(p_A+p_B)^2 = s$, producing an off-shell photon followed by the decay $\gamma^*(q) \to l^- l^+$, and where $X(p_X)$ denotes the unobserved QCD final state. We focus on the cross section differential in the invariant mass and rapidity of the off-shell photon in the collider centre of mass frame

$$Q^2 \equiv q^2, \qquad Y \equiv \frac{1}{2}\log\left(\frac{q^0+q^3}{q^0-q^3}\right), \tag{2.2}$$

which reads

$$\frac{d\sigma}{dQ^2\,dY} = \sigma_0 \sum_{ab} \int_\tau^1 \frac{dz}{z} \int_0^1 dy\, \mathcal{L}_{ab}(z,y)\, \Delta_{ab}(z,y), \tag{2.3}$$

where

$$\sigma_0 = \frac{4\pi\alpha^2 e_q^2}{3N_c Q^2 s}, \tag{2.4}$$

and $\alpha = e^2/4\pi$ is the electromagnetic coupling. Following [73, 74], the luminosity function $\mathcal{L}_{ab}(z,y)$ and the partonic cross section $\Delta_{ab}(z,y)$ are expressed in terms of the variables

$$z \equiv \frac{Q^2}{\hat{s}} = \frac{Q^2}{x_a x_b s} = \frac{\tau}{x_a x_b}, \qquad y = \frac{u-z}{(1-z)(1+u)}, \tag{2.5}$$

where in turn

$$u \equiv e^{-2\hat{Y}}, \qquad \text{with} \qquad \hat{Y} = Y - \frac{1}{2}\log\left(\frac{x_a}{x_b}\right). \tag{2.6}$$

Here $\hat{Y}$ represents the rapidity in the partonic centre of mass frame; similarly, $\hat{s} = (p_a+p_b)^2$ in eq. (2.5) represents the partonic centre of mass energy. The variables x_a, x_b represent the momentum fractions relating the partonic to the hadronic incoming momenta via $p_a = x_a p_A$, $p_b = x_b p_B$. For future reference, we note that the variables above are well-defined in the range

$$\tau \le z \le 1, \qquad \log\sqrt{z} \le \hat{Y} \le \log\frac{1}{\sqrt{z}}, \qquad z \le u \le \frac{1}{z}, \qquad 0 \le y \le 1\,. \tag{2.7}$$

For Born kinematics one has $z = u = 1$, $y = 1/2$. Though eq. (2.3) contains a sum over the partonic channels a, b, we consider here only the leading quark-antiquark annihilation channel $q(p_a) + \bar{q}(p_b) \to \gamma^*(q)$ (plus the symmetric contribution $q \leftrightarrow \bar{q}$), see figure 1, and therefore drop the indices a, b. Given these definitions, the luminosity function in eq. (2.3) reads

$$\mathcal{L}(z,y) = f_{q/A}\left(e^{Y}\sqrt{\frac{\tau}{z}\frac{1-(1-y)(1-z)}{1-y(1-z)}}\right) f_{\bar{q}/B}\left(e^{-Y}\sqrt{\frac{\tau}{z}\frac{1-y(1-z)}{1-(1-y)(1-z)}}\right) + (q \leftrightarrow \bar{q}). \tag{2.8}$$

In the threshold region, defined by the limit $(1-z) \to 0$, the cross section is conveniently approximated by a power expansion in $(1-z)$,

$$\Delta(z,y) = \Delta_{\rm LP}(z,y) + \Delta_{\rm NLP}(z,y) + \mathcal{O}(1-z), \tag{2.9}$$

and it develops large logarithms of $(1-z)$ that need to be resummed. At leading power in $(1-z)$, the resummation of large logarithms in $(1-z)$ for the rapidity distribution is relatively easy because of the factorisation [68, 69, 75]

$$\Delta_{\rm LP}(z,y) = \frac{\delta(y)+\delta(1-y)}{2}\Delta_{\rm LP}(z), \tag{2.10}$$

where $\Delta_{\rm LP}(z)$ represents the partonic cross section integrated over the rapidity, i.e. the partonic invariant mass distribution. Thus, the resummation of large logarithms in $(1-z)$ trivially follows from the resummation obtained for the invariant mass distribution.

Recently, much effort has been devoted to the development of resummation techniques for the NLP term of the invariant mass distribution, $\Delta_{\rm NLP}(z)$, [54, 61]. It is therefore interesting to ask to what extent such techniques may be applicable to $\Delta_{\rm NLP}(z,y)$ in eq. (2.9), provided a factorisation such as the one in eq. (2.10) may be established for $\Delta_{\rm NLP}(z,y)$. In order to pursue this investigation, we start by analysing the perturbative structure of $\Delta_{\rm NLP}(z,y)$.

In general, the partonic cross section $\Delta(z,y)$ in $d = 4-2\epsilon$ spacetime dimensions takes the form

$$\Delta(z,y) = \frac{1}{(2\pi)^d}\,\frac{(-g^{\mu\nu})W_{\mu\nu}}{1-\epsilon}\,\delta(q^2-Q^2)\,\delta\left[y - \frac{p_a\cdot q - z\,p_b\cdot q}{(1-z)(p_a\cdot q + p_b\cdot q)}\right], \tag{2.11}$$

where $W_{\mu\nu}$ represents the Drell-Yan hadronic tensor, and the normalisation is fixed such that the tree-level partonic cross section for the invariant mass reads $\Delta^{(0)}(z) = \delta(1-z)$. Here $\Delta(z)$ is given by the identity

$$\int_0^1 dy\, \mathcal{L}(z,y)\, \Delta(z,y) = \mathcal{L}\left(\frac{\tau}{z}\right) \Delta(z), \tag{2.12}$$

where

$$\mathcal{L}(v) = \int_v^1 \frac{dx}{x}\, f_{q/A}(x)\, f_{\bar{q}/B}\left(\frac{v}{x}\right) + (q \leftrightarrow \bar{q}), \tag{2.13}$$

such that

$$\frac{d\sigma}{dQ^2} = \sigma_0 \int_\tau^1 \frac{dz}{z}\, \mathcal{L}\left(\frac{\tau}{z}\right) \Delta(z). \tag{2.14}$$

The leading order contribution to $\Delta(z,y)$ can be easily calculated in terms of the squared matrix element of the process $q\bar{q} \to \gamma^*$:

$$\Delta^{(0)}(z,y) = \frac{1}{4N_c}\frac{1}{2\pi} \int d\Phi_{\gamma^*} \sum_{\text{s,c,p}} |\mathcal{M}^{(0)}_{q\bar{q}\to\gamma^*}|^2, \tag{2.15}$$

where

$$\sum_{\text{s,c,p}} |\mathcal{M}^{(0)}_{q\bar{q}\to\gamma^*}|^2 = 4(1-\epsilon)\hat{s}N_c, \tag{2.16}$$

where the sum is over spin, colour and polarisation, and the phase space for the production of the off-shell photon $\int d\Phi_{\gamma^*}$ is defined in appendix A. In eq. (2.16) and in what follows we set $e_q = 1$, $e = 1$, given that these factors are already included in σ_0, see eqs. (2.3) and (2.4). Upon integration one easily obtains

$$\Delta^{(0)}(z,y) = \delta(1-z)\, \delta\left(y - \frac{1}{2}\right). \tag{2.17}$$

It can be shown [71] that the tree result in eq. (2.17) is indeed compatible, up to NLP, with the structure in eq. (2.10). Inserting eq. (2.17) into eq. (2.3) gives

$$\frac{d\sigma}{dQ^2\, dY} = \sigma_0 \int_\tau^1 \frac{dz}{z}\, \mathcal{L}\left(z, \frac{1}{2}\right) \Delta^{(0)}(z), \tag{2.18}$$

while using the factorised form of eq. (2.10) into eq. (2.3) leads to

$$\frac{d\sigma}{dQ^2\, dY} = \sigma_0 \int_\tau^1 \frac{dz}{z}\, \frac{\mathcal{L}(z,0) + \mathcal{L}(z,1)}{2}\, \Delta^{(0)}(z). \tag{2.19}$$

As discussed in [71], near threshold the luminosity functions involved in eqs. (2.18) and (2.19) are equivalent, up to corrections starting at NNLP:

$$\frac{\mathcal{L}(z,0) + \mathcal{L}(z,1)}{2} = \mathcal{L}\left(z, \frac{1}{2}\right) + \mathcal{O}[(1-z)^2], \tag{2.20}$$

so that for our purposes, eqs. (2.18) and (2.19) can be considered to be equivalent. For future reference, let us define

$$\Delta^{(0)}(z,y) = \frac{\delta(y) + \delta(1-y)}{2}\, \delta(1-z) \equiv \bar{\Delta}^{(0)}(y)\, \delta(1-z). \tag{2.21}$$

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With these results at hand, we have now the tools to investigate the structure of the perturbative corrections to eq. (2.17). Our goal is to determine the contribution to the partonic differential distribution up to second order in perturbation theory: indeed, this is necessary as the structure of the soft expansion within the method of regions [76] is fully revealed only starting at NNLO. In this regard, let us notice that, to the best of our knowledge, the analytic results presented for the NNLO contribution in section 2.2 are given here for the first time, and provide a useful database for investigations of the Drell-Yan rapidity distribution at NLP, at all logarithmic accuracy.

2.1 Next-to-leading order

At NLO one needs to take into account the emission of a virtual gluon or a real soft gluon. The two contributions take the form

$$\Delta^{(1)}(z,y)|_{\text{virtual}} = \frac{1}{4N_c}\frac{1}{2\pi}\int d\Phi_{\gamma^*}\sum_{\text{s,c,p}} 2\text{Re}\Big[\mathcal{M}^{(0)*}_{q\bar{q}\to\gamma^*}\mathcal{M}^{(2)}_{q\bar{q}\to\gamma^*}\Big], \tag{2.22}$$

$$\Delta^{(1)}(z,y)|_{\text{real}} = \frac{1}{4N_c}\frac{1}{2\pi}\int d\Phi_{\gamma^*g}\sum_{\text{s,c,p}} |\mathcal{M}^{(1)}_{q\bar{q}\to\gamma^*g}|^2, \tag{2.23}$$

where the phase space for the production of the off-shell photon $\int d\Phi_{\gamma^*}$, as well as the phase space involving an additional soft gluon $\int d\Phi_{\gamma^*g}$ are defined in appendix A. In eq. (2.22), $\mathcal{M}^{(0)*}_{q\bar{q}\to\gamma^*}$ represents the Drell-Yan Born amplitude, $\mathcal{M}^{(2)*}_{q\bar{q}\to\gamma^*}$ the one-loop correction involving a virtual gluon, and $\mathcal{M}^{(1)}_{q\bar{q}\to\gamma^*g}$ the tree-level amplitude with emission of a real gluon into the final state. An easy calculation gives

$$\begin{aligned}\Delta^{(1)}(z,y)|_{\text{virtual}} = \delta(1-z)\,\delta\left(y-\frac{1}{2}\right)\frac{\alpha_s C_F}{4\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\Bigg[-\frac{4}{\epsilon^2}-\frac{6}{\epsilon}-16+14\zeta_2\\ +\epsilon\left(-32+21\zeta_2+\frac{28\zeta_3}{3}\right)+\ldots\Bigg]\end{aligned} \tag{2.24}$$

for the virtual contribution, where we introduced $\bar{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$ as the $\overline{\text{MS}}$ renormalisation scale. Concerning now the real emission, a simple calculation gives

$$\sum_{\text{s,c,p}}|\mathcal{M}^{(1)}_{q\bar{q}\to\gamma^*g}|^2 = \frac{\alpha_s}{4\pi}64\pi^2 N_c C_F(1-\epsilon)\frac{\hat{s}^2}{k\cdot p_a k\cdot p_b}\left[1-\frac{2}{\hat{s}}(k\cdot p_a+k\cdot p_b)+\mathcal{O}[(1-z)^2]\right], \tag{2.25}$$

where k is the momentum of the emitted soft gluon. Inserting this into eq. (2.23) and integrating against the phase space, up to NLP one obtains[1]

$$\Delta^{(1)}(z,y)|^{\text{LP}}_{\text{real}} = \frac{\alpha_s C_F}{4\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}(1-z)^{-1-2\epsilon}y^{-1-\epsilon}(1-y)^{-1-\epsilon}\Big[1-\epsilon(1-z)\Big]\Big[4-2\zeta_2\epsilon^2+\mathcal{O}(\epsilon^3)\Big], \tag{2.26}$$

and

$$\Delta^{(1)}(z,y)|^{\text{NLP}}_{\text{real}} = -\frac{\alpha_s C_F}{4\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}(1-z)^{-2\epsilon}y^{-1-\epsilon}(1-y)^{-1-\epsilon}\Big[4-2\zeta_2\epsilon^2+\mathcal{O}(\epsilon^3)\Big]. \tag{2.27}$$

[1] Notice that the term defined as $\Delta^{(1)}(z,y)|^{\text{LP}}_{\text{real}}$ in eq. (2.26) refers to the LP squared matrix element. It actually contains a NLP correction of kinematic origin, due to the expansion of the phase space.

The real emission contribution contains poles that arise upon integration over z and y, at $z = 1$, $y = 0$ and $y = 1$. We can isolate them by means of the standard expansion formula

$$\xi^{-1+a\epsilon} = \frac{\delta(\xi)}{a\epsilon} + \frac{1}{\xi}\bigg|_+ + a\epsilon\frac{\log\xi}{\xi}\bigg|_+ + \frac{(a\epsilon)^2}{2!}\frac{\log^2\xi}{\xi}\bigg|_+ + \mathcal{O}(\epsilon^3), \tag{2.28}$$

with $\xi = 1 - z$, y or $1 - y$. Setting $\bar{\mu}^2 = Q^2$ for simplicity, this leads to

$$\begin{aligned}\Delta^{(1)}(z,y)\big|^{\rm LP}_{\rm real} = \frac{\alpha_s C_F}{4\pi}\bigg\{&\left[\frac{2}{\epsilon^2} - \zeta_2\right]\Big[\delta(y) + \delta(1-y)\Big]\delta(1-z) \\ &+ \Big[\delta(y) + \delta(1-y)\Big]\bigg(-\frac{4}{\epsilon}\frac{1}{1-z}\bigg|_+ + 4 + 8\frac{\log(1-z)}{1-z}\bigg|_+\bigg) \\ &+ \frac{4}{y(1-y)}\bigg|_+\frac{1}{1-z}\bigg|_+ + \dots\bigg\},\end{aligned} \tag{2.29}$$

for the LP squared matrix element. Notice that terms multiplied by $\delta(1-z)$ must have y equal to 0 or 1. At NLP we have

$$\Delta^{(1)}(z,y)\big|^{\rm NLP}_{\rm real} = \frac{\alpha_s C_F}{4\pi}\bigg\{\Big[\delta(y) + \delta(1-y)\Big]\bigg(\frac{4}{\epsilon} - 8\log(1-z)\bigg) - \frac{4}{y(1-y)}\bigg|_+ + \dots\bigg\}, \tag{2.30}$$

from the NLP contribution to the squared matrix element. The virtual and real corrections can be combined, exploiting eq. (2.20). After PDF renormalisation, and setting $\bar{\mu}^2 = Q^2$, one arrives at the finite result

$$\begin{aligned}\Delta^{(1)}(z,y)\big|^{\rm LP}_{\rm ren} = \frac{\alpha_s C_F}{4\pi}\bigg\{&(6\zeta_2 - 8)\Big[\delta(y) + \delta(1-y)\Big]\delta(1-z) \\ &+ \Big[\delta(y) + \delta(1-y)\Big]\bigg(8\frac{\log(1-z)}{1-z}\bigg|_+ + 4\bigg) + \frac{4}{y(1-y)}\bigg|_+\frac{1}{1-z}\bigg|_+\bigg\},\end{aligned} \tag{2.31}$$

and

$$\Delta^{(1)}(z,y)\big|^{\rm NLP}_{\rm ren} = \frac{\alpha_s C_F}{4\pi}\bigg\{\Big[\delta(y) + \delta(1-y)\Big]\Big(-8\log(1-z)\Big) - \frac{4}{y(1-y)}\bigg|_+\bigg\}. \tag{2.32}$$

A few comments are in order. Concerning the LP contribution, we see that it takes the factorised form of eq. (2.10) except for the last term. However, one still needs to take into account that the y-dependence of the parton distribution function in the luminosity defined in eq. (2.8) is indeed subleading in the $z \to 1$ limit [69, 70]:

$$f_{q/A}\left(e^Y\sqrt{\frac{\tau}{z}\frac{1-(1-y)(1-z)}{1-y(1-z)}}\right)\bigg|_{z\to1} \to f_{q/A}(e^Y\sqrt{\tau}) + \mathcal{O}(1-z). \tag{2.33}$$

At LP the PDFs can be approximated with the first term on the r.h.s. of eq. (2.33), and the integration against the last term in eq. (2.31) gives zero. It is thus possible [74] to rearrange eq. (2.31) such that the LP term always take the factorised form of eq. (2.10).[2]

[2]We note that the validity of this argument has been debated, see [77]. In this paper we focus on the LLs in $(1-z)$ at NLP, for which the factorisation formula in eq. (2.34) appears to be valid by explicit computation. We refer to [72] for further discussion.

Another important aspect is that, because of the consideration above, it is evident that the NLP contribution in eq. (2.32) in general does *not* factorise according to eq. (2.10). However, we see that the leading logarithmic contribution at NLP factorises in rapidity, just as the LP contribution. This is because the leading logarithms (LLs) are associated to the maximally soft and collinear momentum configurations, which give rise to the leading poles in the unrenormalised partonic cross section. As a consequence of eq. (2.28), the LLs in $(1-z)$ can only arise from the first term in the expansion of the factor $y^{-1-\epsilon}(1-y)^{-1-\epsilon}$ in eq. (2.27).

Eqs. (2.26) and (2.27) yield the z- and y-dependence in case of a soft gluon emission. In general, it can be shown that a similar structure will appear at higher orders, i.e., such corrections will involve factors of $(1-z)^{-1-(n_1+n_2)\epsilon}$ and $(1-z)^{-(n_1+n_2)\epsilon}$ respectively for the LP and NLP contribution, multiplied by factors of $y^{-1-n_1\epsilon}(1-y)^{-1-n_2\epsilon}$. By the same reasoning as above, we can expect the LLs in $(1-z)$ at NLP to have the same factorised form as the LP contribution:

$$\Delta(z,y) = \frac{\delta(y)+\delta(1-y)}{2}\Big[\Delta_{\rm LP}(z) + \Delta_{\rm NLP,LLs}(z)\Big] + \Delta_{\rm NLP,rest}(z,y) + \mathcal{O}(1-z). \tag{2.34}$$

In the next section we explicitly check this expectation at the next order in perturbation theory.

2.2 Next-to-next-to-leading order

At NNLO one needs to consider three contributions, namely the double-virtual, the virtual-real and the double-real terms. They are given by, respectively,

$$\Delta^{(2)}(z,y)|_{\rm 2v} = \frac{1}{4N_c}\frac{1}{2\pi}\int d\Phi_{\gamma^*}\sum_{\rm s,c,p}\Big\{2\,{\rm Re}\Big[\mathcal{M}^{(0)*}_{q\bar q\to\gamma^*}\mathcal{M}^{(4)}_{q\bar q\to\gamma^*}\Big] + \Big|\mathcal{M}^{(2)}_{q\bar q\to\gamma^*}\Big|^2\Big\}, \tag{2.35}$$

$$\Delta^{(2)}(z,y)|_{\rm 1v1r} = \frac{1}{4N_c}\frac{1}{2\pi}\int d\Phi_{\gamma^* g}\sum_{\rm s,c,p} 2\,{\rm Re}\Big[\mathcal{M}^{(1)*}_{q\bar q\to\gamma^* g}\mathcal{M}^{(3)}_{q\bar q\to\gamma^* g}\Big], \tag{2.36}$$

$$\Delta^{(2)}(z,y)|_{\rm 2r} = \frac{1}{4N_c}\frac{1}{2\pi}\int d\Phi_{\gamma^*(gg+q\bar q)}\sum_{\rm s,c,p}\big|\mathcal{M}^{(2)}_{q\bar q\to\gamma^*(gg+q\bar q)}\big|^2. \tag{2.37}$$

The phase spaces $\int d\Phi_{\gamma^*}$ and $\int d\Phi_{\gamma^* g}$ have already been mentioned in the previous section, while $\int d\Phi_{\gamma^*(gg+q\bar q)}$ represents the phase space for the emission of two gluons (or a quark-antiquark pair) in the final state, in addition to the off-shell photon. It is given in appendix A.3. Of the three contributions above, the first involves the two-loop (virtual) Drell-Yan amplitude $\mathcal{M}^{(4)}_{q\bar q\to\gamma^*}$, and does not present new conceptual issues compared to the corresponding term at one loop, namely $\Delta^{(1)}(z,y)|_{\rm virtual}$ in eqs. (2.22) and (2.24). Starting from the two-loop quark form factor available in literature [78, 79] and setting $\bar\mu^2 = Q^2$ for simplicity, one has

$$\Delta^{(2)}(z,y)|_{\rm 2v} = \left(\frac{\alpha_s}{4\pi}\right)^2\delta(1-z)\,\delta\Big(y-\frac{1}{2}\Big)\bigg\{C_F^2\bigg[\frac{8}{\epsilon^4}+\frac{24}{\epsilon^3}+\frac{82-56\zeta_2}{\epsilon^2}+\frac{1}{\epsilon}\Big(\frac{445}{2}-156\zeta_2 - \frac{184\zeta_3}{3}\Big)+\frac{2303}{4}-516\zeta_2-172\zeta_3+274\zeta_4\bigg]$$

$$
\begin{aligned}
&+C_F C_A\bigg[-\frac{11}{3\epsilon^3}+\frac{1}{\epsilon^2}\bigg(-\frac{166}{9}+2\zeta_2\bigg)+\frac{1}{\epsilon}\bigg(-\frac{4129}{54}+\frac{121\zeta_2}{3}+26\zeta_3\bigg)\\
&\qquad -\frac{89173}{324}+\frac{1754\zeta_2}{9}+\frac{934\zeta_3}{9}-16\zeta_4\bigg]\\
&+C_F n_f\bigg[\frac{2}{3\epsilon^3}+\frac{28}{9\epsilon^2}+\frac{1}{\epsilon}\bigg(\frac{353}{27}-\frac{22\zeta_2}{3}\bigg)+\frac{7541}{162}-\frac{308\zeta_2}{9}-\frac{52\zeta_3}{9}\bigg]\bigg\}. \qquad (2.38)
\end{aligned}
$$

The virtual-real and the double-real contribution, respectively in eqs. (2.36) and (2.37), require more attention. Let us therefore consider them separately in what follows.

Virtual-real contribution. The virtual-real contribution in eq. (2.36) involves the one-loop amplitude $\mathcal{M}^{(3)}_{q\bar{q}\to\gamma^* g}$, with the emission of a soft gluon. It is therefore the first instance where the loop integration involves non-trivial momentum regions. We can cast the squared matrix element for this case in the form

$$
\begin{aligned}
&\sum_{\text{s,c,p}} 2\,\mathrm{Re}\Big[\mathcal{M}^{(1)*}_{q\bar{q}\to\gamma^* g}\mathcal{M}^{(3)}_{q\bar{q}\to\gamma^* g}\Big] = \left(\frac{\alpha_s}{4\pi}\right)^2 256\,\pi^2\,N_c\,(1-\epsilon)\\
&\times\bigg\{C_F^2\bigg[\bigg(\frac{\hat{s}^2}{tu}f_{h_1}(\epsilon)+\frac{\hat{s}(t+u)}{tu}f_{h_2}(\epsilon)\bigg)\left(\frac{\mu^2}{-\hat{s}}\right)^\epsilon+\bigg(\frac{\hat{s}}{t}\left(\frac{\mu^2}{-t}\right)^\epsilon+\frac{\hat{s}}{u}\left(\frac{\mu^2}{-u}\right)^\epsilon\bigg)f_{c_1}(\epsilon)\bigg]\\
&\quad +C_A C_F\bigg[\bigg(\frac{\hat{s}^2}{tu}+\frac{\hat{s}(t+u)}{tu}\bigg)\left(-\frac{\hat{s}\,\mu^2}{t\,u}\right)^\epsilon f_s(\epsilon)+\bigg(\frac{\hat{s}}{t}\left(\frac{\mu^2}{-t}\right)^\epsilon+\frac{\hat{s}}{u}\left(\frac{\mu^2}{-u}\right)^\epsilon\bigg)f_{c_2}(\epsilon)\bigg]\bigg\}, \qquad (2.39)
\end{aligned}
$$

where we have introduced the variables $t=-2p_a\cdot k$, $u=-2p_b\cdot k$, with k being the momentum of the emitted soft gluon, and

$$
\begin{aligned}
f_{h_1}(\epsilon) &= -\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\zeta_2+\epsilon\bigg(-16+\frac{3\zeta_2}{2}+\frac{14\zeta_3}{3}\bigg)\\
&\qquad +\epsilon^2\bigg(-32+4\zeta_2+7\zeta_3+\frac{47\zeta_4}{8}\bigg)+\mathcal{O}(\epsilon^3),\\
f_{h_2}(\epsilon) &= s^\epsilon f_{h_1}(\epsilon)\,\frac{\partial}{\partial s}s^{1-\epsilon} = (1-\epsilon)\,f_{h_1}(\epsilon),\\
f_{c_1}(\epsilon) &= -\frac{2}{\epsilon}-\frac{5}{2}+\epsilon(-3+\zeta_2)+\epsilon^2\bigg(-4+\frac{5\zeta_2}{4}+\frac{14\zeta_3}{3}\bigg)+\mathcal{O}(\epsilon^3),\\
f_{c_2}(\epsilon) &= \frac{5}{2}+\epsilon+\epsilon^2\bigg(4-\frac{5\zeta_2}{4}\bigg)+\mathcal{O}(\epsilon^3),\\
f_s(\epsilon) &= -\frac{1}{\epsilon^2}-\frac{\zeta_2}{2}+\frac{7\zeta_3}{3}\epsilon+\frac{39\zeta_4}{16}\epsilon^2+\mathcal{O}(\epsilon^3). \qquad (2.40)
\end{aligned}
$$

One may readily identify the term proportional to $(-\hat{s})^{-\epsilon}$ as the hard region contribution, the terms proportional to $(-t)^{-\epsilon}$ and $(-u)^{-\epsilon}$ respectively as the collinear and anti-collinear region contribution, and the term proportional to $(-\hat{s}/tu)^\epsilon$ as the soft region contribution. Let us notice that leading poles are present only for the hard and soft region, at the level of the squared matrix element. As such, as already discussed for the invariant mass distribution [54] (see also [27]), we expect that, also for the rapidity distribution, leading logarithms will arise only from the hard and soft region. The phase space integration with measure $d\Phi_{\gamma^* g}$ can be performed with the help of the equations in appendix A.2. Expanding in powers of $1-z$, the integration over the LP squared matrix element gives

$$\Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{LP}} = \Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{LP,h}} + \Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{LP,s}}\,, \tag{2.41}$$

where

$$\begin{aligned}\Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{LP,h}} &= C_F^2\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\bar{\mu}^2}{Q^2}\right)^{2\epsilon} y^{-1-\epsilon}(1-y)^{-1-\epsilon} \\ &\times\left\{(1-z)^{-1-2\epsilon}\left[-\frac{16}{\epsilon^2}-\frac{24}{\epsilon}-64+64\zeta_2+\ldots\right]+(1-z)^{-2\epsilon}\left[\frac{32}{\epsilon}+48+\ldots\right]\right\},\end{aligned} \tag{2.42}$$

and

$$\begin{aligned}\Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{LP,s}} &= C_A C_F\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\bar{\mu}^2}{Q^2}\right)^{2\epsilon} y^{-1-2\epsilon}(1-y)^{-1-2\epsilon} \\ &\times\left\{(1-z)^{-1-4\epsilon}\left[-\frac{8}{\epsilon^2}+24\zeta_2+\ldots\right]+(1-z)^{-4\epsilon}\left[\frac{16}{\epsilon}+\ldots\right]\right\},\end{aligned} \tag{2.43}$$

for the hard and soft region contribution, respectively. We recall that NLP corrections in eqs. (2.42) and (2.43) originate from the power expansion of the phase space. Next, integration of the NLP squared matrix element gives

$$\Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{NLP}} = \Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{NLP,h}} + \Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{NLP,c+\bar{c}}} + \Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{NLP,s}}\,, \tag{2.44}$$

where

$$\begin{aligned}\Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{NLP,h}} &= C_F^2\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\bar{\mu}^2}{Q^2}\right)^{2\epsilon} y^{-1-\epsilon}(1-y)^{-1-\epsilon}(1-z)^{-2\epsilon} \\ &\times\left[\frac{16}{\epsilon^2}+\frac{8}{\epsilon}+40-64\zeta_2+\ldots\right],\end{aligned} \tag{2.45}$$

$$\begin{aligned}\Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{NLP,c+\bar{c}}} &= \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\bar{\mu}^2}{Q^2}\right)^{2\epsilon}\left[y^{-\epsilon}(1-y)^{-1-2\epsilon}+y^{-1-2\epsilon}(1-y)^{-\epsilon}\right](1-z)^{-3\epsilon} \\ &\times\left\{C_F^2\left[\frac{16}{\epsilon}+20+\ldots\right]+C_A C_F\left[-20+\ldots\right]\right\},\end{aligned} \tag{2.46}$$

and

$$\Delta^{(2)}(z,y)|_{1\mathrm{r}1\mathrm{v}}^{\mathrm{NLP,s}} = C_A C_F\left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\bar{\mu}^2}{Q^2}\right)^{2\epsilon} y^{-1-2\epsilon}(1-y)^{-1-2\epsilon}(1-z)^{-4\epsilon}\left[\frac{8}{\epsilon^2}-24\zeta_2+\ldots\right]. \tag{2.47}$$

As already discussed for the single real emission at NLO, note that the y-dependence arises from the typical pattern $y^{-a_1-b_1\epsilon}(1-y)^{-a_2-b_2\epsilon}$, while the dependence on the threshold variable z follows the pattern $z^{-1-(b_1+b_2)\epsilon}$ at LP, and $z^{-(b_1+b_2)\epsilon}$ at NLP. The 1-virtual

1-real correction considered here is the first instance where contributions from different momentum regions of the virtual gluon arise. Inspecting eqs. (2.42)–(2.47), we see that the exponents a_i and b_i are characteristic of each region: at this order, $a_1 = a_2 = 1$, $b_1 = b_2 = 1$ for the hard region, $a_1 = 0$, $a_2 = 1$, $b_1 = 1$, $b_2 = 2$ for the collinear region, $a_1 = 1$, $a_2 = 0$, $b_1 = 2$, $b_2 = 1$ for the anti-collinear region, and $a_1 = a_2 = 1$, $b_1 = b_2 = 2$ for the soft region. Note the quite simple correspondence between regions and the power dependence of the threshold and rapidity variables z and y, in particular involving also the dimensional regularisation parameter. Expansion in powers of ϵ of these factors can be obtained by means of eq. (2.28). Setting again $\bar{\mu}^2 = Q^2$, and introducing the notation

$$\mathcal{D}_n(x) \equiv \frac{\log^n(x)}{x}\bigg|_+ , \qquad \mathcal{L}_n(x) \equiv \log^n(x), \tag{2.48}$$

with $x = \bar{z} \equiv (1-z)$, or $x = y$, or $x = \bar{y} \equiv (1-y)$, we obtain

$$\begin{aligned}
\Delta^{(2)}(z,y)\big|^{\mathrm{LP,h}}_{\mathrm{1r1v}} = C_F^2\left(\frac{\alpha_s}{4\pi}\right)^2 \Bigg\{ \Big[\delta(y)+\delta(1-y)\Big]\Bigg[\delta(1-z)\bigg(-\frac{8}{\epsilon^4}-\frac{12}{\epsilon^3}-\frac{32-32\zeta_2}{\epsilon^2}\\
-\frac{1}{\epsilon}\left(64-48\zeta_2-\frac{64\zeta_3}{3}\right)-128+128\zeta_2+32\zeta_3-72\zeta_4\bigg)+\frac{16\mathcal{D}_0(\bar{z})}{\epsilon^3}\\
+\frac{24\mathcal{D}_0(\bar{z})-32\mathcal{D}_1(\bar{z})-32}{\epsilon^2}+\frac{1}{\epsilon}\bigg(64\mathcal{D}_0(\bar{z})(1-\zeta_2)-48\mathcal{D}_1(\bar{z})+32\mathcal{D}_2(\bar{z})\\
-48+64\mathcal{L}_1(\bar{z})\bigg)+128\mathcal{D}_0(\bar{z})\left(1-\frac{3\zeta_2}{4}-\frac{\zeta_3}{3}\right)-128\mathcal{D}_1(\bar{z})(1-\zeta_2)\\
+48\mathcal{D}_2(\bar{z})-\frac{64}{3}\mathcal{D}_3(\bar{z})+96\mathcal{L}_1(\bar{z})-64\mathcal{L}_2(\bar{z})-128(1-\zeta_2)\Bigg]\\
+\mathcal{D}_0(y)\mathcal{D}_0(\bar{y})\bigg[-\frac{16\mathcal{D}_0(\bar{z})}{\epsilon^2}+\frac{32-24\mathcal{D}_0(\bar{z})+32\mathcal{D}_1(\bar{z})}{\epsilon}+48\\
-64\mathcal{D}_0(\bar{z})(1-\zeta_2)+48\mathcal{D}_1(\bar{z})-32\mathcal{D}_2(\bar{z})-64\mathcal{L}_1(\bar{z})\bigg]\\
+\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar{y})+\mathcal{D}_1(y)\mathcal{D}_0(\bar{y})\Big]\bigg[\frac{16\mathcal{D}_0(\bar{z})}{\epsilon}-32+24\mathcal{D}_0(\bar{z})-32\mathcal{D}_1(\bar{z})\bigg]\\
-16\mathcal{D}_0(\bar{z})\mathcal{D}_1(y)\mathcal{D}_1(\bar{y})-8\mathcal{D}_0(\bar{z})\Big(\mathcal{D}_0(y)\mathcal{D}_2(\bar{y})+\mathcal{D}_2(y)\mathcal{D}_0(\bar{y})\Big)\Bigg\},
\end{aligned} \tag{2.49}$$

and

$$\begin{aligned}
\Delta^{(2)}(z,y)\big|^{\mathrm{LP,s}}_{\mathrm{1r1v}} = C_A C_F\left(\frac{\alpha_s}{4\pi}\right)^2 \Bigg\{ \Big[\delta(y)+\delta(1-y)\Big]\Bigg[\delta(1-z)\bigg(-\frac{1}{\epsilon^4}+\frac{3\zeta_2}{\epsilon^2}+\frac{8\zeta_3}{3\epsilon}-\frac{3\zeta_4}{4}\bigg)\\
+\frac{4\mathcal{D}_0(\bar{z})}{\epsilon^3}-\frac{8+16\mathcal{D}_1(\bar{z})}{\epsilon^2}-\frac{1}{\epsilon}\Big(12\mathcal{D}_0(\bar{z})\zeta_2-32\mathcal{D}_2(\bar{z})-32\mathcal{L}_1(\bar{z})\Big)\\
-\frac{32}{3}\mathcal{D}_0(\bar{z})\zeta_3+48\mathcal{D}_1(\bar{z})\zeta_2-\frac{128}{3}\mathcal{D}_3(\bar{z})+24\zeta_2-64\mathcal{L}_2(\bar{z})\Bigg]\\
+\mathcal{D}_0(y)\mathcal{D}_0(\bar{y})\bigg[-\frac{8\mathcal{D}_0(\bar{z})}{\epsilon^2}+\frac{16+32\mathcal{D}_1(\bar{z})}{\epsilon}+24\zeta_2\mathcal{D}_0(\bar{z})-64\mathcal{D}_2(\bar{z})-64\mathcal{L}_1(\bar{z})\bigg]
\end{aligned}$$

$$
+\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar{y})+\mathcal{D}_0(\bar{y})\mathcal{D}_1(y)\Big]\Big[\frac{16\mathcal{D}_0(\bar{z})}{\epsilon}-32-64\mathcal{D}_1(\bar{z})\Big]
$$

$$
-32\mathcal{D}_0(\bar{z})\mathcal{D}_1(y)\mathcal{D}_1(\bar{y})-16\mathcal{D}_0(\bar{z})\Big(\mathcal{D}_0(y)\mathcal{D}_2(\bar{y})+\mathcal{D}_0(\bar{y})\mathcal{D}_2(y)\Big)\Big\}, \qquad (2.50)
$$

for the hard- and soft-region contribution to the LP squared matrix element. Next, the NLP squared matrix element gives

$$
\begin{aligned}
\Delta^{(2)}(z,y)|^{\rm NLP,h}_{\rm 1r1v} = C_F^2\left(\frac{\alpha_s}{4\pi}\right)^2 \Big\{&\Big[\delta(y)+\delta(1-y)\Big]\Big[-\frac{16}{\epsilon^3}-\frac{8-32\mathcal{L}_1(\bar{z})}{\epsilon^2}-\frac{1}{\epsilon}\Big(40-64\zeta_2 \\
&-16\mathcal{L}_1(\bar{z})+32\mathcal{L}_2(\bar{z})\Big)-64+32\zeta_2+\frac{128\zeta_3}{3}+\mathcal{L}_1(\bar{z})(80-128\zeta_2) \\
&-16\mathcal{L}_2(\bar{z})+\frac{64}{3}\mathcal{L}_3(\bar{z})\Big]+\mathcal{D}_0(\bar{y})\mathcal{D}_0(y)\Big[\frac{16}{\epsilon^2}+\frac{8-32\mathcal{L}_1(\bar{z})}{\epsilon}+40-64\zeta_2 \\
&-16\mathcal{L}_1(\bar{z})+32\mathcal{L}_2(\bar{z})\Big]+\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar{y})+\mathcal{D}_0(\bar{y})\mathcal{D}_1(y)\Big]\Big[-\frac{16}{\epsilon}-8+32\mathcal{L}_1(\bar{z})\Big] \\
&+16\mathcal{D}_1(\bar{y})\mathcal{D}_1(y)+8\Big(\mathcal{D}_0(y)\mathcal{D}_2(\bar{y})+\mathcal{D}_0(\bar{y})\mathcal{D}_2(y)\Big)\Big\},
\end{aligned}
\qquad (2.51)
$$

$$
\begin{aligned}
\Delta^{(2)}(z,y)|^{\rm NLP,c+\bar{c}}_{\rm 1r1v} = \left(\frac{\alpha_s}{4\pi}\right)^2 \Big\{&C_F^2\Big[\Big(\delta(y)+\delta(1-y)\Big)\Big(-\frac{8}{\epsilon^2}-\frac{10-24\mathcal{L}_1(\bar{z})}{\epsilon}-12+30\mathcal{L}_1(\bar{z}) \\
&-36\mathcal{L}_2(\bar{z})+8\zeta_2\Big)+\mathcal{D}_0(y)\Big(\frac{16}{\epsilon}+20-16\mathcal{L}_1(\bar{y})-48\mathcal{L}_1(\bar{z})\Big) \\
&+\mathcal{D}_0(\bar{y})\Big(\frac{16}{\epsilon}+20-16\mathcal{L}_1(y)-48\mathcal{L}_1(\bar{z})\Big)-32\Big(\mathcal{D}_1(y)+\mathcal{D}_1(\bar{y})\Big)\Big] \\
&+C_AC_F\Big[\Big(\delta(y)+\delta(1-y)\Big)\Big(\frac{10}{\epsilon}+4-30\mathcal{L}_1(\bar{z})\Big)-20\Big(\mathcal{D}_0(y)+\mathcal{D}_0(\bar{y})\Big)\Big]\Big\},
\end{aligned}
\qquad (2.52)
$$

and

$$
\begin{aligned}
\Delta^{(2)}(z,y)|^{\rm NLP,s}_{\rm 1r1v} = C_AC_F\left(\frac{\alpha_s}{4\pi}\right)^2 \Big\{&\Big[\delta(y)+\delta(1-y)\Big]\Big[-\frac{4}{\epsilon^3}+\frac{16\mathcal{L}_1(\bar{z})}{\epsilon^2}-\frac{32\mathcal{L}_2(\bar{z})-12\zeta_2}{\epsilon} \\
&+\frac{32\zeta_3}{3}-48\mathcal{L}_1(\bar{z})\zeta_2+\frac{128}{3}\mathcal{L}_3(\bar{z})\Big]+\mathcal{D}_0(\bar{y})\mathcal{D}_0(y)\Big[\frac{8}{\epsilon^2}-\frac{32\mathcal{L}_1(\bar{z})}{\epsilon} \\
&-24\zeta_2+64\mathcal{L}_2(\bar{z})\Big]+\Big(\mathcal{D}_0(y)\mathcal{D}_1(\bar{y})+\mathcal{D}_0(\bar{y})\mathcal{D}_1(y)\Big)\Big[-\frac{16}{\epsilon}+64\mathcal{L}_1(\bar{z})\Big] \\
&+32\mathcal{D}_1(\bar{y})\mathcal{D}_1(y)+16\Big(\mathcal{D}_0(y)\mathcal{D}_2(\bar{y})+\mathcal{D}_0(\bar{y})\mathcal{D}_2(y)\Big)\Big\},
\end{aligned}
\qquad (2.53)
$$

for the hard, collinear plus anti-collinear and soft region, respectively.

Let us comment on these results. First of all, the LP squared matrix element in eqs. (2.49) and (2.50) receives contributions from the hard and the soft region, but no contribution from the collinear region. This is a well-known result for the invariant mass distribution [19], which clearly remains true in case of more differential distributions, like the double-differential distribution in invariant mass and rapidity considered here. This is because, near threshold, contributions from the collinear and anti-collinear regions at LP decouple at the level of the matrix element, regardless of the exact form of the phase space [80–83]. Moreover, we can single out three different types of contributions in the LP squared matrix element, eqs. (2.49) and (2.50): the first is given by terms proportional to $\delta(y)+\delta(1-y)$, which corresponds to the LP factorised terms of eq. (2.34). Then we have terms which are proportional to y- and $\bar{y}$-plus distributions. These do not factorise. However, as discussed around eq. (2.33), the behaviour of the PDFs near $z\to 1$ is such that y- and $\bar{y}$-plus distributions in the LP squared matrix element actually contribute at NLP in $1-z$. In this respect, the important thing to notice concerning our assumption in eq. (2.34) is that such y- and $\bar{y}$-plus distributions in eqs. (2.49) and (2.50) do not contain leading $\mathcal{D}_3(\bar{z})$ distributions, nor leading $\mathcal{L}_3(\bar{z})$ logarithms. As such, these terms contribute to the last term $\Delta_{\rm NLP,rest}(z,y)$ in eq. (2.34). A third type of contribution in eqs. (2.49) and (2.50) is given by logarithms $\mathcal{L}_n(\bar{z})$ arising from the expansion of the phase space. More precisely, the power expansion of the partonic cross section is given by the sum of three terms:

$$\Delta \sim \underbrace{\int d\Phi_{\rm LP}\,|\mathcal{M}|^2_{\rm LP} + \int d\Phi_{\rm NLP}\,|\mathcal{M}|^2_{\rm LP}}_{\Delta_{\rm LP}} + \underbrace{\int d\Phi_{\rm LP}\,|\mathcal{M}|^2_{\rm NLP}}_{\Delta_{\rm NLP}}. \tag{2.54}$$

In eqs. (2.49) and (2.50) we include, with a slight abuse of language, both the first and second term of eq. (2.54) into $\Delta_{\rm LP}$. Thus, $\Delta_{\rm LP}$ contains logarithms $\mathcal{L}_n(\bar{z})$, originating from the second term of eq. (2.54). However, only logarithms with $n=1,2$ appear. Thus, power corrections from the phase space measure do not give rise to leading logarithms at NLP. This result has been already exploited to construct the resummation of LLs in the Drell-Yan invariant mass distribution, see section 3 of [54], and it remains true for the distribution differential in both invariant mass and rapidity. Lastly, in regard to the NLP squared matrix element in eqs. (2.51)–(2.53), we see that leading logarithms $\mathcal{L}_3(\bar{z})$ arise only in the hard and soft region, eqs. (2.51) and (2.53), while the collinear and anti-collinear region contribution in eq. (2.52) contain at most next-to-leading logarithms. Also in this case the result is quite general, i.e. is independent of the particular differential distribution considered, because the collinear momenta configurations contain only subleading poles already at the level of the squared matrix element (cf. eqs. (2.39) and (2.40) with eq. (2.46)), and thus upon integration no LLs can be generated. We conclude that we are left with the NLP LLs $\mathcal{L}_3(\bar{z})$ from the hard and soft region, eqs. (2.51) and (2.53), which indeed have the factorised form of eq. (2.34).

Double-real contribution. We are now left with the double-real correction listed in eq. (2.37):

$$\Delta^{(2)}(z,y)|_{2r} = \frac{1}{4N_c}\frac{1}{2\pi}\int d\Phi_{\gamma^*(gg+q\bar{q})} \sum_{\rm s,c,p} \left|\mathcal{M}^{(2)}_{q\bar{q}\to\gamma^*(gg+q\bar{q})}\right|^2, \tag{2.55}$$

which is given in terms of the tree-level amplitude $\mathcal{M}^{(2)}_{q\bar{q}\to\gamma^*(gg+q\bar{q})}$, describing the emission of two soft gluons (or a quark-antiquark pair) in the final state, and the corresponding phase space $\int d\Phi_{\gamma^*(gg+q\bar{q})}$. The expression of the squared matrix element $|\mathcal{M}^{(2)}_{q\bar{q}\to\gamma^*(gg+q\bar{q})}|^2$ is rather lengthy, and we do not report it here. Instead, it is interesting to spend some words on the phase space integration, which is obviously more involved compared to the case of a single real emission discussed in the previous section. The phase space integral for double-real emission reads

$$\int d\Phi_{\gamma^*(gg+q\bar{q})} = (\mu^2)^{4-d} \int \frac{d^d q}{(2\pi)^{d-1}} \frac{d^d k_1}{(2\pi)^{d-1}} \frac{d^d k_2}{(2\pi)^{d-1}} (2\pi)^d \delta^{(d)}(p_a+p_b-q-k_1-k_2) \\ \times \delta_+(k_1^2)\,\delta_+(k_2^2)\,\delta(q^2-Q^2)\,\delta\left[y-\frac{p_a\cdot q - z\,p_b\cdot q}{(1-z)(p_a\cdot q+p_b\cdot q)}\right]. \tag{2.56}$$

In this case it proves useful to follow the parametrisation used in [23, 84], in which the three-particle phase space is factorised into two two-body phase spaces, one involving the off-shell photon and the other the vector sum of the emitted gluon momenta $K=k_1+k_2$:

$$\int d\Phi(p_a+p_b\to q+k_1+k_2) = \int_0^\infty \frac{dK^2}{2\pi}\int d\Phi(p_a+p_b\to q+K)\times\int d\Phi(K\to k_1+k_2). \tag{2.57}$$

Subsequently, the phase space $\int d\Phi(K\to k_1+k_2)$ is evaluated in the centre of mass frame of the two-gluon system, where the momenta are parameterised as follows:

$$\begin{aligned} p_a &= \frac{\hat{s}-\tilde{t}}{2\sqrt{s_{12}}}(1,0,\ldots,0,1), \\ p_b &= \left(\frac{\tilde{t}+s_{12}-Q^2}{2\sqrt{s_{12}}},0,\ldots,0,|\boldsymbol{q}|\sin\psi,|\boldsymbol{q}|\cos\psi-\frac{\hat{s}-\tilde{t}}{2\sqrt{s_{12}}}\right), \\ q &= \left(\frac{\hat{s}-s_{12}-Q^2}{2\sqrt{s_{12}}},0,\ldots,0,|\boldsymbol{q}|\sin\psi,|\boldsymbol{q}|\cos\psi\right), \\ k_1 &= \frac{\sqrt{s_{12}}}{2}(1,0,\ldots,\sin\theta_2\sin\theta_1,\cos\theta_2\sin\theta_1,\cos\theta_1), \\ k_2 &= \frac{\sqrt{s_{12}}}{2}(1,0,\ldots,-\sin\theta_2\sin\theta_1,-\cos\theta_2\sin\theta_1,-\cos\theta_1), \end{aligned} \tag{2.58}$$

where

$$\tilde{t}=2p_a\cdot q, \qquad \tilde{u}=2p_b\cdot q, \qquad s_{12}=2k_1\cdot k_2=\hat{s}-\tilde{t}-\tilde{u}+Q^2, \tag{2.59}$$

and

$$\cos\psi=\frac{(\hat{s}-Q^2)(\tilde{u}-Q^2)-s_{12}(\tilde{t}+Q^2)}{(\hat{s}-\tilde{t})\sqrt{\Lambda(\hat{s},Q^2,s_{12})}}, \qquad |\boldsymbol{q}|=\frac{\sqrt{\Lambda(\hat{s},Q^2,s_{12})}}{2\sqrt{s_{12}}}, \tag{2.60}$$

where Λ is the standard Källen function $\Lambda(a,b,c)=a^2+b^2+c^2-2ab-2ac-2bc$. The Mandelstam variables $\tilde{t}$ and $\tilde{u}$ can in turn be expressed as functions of the photon energy fraction z and of two further variables $0<v_1<1$ and $0<v_2<1$, such that

$$\tilde{t}=\hat{s}\left[z+v_2(1-z)-\frac{v_2(1-v_2)v_1(1-z)^2}{1-v_2(1-z)}\right], \qquad \tilde{u}=\hat{s}\left[1-v_2(1-z)\right]. \tag{2.61}$$

With this parametrisation, $\int d\Phi_{\gamma^*(gg+q\bar{q})}$ reads

$$\int d\Phi_{\gamma^*(gg+q\bar{q})} = \frac{1}{(4\pi)^d}\frac{\hat{s}^{d-3}(\mu^2)^{4-d}}{\Gamma(d-3)}(1-z)^{2d-5}\int_0^\pi d\theta_1\int_0^\pi d\theta_2\,(\sin\theta_1)^{d-3}(\sin\theta_2)^{d-4}$$
$$\times\int_0^1 dv_1\int_0^1 dv_2\,[v_1(1-v_1)]^{d/2-2}\,[v_2(1-v_2)]^{d-3}\,[1-v_2(1-z)]^{1-d/2}$$
$$\times\,\delta\left\{y-\frac{\hat{s}\Big(z+v_2(1-z)-\frac{v_2(1-v_2)v_1(1-z)^2}{1-v_2(1-z)}\Big)-z\hat{s}(1-v_2(1-z))}{(1-z)\Big[\hat{s}\Big(z+v_2(1-z)-\frac{v_2(1-v_2)v_1(1-z)^2}{1-v_2(1-z)}\Big)+\hat{s}(1-v_2(1-z))\Big]}\right\}. \tag{2.62}$$

The Dirac delta function in the last line is rather involved, but it greatly simplifies in the $z\to 1$ limit. In this case, following [68], we expand

$$\delta\left\{y-\frac{\hat{s}\Big(z+v_2(1-z)-\frac{v_2(1-v_2)v_1(1-z)^2}{1-v_2(1-z)}\Big)-z\hat{s}(1-v_2(1-z))}{(1-z)\Big[\hat{s}\Big(z+v_2(1-z)-\frac{v_2(1-v_2)v_1(1-z)^2}{1-v_2(1-z)}\Big)+\hat{s}(1-v_2(1-z))\Big]}\right\}$$
$$=\delta(y-v_2)+\frac{v_2(1-v_2)v_1}{2}(1-z)\delta'(y-v_2)+\mathcal{O}\big[(1-z)^2\big]. \tag{2.63}$$

Inserting this power expansion into the three-body phase space of eq. (2.62), it is possible to perform the integral of the two-real squared matrix element in eq. (2.55) with the standard methods discussed in [23, 84]. After some elaboration, we obtain the LP and NLP contribution before expanding in ϵ and the scale factors. The contribution of the LP squared matrix element reads

$$\Delta^{(2)}(z,y)\big|_{2\mathrm{r}}^{\mathrm{LP}} = \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\bar{\mu}^2}{Q^2}\right)^{2\epsilon} y^{-1-2\epsilon}(1-y)^{-1-2\epsilon}$$
$$\times\Bigg\{(1-z)^{-1-4\epsilon}\Bigg[C_F^2\left(\frac{32}{\epsilon^2}-96\zeta_2+\dots\right)+C_AC_F\left(\frac{8}{\epsilon^2}+\frac{44}{3\epsilon}+\frac{268}{9}-32\zeta_2+\dots\right)$$
$$+n_fC_F\left(-\frac{8}{3\epsilon}-\frac{40}{9}+\dots\right)\Bigg]$$
$$+(1-z)^{-4\epsilon}\Bigg[C_F^2\left(-\frac{80}{\epsilon}+\dots\right)+C_AC_F\left(-\frac{16}{\epsilon}-\frac{88}{3}+\dots\right)+n_fC_F\left(\frac{16}{3}+\dots\right)\Bigg]\Bigg\}, \tag{2.64}$$

and the NLP contribution reads

$$\Delta^{(2)}(z,y)\big|_{2\mathrm{r}}^{\mathrm{NLP}} = \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\bar{\mu}^2}{Q^2}\right)^{2\epsilon} y^{-1-2\epsilon}(1-y)^{-1-2\epsilon}(1-z)^{-4\epsilon}$$
$$\times\Bigg[C_F^2\left(-\frac{32}{\epsilon^2}-8+96\zeta_2+\dots\right)+C_AC_F\left(-\frac{8}{\epsilon^2}-\frac{44}{3\epsilon}-\frac{220}{9}+32\zeta_2+\dots\right)$$
$$+n_fC_F\left(\frac{8}{3\epsilon}+\frac{64}{9}+\dots\right)\Bigg]. \tag{2.65}$$

We see that the scale factors are consistent with our expectation, discussed below eq. (2.47). Namely, the y-dependence arises from the typical pattern $y^{-a_1-b_1\epsilon}(1-y)^{-a_2-b_2\epsilon}$, with $a_1 =$

$a_2=1$, $b_1=b_2=2$, which is characteristic of the soft region: indeed, in this case we are considering the emission of two soft gluons, and the same pattern arises in case of the virtual-real contribution, where both the virtual and the real gluon are taken to be soft, cf. eq. (2.47). This immediately allows us to conclude that, upon expanding in ϵ, the double-real contribution, too, has the structure of eq. (2.34), as we have already verified in case of eq. (2.47), compare with eqs. (2.50) and (2.53). In this case, expanding the scale factors in eqs. (2.64) and (2.65) in ϵ by means of eq. (2.28) and setting $\bar{\mu}^2=Q^2$ we explicitly obtain

$$
\begin{aligned}
\Delta^{(2)}(z,y)|^{\rm LP}_{\rm 2r} = \left(\frac{\alpha_s}{4\pi}\right)^2 \Bigg\{ & C_F^2\bigg[\Big[\delta(y)+\delta(1-y)\Big]\bigg[\delta(1-z)\left(\frac{4}{\epsilon^4}-\frac{12\zeta_2}{\epsilon^2}-\frac{56\zeta_3}{3\epsilon}+15\zeta_4\right) \\
&-\frac{16\mathcal{D}_0(\bar z)}{\epsilon^3}+\frac{64\mathcal{D}_1(\bar z)+40}{\epsilon^2}+\frac{1}{\epsilon}\Big(-128\mathcal{D}_2(\bar z)+48\zeta_2\mathcal{D}_0(\bar z)-160\mathcal{L}_1(\bar z)\Big) \\
&+\frac{512}{3}\mathcal{D}_3(\bar z)-192\zeta_2\mathcal{D}_1(\bar z)+\frac{224}{3}\zeta_3\mathcal{D}_0(\bar z)+120\zeta_2+320\mathcal{L}_2(\bar z)\bigg] \\
&+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\bigg[\frac{32\mathcal{D}_0(\bar z)}{\epsilon^2}-\frac{128\mathcal{D}_0(\bar z)+80}{\epsilon}+256\mathcal{D}_2(\bar z)-96\zeta_2\mathcal{D}_0(\bar z)+230\mathcal{L}_1(\bar z)\bigg] \\
&+\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar y)+\mathcal{D}_1(y)\mathcal{D}_0(\bar y)\Big]\bigg[-\frac{64\mathcal{D}_0(\bar z)}{\epsilon}+256\mathcal{D}_1(\bar z)+160\bigg] \\
&-128\mathcal{D}_0(\bar z)\mathcal{D}_1(y)\mathcal{D}_1(\bar y)+64\mathcal{D}_0(\bar z)\Big[\mathcal{D}_0(y)\mathcal{D}_2(\bar y)+\mathcal{D}_2(y)\mathcal{D}_0(\bar y)\Big]\bigg] \\
&+C_AC_F\bigg[\Big[\delta(y)+\delta(1-y)\Big]\bigg[\delta(1-z)\left(\frac{1}{\epsilon^4}+\frac{11}{6\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{67}{18}-4\zeta_2\right)\right. \\
&\left.+\frac{1}{\epsilon}\left(\frac{202}{27}-\frac{11\zeta_2}{2}-\frac{29\zeta_3}{3}\right)+\frac{1214}{81}-\frac{67\zeta_2}{6}-\frac{77\zeta_3}{9}-\frac{17\zeta_4}{4}\right]-\frac{4\mathcal{D}_0(\bar z)}{\epsilon^3} \\
&+\frac{1}{\epsilon^2}\left(16\mathcal{D}_1(\bar z)-\frac{22}{3}\mathcal{D}_0(\bar z)+8\right)+\frac{1}{\epsilon}\left(-32\mathcal{D}_2(\bar z)+\frac{88}{3}\mathcal{D}_1(\bar z)+\left(16\zeta_2-\frac{134}{9}\right)\mathcal{D}_0(\bar z)\right. \\
&\left.+\frac{44}{3}-32\mathcal{L}_1(\bar z)\right)+\frac{128}{3}\mathcal{D}_3(\bar z)-\frac{176}{3}\mathcal{D}_2(\bar z)+\left(\frac{536}{9}-64\zeta_2\right)\mathcal{D}_1(\bar z) \\
&+\left(\frac{116}{3}\zeta_3+22\zeta_2-\frac{808}{27}\right)\mathcal{D}_0(\bar z)+\frac{238}{9}-32\zeta_2+64\mathcal{L}_2(\bar z)-\frac{176}{3}\mathcal{L}_1(\bar z)\bigg] \\
&+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\bigg[\frac{8\mathcal{D}_0(\bar z)}{\epsilon^2}+\frac{1}{\epsilon}\left(-32\mathcal{D}_1(\bar z)+\frac{44}{3}\mathcal{D}_0(\bar z)-16\right)+64\mathcal{D}_2(\bar z) \\
&-\frac{176}{3}\mathcal{D}_1(\bar z)+\left(\frac{268}{9}-32\zeta_2\right)\mathcal{D}_0(\bar z)-\frac{88}{3}+64\mathcal{L}_1(\bar z)\bigg] \\
&+\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar y)+\mathcal{D}_1(y)\mathcal{D}_0(\bar y)\Big]\bigg[-\frac{16\mathcal{D}_0(\bar z)}{\epsilon}+64\mathcal{D}_1(\bar z)-\frac{88}{3}\mathcal{D}_0(\bar z)+32\bigg] \\
&+32\mathcal{D}_0(\bar z)\mathcal{D}_1(y)\mathcal{D}_1(\bar y)+16\mathcal{D}_0(\bar z)\Big[\mathcal{D}_0(y)\mathcal{D}_2(\bar y)+\mathcal{D}_2(y)\mathcal{D}_0(\bar y)\Big]\bigg] \\
&+n_fC_F\bigg[\Big[\delta(y)+\delta(1-y)\Big]\bigg[\delta(1-z)\left(-\frac{1}{3\epsilon^3}-\frac{5}{9\epsilon^2}+\frac{1}{\epsilon}\left(\zeta_2-\frac{28}{27}\right)-\frac{164}{81}+\frac{5\zeta_2}{3}\right.
\end{aligned}
$$

$$
+\frac{14\zeta_3}{9}\Big)+\frac{4\mathcal{D}_0(\bar z)}{3\epsilon^2}+\frac{1}{\epsilon}\Big(-\frac{16}{3}\mathcal{D}_1(\bar z)+\frac{20}{9}\mathcal{D}_0(\bar z)-\frac{8}{3}\Big)+\frac{32}{3}\mathcal{D}_2(\bar z)-\frac{80}{9}\mathcal{D}_1(\bar z)
+\Big(\frac{112}{27}-4\zeta_2\Big)\mathcal{D}_0(\bar z)-\frac{28}{9}+\frac{32}{3}\mathcal{L}_1(\bar z)\Big]+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\Big[-\frac{8\mathcal{D}_0(\bar z)}{3\epsilon}+\frac{32}{3}\mathcal{D}_1(\bar z)
-\frac{40}{9}\mathcal{D}_0(\bar z)+\frac{16}{3}\Big]+\frac{16\mathcal{D}_0(\bar z)}{3}\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar y)+\mathcal{D}_1(y)\mathcal{D}_0(\bar y)\Big]\Big]\Big\}, \tag{2.66}
$$

at leading power, and

$$
\Delta^{(2)}(z,y)|_{2r}^{\rm NLP} = \Big(\frac{\alpha_s}{4\pi}\Big)^2\Big\{C_F^2\Big[\Big[\delta(y)+\delta(1-y)\Big]\Big[\frac{16}{\epsilon^3}-\frac{64\mathcal{L}_1(\bar z)}{\epsilon^2}+\frac{128\mathcal{L}_2(\bar z)+4-48\zeta_2}{\epsilon}
-\frac{512}{3}\mathcal{L}_3(\bar z)+\Big(192\zeta_2-16\Big)\mathcal{L}_1(\bar z)+12-\frac{224\zeta_3}{3}\Big]+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\Big[-\frac{32}{\epsilon^2}+\frac{128\mathcal{L}_1(\bar z)}{\epsilon}
-8+96\zeta_2-256\mathcal{L}_2(\bar z)\Big]+\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar y)+\mathcal{D}_1(y)\mathcal{D}_0(\bar y)\Big]\Big[\frac{64}{\epsilon}-256\mathcal{L}_1(\bar z)\Big]
-128\mathcal{D}_1(y)\mathcal{D}_1(\bar y)-64\Big[\mathcal{D}_0(y)\mathcal{D}_2(\bar y)+\mathcal{D}_2(y)\mathcal{D}_0(\bar y)\Big]\Big]+C_AC_F\Big[\Big[\delta(y)+\delta(1-y)\Big]\Big[\frac{4}{\epsilon^3}
+\frac{1}{\epsilon^2}\Big(\frac{22}{3}-16\mathcal{L}_1(\bar z)\Big)+\frac{1}{\epsilon}\Big(32\mathcal{L}_2(\bar z)-\frac{88}{3}\mathcal{L}_1(\bar z)+\frac{110}{9}-16\zeta_2\Big)-\frac{128}{3}\mathcal{L}_3(\bar z)
+\frac{176}{3}\mathcal{L}_2(\bar z)+\Big(64\zeta_2-\frac{440}{9}\Big)\mathcal{L}_1(\bar z)+\frac{580}{27}-22\zeta_2-\frac{116\zeta_3}{3}\Big]+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\Big[-\frac{8}{\epsilon^2}
+\frac{1}{\epsilon}\Big(32\mathcal{L}_1(\bar z)-\frac{44}{3}\Big)-64\mathcal{L}_2(\bar z)+\frac{176}{3}\mathcal{L}_1(\bar z)-\frac{220}{9}+32\zeta_2\Big]+\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar y)
+\mathcal{D}_1(y)\mathcal{D}_0(\bar y)\Big]\Big[\frac{16}{\epsilon}-64\mathcal{L}_1(\bar z)+\frac{88}{3}\Big]-32\mathcal{D}_1(y)\mathcal{D}_1(\bar y)-16\Big[\mathcal{D}_0(y)\mathcal{D}_2(\bar y)+\mathcal{D}_2(y)\mathcal{D}_0(\bar y)\Big]\Big]
+n_fC_F\Big[\Big[\delta(y)+\delta(1-y)\Big]\Big[-\frac{4}{3\epsilon^2}+\frac{1}{\epsilon}\Big(\frac{16}{3}\mathcal{L}_1(\bar z)-\frac{32}{9}\Big)-\frac{32}{3}\mathcal{L}_2(\bar z)+\frac{128}{9}\mathcal{L}_1(\bar z)
-\frac{244}{27}+4\zeta_2\Big]+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\Big[\frac{8}{3\epsilon}-\frac{32}{3}\mathcal{L}_1(\bar z)+\frac{64}{9}\Big]-\frac{16}{3}\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar y)+\mathcal{D}_1(y)\mathcal{D}_0(\bar y)\Big]\Big]\Big\}, \tag{2.67}
$$

at NLP.

Sum. The double-real correction completes our calculation of the Drell-Yan rapidity distribution at NNLO. Summing eqs. (2.38), (2.49), (2.50), (2.51), (2.52), (2.53), (2.66) and (2.67) we obtain

$$
\Delta^{(2)}(z,y)|^{\rm LP} = \Big(\frac{\alpha_s}{4\pi}\Big)^2\Big\{C_F^2\Big[\Big[\delta(y)+\delta(1-y)\Big]\Big[\delta(1-z)\Big(\frac{9-8\zeta_2}{\epsilon^2}+\frac{1}{\epsilon}\Big(\frac{189}{4}-30\zeta_2-28\zeta_3\Big)
+\frac{1279}{8}-130\zeta_2-54\zeta_3+80\zeta_4\Big)+\mathcal{D}_0(\bar z)\Big(\frac{24}{\epsilon^2}+\frac{64-16\zeta_2}{\epsilon}+128-96\zeta_2+32\zeta_3\Big)
+\mathcal{D}_1(\bar z)\Big(\frac{32}{\epsilon^2}-\frac{48}{\epsilon}-128-64\zeta_2\Big)+\mathcal{D}_2(\bar z)\Big(48-\frac{96}{\epsilon}\Big)+\frac{448}{3}\mathcal{D}_3(\bar z)+\frac{8}{\epsilon^2}-\frac{48}{\epsilon}-128+8\zeta_2
$$

$$
\begin{aligned}
&+\mathcal{L}_1(\bar z)\Big(96-\frac{96}{\epsilon}\Big)+256\mathcal{L}_2(\bar z)\Big]+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\Big[\mathcal{D}_0(\bar z)\Big(\frac{16}{\epsilon^2}-\frac{24}{\epsilon}-64-32\zeta_2\Big)\\
&+\mathcal{D}_1(\bar z)\Big(48-\frac{96}{\epsilon}\Big)+224\mathcal{D}_2(\bar z)-\frac{48}{\epsilon}+48+256\mathcal{L}_1(\bar z)\Big]+\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar y)\\
&+\mathcal{D}_1(y)\mathcal{D}_0(\bar y)\Big]\Big[\mathcal{D}_0(\bar z)\Big(24-\frac{48}{\epsilon}\Big)+224\mathcal{D}_1(\bar z)+128\Big]+112\mathcal{D}_0(\bar z)\mathcal{D}_1(y)\mathcal{D}_1(\bar y)\\
&+56\mathcal{D}_0(\bar z)\Big[\mathcal{D}_0(y)\mathcal{D}_2(\bar y)+\mathcal{D}_2(y)\mathcal{D}_0(\bar y)\Big]\Big]+C_AC_F\Big[\Big[\delta(y)+\delta(1-y)\Big]\Big[\delta(1-z)\Big(-\frac{11}{2\epsilon^2}\\
&+\frac{1}{\epsilon}\Big(-\frac{123}{4}+\frac{44\zeta_2}{3}+6\zeta_3\Big)-\frac{981}{8}+\frac{1553\zeta_2}{18}+\frac{130\zeta_3}{3}-13\zeta_4\Big)+\mathcal{D}_0(\bar z)\Big(-\frac{22}{3\epsilon^2}\\
&+\frac{1}{\epsilon}\Big(-\frac{134}{9}+4\zeta_2\Big)-\frac{808}{27}+22\zeta_2+28\zeta_3\Big)+\mathcal{D}_1(\bar z)\Big(\frac{88}{3\epsilon}+\frac{536}{9}-16\zeta_2\Big)-\frac{176}{3}\mathcal{D}_2(\bar z)\\
&+\frac{44}{3\epsilon}+\frac{238}{9}-8\zeta_2-\frac{176}{3}\mathcal{L}_1(\bar z)\Big]+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\Big[\mathcal{D}_0(\bar z)\Big(\frac{44}{3\epsilon}+\frac{268}{9}-8\zeta_2\Big)-\frac{176}{3}\mathcal{D}_1(\bar z)-\frac{88}{3}\Big]\\
&-\frac{88}{3}\mathcal{D}_0(\bar z)\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar y)+\mathcal{D}_1(y)\mathcal{D}_0(\bar y)\Big]\Big]+n_fC_F\Big[\Big[\delta(y)+\delta(1-y)\Big]\Big[\delta(1-z)\Big(\frac{1}{\epsilon^2}\\
&+\frac{1}{\epsilon}\Big(\frac{11}{2}-\frac{8\zeta_2}{3}\Big)+\frac{85}{4}-\frac{139\zeta_2}{9}-\frac{4\zeta_3}{3}\Big)+\mathcal{D}_0(\bar z)\Big(\frac{4}{3\epsilon^2}+\frac{20}{9\epsilon}+\frac{112}{27}-4\zeta_2\Big)\\
&+\mathcal{D}_1(\bar z)\Big(-\frac{16}{3\epsilon}-\frac{80}{9}\Big)+\frac{32}{3}\mathcal{D}_2(\bar z)-\frac{8}{3\epsilon}-\frac{28}{9}+\frac{32}{3}\mathcal{L}_1(\bar z)\Big]+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\Big[\mathcal{D}_0(\bar z)\Big(-\frac{8}{3\epsilon}\\
&-\frac{40}{9}\Big)+\frac{32}{3}\mathcal{D}_1(\bar z)+\frac{16}{3}\Big]+\frac{16}{3}\mathcal{D}_0(\bar z)\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar y)+\mathcal{D}_1(y)\mathcal{D}_0(\bar y)\Big]\Big]\Big\}, \qquad (2.68)
\end{aligned}
$$

at leading power, and

$$
\begin{aligned}
\Delta^{(2)}(z,y)|^{\rm NLP} = {}& \Big(\frac{\alpha_s}{4\pi}\Big)^2\Big\{C_F^2\Big[\Big[\delta(y)+\delta(1-y)\Big]\Big[-\frac{32\mathcal{L}_1(\bar z)+16}{\epsilon^2}\\
&+\frac{96\mathcal{L}_2(\bar z)+40\mathcal{L}_1(\bar z)-46+16\zeta_2}{\epsilon}-\frac{448}{3}\mathcal{L}_3(\bar z)-52\mathcal{L}_2(\bar z)+\Big(94+64\zeta_2\Big)\mathcal{L}_1(\bar z)\\
&-64+40\zeta_2-32\zeta_3\Big]+\mathcal{D}_0(y)\Big(\frac{16}{\epsilon}+20-16\mathcal{L}_1(\bar y)-48\mathcal{L}_1(\bar z)\Big)+\mathcal{D}_0(\bar y)\Big(\frac{16}{\epsilon}+20\\
&-16\mathcal{L}_1(y)-48\mathcal{L}_1(\bar z)\Big)-32\Big(\mathcal{D}_1(y)+\mathcal{D}_1(\bar y)\Big)+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\Big[-\frac{16}{\epsilon^2}+\frac{96\mathcal{L}_1(\bar z)+8}{\epsilon}\\
&-224\mathcal{L}_2(\bar z)-16\mathcal{L}_1(\bar z)+32+32\zeta_2\Big]+\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar y)+\mathcal{D}_1(y)\mathcal{D}_0(\bar y)\Big]\Big[\frac{48}{\epsilon}-224\mathcal{L}_1(\bar z)-8\Big]\\
&-112\mathcal{D}_1(y)\mathcal{D}_1(\bar y)-56\Big[\mathcal{D}_0(y)\mathcal{D}_2(\bar y)+\mathcal{D}_2(y)\mathcal{D}_0(\bar y)\Big]\Big]+C_AC_F\Big[\Big[\delta(y)+\delta(1-y)\Big]\Big[\frac{22}{3\epsilon^2}\\
&+\frac{1}{\epsilon}\Big(-\frac{88}{3}\mathcal{L}_1(\bar z)+\frac{200}{9}-4\zeta_2\Big)+\frac{176}{3}\mathcal{L}_2(\bar z)+\Big(16\zeta_2-\frac{710}{9}\Big)\mathcal{L}_1(\bar z)+\frac{688}{27}-22\zeta_2-28\zeta_3\Big]\\
&-20\Big(\mathcal{D}_0(y)+\mathcal{D}_0(\bar y)\Big)+\mathcal{D}_0(y)\mathcal{D}_0(\bar y)\Big[-\frac{44}{3\epsilon}+\frac{176}{3}\mathcal{L}_1(\bar z)-\frac{220}{9}+8\zeta_2\Big]
\end{aligned}
$$

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$$
\begin{aligned}
&+\frac{88}{3}\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar{y})+\mathcal{D}_1(y)\mathcal{D}_0(\bar{y})\Big]\Bigg]+n_fC_F\Bigg[\Big[\delta(y)+\delta(1-y)\Big]\Bigg[-\frac{4}{3\epsilon^2}+\frac{1}{\epsilon}\bigg(\frac{16}{3}\mathcal{L}_1(\bar{z})\\
&-\frac{32}{9}\bigg)-\frac{32}{3}\mathcal{L}_2(\bar{z})+\frac{128}{9}\mathcal{L}_1(\bar{z})-\frac{244}{27}+4\zeta_2\Bigg]+\mathcal{D}_0(y)\mathcal{D}_0(\bar{y})\bigg[\frac{8}{3\epsilon}-\frac{32}{3}\mathcal{L}_1(\bar{z})+\frac{64}{9}\bigg]\\
&-\frac{16}{3}\Big[\mathcal{D}_0(y)\mathcal{D}_1(\bar{y})+\mathcal{D}_1(y)\mathcal{D}_0(\bar{y})\Big]\Bigg]\Bigg\}, \qquad (2.69)
\end{aligned}
$$

at NLP. Eqs. (2.68) and (2.69) still contain ϵ poles, which are eventually removed by PDF renormalisation. However, this result already allows us to conclude that the partonic cross section $\Delta(z,y)$ has the structure of eq. (2.34) also at NNLO, i.e.

$$
\Delta^{(2)}(z,y)=\frac{\delta(y)+\delta(1-y)}{2}\Big[\Delta^{(2)}_{\mathrm{LP}}(z)+\Delta^{(2)}_{\mathrm{NLP,LLs}}(z)\Big]+\Delta^{(2)}_{\mathrm{NLP,rest}}(z,y)+\mathcal{O}(1-z). \qquad (2.70)
$$

Indeed, y- and $(1-y)$-plus distributions appearing in the LP term of eq. (2.68) are effectively contributing at NLP, after the simplification in eq. (2.33) is taken into account. Furthermore, such terms do not contain leading logarithms or leading plus distributions in $(1-z)$. These terms can therefore be rearranged to be part of $\Delta_{\mathrm{NLP,rest}}(z,y)$ in eq. (2.70). Hence the only term remaining at LP is the term proportional to $(\delta(y)+\delta(1-y))/2$ in eq. (2.68). Concerning now the NLP result in eq. (2.69), we see that leading logarithms in $(1-z)$ do arise only in the $(\delta(y)+\delta(1-y))/2$ term, and do not contribute to the y- and $(1-y)$-plus distribution terms. Thus also eq. (2.69) is consistent with eq. (2.70).

2.3 Rapidity distribution and kinematic shifts

The result in eq. (2.70) allows us to conclude that near threshold the y-dependence in the partonic cross section factorises into a universal factor $[\delta(y)+\delta(1-y)]/2$, provided one restricts to the LP term and the LLs at NLP. Although proven explicitly up to two loops, the discussion in the previous sections suggests that the result in eq. (2.70) should extend to all orders, as in the ansatz of eq. (2.34). The z-dependence is entirely contained into the factors $\Delta^{(2)}_{\mathrm{LP}}(z)$ and $\Delta^{(2)}_{\mathrm{NLP,LLs}}(z)$, which are thus proportional to the corresponding terms in the partonic invariant mass distribution. This result suggest that it may be possible to exploit methods developed in [25], and obtain the NLP NLO partonic cross section not by direct calculation, but rather by means of kinematic shifts. To be more specific, given a colourless final state produced by the annihilation of an initial $q\bar{q}$ pair with momenta p_a, p_b, the squared matrix element should be given by

$$
\sum_{\mathrm{s,c,p}}|\mathcal{M}|^2_{\mathrm{NLO,NLP}}=g_s^2C_F\frac{\hat{s}}{(p_a\cdot k)(p_b\cdot k)}\sum_{\mathrm{s,c,p}}\Big|\mathcal{M}^{(0)}_{q\bar{q}\to\gamma^*}(p_a+\delta p_a,p_b+\delta p_b)\Big|^2, \qquad (2.71)
$$

where

$$
\delta p_a^\mu=-\frac{1}{2}\bigg(\frac{k\cdot p_b}{p_a\cdot p_b}p_a^\mu-\frac{k\cdot p_a}{p_a\cdot p_b}p_b^\mu+k^\mu\bigg),\quad \delta p_b^\mu=-\frac{1}{2}\bigg(\frac{k\cdot p_a}{p_a\cdot p_b}p_b^\mu-\frac{k\cdot p_b}{p_a\cdot p_b}p_a^\mu+k^\mu\bigg), \qquad (2.72)
$$

which implies $\hat{s}\to z\hat{s}$. Given the simple structure of $|\mathcal{M}^{(0)}_{q\bar{q}\to\gamma^*}|^2$, see eq. (2.16), which depends on the single scale $\hat{s}$, we immediately have

$$
\sum_{\mathrm{s,c,p}}\Big|\mathcal{M}^{(0)}_{q\bar{q}\to\gamma^*}(p_a+\delta p_a,p_b+\delta p_b)\Big|^2=4(1-\epsilon)z\hat{s}N_c. \qquad (2.73)
$$

Inserting this result into eq. (2.71) we get

$$\begin{aligned}\sum_{\mathrm{s,c,p}}|\mathcal{M}|^2_{\mathrm{NLO,NLP}} &= \frac{\alpha_s}{4\pi}64\pi^2 N_c C_F(1-\epsilon)\frac{z\hat{s}^2}{k\cdot p_a\, k\cdot p_b}\\ &= \frac{\alpha_s}{4\pi}64\pi^2 N_c C_F(1-\epsilon)\frac{\hat{s}^2}{k\cdot p_a\, k\cdot p_b}\Big(1-(1-z)\Big).\end{aligned} \tag{2.74}$$

This result has to be compared with the exact result in eq. (2.25). We see that the two forms of the NLO correction are indeed equal up to the factor $1-2/\hat{s}(k\cdot p_a+k\cdot p_b)$, appearing in the exact result of eq. (2.25), vs the factor $1-(1-z)$, obtained in the shifted result in eq. (2.74). Indeed, the two factors coincide, given that upon phase space integration one has $(k\cdot p_a+k\cdot p_b)=(\hat{s}-Q^2)/2=\hat{s}(1-z)/2$. Furthermore, the rapidity distribution is symmetric w.r.t. the exchange $y\leftrightarrow(1-y)$, a feature which is not altered between LP and NLP. Indeed, near threshold additional gluon radiation is constrained to be soft, but remains isotropic. Thus phase space integration gives rise to the same factor $y^{-1-\epsilon}(1-y)^{-1-\epsilon}$ both at LP and NLP, as can be seen explicitly in eqs. (2.26) and (2.27), and the shift procedure gives rise to the correct relation between the LP and NLP contribution, which within the exact result of eqs. (2.26) and (2.27) arises after phase space integration. To be more specific, integrating eq. (2.25) *or* eq. (2.74) against the phase space measure $d\Phi_{\gamma^* g}$ as in eq. (2.23) gives rise to the same NLP LL result

$$\Delta^{(1)}(z,y)\big|^{\mathrm{LP+NLP,LL}}_{\mathrm{real}} = \frac{\alpha_s C_F}{4\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon} y^{-1-\epsilon}(1-y)^{-1-\epsilon}(1-z)^{-1-2\epsilon}\Big[1-(1-z)\Big]\Big[4+\mathcal{O}(\epsilon)\Big]. \tag{2.75}$$

In light of using this result for resummation, let us keep the exact phase space dependence, and expand the y-dependent part in powers of ϵ. One has

$$\begin{aligned}\Delta^{(1)}(z,y)\big|^{\mathrm{LP+NLP,LL}}_{\mathrm{real}} &= \frac{\alpha_s C_F}{\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left[-\frac{\delta(y)+\delta(1-y)}{\epsilon}+\mathcal{O}(\epsilon^0)\right]z(1-z)^{-1-2\epsilon}\\ &= \frac{\Gamma(-2\epsilon)}{\Gamma^2(-\epsilon)}\left[-\frac{\delta(y)+\delta(1-y)}{\epsilon}+\mathcal{O}(\epsilon^0)\right]K_{\mathrm{NLP}}(z,\epsilon)\\ &= \Big[\bar{\Delta}^{(0)}(y)+\mathcal{O}(\epsilon)\Big]K_{\mathrm{NLP}}(z,\epsilon),\end{aligned} \tag{2.76}$$

where the factor $\bar{\Delta}^{(0)}(y)$ has been defined in eq. (2.21), and the factor

$$K_{\mathrm{NLP}}(z,\epsilon)=\frac{\alpha_s C_F}{\pi}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^{\epsilon} z(1-z)^{-1-2\epsilon}\frac{e^{\epsilon\gamma_E}\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)\Gamma(1-\epsilon)}, \tag{2.77}$$

has been introduced in eq. (3.62) of [54]. In this respect, the last line of eq. (2.76) constitutes the generalisation of eq. (3.61) of [54] for the DY rapidity distribution. In section 4.2.2 we will use this result to sum the large logarithms in $1-z$ to all order in α_s at NLP, at LL accuracy.

3 Rapidity distribution for fixed order diphoton production

In this section we move away from the discussion of the Drell-Yan process and look instead at QCD-induced diphoton production. Also for this process we only consider the leading

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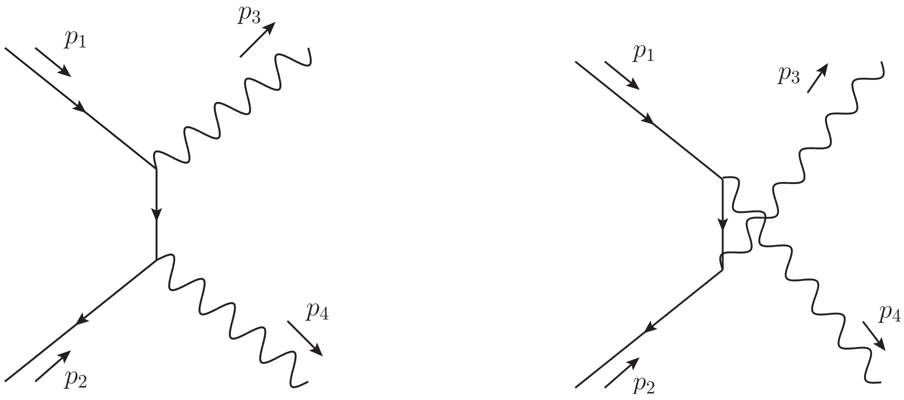

Figure 2. Tree-level contributions to diphoton production.

production channel, given by an incoming quark-antiquark pair. Whereas in the previous section we considered the rapidity of the whole final state, i.e. the virtual photon, here we select the rapidity of one of the final state particles. The production of two photons is important in particle physics, since one of the most important decay channels for the Higgs boson is $H\to\gamma\gamma$. The QCD production of two photons forms a large irreducible background in the $H\to\gamma\gamma$ analysis [85, 86]. In this section we look at diphoton production at fixed order, similar to the Drell-Yan process in the previous section. This will also serve as preparation for the calculation of the corresponding resummed cross section, which we perform in section 4.

3.1 Leading order

We consider diphoton production by an incoming quark-antiquark pair

$$q(p_1)+\bar{q}(p_2)\to\gamma(p_3)+\gamma(p_4)\,. \tag{3.1}$$

At LO the process is represented by the two diagrams in figure 2. Taking into account a symmetry factor of $\frac{1}{2}$ since the final-state photons are indistinguishable, the corresponding squared matrix element, respectively averaged and summed over the spin, colour and polarisation degrees of freedom of the initial and final state, indicated by the bar, is given by

$$\overline{|\mathcal{M}|^2_{\mathrm{LO}}}=\frac{2(ee_q)^4}{N_c}\left[\frac{1+\cos^2\theta}{1-\cos^2\theta}(1-\epsilon)^2-\epsilon(1-\epsilon)\right], \tag{3.2}$$

where θ is the angle between the incoming quark and the photon with momentum p_3 in the centre of mass frame of the incoming quarks, and where e_q is the quark fractional electric charge. For simplicity we set $e_q=1$ and $e^2=4\pi\alpha$. At lowest order it does not matter which of the two photons is chosen, but at higher orders we choose the photon with momentum p_3 as the one whose rapidity we record. To calculate the cross section, one must integrate eq. (3.2) over the phase space of the two outgoing photons, which involves integrating θ between $\theta=0$ and $\theta=\pi$. This integrand diverges at both the endpoints. However, since detectors have no coverage at these extremes one may restrict the integration to a range from $\theta = \delta$ to $\theta = \pi - \delta$, with δ fixed by the experimental set-up. More convenient for this purpose is the pseudorapidity η

$$\eta \equiv -\log\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\log\left(\frac{p_3^{(0)}+p_3^{(3)}}{p_3^{(0)}-p_3^{(3)}}\right), \tag{3.3}$$

with $p_3^2 = 0$ and $p_3^{(3)} = p_3^{(0)}\cos\theta$. Integrating the pseudorapidity in the range $(-\infty,\infty)$ still yields a divergent integral, but the cut-off δ on the angle θ, which we have introduced above, implies a corresponding cut-off on the pseudorapidity, which then also serves as a regulator of the integral. The squared amplitude in eq. (3.2) reads then

$$\overline{|\mathcal{M}|^2_{\rm LO}} = \frac{2(4\pi\alpha)^2}{N_c}\Big[\cosh(2\eta)(1-\epsilon)^2 - \epsilon(1-\epsilon)\Big]\,. \tag{3.4}$$

The double differential cross section is therefore given by

$$\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}}{dz d\eta}(\hat{s},z,\eta,\epsilon,\bar{\mu}^2) = \frac{1}{2\hat{s}}\frac{d}{dz d\eta}\int dR_2\,\overline{|\mathcal{M}|^2_{\rm LO}} = \frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dz d\eta}(\hat{s},\eta,\epsilon,\bar{\mu}^2)\,\delta(1-z)\,, \tag{3.5}$$

where the phase space integration $\int dR_2$ is given in appendix B.1 and

$$\begin{aligned}\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dz d\eta}(\hat{s},\eta,\epsilon,\bar{\mu}^2) = {} & \frac{\pi\alpha^2}{N_c\,\hat{s}}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^{\epsilon}(4\cosh^2\eta)^{\epsilon}\\ & \times\left[(1+\tanh^2\eta)(1-\epsilon)^2 - \frac{\epsilon(1-\epsilon)}{\cosh^2\eta}\right].\end{aligned} \tag{3.6}$$

The partonic centre of mass energy is $\hat{s} = (p_1+p_2)^2$, we denote the invariant mass of the final state by $Q^2 = (p_3+p_4)^2$ and define the corresponding ratio $z = Q^2/\hat{s}$. The double differential cross section in $d = 4$ dimensions is therefore given by

$$\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}}{dz d\eta}(\hat{s},z,\eta) = \frac{\pi\alpha^2}{N_c\,\hat{s}}(1+\tanh^2\eta)\,\delta(1-z)\,. \tag{3.7}$$

This partonic cross section can be translated to a hadronic one. We define the hadronic rapidity

$$Y = \eta + \frac{1}{2}\log\left(\frac{x_a}{x_b}\right). \tag{3.8}$$

The hadronic cross section then reads

$$\frac{d\sigma}{dQ^2 dY} = \frac{1}{s}\int_\tau^1\frac{dz}{z}\int_{\log\left(\sqrt{\frac{\tau}{z}}e^Y\right)}^{\log\left(\sqrt{\frac{z}{\tau}}e^Y\right)} d\eta\,\mathcal{L}(z,\eta)\frac{d\hat{\sigma}}{dz d\eta}(\hat{s},z,\eta)\,, \tag{3.9}$$

where the luminosity function is given by

$$\mathcal{L}(z,\eta) = f_{q/A}\left(e^Y e^{-\eta}\sqrt{\frac{\tau}{z}}\right) f_{\bar{q}/B}\left(e^{-Y}e^{\eta}\sqrt{\frac{\tau}{z}}\right) + (q\leftrightarrow\bar{q}), \tag{3.10}$$

in analogy with eq. (2.8). Note that there is a bound on the integration over the rapidity η for finite Y. The divergence that we observed for the partonic cross section when one would integrate over the full range of η is now transformed to a divergence if we would integrate over the full range of the hadronic rapidity Y. In reality there is a finite rapidity range for the hadronic rapidity Y. The partonic differential cross section $d\hat{\sigma}/dzd\eta$ can be calculated perturbatively and its LO contribution is given by eq. (3.7).

Comparing eq. (2.17) with eq. (3.7), we see that the main difference between a distribution differential in the total rapidity of the final state (eq. (2.17)) and a distribution differential in the rapidity of a given particle in the final state (eq. (3.7)) is that the latter exhibits a non-trivial dependence on the rapidity considered, given e.g. by the factor $1+\tanh^2\eta$ in case of diphoton production. Therefore, a priori it is not clear whether for $z\to 1$ one may express the NLO NLP correction to eq. (3.7) as obtained in [25], namely, in terms of the LO partonic distribution with shifted kinematics times a universal factor. The purpose of this section is indeed to investigate this issue. To this end we will now proceed in two different ways, both aimed at obtaining the NLP partonic distribution at NLO. At first we will derive this result from a direct calculation in the soft limit at NLO. We will then try to obtain the same NLP cross section at NLO by expressing it in terms of a universal factor times the LO partonic differential distribution with shifted kinematics. The calculation that we perform serves as an extension of the method developed in [25], which so far has been developed only for total cross sections, invariant mass distributions and (in section 2.3 of the present paper) for distributions differential in the total rapidity of the final state.

3.2 Next-to-leading order

At NLO, the partonic process with one gluon emission is given by

$$q(p_1)+\bar{q}(p_2)\to\gamma(p_3)+\gamma(p_4)+g(k)\,, \tag{3.11}$$

with $k^2=0$. We define the following invariant variables

$$\begin{aligned} s_{ij}&=(\sigma_i p_i+\sigma_j p_j)^2\,, && i,j=1,\dots,4,\\ s_i&=(\sigma_i p_i-k)^2\,, && i=1,\dots,4, \end{aligned} \tag{3.12}$$

where $\sigma_i=+1$ if the momenta p_i are incoming, and $\sigma_i=-1$ otherwise. First we use the direct calculation method. We calculate the squared matrix element straightforwardly and perform the power expansion in the soft limit $s_i\ll s_{ij}$. Note that there are five independent variables in the massless $2\to 3$ process. Using momentum conservation we can express s_{i4} and s_4 as linear combinations of $s_{ij}\,(i,j=1,2,3)$ and $s_i\,(i=1,2,3)$. We can expand the NLO squared matrix element in the limit $s_i\ll s_{ij}$. After the expansion, s_3 can be further removed by using the on-shell condition $p_4^2=(p_1+p_2-p_3-k)^2=0$. As a result, we have the squared matrix element in the following form

$$\overline{|\mathcal{M}|^2_{\rm real}}(s_{12},s_{13},s_{23},s_1,s_2)=\sum_l c_l(\epsilon)s_{12}^{\alpha_l}s_{13}^{\beta_l}s_{23}^{\gamma_l}s_1^{\kappa_l}s_2^{-1-\alpha_l-\beta_l-\gamma_l-\kappa_l}\,. \tag{3.13}$$

Note that the total power of s_{ij} and s_i is limited by the mass dimension of the squared amplitude. In terms of the light-cone coordinates defined in appendix A we have

$$\begin{aligned} s_{13} &= -2\sqrt{\frac{\hat{s}}{2}}p_3^-\,, & s_{23} &= -2\sqrt{\frac{\hat{s}}{2}}p_3^+\,, \\ s_1 &= -2\sqrt{\frac{\hat{s}}{2}}k^-\,, & s_2 &= -2\sqrt{\frac{\hat{s}}{2}}k^+\,, \end{aligned} \tag{3.14}$$

where $\hat{s}=s_{12}$. To obtain the differential cross section at NLO, we need to calculate the three-body phase space integral, which is given in appendix B.2. Up to NLP, the contribution from the NLO real emission to the differential cross section is given by

$$\begin{aligned} \frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta,\epsilon,\bar{\mu}^2)}{dz d\eta}\bigg|^{\rm NLP}_{\rm real} &= \frac{\alpha_s C_F}{\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\frac{2z^{1+\epsilon}(1-z)^{-1-2\epsilon}e^{\epsilon\gamma_E}\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dz d\eta}(Q^2,\eta,\epsilon,\bar{\mu}^2) \\ &= \frac{\alpha_s C_F}{\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\left[\delta(1-z)\left(\frac{1}{\epsilon^2}-\frac{\pi^2}{4}\right)+\frac{2}{\epsilon}\left(1-\frac{1}{1-z}\bigg|_+\right)\right. \\ &\quad \left. +4\frac{\log(1-z)}{1-z}\bigg|_+ -4\log(1-z)+2+\mathcal{O}(\epsilon)\right]\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dz d\eta}(Q^2,\eta,\epsilon,\bar{\mu}^2) \end{aligned} \tag{3.15}$$

where the factor $z^{1+\epsilon}$ follows from

$$\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dz d\eta}(\hat{s},\eta,\epsilon,\bar{\mu}^2) \to \frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dz d\eta}(Q^2,\eta,\epsilon,\bar{\mu}^2)\,. \tag{3.16}$$

In the second line of eq. (3.15) we expanded the result to NLP in $(1-z)$ and to finite order in ϵ. In fact, the NLO differential cross section can be calculated up to arbitrary power in $(1-z)$ by using eq. (B.9). We will demonstrate this in section 3.2.1. Upon adding the virtual contribution at one loop, the $1/\epsilon^2$ pole from eq. (3.15) cancels with the virtual contribution, leaving a $1/\epsilon$ pole that is removed by so-called mass factorisation, i.e. PDF renormalisation. This subtraction term arises from eq. (3.9) when we replace the hadrons by the quarks. The left-hand side of that equation becomes the partonic cross section with $1/\epsilon$ poles, and the PDFs become parton-in-parton distributions, which are directly related to the Altarelli-Parisi splitting kernels. From this one obtains the finite partonic cross section at $\mathcal{O}(\alpha_s)$ accuracy

$$\begin{aligned} \frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta,\epsilon,\bar{\mu}^2)}{dz d\eta}\bigg|_{\rm ren} &= \frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta,\epsilon,\bar{\mu}^2)}{dz d\eta}\bigg|_{\rm real+virtual} \\ &\quad -\int_0^1 d\xi\,\Gamma^{(1)}_{qq}(\xi)\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}(\xi\hat{s},\xi,z,\eta,\epsilon,\bar{\mu}^2)}{dz d\eta}\,, \end{aligned} \tag{3.17}$$

where $\Gamma^{(1)}_{qq}(\xi)$ is the well-known splitting function kernel, given by

$$\Gamma^{(1)}_{qq}(\xi) = -\frac{\alpha_s C_F}{2\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\frac{1}{\epsilon}\left[2\frac{1}{1-\xi}\bigg|_+ -2+(1-\xi)+\frac{3}{2}\delta(1-\xi)\right]. \tag{3.18}$$

The LO cross section entering the subtraction term in eq. (3.17) is defined as

$$\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}(\xi\hat{s},\xi,z,\eta,\epsilon,\bar{\mu}^2)}{dzd\eta} = \frac{1}{8N_c^2\xi\hat{s}}\left\{\int d\Phi_2(\xi p_1+p_2;p_3,p_4)\left|\mathcal{M}_{\rm LO}(\xi p_1+p_2;p_3,p_4)\right|^2 \right.$$
$$\left. +\int d\Phi_2(p_1+\xi p_2;p_3,p_4)\left|\mathcal{M}_{\rm LO}(p_1+\xi p_2;p_3,p_4)\right|^2\right\}\delta\left[\eta-\frac{1}{2}\log\left(\frac{p_3^+}{p_3^-}\right)\right]\delta\left[\xi-\frac{Q^2}{\hat{s}}\right]. \tag{3.19}$$

The calculation of squared matrix elements and the two-body phase space is straightforward and is similar to eq. (3.5). Note that only LP contributes to eq. (3.5). The LP result of eq. (3.19) is proportional to eq. (3.6), while higher power corrections appear due to $p_1\to\xi p_1$ or $p_2\to\xi p_2$. When combining the two terms in eq. (3.19) NLP corrections cancel each other, while even higher power corrections remain. As a result, up to NLP, eq. (3.19) is given by

$$\left.\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}(\xi\hat{s},\xi,z,\eta,\epsilon,\bar{\mu}^2)}{dzd\eta}\right|^{\rm NLP} = 2\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon,\bar{\mu}^2)\delta(\xi-z). \tag{3.20}$$

We therefore get the subtraction term in eq. (3.17) by combining eqs. (3.18) and (3.20) to be of the following form

$$\int_0^1 d\xi\,\Gamma^{(1)}_{qq}(\xi)\left.\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}(\xi\hat{s},\xi,z,\eta,\epsilon,\bar{\mu}^2)}{dzd\eta}\right|^{\rm NLP}$$
$$=\frac{\alpha_s C_F}{\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\frac{1}{\epsilon}\left[2\left(1-\left.\frac{1}{1-z}\right|_+\right)-\frac{3}{2}\delta(1-z)\right]\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon,\bar{\mu}^2). \tag{3.21}$$

Focusing on the real contribution of eq. (3.15), one immediately sees that the part of the subtraction term in eq. (3.21) not proportional to $\delta(1-z)$ does remove the single pole in eq. (3.15), while the $\delta(1-z)$ term removes the single pole in the virtual contribution. The latter is not discussed explicitly here since we focus on the LL contributions up to NLP at NLO. In the end, we obtain the finite part of the NLO differential cross section with one gluon emission up to NLP LL accuracy, and it reads

$$\left.\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta)}{dzd\eta}\right|^{\rm NLP,\,LL}_{\rm ren} = \frac{\alpha_s}{4\pi}C_F\frac{(4\pi\alpha)^2}{\pi N_c Q^2}\left(1+\tanh^2\eta\right)\left\{\left.\frac{\log(1-z)}{1-z}\right|_+ -\log(1-z)\right\}. \tag{3.22}$$

If we compare eq. (3.22) to eq. (3.7), we note that the dependence on the rapidity η is the same, i.e. it completely factorises from the NLO contribution from emissions. This is in complete analogy with Drell-Yan, cf. in particular eq. (2.34). Even though we used a different definition for the rapidity compared to the one used in section 2, the conclusion is unchanged: the diphoton cross section again takes the form of eq. (2.34). We will show in section 3.2.1 that this factorised structure no longer holds beyond NLP LL, as indeed it did not for Drell-Yan.

We will now attempt to reproduce the result in eq. (3.22) by means of the method of shifted kinematics. We are interested in a cross section that is differential in both the invariant mass and the rapidity of one of the final state photons. The NLO real emission term up to NLP in the soft limit is expressed in terms of the LO squared matrix element, with initial state momenta shifted according to[3] eq. (2.72), integrated over the *full* 3-particle phase space $\int dR_3$, as defined in eq. (B.4):

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta,\epsilon,\bar{\mu}^2)}{dz d\eta}\Bigg|^{\rm NLP}_{\rm real} = \frac{1}{2\hat{s}}\int dR_3\, \overline{|\mathcal{M}|^2_{\rm NLO,NLP}}\,, \tag{3.23}$$

where we keep only terms up to NLP on the right hand side, and where

$$\overline{|\mathcal{M}|^2_{\rm NLO,NLP}} = g_s^2 C_F \frac{2p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}\overline{|\mathcal{M}(p_1+\delta p_1, p_2+\delta p_2)|^2_{\rm LO}}. \tag{3.24}$$

The shift of momenta in eq. (3.24) implies the partonic centre of mass energy shifts as $\hat{s}\to z\hat{s}$. It also induces a shift in the rapidity: using momentum conservation within the Born approximation, the shift gives

$$\begin{aligned}\eta &= \frac{1}{2}\log\left(\frac{p_3^+}{p_3^-}\right) = \frac{1}{2}\log\left(\frac{(p_1+p_2)^+ - p_4^+}{(p_1+p_2)^- - p_4^-}\right)\\ &\to \frac{1}{2}\log\left(\frac{(p_1+\delta p_1+p_2+\delta p_2)^+ - p_4^+}{(p_1+\delta p_1+p_2+\delta p_2)^- - p_4^-}\right)\\ &= \frac{1}{2}\log\left(\frac{(p_1+p_2)^+ - p_4^+ - k^+}{(p_1+p_2)^- - p_4^- - k^-}\right).\end{aligned} \tag{3.25}$$

On the other hand, the delta function in $\int dR_3$ in the phase space eq. (B.4) fixes the rapidity to $\eta = 1/2\log\left(p_3^+/p_3^-\right)$. However, within $\int dR_3$ we need to use momentum conservation for the $2\to 3$ process $q(p_1)+\bar{q}(p_2)\to\gamma(p_3)+\gamma(p_4)+g(k)$, which implies

$$\eta = \frac{1}{2}\log\left(\frac{p_3^+}{p_3^-}\right) = \frac{1}{2}\log\left(\frac{(p_1+p_2)^+ - p_4^+ - k^+}{(p_1+p_2)^- - p_4^- - k^-}\right). \tag{3.26}$$

The two expressions do indeed coincide! Furthermore, upon closer inspection, we see that the rapidity can be expanded for soft k, with the leading power term being equal to its Born value:

$$\begin{aligned}\eta &= \frac{1}{2}\log\left(\frac{(p_1+p_2-p_4)^+}{(p_1+p_2-p_4)^-}\right) + \frac{1}{2}\left(\frac{k^-}{p_3^-} - \frac{k^+}{p_3^+}\right) + \mathcal{O}(k^2)\\ &\equiv \bar{\eta} + \delta\eta + \mathcal{O}\left((1-z)^2\right),\end{aligned} \tag{3.27}$$

where p_1, p_2 and p_4 are now the Born momenta, and where we defined $\bar{\eta}$ to be the rapidity at lowest order, and $\delta\eta \sim \mathcal{O}(1-z)$ its NLP correction. Both the fact that the shifted rapidity coincides with its NLO exact value, and that upon expansion in powers of $1-z$ the first term of the NLO rapidity is given by its Born value are relevant for our discussion. Indeed, we can use eq. (3.27) and expand the shifted squared matrix element around $\bar{\eta}$:

$$|\mathcal{M}(p_1+\delta p_1, p_2+\delta p_2, \bar{\eta}+\delta\eta)|^2_{\rm LO} = |\mathcal{M}(z\hat{s},\bar{\eta})|^2_{\rm LO} + \frac{\partial|\mathcal{M}(p_1,p_2,\bar{\eta})|^2_{\rm LO}}{\partial\bar{\eta}}\delta\eta + \mathcal{O}(\delta\eta^2). \tag{3.28}$$

[3]Note that $p_a\to p_1$ and $p_b\to p_2$, see figures 1 and 2.

Furthermore, following ref. [68], we can expand the Dirac delta function defining the rapidity in the phase space $\int dR_3$ around its Born value, according to eq. (3.27):

$$\delta\left(\eta-\frac{1}{2}\log\left(\frac{p_3^+}{p_3^-}\right)\right)=\delta(\eta-\bar{\eta})+\delta\eta\frac{\partial}{\partial\eta}\delta(\eta-\bar{\eta})+\mathcal{O}(\delta\eta^2). \tag{3.29}$$

Next, we insert both eqs. (3.28) and (3.29) into eq. (3.23). Focusing on the integration over η, we integrate by parts the term involving the derivative of the Dirac delta in eq. (3.29), and arrive at

$$\begin{aligned}&\int d\eta\left[|\mathcal{M}(z\hat{s},\bar{\eta})|^2_{\rm LO}+\frac{\partial|\mathcal{M}(p_1,p_2,\bar{\eta})|^2_{\rm LO}}{\partial\bar{\eta}}\delta\eta+\mathcal{O}(\delta\eta)^2\right]\left[\delta(\eta-\bar{\eta})+\delta\eta\frac{\partial}{\partial\eta}\delta(\eta-\bar{\eta})+\mathcal{O}(\delta\eta^2)\right]\\&=\int d\eta\left[|\mathcal{M}(z\hat{s},\bar{\eta})|^2_{\rm LO}+\delta\eta\left(\frac{\partial|\mathcal{M}(\hat{s},\eta)|^2_{\rm LO}}{\partial\eta}-\frac{\partial|\mathcal{M}(z\hat{s},\eta)|^2_{\rm LO}}{\partial\eta}\right)+\mathcal{O}(\delta\eta^2)\right]\delta(\eta-\bar{\eta}). \end{aligned}\tag{3.30}$$

The difference of the derivatives in the second term of eq. (3.30) is at least $\mathcal{O}(1-z)$. Since it is multiplied by $\delta\eta$, this whole term is beyond NLP accuracy and can therefore be neglected. We can now proceed with the complete phase space integration from eq. (3.23). Using the results in eq. (3.30), it reads

$$\begin{aligned}\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta,\epsilon,\bar{\mu}^2)}{dzd\eta}\bigg|^{\rm NLP}_{\rm real}&=\frac{1}{2\hat{s}}\int dQ^2\,d\Phi_3(p_1+p_2;p_3,p_4,k)\,\delta\left(z-\frac{Q^2}{\hat{s}}\right)\\&\times\delta\left[Q^2-(p_3+p_4)^2\right]\,\delta(\eta-\bar{\eta})\,g_s^2C_F\frac{2p_1\cdot p_2}{(p_1\cdot k)(p_2\cdot k)}\overline{|\mathcal{M}(z\hat{s},\bar{\eta})|^2_{\rm LO}}.\end{aligned}\tag{3.31}$$

The phase space integration involves only the eikonal factor $2p_1\cdot p_2/(p_1\cdot k\,p_2\cdot k)$, which, using the result for the phase space integration of eq. (B.9), gives

$$\begin{aligned}\int dR_3\frac{\hat{s}}{(p_1\cdot k)(p_2\cdot k)}=\int dR_3\frac{2}{k^+k^-}=\int dz\,d\eta\,\frac{1}{32\pi^3}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^{2\epsilon}\frac{1}{\cosh^2\eta}(4\cosh^2\eta)^{\epsilon}\\\times\frac{e^{2\epsilon\gamma_E}\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)}(1-z)^{-1-2\epsilon}(1+\epsilon(1-z)+\ldots),\end{aligned}\tag{3.32}$$

such that eq. (3.31) finally gives

$$\begin{aligned}\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta,\epsilon,\bar{\mu}^2)}{dzd\eta}\bigg|^{\rm NLP}_{\rm real}&=g_s^2C_F\frac{\alpha^2}{2\pi N_c}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^{\epsilon}\frac{e^{2\epsilon\gamma_E}\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)}\frac{1}{z\hat{s}}\left(\frac{\bar{\mu}^2}{z\hat{s}}\right)^{\epsilon}\\&\times\left[(1+\tanh^2\eta)(1-\epsilon)^2-\frac{\epsilon(1-\epsilon)}{\cosh^2\eta}\right](4\cosh^2\eta)^{\epsilon}z(1-z)^{-1-2\epsilon}+\ldots.\end{aligned}\tag{3.33}$$

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The NLO cross section up to NLP LL accuracy is then given by

$$\begin{aligned}\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta,\epsilon,\bar{\mu}^2)}{dzd\eta}\Bigg|^{\rm NLP}_{\rm real} &= \frac{\alpha_s}{4\pi}C_F\frac{4\pi\alpha^2}{N_c z\hat{s}}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^{\epsilon}\left(\frac{\bar{\mu}^2}{z\hat{s}}\right)^{\epsilon}\frac{2e^{2\epsilon\gamma_E}\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1-\epsilon)}\\ &\quad\times(1+\tanh^2\eta)z(1-z)^{-1-2\epsilon}\\ &= K_{\rm NLP}(z,\epsilon)\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(z\hat{s},\eta),\end{aligned} \tag{3.34}$$

which is the same as eq. (3.15). We introduced here again a K-factor, as in eq. (2.77). We note that this cross section is precisely as was advocated in ref. [25], even though the cross section is now double differential, whereas that paper only considered cross sections that are differential in the threshold variable z. The divergences in the form of poles in ϵ can now be removed by adding the virtual contribution and by means of mass factorisation. Starting from eq. (3.17), we perform the same steps to arrive at the desired finite cross section, namely eq. (3.22).

Summarising, from the first calculation, i.e. eq. (3.22), we have already concluded that the differential distribution is of the form of eq. (2.34), i.e. that the dependence on the rapidity variable for the NLP LL contribution is the same as in the LO result. We then noted that there exists another prescription to obtain the NLP NLO contribution, namely the shifted kinematics method of [25], which, before phase space integration, is given by eq. (3.24). This presents a problem however, since the shift in kinematics also affects the rapidity dependence, while we know that this should stay the same as the LO contribution. We could show explicitly that the effect of the shift on the rapidity variable is actually of NNLP accuracy, and therefore does not affect the general NLO-NLP formula eq. (4.17) of [25]. We can hence safely apply this formula and obtain eq. (3.34), which is now also differential in the rapidity. The advantage of rephrasing the derivation in terms of this general NLO-NLP formula is that we can profit from the result of [54] and upgrade the NLO result to a resummed result quite straightforwardly. This will be discussed in detail in section 4.

3.2.1 NLO result beyond NLP

Beyond NLP, the contribution is suppressed by factors of $(1-z)$ in the limit $z\to 1$. A logarithmic term $\log(1-z)$ multiplied by $(1-z)$ appears at N^{2}LP, while the cross section is free of this logarithm starting from N^{3}LP, which can be seen from the splitting kernel function in eq. (3.18). While we have investigated the universal structure up to NLP LL at NLO in section 3.2, it is still of interest to examine contributions beyond NLP at NLO. Due to the simple structure of the NLO squared amplitude in eq. (3.13) and our generalised phase space integration formula eq. (B.9) in the soft limit, we can straightforwardly calculate the NLO cross section up to arbitrary powers of $(1-z)$. In this section we present the full analytic results valid up to N^{3}LP at NLO. Recalling the mass factorisation formula eq. (3.17), the finite NLO differential cross section receives contributions from two parts: the NLO correction and the subtraction term which is a convolution of the splitting kernel function and the LO cross section. Up to N^{3}LP, the LO cross section entering the subtraction term is

$$\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}(\xi\hat{s},\xi,z,\eta,\epsilon,\bar{\mu}^2)}{dzd\eta}\bigg|^{\mathrm{N^3LP}} = 2\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon,\bar{\mu}^2)\delta(\xi-z)$$
$$+\frac{\pi\alpha^2 e^{\epsilon\gamma_E}}{N_c Q^2\Gamma(1-\epsilon)}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}(4\cosh^2\eta)^{\epsilon}\frac{(1-\epsilon)^2}{8\cosh^4\eta}(1-z)^2\left[1+(1-z)\right]$$
$$\times\Big[4(2-\cosh 2\eta)+8\epsilon\cosh 2\eta+2\epsilon^2(\cosh 4\eta-1)\Big]\delta(\xi-z)\,. \tag{3.35}$$

We note that the $\mathrm{N^2LP}$ and $\mathrm{N^3LP}$ contributions have the same structure. Combining eq. (3.35) and eq. (3.18), the subtraction term is given by

$$\int_0^1 d\xi\,\Gamma^{(1)}_{qq}(\xi)\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}(\xi\hat{s},\xi,z,\eta,\bar{\mu}^2)}{dzd\eta} = -\frac{\alpha_s C_F}{\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}$$
$$\times\frac{1}{\epsilon}\left[2\frac{1}{1-z}\bigg|_+ -2+(1-z)+\frac{3}{2}\delta(1-z)\right]\left\{\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon,\bar{\mu}^2)\right.$$
$$+\frac{\pi\alpha^2}{N_c Q^2\Gamma(1-\epsilon)}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}(4\cosh^2\eta)^{\epsilon}\frac{(1-\epsilon)^2 e^{\epsilon\gamma_E}}{16\cosh^4\eta}(1-z)^2\left[1+(1-z)\right]$$
$$\left.\times\Big[4(2-\cosh 2\eta)+8\epsilon\cosh 2\eta+2\epsilon^2(\cosh 4\eta-1)\Big]\right\}. \tag{3.36}$$

Note that the contributions of $\mathrm{N^3LP}$ and beyond in $\Gamma^{(1)}_{qq}(\xi)$ are zero. Up to $\mathrm{N^3LP}$, the NLO real correction is given by

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta,\epsilon,\bar{\mu}^2)}{dzd\eta}\bigg|^{\mathrm{N^3LP}}_{\mathrm{real}} = \frac{\alpha_s C_F}{\pi}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\frac{e^{\epsilon\gamma_E}(1+z^2)\,z^{\epsilon}(1-z)^{-1-2\epsilon}\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}$$
$$\times\left\{\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon,\bar{\mu}^2)+\frac{\pi\alpha^2 e^{\epsilon\gamma_E}}{N_c Q^2\Gamma(1-\epsilon)}\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}(4\cosh^2\eta)^{\epsilon}(1-z)^2\frac{1+(1-z)}{16\cosh^4\eta}\right.$$
$$\left.\times\Big[4(2-\cosh 2\eta)-\epsilon(4+11\cosh 2\eta+4\cosh 4\eta+\cosh 6\eta)+\mathcal{O}(\epsilon^2)\Big]\right\}, \tag{3.37}$$

Up to the chosen prefactor, the $\mathrm{N^2LP}$ and $\mathrm{N^3LP}$ contributions have the same structure. The prefactor is chosen to give the same $1/\epsilon$ divergent terms as eq. (3.18), namely

$$\frac{e^{\epsilon\gamma_E}(1+z^2)\,z^{\epsilon}(1-z)^{-1-2\epsilon}\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} = \frac{1}{\epsilon^2}\delta(1-z)-\frac{1}{\epsilon}\left[2\frac{1}{1-z}\bigg|_+ -2+(1-z)\right]$$
$$+4\frac{\log(1-z)}{1-z}\bigg|_+ -4\log(1-z)+2-\frac{\pi^2}{4}\delta(1-z)$$
$$+(1-z)\left[2\log(1-z)-1\right]+\frac{2}{3}(1-z)^2+\mathcal{O}(\epsilon,(1-z)^3)\,, \tag{3.38}$$

Combining eqs. (3.36) and (3.37), and ignoring for now the Dirac delta function $\delta(1-z)$ terms which will be given together with the NLO virtual corrections in the following, we

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find that the $\mathcal{O}(1/\epsilon)$ pole has been cancelled. The finite part up to N^{3}LP is then given by

$$\begin{aligned}\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta)}{dzd\eta}\bigg|^{\mathrm{N^3LP}}_{\mathrm{ren}} &= \frac{\alpha_s C_F}{\pi}\frac{\pi\alpha^2}{N_c Q^2}\bigg\{4\left(1+\tanh^2\eta\right)\frac{\log(1-z)}{1-z}\bigg|_+ \\ &+2\left(1+\tanh^2\eta\right)[1-2\log(1-z)] \\ &+\frac{1-z}{8\cosh^4\eta}[4(5+\cosh 4\eta)\log(1-z)-14+23\cosh 2\eta+2\cosh 4\eta+\cosh 6\eta] \\ &+\frac{(1-z)^2}{6\cosh^4\eta}[7-\cosh 2\eta+\cosh 4\eta]+\mathcal{O}\left[(1-z)^3\right]\bigg\}. \end{aligned} \tag{3.39}$$

As mentioned before, there is a logarithmic term $\log(1-z)$ at N^{2}LP, while there are no such logarithms at N^{3}LP and beyond. It is clear that the η-structure of the coefficients of $\log(1-z)$ at N^{2}LP and beyond in this double differential cross section is different from those at LP and NLP, such that the LL resummation of threshold logarithms is non-trivial. In order to get the full analytic results valid up to N^{3}LP at NLO, we need to keep track of the terms proportional to $\delta(1-z)$ when combining eqs. (3.36) and (3.37), and add the contribution from the NLO virtual correction. We combine all these contributions proportional to $\delta(1-z)$ in the form

$$\begin{aligned}\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}(Q^2,z,\eta)}{dzd\eta}\bigg|_{\mathrm{ren},\delta(1-z)} &= \frac{\alpha_s C_F}{\pi}\frac{\pi\alpha^2}{N_c Q^2}\delta(1-z)\bigg[\frac{1}{6}\left(2\pi^2-21\right)\left(1+\tanh^2\eta\right) \\ &+\frac{1}{4}(1+\tanh\eta)(5+\tanh\eta)\log\left(\frac{1-\tanh\eta}{2}\right) \\ &+\frac{1}{4}(1-\tanh\eta)(5-\tanh\eta)\log\left(\frac{1+\tanh\eta}{2}\right) \\ &+\frac{1}{4}\left(6-2\tanh\eta-\frac{1}{\cosh^2\eta}\right)\log^2\left(\frac{1-\tanh\eta}{2}\right) \\ &+\frac{1}{4}\left(6+2\tanh\eta-\frac{1}{\cosh^2\eta}\right)\log^2\left(\frac{1+\tanh\eta}{2}\right)\bigg]. \end{aligned} \tag{3.40}$$

The combination of eqs. (3.39) and (3.40) is the analytic expression for diphoton production, up to N^{3}LP, for the finite NLO cross section that is double differential in the diphoton invariant mass and single photon rapidity.

4 Threshold resummation for rapidity distributions

In section 2 and 3 we discussed fixed order computations of the Drell-Yan process and QCD-induced diphoton production. In this section we will resum the leading threshold logarithms at both leading and next-to-leading power. Our method was previously only used to obtain resummed cross sections that are differential in the threshold variable z. In this section we will investigate whether we can also resum cross sections that are additionally differential in the rapidity. We start by discussing the case of diphoton production, where we consider the rapidity of one of the photons, using results of section 3. Subsequently we look at the Drell-Yan process. As a direct application of the diphoton case, we first develop resummation at

NLP LL accuracy for the distribution differential in the rapidity of one of the final state leptons. Finally, using results of section 2, we consider the distribution differential in the total rapidity of the final state (of the off-shell photon), and develop the resummation of large logarithms of $1-z$ at NLP LL accuracy for this case, too.

4.1 NLP resummation of diphoton production

Let us recall that in terms of the soft power expansion the partonic cross section can be written as

$$\hat{\sigma} \sim \frac{1}{2\hat{s}} \left[\int d\Phi_{\rm LP} \, |\mathcal{M}|^2_{\rm LP} + \int d\Phi_{\rm LP} \, |\mathcal{M}|^2_{\rm NLP} + \int d\Phi_{\rm NLP} \, |\mathcal{M}|^2_{\rm LP} + \mathcal{O}({\rm NNLP}) \right], \tag{4.1}$$

i.e. NLP large logarithms arise both from the squared matrix element expanded to NLP and from the expansion of the phase space. The results of section 3 allow us to deal with the LLs in the second term of eq. (4.1). Therefore, as discussed in [54] for the case of invariant mass distribution, we need first to assess the influence of the third terms of eq. (4.1), namely, we need to determine whether LLs can arise from the NLP contribution of the phase space.

To this end, we need the squared matrix element involving n gluons in the final state at LP and the corresponding $(n+2)$-particle phase space measure. For LL accuracy, the former has the simple eikonal form

$$|\mathcal{M}|^2_{{\rm LP},n} = f(\alpha_s, \epsilon, \bar{\mu}^2, \eta) \prod_{i=1}^{n} \frac{p_1 \cdot p_2}{p_1 \cdot k_i p_2 \cdot k_i}, \tag{4.2}$$

where $f(\alpha_s, \epsilon, \bar{\mu}^2, \eta)$ is a general function that collects all factors not involved in the phase space integration. The $(n+2)$-particle phase space measure is given by

$$\begin{aligned} \int d\Phi_{n+2} = & \int dQ^2 \int d\eta \int \frac{d^d p_3}{(2\pi)^{d-1}} \delta_+(p_3^2) \int \frac{d^d p_4}{(2\pi)^{d-1}} \delta_+(p_4^2) \left[\prod_{i=1}^{n} \int \frac{d^d k_i}{(2\pi)^{d-1}} \delta_+(k_i^2) \right] \\ & \times \delta(Q^2 - (p_3+p_4)^2)\, \delta\left(\eta - \frac{1}{2} \log\left(\frac{p_3^+}{p_3^-} \right) \right) (2\pi)^d \delta^{(d)} \left(p_1 + p_2 - p_3 - p_4 - \sum_{i=1}^{n} k_i \right), \end{aligned} \tag{4.3}$$

where we put any $\bar{\mu}$-dependence in the general function f. We can integrate out p_4 using the momentum-conserving delta function. We then have

$$\begin{aligned} \delta(p_4^2) &= \frac{1}{\hat{s}} \delta \left(1 - \frac{2p_3 \cdot (p_1+p_2)}{\hat{s}} - \frac{2\sum_i k_i \cdot (p_1+p_2)}{\hat{s}} + \frac{2p_3 \cdot \sum_i k_i}{\hat{s}} + \frac{2\sum_{i<j} k_i \cdot k_j}{\hat{s}} \right), \\ \delta(Q^2 - (p_3+p_4)^2) &= \frac{1}{\hat{s}} \delta \left(1 - z - \frac{2\sum_i k_i \cdot (p_1+p_2)}{\hat{s}} + \frac{2\sum_{i<j} k_i \cdot k_j}{\hat{s}} \right). \end{aligned} \tag{4.4}$$

In order to perform all these integrals we use a representation of the delta function as a Laplace transform, given by

$$\delta(x) = \int_{-i\infty}^{i\infty} \frac{dT}{2\pi i} e^{Tx}. \tag{4.5}$$

Turning to light-cone coordinates, the integrated squared matrix element reads

$$\begin{aligned}\int d\Phi_{n+2}|\mathcal{M}|^2_{\mathrm{LP},n} =& \frac{2\pi}{\hat{s}^2}\frac{\Omega^{n+1}_{d-2}}{2^{n+1}}\int\frac{dp_3^+dp_3^-}{(2\pi)^{d-1}}(2p_3^+p_3^-)^{\frac{d-4}{2}}\left[\prod_{i=1}^n\int\frac{dk_i^+dk_i^-}{(2\pi)^{d-1}}(2k_i^+k_i^-)^{\frac{d-4}{2}}\frac{1}{k_i^+k_i^-}\right]\\ &\times\int dQ^2d\eta\frac{dT}{2\pi i}\frac{d\tau}{2\pi i}2p_3^+\delta(p_3^+-e^{2\eta}p_3^-)f(\alpha_s,\epsilon,\mu^2,\eta)e^{T(1-z)+\tau}\\ &\times\prod_{i=1}^n e^{-\sqrt{\frac{2}{\hat{s}}}(T+\tau)(k_i^++k_i^-)}e^{-\sqrt{\frac{2}{\hat{s}}}\tau(p_3^++p_3^-)}\\ &\times\left(1+\frac{2(T+\tau)}{\hat{s}}\sum_{i<j}k_i\cdot k_j+\sum_{i=1}^n\frac{2\tau}{\hat{s}}(p_3^+k_i^-+p_3^-k_i^+)+\dots\right),\end{aligned}\tag{4.6}$$

where $\Omega_{2d}=2\pi^d/\Gamma(d)$. The last line originates from expanding the exponential in the Laplace transform of the delta functions in eq. (4.4): this is possible because the terms proportional to $k_i\cdot k_j$ and $(p_3^+k_i^-+p_3^-k_i^+)$ are subleading in the small k_i-expansion with respect to the terms in the third line of eq. (4.6). Terms that involve the perpendicular components, which would contribute at NLP, are odd and vanish upon integration. Subleading terms represented by the ellipsis in the last line of eq. (4.6) will be beyond NLP, as they involve higher powers of the soft momentum k_i. We can then integrate out p_3^+ using the delta function, and integrate over p_3^- and all the $k_i^\pm$. In order to do the T- and τ-integrals, we use that these integrals are of the form of an inverse Laplace transform, viz.

$$\int_{-i\infty}^{i\infty}\frac{dT}{2\pi i}e^{T(1-x)}\left(\frac{1}{T}\right)^\alpha=\frac{(1-x)^{\alpha-1}}{\Gamma(\alpha)}.\tag{4.7}$$

Collecting all the terms, we find that the phase space integral is given by

$$\begin{aligned}\int d\Phi_{n+2}|\mathcal{M}|^2_{\mathrm{LP},n} =& \frac{2\pi}{\hat{s}^2}\frac{\Omega^{n+1}_{d-2}}{2^{n+1}}\frac{2^{n(\frac{d-4}{2})}}{(2\pi)^{(n+1)(d-1)}}2^{\frac{d-2}{2}}\left(\frac{\hat{s}}{2}\right)^{1-(n+1)\epsilon}\frac{\Gamma\left(\frac{d-4}{2}\right)^{2n}}{\Gamma(n(d-4))}\int dQ^2\int d\eta\\ &\times\frac{1}{4\cosh^2\eta}(4\cosh^2\eta)^\epsilon f(\alpha_s,\epsilon,\mu^2,\eta)(1-z)^{n(d-4)-1}z^{d-3}\\ &\times\left(1+\frac{(n-1)(d-4)}{4}(1-z)+\frac{d-2}{2}(1-z)+\mathcal{O}\left((1-z)^2\right)\right)\\ =&\int dzd\eta\tilde{f}(\alpha_s,\epsilon,\mu^2,\eta,\hat{s},n)\frac{\Gamma(\frac{d-4}{2})^{2n}}{\Gamma(n(d-4))}(1-z)^{n(d-4)-1}\\ &\times\left(1-\epsilon\frac{n-3}{2}(1-z)+\mathcal{O}\left((1-z)^2\right)\right).\end{aligned}\tag{4.8}$$

We immediately see that the NLP term does not contribute at LL, since it is multiplied by a factor of ϵ.[4] We conclude that there is no NLP effect due to the phase space measure at leading logarithmic accuracy.

We can now focus on the second term of eq. (4.1). It was already shown in ref. [54] how to perform threshold resummation at NLP, which essentially relies on generalising the

[4]We absorbed some irrelevant factors into a new function $\tilde{f}$.

K-factor at next-to-leading order to a general leading power soft function. At NLO we found

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon)=K_{\rm NLP}(z,\epsilon)\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon), \tag{4.9}$$

up to NLP, where we used the fact that $z\hat{s}=Q^2$. We generalise the K-factor straightforwardly to a leading power soft function, and we find

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon)=z\mathcal{S}_{\rm LP}(z,\epsilon)\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon). \tag{4.10}$$

This can be transformed into Mellin space

$$\int dz\, z^{N-1}\frac{d\hat{\sigma}^{(1)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon)=\mathcal{S}_{\rm LP}(N+1)\frac{d\bar{\sigma}^{(0)}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon). \tag{4.11}$$

The LP soft function at order $\mathcal{O}(\alpha_s)$, expanded consistently in N-space up to next-to-leading power terms in $1/N$, is given by [54]

$$\mathcal{S}_{\rm LP}(N)=\left(\frac{\bar{\mu}^2}{Q^2}\right)^{\epsilon}\frac{2\alpha_sC_F}{\pi}\left[\frac{1}{\epsilon}\left(\log N-\frac{1}{2N}\right)+\log^2 N-\frac{\log N}{N}\right]. \tag{4.12}$$

The LP soft function is known to exponentiate, which can be proven for instance by the replica trick [87]. Setting $\bar{\mu}^2=Q^2$ we find

$$\begin{aligned}\int dz\, z^{N-1}\frac{d\hat{\sigma}_{q\bar{q}}}{dzd\eta}(Q^2,\eta,\epsilon)=&\frac{\pi\alpha^2C_F}{N_cQ^2}(1+\tanh^2\eta)\exp\left[\frac{2\alpha_sC_F}{\pi}\frac{1}{\epsilon}\log N\right]\\&\times\exp\left[\frac{2\alpha_sC_F}{\pi}\log^2 N\right]\left(1+\frac{2\alpha_sC_F}{\pi}\frac{1}{\epsilon}\frac{1}{2N}+\frac{2\alpha_sC_F}{\pi}\frac{\log N}{N}\right).\end{aligned} \tag{4.13}$$

The poles in ϵ are now subtracted by the parton distribution functions by defining

$$q_{\rm LL,NLP}(N,Q^2)=q(N,Q^2)\exp\left[\frac{\alpha_sC_F}{\pi}\frac{\log N}{\epsilon}\right]\left(1+\frac{\alpha_sC_F}{\pi}\frac{1}{\epsilon}\frac{1}{2N}\right). \tag{4.14}$$

and similarly for the antiquark $\bar{q}$. We then find

$$\int_0^1 dz\, z^{N-1}\frac{d\hat{\sigma}_{q\bar{q}}}{dzd\eta}=\frac{\pi\alpha^2}{N_cQ^2}(1+\tanh^2\eta)\exp\left[\frac{2\alpha_sC_F}{\pi}\log^2 N\right]\left(1+\frac{2\alpha_sC_F}{\pi}\frac{\log N}{N}\right), \tag{4.15}$$

which is the resummed cross section for the leading logarithms at both LP and NLP.

4.2 NLP resummation of the Drell-Yan process

As a last application we consider again the Drell-Yan process. In section 2 we have discussed the fixed-order calculation of the distribution differential in the total rapidity of the final state, i.e., the rapidity of the produced off-shell photon. However, in the previous section we have derived the resummed distribution for the diphoton cross section, differential in the rapidity of one of the final state photons. It is then straightforward to apply the same result to Drell-Yan, and derive the resummed distribution differential in the rapidity of one of the two leptons. We will then conclude by considering again the Drell-Yan rapidity distribution differential in the total rapidity of the final state, obtaining the corresponding resummed result at NLP, with LL accuracy.

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4.2.1 Drell-Yan process for a final state lepton-antilepton pair

Using the definition of the rapidity from eq. (3.3), one readily finds at leading order in $d=4$ dimensions that the cross section of the Drell-Yan process, producing an lepton-antilepton pair, is given by

$$\frac{d\hat{\sigma}^{(0)}_{q\bar{q}}}{dz d\eta}(\hat{s},\eta) = \frac{\hat{\sigma}_0(\hat{s})}{2\cosh^2\eta}\delta(1-z)\,, \tag{4.16}$$

where $\hat{\sigma}_0(\hat{s})$ is given by

$$\hat{\sigma}_0(\hat{s}) = \frac{4\pi\alpha^2 e_q^2}{3N_c\hat{s}}\,. \tag{4.17}$$

The squared matrix element is independent of the rapidity η, such that the shift in kinematics only induces a shift in the centre of mass energy $\hat{s}$, namely $\hat{s}\to z\hat{s}$. Using the method of shifted kinematics, one finds a NLO cross section up to next to leading power that reads

$$\begin{aligned}\frac{d\hat{\sigma}^{(1)}_{q\bar{q},\,\mathrm{NLP}}}{dz d\eta}(z,Q^2,\eta,\epsilon) = \frac{\alpha_s}{4\pi}C_F\frac{\hat{\sigma}_0(z\hat{s})}{2\cosh^2\eta}\Bigg[\frac{1}{\epsilon}\left(8-8\left.\frac{1}{1-z}\right|_+\right) \\ +16\left.\frac{\log(1-z)}{1-z}\right|_+ -16\log(1-z)+8\Bigg].\end{aligned} \tag{4.18}$$

Resummation is obtained as usual now, as discussed in the case of diphoton production. After removing the collinear divergences through mass factorisation, we immediately obtain

$$\int_0^1 dz\, z^{N-1}\frac{d\hat{\sigma}_{q\bar{q}}}{dz d\eta}(Q^2,\eta) = \frac{\hat{\sigma}_0(Q^2)}{2\cosh^2\eta}\exp\left[\frac{2\alpha_s C_F}{\pi}\log^2 N\right]\left(1+\frac{2\alpha_s C_F}{\pi}\frac{\log N}{N}\right). \tag{4.19}$$

We were hence again able to obtain a resummed cross section at leading logarithmic accuracy at LP and NLP.

4.2.2 Drell-Yan process for a final state off-shell photon

The derivation of the resummed distribution differential in the total rapidity of the final state easily follows from the results of section 2. First of all, as shown in appendix C, integration of the LP squared matrix element against the NLP phase space does not generate LLs at NLP. Therefore, also in this case we can neglect the third term in eq. (4.1) and focus on the second term. In section 2.3 we found

$$\frac{d\hat{\sigma}^{(1)}_{q\bar{q},\mathrm{NLP}}}{dz dy}(z,y,\epsilon,\bar{\mu}^2) = \hat{\sigma}_0\left(\frac{\delta(y)+\delta(1-y)}{2}+\mathcal{O}(\epsilon)\right)K_{\mathrm{NLP}}(z,\epsilon), \tag{4.20}$$

This result is already in the right form. We can proceed as before and generalise the K-factor to a leading power soft function, and then use the fact that this soft function can be exponentiated. The resummed result then yields, after removing the collinear divergences through mass factorisation,

$$\int dz\, z^{N-1}\frac{d\hat{\sigma}_{q\bar{q}}}{dz dy}(z,y) = \hat{\sigma}_0\left(\frac{\delta(y)+\delta(1-y)}{2}\right)\exp\left[\frac{2\alpha_s C_F}{\pi}\log^2 N\right]\left(1+\frac{2\alpha_s C_F}{\pi}\frac{\log N}{N}\right). \tag{4.21}$$

Note that this resummed cross section has the same structure as for the other processes, even though we are considering a different rapidity variable in this case. Upon converting eq. (4.21) back to z-space, and expanding the result in powers of α_s, it is possible to check that the leading logarithmic contribution up to NLP does agree with the LLs appearing in eqs. (2.68) and (2.69), after PDF renormalisation has been taken into account.

5 Conclusions

In this paper we have extended the treatment of next-to-leading power corrections from single differential cross sections to double differential cross sections, in particular including rapidity dependence for a number of observables. We examined the NLP structure for fixed order results and derived resummed cross sections, to leading logarithmic accuracy at NLP.

In section 2 we gave explicit NNLO partonic cross section expressions for the Drell-Yan process, with dependence on both the threshold and rapidity variable up to NLP accuracy. We showed that at LL accuracy at NLP the rapidity dependence indeed factorises from the dependence on the threshold variable, as was already known for LP contributions. Beyond LL accuracy the z- and y-dependence is more entangled. At NLO, we achieved the same result by using the method of shifted kinematics [25]. We then examined the case of diphoton production differential in the diphoton invariant mass and single-photon rapidity in section 3. We constructed the NLO cross section up to NLP terms by generalising a method involving momentum shifts for single-differential cross sections. We also presented analytical results of the NLO cross section up to N^{3}LP. Generalising the analysis in ref. [54], we then derived in section 4 a result for the resummed cross section for diphoton production for the leading logarithms at NLP, differential in both the threshold variable and the rapidity. Using the same methods, we also resummed the NLP leading logarithms for the Drell-Yan process, both for the lepton pair and single lepton inclusive case.

Extension beyond leading logarithms would clearly be interesting, though our results show that this is very challenging due to the non-factorising structure of rapidity and threshold logarithms for that case. Extension to different differential observables would likewise be desirable, as would the inclusion of off-diagonal channels [56, 63]. Our present results are, we believe, useful to further the insight into and use of next-to-leading power corrections for phenomenological studies.

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A Phase space integrals for fixed-order Drell-Yan

In this appendix we compute the relevant phase space integrals for doubly differential distributions in both the invariant mass and the rapidity that we use for the Drell-Yan process. For the three-body phase space integrals at NNLO, it is hard to obtain complete results at NLP with this approach. The result in this appendix for the three-body phase space integral can only be used to calculate the contributions proportional to C_F^2 up to LL at NLP. However, this approach can be used in other processes, e.g. pure quantum electrodynamics corrections. Note that we have used a more powerful method with a different parametrisation of external momenta to calculate the three-body phase space integrals in section 2.2.

Before we give the results, let us first introduce a pair of light-like vectors

$$n^\mu = \frac{1}{\sqrt{2}}(1,0,0,1)\,, \quad \bar{n}^\mu = \frac{1}{\sqrt{2}}(1,0,0,-1)\,. \tag{A.1}$$

For any momenta p and q, we have

$$\begin{aligned} &p^+ = \bar{n}\cdot p = \frac{1}{\sqrt{2}}(p^0+p^3)\,, \qquad p^- = n\cdot p = \frac{1}{\sqrt{2}}(p^0-p^3)\,, \\ &p\cdot q = p^+q^- + p^-q^+ + \boldsymbol{p}_\perp\cdot\boldsymbol{q}_\perp\,. \end{aligned} \tag{A.2}$$

In the centre of mass frame of initial states, we then have

$$p_a^\mu = \sqrt{\frac{\hat{s}}{2}}\,n^\mu\,, \quad p_b^\mu = \sqrt{\frac{\hat{s}}{2}}\,\bar{n}^\mu\,. \tag{A.3}$$

In this coordinate system the integration measure reads

$$d^dp = dp^+dp^-d^{d-2}\boldsymbol{p}_\perp\,. \tag{A.4}$$

A.1 One-body phase space integral

The one-body phase space integral is trivial, we have

$$\begin{aligned} \int d\Phi_{\gamma^*} &= \int \frac{d^dq}{(2\pi)^{d-1}}\,(2\pi)^d\delta^{(d)}(p_a+p_b-q)\,\delta(q^2-Q^2)\,\delta\left[y-\frac{p_a\cdot q - z\,p_b\cdot q}{(1-z)(p_a\cdot q+p_b\cdot q)}\right] \\ &= \frac{2\pi}{Q^2}\,\delta\,(1-z)\,\delta\left(y-\frac{1}{2}\right)\,. \end{aligned} \tag{A.5}$$

A.2 Two-body phase space integral

We define the two-body phase space integral in both invariant mass and rapidity y as

$$\begin{aligned} \int d\Phi_{\gamma^* g} = \left(\frac{\bar{\mu}^2 e^{\gamma_E}}{4\pi}\right)^\epsilon \int \frac{d^dq}{(2\pi)^{d-1}}\frac{d^dk}{(2\pi)^{d-1}}\,(2\pi)^d\,\delta^{(d)}(p_a+p_b-q-k)\,\delta_+(k^2)\,\delta(q^2-Q^2) \\ \times\,\delta\left[y-\frac{p_a\cdot q - z\,p_b\cdot q}{(1-z)(p_a\cdot q+p_b\cdot q)}\right], \end{aligned} \tag{A.6}$$

and $\delta_+\left(k^2\right)=\delta\left(k^2\right)\theta\left(k^{(0)}\right)$. We first use the momentum-conserving delta function to integrate out q. We obtain

$$\delta(q^2-Q^2)=\frac{1}{\hat{s}}\delta\left[(1-z)-\frac{2}{\hat{s}}(p_a+p_b)\cdot k\right], \tag{A.7}$$

$$\delta\left[y-\frac{p_a\cdot q-z\,p_b\cdot q}{(1-z)(p_a\cdot q+p_b\cdot q)}\right]=(1-z)\,\delta\left[(1-z)(1-y)-\frac{2}{\hat{s}}p_a\cdot k\right], \tag{A.8}$$

where we have used eq. (A.7) to remove $p_b\cdot k$ in eq. (A.8). When we consider NLO real corrections and NNLO real-virtual corrections up to NLP, the squared amplitudes depends on k in the following general form

$$\frac{\hat{s}^{\alpha+\beta}}{(2p_a\cdot k)^\alpha(2p_b\cdot k)^\beta}=\left(\sqrt{\frac{\hat{s}}{2}}\right)^{\alpha+\beta}\frac{1}{(k^-)^\alpha(k^+)^\beta}. \tag{A.9}$$

After inserting eq. (A.7) and eq. (A.8) into eq. (A.6) and combining it with eq. (A.9), we can perform the integration straightforwardly, such that we have

$$\int d\Phi_{\gamma^* g}\frac{\hat{s}^{\alpha+\beta}}{(2p_a\cdot k)^\alpha(2p_b\cdot k)^\beta}=\frac{1}{8\pi}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^\epsilon\frac{e^{\epsilon\gamma_E}(1-y)^{-\epsilon-\alpha}y^{-\epsilon-\beta}}{\Gamma(1-\epsilon)}(1-z)^{1-2\epsilon-\alpha-\beta}. \tag{A.10}$$

A.3 Three-body phase space integral

We give an alternative approach to calculate the three-body phase space integral, instead of the novel method of parametrisation of external momenta used in section 2.2. The three-body phase space integral for distributions differential in both the invariant mass and rapidity y is defined as

$$\begin{aligned}\int d\Phi_{\gamma^*(gg+q\bar{q})}=&\left(\frac{\bar{\mu}^2e^{\gamma_E}}{4\pi}\right)^{2\epsilon}\int\frac{d^dq}{(2\pi)^{d-1}}\frac{d^dk_1}{(2\pi)^{d-1}}\frac{d^dk_2}{(2\pi)^{d-1}}(2\pi)^d\,\delta^{(d)}(p_a+p_b-q-k_1-k_2)\\&\times\delta_+(k_1^2)\,\delta_+(k_2^2)\,\delta(q^2-Q^2)\,\delta\left[y-\frac{p_a\cdot q-z\,p_b\cdot q}{(1-z)(p_a\cdot q+p_b\cdot q)}\right].\end{aligned} \tag{A.11}$$

We integrate out q by using the momentum-conserving delta function, and obtain

$$\delta(q^2-Q^2)=\frac{1}{\hat{s}}\delta\left[(1-z)-\frac{2}{\hat{s}}(p_a+p_b)\cdot(k_1+k_2)+\frac{2}{\hat{s}}k_1\cdot k_2\right], \tag{A.12}$$

$$\begin{aligned}\delta\left[y-\frac{p_a\cdot q-z\,p_b\cdot q}{(1-z)(p_a\cdot q+p_b\cdot q)}\right]=&(1-z)\left[(1+z)-\frac{2}{\hat{s}}k_1\cdot k_2\right]\\&\times\delta\left[(1+z)(1-z)(1-y)-\frac{2}{\hat{s}}(1+z)p_a\cdot(k_1+k_2)+\frac{2}{\hat{s}}(y+z-yz)k_1\cdot k_2\right],\end{aligned} \tag{A.13}$$

where we have used eq. (A.12) to remove $p_b\cdot k_1$ in eq. (A.13). Note that $k_1\cdot k_2$ is a power suppressed term in eq. (A.12). We apply eq. (4.5) to eq. (A.12) and expand in powers of T, which yields

$$\begin{aligned}\delta(q^2-Q^2)=&\frac{1}{\hat{s}}\int\frac{dT}{2\pi i}\exp\left\{T\left[(1-z)-\frac{2}{\hat{s}}(p_a+p_b)\cdot(k_1+k_2)\right]\right\}\\&\times\left[1+T\frac{2}{\hat{s}}k_1\cdot k_2+\mathcal{O}\left(T^2\right)\right].\end{aligned} \tag{A.14}$$

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The term $k_1\cdot k_2$ is also power suppressed in eq. (A.13). We apply eq. (4.5) to eq. (A.13) and expand in powers of τ, which yields

$$\begin{aligned}\delta\left[y-\frac{p_a\cdot q-z\,p_b\cdot q}{(1-z)(p_a\cdot q+p_b\cdot q)}\right]=&(1-z)\left[(1+z)-\frac{2}{\hat{s}}k_1\cdot k_2\right]\\&\times\int\frac{d\tau}{2\pi i}\exp\left\{\tau\left[(1+z)(1-z)(1-y)-\frac{2}{\hat{s}}(1+z)p_a\cdot(k_1+k_2)\right]\right\}\\&\times\left[1+\tau\frac{2}{\hat{s}}(y+z-yz)k_1\cdot k_2+\mathcal{O}\left(\tau^2\right)\right],\end{aligned}\tag{A.15}$$

We now substitute eq. (A.14) and eq. (A.15) into eq. (A.11) and find that the phase space measure depends on $k_1\cdot k_2=k_1^-k_2^++k_1^+k_2^-+k_{1\perp}\cdot k_{2\perp}$ linearly at NLP.

If we only consider the contributions proportional to C_F^2 up to NLP LL in eq. (2.55), the squared amplitudes do not involve the term $k_1\cdot k_2$. As a result, the $k_{1\perp}\cdot k_{2\perp}$ term does not contribute by symmetry. The remaining integrations over k_1^+, k_1^-, k_2^+ and k_2^-, as well as the inverse Laplace transformations over T and τ, are straightforward. Up to LL, at LP, we have

$$\begin{aligned}\int d\Phi_{\gamma^*(gg+q\bar{q})}\frac{\hat{s}^2}{(2p_a\cdot k_1)(2p_a\cdot k_2)(2p_b\cdot k_1)(2p_b\cdot k_2)}=&\frac{1}{32\pi^3\hat{s}}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^{2\epsilon}\frac{e^{2\epsilon\gamma_E}\Gamma^2(-\epsilon)}{\Gamma^2(1-2\epsilon)}\\&\times y^{-1-2\epsilon}(1-y)^{-1-2\epsilon}(1-z)^{-1-4\epsilon},\end{aligned}\tag{A.16}$$

and at NLP we have

$$\begin{aligned}\int d\Phi_{\gamma^*(gg+q\bar{q})}\frac{\hat{s}(2p_a\cdot k_1+2p_a\cdot k_2+2p_b\cdot k_1+2p_b\cdot k_2)}{(2p_a\cdot k_1)(2p_a\cdot k_2)(2p_b\cdot k_1)(2p_b\cdot k_2)}=&\frac{1}{32\pi^3\hat{s}}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^{2\epsilon}\frac{e^{2\epsilon\gamma_E}\Gamma^2(-\epsilon)}{\Gamma^2(1-2\epsilon)}\\&\times y^{-1-2\epsilon}(1-y)^{-1-2\epsilon}(1-z)^{-4\epsilon}.\end{aligned}\tag{A.17}$$

We have checked that the above results are consistent with the corresponding ones given by using the method in section 2.2.

B Phase space integral for fixed-order diphoton production

In section 3 we calculate the diphoton rapidity distribution at LO and NLO, for which we need respectively the two- and three-particle phase space.

B.1 Two-body phase space integral

The two-body phase space is defined as follows:

$$\int dR_2=\int dQ^2\,d\eta\,d\Phi_2(p_1+p_2;p_3,p_4)\,\delta\left[Q^2-(p_3+p_4)^2\right]\,\delta\left[\eta-\frac{1}{2}\log\left(\frac{p_3^+}{p_3^-}\right)\right],\tag{B.1}$$

where

$$\begin{aligned}\int d\Phi_2(p_1+p_2;p_3,p_4)=&\left(\frac{\bar{\mu}^2e^{\gamma_E}}{4\pi}\right)^{\epsilon}\int\frac{d^dp_3}{(2\pi)^{d-1}}\frac{d^dp_4}{(2\pi)^{d-1}}\\&\times\delta_+\left(p_3^2\right)\delta_+\left(p_4^2\right)(2\pi)^d\delta^{(d)}(p_1+p_2-p_3-p_4),\end{aligned}\tag{B.2}$$

and the decomposition into light-cone momentum components $p_{i\pm}$ follows the definition in eq. (A.2). The two-body phase space is easy to evaluate, and one gets

$$\int dR_2 = \frac{e^{\epsilon\gamma_E}}{4\pi\Gamma(1-\epsilon)}\left(\frac{\bar{\mu}^2}{\hat{s}}\right)^{\epsilon}\int dz\, d\eta\, (2\cosh\eta)^{-2+2\epsilon}\,\delta(1-z). \tag{B.3}$$

B.2 Three-body phase space integral

The three-body phase space reads

$$\int dR_3 = \int dQ^2\, d\eta\, d\Phi_3(p_1+p_2;p_3,p_4,k)\,\delta\left[Q^2-(p_3+p_4)^2\right]\,\delta\left[\eta-\frac{1}{2}\log\left(\frac{p_3^+}{p_3^-}\right)\right], \tag{B.4}$$

where

$$\begin{aligned}\int d\Phi_3(p_1+p_2;p_3,p_4,k) = \left(\frac{\bar{\mu}^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon}\int \frac{d^d p_3}{(2\pi)^{d-1}}\frac{d^d p_4}{(2\pi)^{d-1}}\frac{d^d k}{(2\pi)^{d-1}}\,\delta_+(p_3^2)\,\delta_+(p_4^2)\,\delta_+(k^2)\\ \times (2\pi)^d\delta^{(d)}(p_1+p_2-p_3-p_4-k)\,.\end{aligned} \tag{B.5}$$

We start by performing the integration over p_4 with the momentum-conserving delta function, obtaining

$$\begin{aligned}\int dR_3 &= (2\pi)\left(\frac{\bar{\mu}^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon}\int dQ^2\, d\eta\, \frac{dp_3^+ dp_3^-\, d^{d-2}\boldsymbol{p}_{3,\perp}}{(2\pi)^{d-1}}\,\frac{dk^+ dk^-\, d^{d-2}\boldsymbol{k}_\perp}{(2\pi)^{d-1}}\\ &\quad\times \delta_+\left(2p_3^+p_3^- - \boldsymbol{p}_{3,\perp}^2\right)\,\delta_+\left(2k^+k^- - \boldsymbol{k}_\perp^2\right)\,\delta\left[\hat{s}-2p_3\cdot(p_1+p_2)-2k\cdot(p_1+p_2)+2p_3\cdot k\right]\\ &\quad\times\delta\left[Q^2-\hat{s}+2k\cdot(p_1+p_2)\right]\,\delta\left[\eta-\frac{1}{2}\log\left(\frac{p_3^+}{p_3^-}\right)\right]\\ &= \left(\frac{\bar{\mu}^2 e^{\gamma_E}}{4\pi}\right)^{2\epsilon}\frac{\Omega_{d-2}\Omega_{d-3}}{2(2\pi)^{2d-3}\hat{s}}\int dz d\eta\, dp_3^+ dp_3^-\, dp_{3,T}^2\, dk^+ dk^- dk_T^2\, d\cos\alpha\, p_3^+\left(p_{3,T}^2\right)^{\frac{d-4}{2}}\left(k_T^2\right)^{\frac{d-4}{2}}\\ &\quad\times\left(1-\cos^2\alpha\right)^{\frac{d-5}{2}}\,\delta_+\left(2p_3^+p_3^- - p_{3,T}^2\right)\,\delta_+\left(2k^+k^- - k_T^2\right)\,\delta\left[(1-z)-\sqrt{\frac{2}{\hat{s}}}(k^+ + k^-)\right]\\ &\quad\times\delta\left(p_3^+ - p_3^- e^{2\eta}\right)\,\delta\left[z-\sqrt{\frac{2}{\hat{s}}}(p_3^+ + p_3^-)+\frac{2}{\hat{s}}(p_3^+k^- + p_3^-k^+ - p_{3,T}k_T\cos\alpha)\right],\end{aligned} \tag{B.6}$$

where $\Omega_{2d} = 2\pi^d/\Gamma(d)$. Applying eq. (4.5) to the last delta function, we can integrate out the angle α by using

$$\begin{aligned}&\int_{-1}^{1} d\cos\alpha\left(1-\cos^2\alpha\right)^{\frac{-1-2\epsilon}{2}}\exp\left(-\frac{2}{\hat{s}}T p_{3,T}k_T\cos\alpha\right)\\ &\quad= \frac{4^\epsilon\pi\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)}\,{}_0F_1\left(1-\epsilon,\frac{T^2 p_{3,T}^2 k_T^2}{\hat{s}^2}\right)\\ &\quad= \frac{4^\epsilon\pi\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\sum_{j=0}^{\infty}\frac{1}{\Gamma(1-\epsilon+j)}\left(\frac{T^2 p_{3,T}^2 k_T^2}{\hat{s}^2}\right)^j\frac{1}{j!}\,,\end{aligned} \tag{B.7}$$

where we have used the series representation of the hypergeometric function ${}_0F_1(a,z)$ in the last line. Now, we have

$$\int dR_3 = \frac{\bar{\mu}^{4\epsilon} e^{2\epsilon\gamma_E}}{(4\pi)^2\pi\hat{s}\Gamma(1-\epsilon)} \int dz d\eta \sum_{j=0}^{\infty} \int \frac{dT}{2\pi i} \frac{T^{2j}}{\Gamma(1-\epsilon+j)\hat{s}^{2j} j!} \int dp_3^+ dp_3^- dp_{3,T}^2 dk^+ dk^- dk_T^2 \, p_3^+$$
$$\times \left(p_{3,T}^2 k_T^2\right)^{j-\epsilon} \delta_+\left(2p_3^+ p_3^- - p_{3,T}^2\right) \delta_+\left(2k^+k^- - k_T^2\right) \delta\left[(1-z) - \sqrt{\frac{2}{\hat{s}}}(k^+ + k^-)\right]$$
$$\times \delta\left(p_3^+ - p_3^- e^{2\eta}\right) \exp\left\{T\left[z - \sqrt{\frac{2}{\hat{s}}}(p_3^+ + p_3^-) + \frac{2}{\hat{s}}(p_3^+ k^- + p_3^- k^+)\right]\right\}. \tag{B.8}$$

Combining the above $\int dR_3$ with the squared amplitudes in eq. (3.13), we use the four delta functions to integrate over $p_3^+, p_{3,T}, k^+$ and k_T. The integration over p_3^- and k^- and the inverse Laplace transformation would be straightforward. Finally, we have

$$\int dR_3 (k^+)^\alpha (k^-)^\beta (p_3^+)^\gamma (p_3^-)^\kappa = \bar{\mu}^{4\epsilon} \int dz d\eta \frac{2^{-6-\frac{1}{2}(\alpha+\beta+\gamma+\kappa)} z^{1+\gamma+\kappa-2\epsilon}}{\pi^3 \hat{s}^{-1+2\epsilon-\frac{1}{2}(\alpha+\beta+\gamma+\kappa)}}$$
$$\times \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} (1-z)^{1+\alpha+\beta-2\epsilon+2j+n} (e^{2\eta})^{1+\gamma-\epsilon+j} (e^{2\eta}-1)^n (e^{2\eta}+z)^{-2-\gamma-\kappa+2\epsilon-2j-n}$$
$$\times \frac{e^{2\epsilon\gamma_E}\Gamma(1+\alpha-\epsilon+j)\Gamma(1+\beta-\epsilon+j+n)\Gamma(2+\gamma+\kappa-2\epsilon+2j+n)}{j!n!\Gamma(1-\epsilon)\Gamma(2+\gamma+\kappa-2\epsilon)\Gamma(1-\epsilon+j)\Gamma(2+\alpha+\beta-2\epsilon+2j+n)}. \tag{B.9}$$

Note that this result is valid up to arbitrary power of $1-z$. Only the $j=0, n=0$ and $j=0, n=1$ terms are necessary at NLP because $\alpha+\beta \geq -2$ in the NLO squared amplitudes of eq. (3.13).

C NLP phase space contribution for Drell-Yan

In this appendix we discuss the NLP contribution of the phase space for the Drell-Yan production of a virtual photon, i.e. $q(p_a)\bar{q}(p_b) \to \gamma^*(q)$. This result is needed in section 4.2.2. The $(n+1)$-particle phase space integral is given by

$$\int d\Phi_{n+1} = \int \frac{d^d q}{(2\pi)^{d-1}} \delta_+(q^2 - Q^2) \int \prod_{i=1}^{n} \left[\frac{d^d k_i}{(2\pi)^{d-1}} \delta_+(k_i^2)\right]$$
$$\times \int d\eta \, \delta\left(\eta - \frac{1}{2}\log\left(\frac{q^+}{q^-}\right)\right) (2\pi)^d \delta^{(d)}\left(p_a + p_b - q - \sum_{i=1}^{n} k_i\right)$$
$$= \frac{2\pi}{\hat{s}} \int \prod_{i=1}^{n} \left[\frac{d^d k_i}{(2\pi)^{d-1}} \delta_+(k_i^2)\right] \delta\left(1 - z - \sqrt{\frac{2}{\hat{s}}} \sum_i (k_i^+ + k_i^-) + \frac{2}{\hat{s}} \sum_{i<j} k_i \cdot k_j\right)$$
$$\times \int d\eta \, \delta\left(\eta - \frac{1}{2}\log\left(\frac{\sqrt{\hat{s}/2} - \sum_i k_i^+}{\sqrt{\hat{s}/2} - \sum_i k_i^-}\right)\right), \tag{C.1}$$

where we used the rapidity variable η for the moment, and that $(p_a+p_b)^+=(p_a+p_b)^-=\sqrt{\hat{s}/2}$. We can manipulate the delta function of the rapidity η as follows:

$$\delta\left(\eta-\frac{1}{2}\log\left(\frac{\sqrt{\hat{s}/2}-\sum_i k_i^+}{\sqrt{\hat{s}/2}-\sum_i k_i^-}\right)\right)=2\left(\sqrt{\frac{\hat{s}}{2}}-\sum_i k_i^+\right) \\ \times\delta\left(\sum_i k_i^+-\sqrt{\frac{\hat{s}}{2}}-e^{2\eta}\left(\sum_i k_i^--\sqrt{\frac{\hat{s}}{2}}\right)\right), \tag{C.2}$$

likewise as was done in the analogous derivation for diphoton production, where we had p_3^+ and p_3^- as inputs of the logarithm. As was done in section 4.2.2, we introduce the rapidity variable y. It is related to η via

$$y=\frac{1-e^{2\eta}z}{(e^{2\eta}+1)(1-z)}. \tag{C.3}$$

Using the expression for the squared matrix element at leading power from eq. (4.2), we write

$$\int d\Phi_{n+1}|\mathcal{M}|^2_{\mathrm{LP},n}=\frac{4\pi}{\hat{s}}\frac{\Omega^n_{d-2}}{2^n}\int\left[\prod_{i=1}^n\frac{dk_i^+dk_i^-}{(2\pi)^{d-1}}(2k_i^+k_i^-)^{-\epsilon}\frac{1}{k_i^+k_i^-}\right] \\ \times\int dy\left(-\frac{2(1-z)}{1+z}\right)\frac{dT}{2\pi i}\frac{d\tau}{2\pi i}\left(\sqrt{\frac{\hat{s}}{2}}-\sum_{i=1}^n k_i^+\right)e^{T(1-z)}e^{\tau\sqrt{\hat{s}/2}(e^{2\eta(y)}-1)}f(\alpha_s,\epsilon,y,\bar{\mu}^2) \\ \times\prod_{i=1}^n e^{k_i^+(-\sqrt{2/\hat{s}}T+\tau)}e^{k_i^-(-\sqrt{2/\hat{s}}T-e^{2\eta(y)}\tau)}\left(1+\frac{2T}{\hat{s}}\sum_{i<j}(k_i^+k_j^-+k_i^-k_j^+)+\mathcal{O}(T^2)\right). \tag{C.4}$$

For the moment, we will compactly write $\eta(y)$ instead of the expression in eq. (C.3). Using the expression for the squared matrix element at leading power from eq. (4.2), we write

$$\int d\Phi_{n+1}|\mathcal{M}|^2_{\mathrm{LP},n}=\frac{4\pi}{\hat{s}}\frac{\Omega^n_{d-2}}{2^n}\int\left[\prod_{i=1}^n\frac{dk_i^+dk_i^-}{(2\pi)^{d-1}}(2k_i^+k_i^-)^{-\epsilon}\frac{1}{k_i^+k_i^-}\right] \\ \times\int dy\left(-\frac{2(1-z)}{1+z}\right)\frac{dT}{2\pi i}\frac{d\tau}{2\pi i}\left(\sqrt{\frac{\hat{s}}{2}}-\sum_{i=1}^n k_i^+\right)e^{T(1-z)}e^{\tau\sqrt{\hat{s}/2}(e^{2\eta(y)}-1)}f(\alpha_s,\epsilon,v,\mu^2) \\ \times\prod_{i=1}^n e^{k_i^+(-\sqrt{2/\hat{s}}T+\tau)}e^{k_i^-(-\sqrt{2/\hat{s}}T-e^{2\eta}\tau)}\left(1+\frac{2T}{\hat{s}}\sum_{i<j}(k_i^+k_j^-+k_i^-k_j^+)+\mathcal{O}(T^2)\right). \tag{C.5}$$

The first contribution we will evaluate is given by

$$I_1\equiv\int\left[\prod_{i=1}^n dk_i^+dk_i^-(k_i^+k_i^-)^{\frac{d-6}{2}}\right]\int dy\left(-\frac{2(1-z)}{1+z}\right)\frac{dT}{2\pi i}\frac{d\tau}{2\pi i}\sqrt{\frac{\hat{s}}{2}}e^{T(1-z)}e^{\tau\sqrt{\hat{s}/2}(e^{2\eta(y)}-1)} \\ \times\prod_{i=1}^n e^{k_i^+(-\sqrt{2/\hat{s}}T+\tau)}e^{k_i^-(-\sqrt{2/\hat{s}}T-e^{2\eta(y)}\tau)}. \tag{C.6}$$

We then integrate over all the momenta $\{k_i\}_{i=1}^n$, which all become Gamma functions, i.e.

$$
\begin{aligned}
I_1 = &\left(-\frac{2(1-z)}{1+z}\right)\sqrt{\frac{\hat{s}}{2}}\int dy \frac{dT}{2\pi i}\frac{d\tau}{2\pi i}\, e^{T(1-z)} e^{\tau\sqrt{\hat{s}/2}(e^{2\eta(y)}-1)} \\
&\times\left(\frac{1}{\sqrt{\frac{2}{\hat{s}}}T-\tau}\right)^{n\left(\frac{d-4}{2}\right)}\left(\frac{1}{\sqrt{\frac{2}{\hat{s}}}T+e^{2\eta(y)}\tau}\right)^{n\left(\frac{d-4}{2}\right)}\Gamma\left(\frac{d-4}{2}\right)^{2n}.
\end{aligned}
\tag{C.7}
$$

Upon subsequent coordinate transformations

$$
\tilde{T} = \sqrt{\frac{2}{\hat{s}}}T-\tau, \qquad \tilde{\tau} = \tilde{T}+\tau(1+e^{2\eta(y)}),
$$

one finds the following expression for I_1, namely

$$
I_1 = -2^{-1+n\epsilon}\hat{s}^{-n\epsilon}\frac{\Gamma^{2n}(-\epsilon)}{\Gamma^2(-n\epsilon)}\int dy\, y^{-1-n\epsilon}(1-y)^{-1-n\epsilon}\Big(1+y(1-z)+\mathcal{O}((1-z)^2)\Big), \tag{C.8}
$$

where we now explicitly used eq. (C.3) for the rapidity y. Using similar methods, the second contribution that we calculate reads

$$
\begin{aligned}
I_2 \equiv &\int\left[\prod_{i=1}^n dk_i^+ dk_i^- (k_i^+k_i^-)^{\frac{d-6}{2}}\right]\int dy\left(-\frac{2(1-z)}{1+z}\right)\frac{dT}{2\pi i}\frac{d\tau}{2\pi i}\left(-\sum_{i=1}^n k_i^+\right) \\
&\times e^{T(1-z)} e^{\tau\sqrt{\hat{s}/2}(e^{2\eta(y)}-1)}\prod_{i=1}^n e^{k_i^+(-\sqrt{2/\hat{s}}T+\tau)} e^{k_i^-(-\sqrt{2/\hat{s}}T-e^{2\eta(y)}\tau)} \\
= &\left(\frac{2(1-z)}{1+z}\right)\int dy \frac{dT}{2\pi i}\frac{d\tau}{2\pi i} e^{T(1-z)} e^{\tau\sqrt{\hat{s}/2}(e^{2\eta(y)}-1)} \\
&\times n\left[\int dk^+ dk^- (k^+)^{\frac{d-4}{2}}(k^-)^{\frac{d-6}{2}} e^{k^+(-\sqrt{2/\hat{s}}T+\tau)} e^{k^-(-\sqrt{2/\hat{s}}T-e^{2\eta(y)}\tau)}\right] \\
&\times\left[\int dk^+ dk^- (k^+k^-)^{\frac{d-6}{2}} e^{k^+(-\sqrt{2/\hat{s}}T+\tau)} e^{k^-(-\sqrt{2/\hat{s}}T-e^{2\eta(y)}\tau)}\right]^{n-1} \\
= &\,2^{-1+n\epsilon}\hat{s}^{-n\epsilon}\frac{\Gamma^{2n}(-\epsilon)}{\Gamma^2(-n\epsilon)}\int dy\, y^{-1-n\epsilon}(1-y)^{-1-n\epsilon}\Big(y(1-z)^{-2n\epsilon}+\mathcal{O}((1-z))\Big),
\end{aligned}
\tag{C.9}
$$

where we rewrote the expression in such a way that it has a recognisable common prefactor with I_1. We now turn our attention to the third integral, which reads

$$
\begin{aligned}
I_3 \equiv &\int\left[\prod_{i=1}^n dk_i^+ dk_i^- (k_i^+k_i^-)^{\frac{d-6}{2}}\right]\int dy\left(-\frac{2(1-z)}{1+z}\right)\frac{dT}{2\pi i}\frac{d\tau}{2\pi i}\sqrt{\frac{\hat{s}}{2}} e^{T(1-z)} e^{\tau\sqrt{\hat{s}/2}(e^{2\eta(y)}-1)} \\
&\times\prod_{i=1}^n e^{k_i^+(-\sqrt{2/\hat{s}}T+\tau)} e^{k_i^-(-\sqrt{2/\hat{s}}T-e^{2\eta(y)}\tau)}\left(\frac{2T}{\hat{s}}\sum_{i<j}(k_i^+k_j^-+k_i^-k_j^+)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(-\frac{2(1-z)}{1+z}\right)\sqrt{\frac{2}{\hat{s}}}\int dy \frac{dT}{2\pi i}\frac{d\tau}{2\pi i} e^{T(1-z)} e^{\tau\sqrt{\hat{s}/2}(e^{2\eta(y)}-1)T} \\
&\quad \times n(n-1)\left[\int dk^+ dk^- (k^+)^{\frac{d-4}{2}} (k^-)^{\frac{d-6}{2}} e^{k^+(-\sqrt{2/\hat{s}}T+\tau)} e^{k^-(-\sqrt{2/\hat{s}}T-e^{2\eta(y)}\tau)}\right] \\
&\quad \times \left[\int dk^+ dk^- (k^+)^{\frac{d-6}{2}} (k^-)^{\frac{d-4}{2}} e^{k^+(-\sqrt{2/\hat{s}}T+\tau)} e^{k^-(-\sqrt{2/\hat{s}}T-e^{2\eta(y)}\tau)}\right] \\
&\quad \times \left[\int dk^+ dk^- (k^+k^-)^{\frac{d-6}{2}} e^{k^+(-\sqrt{2/\hat{s}}T+\tau)} e^{k^-(-\sqrt{2/\hat{s}}T-e^{2\eta(y)}\tau)}\right]^{n-2} \\
&= 2^{-1+n\epsilon}\hat{s}^{-n\epsilon}\frac{\Gamma^{2n}(-\epsilon)}{\Gamma^2(-n\epsilon)}\frac{(n-1)}{2}\epsilon\int dy\, y^{-1-n\epsilon}(1-y)^{-1-n\epsilon}\Big((1-z)^{-2n\epsilon}+\mathcal{O}((1-z))\Big).
\end{aligned}
\tag{C.10}
$$

We again extracted a common prefactor with I_1 and I_2 and we observe that the expression is proportional to ϵ, which means that this expression is subleading in the logarithmic expansion.

We potentially have a fourth integral that contributes and it is given by

$$
\begin{aligned}
I_4 \equiv &\int \left[\prod_{i=1}^{n} dk_i^+ dk_i^- (k_i^+ k_i^-)^{\frac{d-6}{2}}\right] \int dy \left(-\frac{2(1-z)}{1+z}\right)\frac{dT}{2\pi i}\frac{d\tau}{2\pi i}\left(-\sum_{i=1}^{n} k_i^+\right) \\
&\times e^{T(1-z)} e^{\tau\sqrt{\hat{s}/2}(e^{2\eta(y)}-1)} \prod_{i=1}^{n} e^{k_i^+(-\sqrt{2/\hat{s}}T+\tau)} e^{k_i^-(-\sqrt{2/\hat{s}}T-e^{2\eta(y)}\tau)} \left(\frac{2T}{\hat{s}}\sum_{i<j}(k_i^+k_j^-+k_i^-k_j^+)\right).
\end{aligned}
\tag{C.11}
$$

Looking at the delta function in eq. (C.1), which reads

$$
\delta\left(1-z-\sqrt{\frac{2}{\hat{s}}}\sum_i (k_i^+ + k_i^-) + \frac{2}{\hat{s}}\sum_{i<j} k_i\cdot k_j\right), \tag{C.12}
$$

we argue that both $\sum_i k_i^+$ and $\sum_{i<j} k_i\cdot k_j$ are at least of order $\mathcal{O}(1-z)$, which means that the result from integral I_4 is of order $\mathcal{O}\left((1-z)^2\right)$, and hence that it does not contribute at the required accuracy.[5] Adding the three NLP contributions, one obtains

$$
\begin{aligned}
I_1+I_2+I_3 = &-2^{-1+n\epsilon}\hat{s}^{-n\epsilon}\frac{\Gamma^{2n}(-\epsilon)}{\Gamma^2(-n\epsilon)}\int dy\, y^{-1-n\epsilon}(1-y)^{-1-n\epsilon}(1-z)^{-1-2n\epsilon} \\
&\times\left(1+y(1-z)-y(1-z)-\epsilon\frac{(n-1)}{2}(1-z)+\mathcal{O}\left((1-z)^2\right)\right).
\end{aligned}
\tag{C.13}
$$

We immediately see that the NLP contribution of the phase space integral vanishes for the leading logarithms, which is the reason that we can resum the cross section in section 4.2.2 using the steps performed there.

[5] Upon carrying out the calculation, we indeed found a contribution starting from NNLP.

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