# Supplementary Material:
# Lift force in odd compressible fluids


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## I. INCOMPRESSIBLE DRAG FORCE

In this section we explicitly compute the response matrix in two simple incompressible scenarios. In the absence of relaxation ($\tilde{\tau}^{-1} \to 0$), the incompressible response matrix is given by

$$\begin{aligned}(\mathbb{M}^{-1})_{ij}(\tilde{\omega}) &= \frac{\delta_{ij}}{4\pi\eta_{\mathrm{s}}}\int \mathrm{d}z\, \frac{zJ_0(z)}{z^2 - \mathrm{i}\tilde{\omega}} \\ &= -\frac{\delta_{ij}}{8\pi\eta_{\mathrm{s}}}\left[\log\left(\frac{\tilde{\omega}}{4}\right) + 2\gamma_{\mathrm{EM}} - \mathrm{i}\frac{\pi}{2}\right] + \mathcal{O}(\tilde{\omega})\,,\end{aligned} \tag{S1}$$

where $\gamma_{\mathrm{EM}}$ is the Euler-Mascheroni constant. We see that this result is divergent in the steady-state limit ($\tilde{\omega} \to 0$), which is a signature of the Stokes paradox [1]. Note that the shell localization result given in Eq. (S1) matches the drag force that one would obtain from solving explicitly the Stokes equation with no-slip boundary conditions [2, 3].

A second scenario is the steady-steady case $\tilde{\omega} \to 0$ with a finite relaxation rate $\tilde{\tau}^{-1} \neq 0$. The incompressible response matrix now reads

$$\begin{aligned}(\mathbb{M}^{-1})_{ij}(0) &= \frac{\delta_{ij}}{4\pi\eta_{\mathrm{s}}}\int \mathrm{d}z\, \frac{zJ_0(z)}{z^2 + 1/\tilde{\tau}} \\ &= \frac{\delta_{ij}}{4\pi\eta_{\mathrm{s}}}\left[\log\left(2\sqrt{\tilde{\tau}}\right) - \gamma_{\mathrm{EM}}\right] + \mathcal{O}(\tilde{\tau}^{-1})\ .\end{aligned} \tag{S2}$$

In this case, the response matrix is non-divergent thanks to the momentum relaxation circumventing the Stokes paradox [4–7]. The result in Eq. (S2) can be compared to the result from works of Saffman and Delbrück [8, 9], if one matches the relaxation $\tilde{\tau}$ as [5]

$$\tilde{\tau} = \left(\frac{\eta_{\mathrm{s}}}{2a\eta'_{\mathrm{s}}}\right)^2, \tag{S3}$$

where $\eta'_{\mathrm{s}}$ is the shear viscosity of the surrounding bulk fluid that is tied to the substrate in Refs. [8, 9][1]. We thus find that in these two instances, the shell localization approach yields the same results as in previous works where the fluid velocity profile is computed over the entire two-dimensional surface [8, 9].

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[1] Note in Refs. [8, 9] the shear viscosity in the substrate $\eta_{\mathrm{s}}^{(SD)}$ is three-dimensional and therefore it has different units from the $\eta_{\mathrm{s}}$ appearing in this letter. In Eq. (S3) the two viscosities are related by taking $\eta_{\mathrm{s}}^{(SD)} \to \eta_{\mathrm{s}}/h$, with $h$ being the height of the substrate.

## II. ANALYTICAL COMPUTATION OF THE RESPONSE MATRIX

In this section, we show how the integrals performed throughout this Letter can be performed using the method of residues. For a compressible fluid as described in the main text, the response coefficients are obtained by performing momentum integrals that take the form

$$I[R] = \int_0^\infty \mathrm{d}z\, R(z) J_0(z)\,, \tag{S4}$$

where $R(z) = A(z)/B(z)$ is an odd function of $z$, and where $A$ and $B$ are polynomials in $z$. We call $z_n$ the $n^{\text{th}}$ root of $B(z)$, such that $B(z_n) = 0$. Following Ref. [10], the integral $I[R]$ can be computed analytically in terms of the Hankel functions of the first kind $H_\nu^{(1)}$ and the Bessel functions of the second kind $Y_\nu$. It reads:

$$\begin{aligned} I[R] = \mathrm{i}\pi \sum_{z_n \in \mathbb{C}^+ \backslash \mathbb{R}} &\mathrm{Res}\left(R(z) H_0^{(1)}(z), z_n\right) \\ - \pi \sum_{z_n \in \mathbb{R}^+} &\mathrm{Res}\left(R(z) Y_0(z), z_n\right)\,, \end{aligned} \tag{S5}$$

where the first sum is over the roots of $B(z)$ whose imaginary part is strictly positive, and the second one is over the positive real roots of $B(z)$. We denote by $\mathrm{Res}(f(z), z_n)$ the residue of $f$ at point $z_n$.

As an illustration, we consider the oscillatory incompressible case in the absence of relaxation ($\tilde{\tau}^{-1} \to 0$), for which one has

$$R(z) = \frac{z}{z^2 - \mathrm{i}\tilde{\omega}}\,, \tag{S6}$$

and thus for which $A(z) = z$ and $B(z) = z^2 - \mathrm{i}\tilde{\omega}$ with the roots $z_{1,2} = \pm\sqrt{\mathrm{i}\tilde{\omega}}$. In this case, only the first term in the right-hand side of Eq. (S5) contributes, and since the Hankel function $H_0^{(1)}$ has no pole in $z_1 = \sqrt{\mathrm{i}\tilde{\omega}}$, it yields

$$I\left[z/(z^2 - \mathrm{i}\tilde{\omega})\right] = \frac{\mathrm{i}\pi}{2} H_0^{(1)}(\sqrt{\mathrm{i}\tilde{\omega}}) = K_0(-\mathrm{i}\sqrt{\mathrm{i}\tilde{\omega}})\quad, \tag{S7}$$

where $K_\nu(x)$ is the $\nu^{\text{th}}$ modified Bessel function of the second kind. An expansion of Eq. (S7) in series of $\tilde{\omega}$ yields the result given in Eq. (S1).

The same procedure can be applied for the compressible case, and was used the main text.


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