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Complete Insecurity of Quantum Protocols for Classical Two-Party Computation

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A fundamental task in modern cryptography is the joint computation of a function which has two inputs, one from Alice and one from Bob, such that neither of the two can learn more about the other’s input than what is implied by the value of the function. In this Letter, we show that any quantum protocol for the computation of a classical deterministic function that outputs the result to both parties (two-sided computation) and that is secure against a cheating Bob can be completely broken by a cheating Alice.

Whereas it is known that quantum protocols for this task cannot be completely secure, our result implies that security for one party implies complete insecurity for the other. Our findings stand in stark contrast to recent protocols for weak coin tossing and highlight the limits of cryptography within quantum mechanics. We remark that our conclusions remain valid, even if security is only required to be approximate and if the function that is computed for Bob is different from that of Alice.

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Traditionally, cryptography has been understood as the art of “secret writing,” i.e., of sending messages securely from one party to another. Today, the research field of cryptography comprises much more than encryption and studies all aspects of secure communication and computation among players that do not trust each other, including tasks such as electronic voting and auctioning. Following the excitement that the exchange of quantum particles may allow for the distribution of a key that is unconditionally secure [1,2], a level of security unattainable by classical means, the question arose whether other fundamental cryptographic tasks could be implemented with the same level of security using quantum mechanical effects. For oblivious transfer and bit commitment, it was shown that the excitement that the exchange of quantum particles may allow for the distribution of a key that is unconditionally secure [1,2], a level of security unattainable by classical means, the question arose whether other fundamental cryptographic tasks could be implemented with the same level of security using quant...
and only connected with a quantum channel, wish to

Likewise, every ideal adversary interacting with the ideal

protocol by the honest and dishonest players is modeled by

All these notions have in common that the execution of the

informal notion of the real–ideal-world paradigm precise.

classical and quantum protocols, respectively.

information about this concept of security in the context of

protocols in a modular fashion. See Refs. [9–15] for further

possible to construct and prove secure more complicated

functionality (which is a CPTP map itself). A desirable

notion of security is the following: for every real adversary

there exists an ideal adversary such that the corresponding

CPTP maps are (approximately) indistinguishable. The

natural measure of distinguishability of CPTP maps in

this context is the diamond norm, since it can be viewed

as the maximal bias of distinguishing real and ideal world

by supplying inputs to the CPTP maps and attempting to
distinguish the outputs by measurements (i.e., by interact-
ing with an environment). This rather strong notion of

security naturally embeds into a composable framework

for security in which also quantum key distribution can be

proven secure (see, e.g., Ref. [16]).

Since our goal is the establishment of a no-go theorem,

we consider a notion of security which is weaker than the

above in two respects. First, we do not allow the environ-

ment to supply an arbitrary input state but only the purifi-
cation of a classical input (see definition of $\rho_{UVR}$ below),

and second, we consider a different order of quantifiers:

instead of “$\forall$ real adversary $\exists$ ideal adversary $\forall$ input, the

output states are indistinguishable” as a security require-

ment we only require “$\forall$ real adversary $\forall$ input $\exists$ ideal

adversary, the outputs states are indistinguishable.” This

notion of security is closely related to notions of security

considered in Ref. [13,15] and is further discussed in the

Supplemental Material [17].

We will now give a formal definition of security.

Following the notation of Ref. [15], we denote by $A$ and
$B$ the real honest Alice and Bob and add a prime to denote
dishonest players $A'$, $B'$ and a hat for the ideal versions $\hat{A}$,
$\hat{B}$. The CPTP map corresponding to the protocol for honest
Alice and dishonest Bob is denoted by $\sigma_{A'B'}$. Both honest
and dishonest players obtain an input, in Alice’s case $u$
in (register $U$) and in Bob’s case $v$ (in register $V$) drawn
from the joint distribution $p(u,v)$. The output state of the
protocol, augmented by the reference $R$, takes the form $\sigma_{u,v} \sigma_{A'B'}(\rho_{UVR})$, where $\rho_{UVR}$ is a purification of
$\sum_{u,v} p(u,v)\rho_{u|v}(u|v)_{V}$.

Since we are faced with the task of the secure evalua-
tion of a classical deterministic function, we con-
sider an ideal functionality $\mathcal{F}$ which measures the
inputs in registers $\tilde{U}$ and $\tilde{V}$ and outputs orthogonal
states in registers $\tilde{X}$ and $\tilde{Y}$ that correspond to the function
values. Formally, $\mathcal{F}(\rho_{u,v})_{U|V} := \delta_{u,v} \delta_{\rho_{u,v}} [f(u,v)]_{X} [f(u,v)]_{Y}$, where $\delta$
denotes the Kronecker delta function. When an ideal
honest $\hat{A}$ and an ideal adversary $\hat{B}$ interact with the ideal
functionality, we denote the joint map by $\mathcal{F}_{\hat{A}\hat{B}}:UV \to
XY'$ (see Fig. 1). $\hat{A}$ just forwards the in- and outputs to and
from the functionality, whereas $\hat{B}$ pre- and postprocesses
them with CPTP maps $\Lambda_{\tilde{V} \to \tilde{Y}_{K}}$ and $\Lambda_{\tilde{X} \to \tilde{Y}}$ resulting in a
joint map $\mathcal{F}_{\hat{A}\hat{B}} = [\text{id}_{U} \otimes \Lambda_{\tilde{V} \to \tilde{Y}_{K}}] \circ [\mathcal{F}_{\tilde{U} \tilde{V} \to \tilde{X} \tilde{Y}} \otimes \text{id}_{\tilde{K}}] \circ [\text{id}_{U} \otimes \Lambda_{\tilde{V} \to \tilde{Y}_{K}}]$, where $\circ$ denotes sequential application
of CPTP maps.
We say that a (two-party quantum) protocol \( \rho =_e \sigma \) if \( C(\rho, \sigma) \leq e \). \( C(\rho, \sigma) \) is the purified distance, defined as \( \sqrt{1 - F(\rho, \sigma)}^2 \) for \( F(\rho, \sigma) := \text{tr}\sqrt{\rho \sigma} \sqrt{\rho \sigma} \) the fidelity.

We say that a (two-party quantum) protocol \( \pi \) for \( f \) is \( \varepsilon \)-correct if for any distribution \( p(u, v) \) of the inputs \([id_R \otimes \pi_{AB}]|\rho_{UVR}\rangle =_\varepsilon [id_R \otimes F_{AB}]|\rho_{UVR}\rangle \) and \( \varepsilon \)-secure against dishonest Bob if for any \( p(u, v) \) and for any real adversary \( B' \) there exists an ideal adversary \( \tilde{B}' \) such that \([id_R \otimes \pi_{AB}]|\rho_{UVR}\rangle =_\varepsilon [id_R \otimes F_{AB}]|\rho_{UVR}\rangle \) and \( \varepsilon \)-security against dishonest Alice is defined analogously.

Since \( F \) is classical, we can augment it so that it outputs \( \hat{v} \) in addition. More precisely, we define \( F_{\text{aug}}: \tilde{U} \tilde{V} \rightarrow \tilde{X} \tilde{Y} \tilde{V} \) by \( F_{\text{aug}}(u|x|v) = \delta_{uv,v'}[f(u,v)](f(u,v)|v|v') \) which has the property that \( F = \text{tr} F_{\text{aug}} \) for any distribution \( \sigma_{\text{RXYY'}} \) which satisfies \( \sigma_{\text{RXYY'}} =_\varepsilon \rho_{\text{RXYY'}} \) for \( \rho_{\text{RXYY'}} = [id_R \otimes \pi_{AB}]|\rho_{UVR}\rangle \) if the protocol is secure against cheating Bob. We call \( \sigma_{\text{RXYY'}} \) an ideal state for \( p(u,v) \).

**Main results.**—The proof of our main results build upon the following lemma which constructs a cheating strategy for Alice that works on average over the input distribution \( p(u,v) \).

**Lemma.**—If a protocol \( \pi \) for the evaluation of \( f \) is \( \varepsilon \)-correct and \( \varepsilon \)-secure against Bob, then for all input distributions \( p(u,v) \) there is a cheating strategy for Alice such that she obtains \( \hat{v} \) with some probability distribution \( q(\hat{v}|u,v) \) satisfying \( \sum u,v,q(p(u,v)q(\hat{v}|u,v)\delta_{f(u,v),f(\hat{u},v)} \geq 1 - 6\varepsilon \). Furthermore, \( q(\hat{v}|u,v) \) is almost independent of \( u \); i.e., there exists a distribution \( \tilde{q}(\hat{v}|v) \) such that \( \sum u,v,p(u,v)q(\hat{v}|u,v) - \tilde{q}(\hat{v}|v) \leq 6\varepsilon \).

**Proof.**—We first construct a “cheating unitary” \( F \) for Alice and then show how Alice can use it to cheat successfully.

Let Alice and Bob play honestly, but let them purify their protocol with purifying registers \( X'_1 \) and \( Y'_1 \), respectively. We assume without loss of generality that honest parties measure their classical input, and hence, \( X'_1 \) and \( Y'_1 \) contain copies of \( u \) and \( v \), respectively. We denote by \( |\Phi\rangle_{\text{RXXY'}Y'} \) the state of all registers at the end of the protocol. Notice that tracing out \( X'_1 \) from \( |\Phi\rangle_{\text{RXXY'}Y'} \) results in a state \( |\Phi\rangle_{\text{RXYY'}} \), which is exactly the final state when Alice played honestly and Bob played dishonestly with the following strategy: he plays the honest but purified strategy and outputs the purification of the protocol (register \( Y'_1 \)) and the output values \( f(u,v) \) (register \( Y \)). His combined dishonest register is \( Y' = Y'_1 Y \). Since the protocol is \( \varepsilon \)-secure against Bob by assumption, there exists a secure state \( \sigma_{\text{RXYY'}} \) satisfying \( \sigma_{\text{RXYY'}} =_\varepsilon \rho_{\text{RXYY'}} \). Let \( |\Psi\rangle_{\text{RXYY'}} \) be a purification of \( \sigma_{\text{RXYY'}} \) with purifying register \( P \). Note that \( |\Psi\rangle_{\text{RXYY'}} \) is also a purification of \( \sigma_{\text{RXYY}} \), this time with purifying registers \( PV \). Recall that \( |\Phi\rangle_{\text{RXXY'}} \) purifies \( \rho_{\text{RXYY'}} \) with purifying register \( X'_1 \). Since \( \sigma_{\text{RXYY'}} =_\varepsilon \rho_{\text{RXYY'}} \), we can use Uhlmann’s theorem [18] to conclude that there exists an isometry \( T = T_{X'_1 \rightarrow PV} \) (with induced CPTP map \( T = T_{X'_1 \rightarrow PV} \)) such that

\[
[T_{X'_1 \rightarrow PV} \otimes \text{id}_{RXY'}]|\Phi\rangle_{\text{RXXY'}} =_\varepsilon |\Psi\rangle_{\text{RXYY'}}.
\]

We will now show how Alice can use \( T \) to cheat. Notice that tracing out \( Y'_1 \) from \( |\Phi\rangle_{\text{RXXY'}Y'} \) results exactly in the final state when Bob played honestly and Alice played dishonestly with the following strategy: she plays the honest but purified strategy and outputs the purification of the protocol (register \( X'_1 \)) and the output values \( f(u,v) \) (register \( X \)). She then applies \( T_{X'_1 \rightarrow PV} \), measures register \( V \) in the computational basis, and obtains a value \( \hat{v} \). It remains to argue that Alice can compute \( f(u,v) \) with good probability based on the value \( \hat{v} \) that she has obtained from measuring register \( V \).

Let \( \mathcal{M}_{RX} \) be the CPTP map that measures registers \( R, X, \) and \( V \) in the computational basis. Tracing over \( PV \) and applying \( \mathcal{M}_{RX} \) on both sides of Eq. (1), we find

\[
|\mathcal{M}_{RX} \otimes \text{id}_{PV}|(T_{X'_1 \rightarrow PV} \otimes \text{id}_{RXY'})(|\Phi\rangle_{\text{RXXY'}}) =_\varepsilon |\mathcal{M}_{RX} \otimes \text{id}_{PV}|(\Psi_{\text{RXYY'}})
\]

by the monotonicity of the purified distance under CPTP maps. The right-hand side of Eq. (2) equals \( \sum u,v,p(u,v)q(\hat{v}|u,v)|uv|\hat{v}|u|v|f(u,v)\) for some probability distribution \( q(\hat{v}|u,v) \) that is conditioned only on Bob’s input \( v \), since \( |\Psi_{\text{RXYY'}}\rangle \) is a purification of the secure state \( \sigma_{\text{RXYY'}} \). The left-hand side of Eq. (2) equals \( \sum u,v,p(u,v)q(\hat{v}|u,v)|uv|\hat{v}|u|v|r(x|u,v|v')x|x' \) for some conditional probability distributions \( q(\hat{v}|u,v) \) and \( r(x|u,v,\hat{v}) \). Because of the correctness of the protocol, this state is \( \varepsilon \)-close to

\[
\sum u,v,p(u,v)q(\hat{v}|u,v)|uv|\hat{v}|u|v|f(u,v)\) for some conditional probability distribution \( \tilde{q}(\hat{v}|u,v) \). Noting that therefore also \( p(\cdot,\cdot)q(\cdot|\cdot,\cdot) \) and \( p(\cdot,\cdot)\tilde{q}(\cdot|\cdot,\cdot) \) (when interpreted as quantum states) are \( \varepsilon \)-close in purified distance, we can replace \( p(\cdot,\cdot)q(\cdot|\cdot,\cdot) \) in Eq. (3) by \( p(\cdot,\cdot)\tilde{q}(\cdot|\cdot,\cdot) \) increasing the purified distance to the left-hand side of Eq. (2) only to \( 2\varepsilon \). Putting things together, Eq. (2) implies

\[
\sum u,v,p(u,v)\tilde{q}(\hat{v}|u,v)|uv|\hat{v}|u|v|f(u,v)\) for some conditional probability distribution \( \tilde{q}(\hat{v}|u,v) \).
\[\sum_{u,v,\tilde{v}} p(u,v)g(\tilde{v}|u,v)\mu(v,u)\rho(\tilde{v}|v)f(u,v)\delta(f(u,v))\mid X = 3e \sum_{u,v,\tilde{v}} p(u,v)\tilde{g}(\tilde{v}|u,v)\mu(v,u)\rho(\tilde{v}|v)f(u,v)\delta(f(u,v))\mid X.\]

Sandwiching both sides with \(\text{tr}[Z^+\cdot]\), where \(Z = \sum_{u,v,\tilde{v}} p(u,v)\mu(v,u)\rho(\tilde{v}|v)f(u,v)\cdot f(u,v)\delta(f(u,v))\mid X\), we find the first claim since the purified distance of two distributions upper bounds their total variation distance and since the latter does not increase under \(\text{tr}[Z^+\cdot]\). The second claim follows similarly by tracing out register \(X\) from the last displayed equation.

Applying the lemma to the uniform distribution we immediately obtain our impossibility result for perfectly secure protocols.

**Theorem 1.**—If a protocol \(\pi\) for the evaluation of \(f\) is perfectly correct and perfectly secure (\(e = 0\)) against Bob, then, if Bob holds input \(v\), Alice can compute \(f(u,v)\) for all \(u\).

We note that this notion of insecurity implies that Alice can completely break the security for nontrivial functions \(f\).

**Proof.** Letting \(p(u,v) = \frac{1}{|\mathcal{U}|}\) and \(e = 0\) in the lemma results in the statement that if Alice has input \(u_0\), then she will obtain \(\tilde{v}\) from the distribution \(q(\tilde{v}|u_0, v)\) which equals \(\tilde{q}(\tilde{v}|v)\). But since also \(q(\tilde{v}|u, v) = \tilde{q}(\tilde{v}|v)\) for all \(u\), we have \(\frac{1}{|\mathcal{U}|}\sum_{u,v,\tilde{v}} q(\tilde{v}|u_0, v)\delta(f(u,v),f(u,\tilde{v})) = 1\). In other words, all \(\tilde{v}\) that occur (i.e., that have \(\tilde{q}(\tilde{v}|v) > 0\)) satisfy for all \(u\), \(f(u,v) = f(u,\tilde{v})\). Alice can therefore compute the function for all \(u\).

The impossibility result for the case of imperfect protocols is also based on the lemma but requires a subtle swap in the order of quantifiers (from “\(\forall v\) input \(\exists\) ideal adversary” to “\(\exists\) ideal adversary \(\forall v\) input”) which we achieve by use of von Neumann’s minimax theorem.

**Theorem 2.**—If a protocol \(\pi\) for the evaluation of \(f\) is \(e\)-correct and \(e\)-secure against Bob, then there is a cheating strategy for Alice (where she uses input \(u_0\) while Bob has input \(v\)) which gives her \(\tilde{v}\) distributed according to some distribution \(Q(\tilde{v}|u_0, v)\) such that for all \(u\): 
\[P_{\rho_{\tilde{v}-Q}}[f(u,v) = f(u,\tilde{v})] \geq 1 - 28e.\]

**Proof.**—The argument is inspired by Ref. [19]. For a finite set \(S\), we denote by \(\Delta(S)\) the simplex of probability distributions over \(S\). Denote by \(\mathcal{W}\) the set of pairs \((u,v)\). Consider a finite \(e\)-net \(\mathcal{D}\) of \(\Delta(\mathcal{W})\) in total variation distance and to each distribution \(\rho\) in \(\mathcal{D}\) the corresponding cheating unitary \(T\) constructed in the proof of the lemma. We collect all these unitaries in the (finite) set \(\mathcal{E}\) and assume that \(T\) determines \(p\) uniquely, as we could include the value \(p\) into \(T\). For each such \(T\), let \(q(\tilde{v}|u,v,T)\) and \(\tilde{q}(\tilde{v}|v)\) be the distributions from the lemma. Define the payoff function \(g(u,v,T) = \sum_{\tilde{v}} q(\tilde{v}|u,v,T)\delta(f(u,v),f(u,\tilde{v})) - \sum_{\tilde{v}} \tilde{q}(\tilde{v}|u,v,T)\delta(f(u,v),f(u,\tilde{v}))\mid T\). The lemma then yields \(1 - 12e \leq \min_{\rho} \max_{T} \sum_{u,v} p(u,v)g(u,v,T)\) which is at most \(2e + \min_{\rho} \max_{T} \sum_{u,v} p(u,v)\rho(u,v)g(u,v,T)\), since replacing \(p\) by \(p'\) incurs only an overall change in the value by \(2e\) [as \(-1 \leq g(u,v,T) \leq 1\)]. By von Neumann’s minimax theorem, this last term equals \(2e + \max_{\rho} \min_{T} \sum_{u,v} p(u,v,T)\rho(u,v,T)\mid T\).

Hence, we have shown that there is a strategy for Alice, where she chooses her cheating unitary \(T\) with probability \(p''(T)\), such that (for some \(e_1 + e_2 \leq 14e\)) for all \(u,v\),
\[\sum_{T} Q(\tilde{v}|u,v)\delta(f(u,v),f(u,\tilde{v})) \geq 1 - e_1 - e_2.\]

One might wonder whether Theorem 2 can be strengthened to obtain, with probability \(1 - O(e)\), a \(\bar{v}\) such that for all \(u,v\): \(f(u,v) = f(u,\bar{v})\). It turns out that this depends on the function \(f\): when \(f\) is equality \(EQ(u,v) = 1\) if \(u = v\) and inner product \([IP(u,v) = \sum_{u} u_v \cdot v_\text{mod2}\], the stronger conclusion is possible. However, for disjointness \([DISJ(u,v) = 0\) if \(\exists i: u_i = v_i = 1\)] such a strengthening is not possible showing that our result is tight in general.

For \(EQ\), we reason as follows. Set \(u = v\) in Theorem 2. Alice is able to sample a \(\tilde{v}\) such that \(\sum_{\rho} Q(\tilde{v}|u_0, \bar{v})\delta_{EQ}(\tilde{v}|u_0, v) \geq 1 - 28e\). Since \(\delta_{EQ}(\tilde{v}|u_0, v) = 1\) if \(v = \bar{v}\), we find \(Q(\tilde{v}|u_0, v) \geq 1 - 28e\). When \(f\) is IP, we pick \(u\) uniform at random and obtain \(Q(\tilde{v}|u_0, v)(2^{-n} \sum_{\rho} \delta_{IP}(u,v)\delta_{IP}(u_0,v)) \geq 1 - 28e\). Using \(2^{-n} \sum_{v} \delta_{IP}(u,v)\delta_{IP}(u_0,v) = 1\) if \(v = \bar{v}\), and \(\frac{1}{2}\) if \(v \neq \bar{v}\), we find \(Q(\tilde{v}|u_0, v) \geq 1 - 56e\). Interestingly, for \(DISJ\) such an argument is not possible. Assume that we have a protocol that is \(e\)-secure against Bob. Bob could now run the protocol normally on strings \(u,v\) with Hamming weight \(|v| \leq n/2\), but on inputs \(v\) with \(|v| > n/2\) he could flip, at random, \(\sqrt{n}\) of \(v\)’s bits that are 1. It is not hard to see that this new protocol is still \(e\)-secure and \(e + O(\frac{1}{\sqrt{n}})\)-correct. The loss in the correctness is due to the fact that, on high Hamming-weight strings, the protocol may, with a small probability, not be correct. On the other hand, on high Hamming-weight inputs, the protocol cannot transmit or leak the complete input \(v\) to Alice, simply because Bob does not use it.

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[20] In order to apply von Neumann’s theorem, note that the initial term equals \(\min_{p' \in \Delta(W)} \max_{p \in \Delta(S)} \sum_{u,v} p'(u,v) \times g(u,v,T) \) since the maximal value of the latter is attained at an extreme point. Von Neumann’s minimax theorem [21] allows us to swap minimization and maximization leading to \(\max_{p \in \Delta(S)} \min_{p' \in \Delta(W)} \sum_{u,v,T} p(u,v) g(u,v,T) p'(T)\). This expression corresponds to the final term since the minimization can without loss of generality be restricted to its extreme points.