UNIVERSITY OF AMSTERDAM

UvA-DARE (Digital Academic Repository)

Two-sided reflection of Markov-modulated Brownian motion

D'Auria, B.; Ivanovs, J.; Kella, O.; Mandjes, M.

DOI
10.1080/15326349.2012.672285

Publication date
2012

Published in
Stochastic Models

Link to publication

Citation for published version (APA):
D'Auria, B., Ivanovs, J., Kella, O., & Mandjes, M. (2012). Two-sided reflection of Markov-modulated Brownian motion. *Stochastic Models*, *28*(2), 316-332. https://doi.org/10.1080/15326349.2012.672285

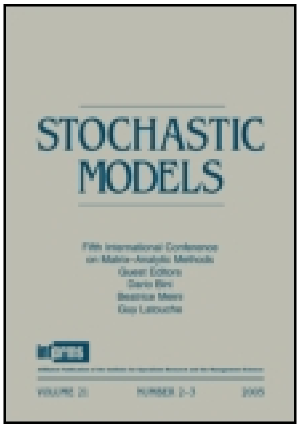
General rights
It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

Disclaimer/Complaints regulations
If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: https://uba.uva.nl/en/contact, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

UvA-DARE is a service provided by the library of the University of Amsterdam (https://dare.uva.nl)

Download date:29 Nov 2021

This article was downloaded by: [UVA Universiteitsbibliotheek SZ]
On: 22 January 2013, At: 06:29
Publisher: Taylor & Francis
Informa Ltd Registered in England and Wales Registered Number: 1072954 Registered office: Mortimer House, 37-41 Mortimer Street, London W1T 3JH, UK



Stochastic Models
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lstm20

Two-Sided Reflection of Markov-Modulated Brownian Motion

B. D'Auria [a], J. Ivanovs [b], O. Kella [c] & M. Mandjes [b]
[a] Universidad Carlos III de Madrid, Madrid, Spain
[b] Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, The Netherlands
[c] Department of Statistics, The Hebrew University of Jerusalem, Jerusalem, Israel
Version of record first published: 02 May 2012.

To cite this article: B. D'Auria, J. Ivanovs, O. Kella & M. Mandjes (2012): Two-Sided Reflection of Markov-Modulated Brownian Motion, Stochastic Models, 28:2, 316-332

To link to this article: http://dx.doi.org/10.1080/15326349.2012.672285

PLEASE SCROLL DOWN FOR ARTICLE

Full terms and conditions of use: http://www.tandfonline.com/page/terms-and-conditions

This article may be used for research, teaching, and private study purposes. Any substantial or systematic reproduction, redistribution, reselling, loan, sub-licensing, systematic supply, or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae, and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings, demand, or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.

Stochastic Models, 28:316–332, 2012
Copyright © Taylor & Francis Group, LLC
ISSN: 1532-6349 print/1532-4214 online
DOI: 10.1080/15326349.2012.672285

TWO-SIDED REFLECTION OF MARKOV-MODULATED BROWNIAN MOTION

B. D'Auria[1], J. Ivanovs[2], O. Kella[3], and M. Mandjes[2]

[1] *Universidad Carlos III de Madrid, Madrid, Spain*
[2] *Korteweg-de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, The Netherlands*
[3] *Department of Statistics, The Hebrew University of Jerusalem, Jerusalem, Israel*

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

□ *This article considers a Markov-modulated Brownian motion with a two-sided reflection. For this doubly-reflected process we compute the Laplace transform of the stationary distribution, as well as the average loss rates at both barriers. Our approach relies on spectral properties of the matrix polynomial associated with the generator of the free (that is, non-reflected) process. This work generalizes previous partial results allowing the spectrum of the generator to be non-semi-simple and also covers the delicate case where the asymptotic drift of the free process is zero.*

Keywords Markov additive process; Markov-modulated Brownian motion; Skorohod reflection; Two-sided reflection.

Mathematics Subject Classification Primary 60K25; Secondary 60K37.

1. INTRODUCTION

Fluid queues have been introduced in the late 1970s and the 1980s to model various types of networking technologies; the leading example at that time was ATM (Asynchronous Transfer Mode). In these technologies information packets were relatively small, thus justifying modeling the buffer dynamics by streams of fluid. In the original model it was assumed that the queue's input rate equals a constant number a_i when the state of a Markovian background process was i, see for instance Refs.[2,10,11,22,25] In later article this model with deterministic input rates was extended to

Received October 2010; Accepted June 2011
Address correspondence to B. D'Auria, Universidad Carlos III de Madrid, Avda. Universidad 30, Madrid, Leganes 28911, Spain; E-mail: bernardo.dauria@uc3m.es

include a Brownian component; such an input process is often referred to as Markov-modulated Brownian motion (MMBM) input. The seminal papers in this area are the works of Karandikar and Kulkarni,[16] Rogers,[24] Asmussen,[3] and Pacheco and Prabhu;[23] they primarily focus on the analysis of the infinite-buffer systems, that is, MMBM with one-sided reflection. It is noted that MMBM can be used as an approximation of various Markov-modulated models.

Rogers[24] studies MMBM with two-sided reflection, which can serve as an approximation of a finite-buffer queue in a Markov-modulated environment as well. It was shown how to derive the stationary workload distribution under the (restrictive) assumption that the Brownian component is *not* subject to modulation. Relaxing this assumption, Asmussen and Kella[5] applied a martingale technique to express the Laplace transform of the stationary distribution in terms of the average loss rates at the two boundaries. They managed to compute these average loss rates under the assumption that the spectrum of the matrix polynomial associated to the underlying generator (in the sequel referred to as the *matrix exponent*) is semi-simple (see Ref.[12]). Asmussen and Pihlsgård[6] computed under similar assumptions the average loss rate for the more general class of Markov additive processes.

The contribution of the present paper is to extend the theory developed in Ref.[5] by determining the stationary distribution of doubly-reflected MMBM in full generality. This will be done relying on a detailed understanding of the structure of the spectrum of the matrix exponent. More specifically, we use the methodology of *generalized Jordan chains*, which we introduced in Ref.[8], as a tool to analyze the first passage times of spectrally one-sided Markov additive processes (has no positive or no negative jumps). We show how to derive a set of independent linear equations whose solution yields the average loss rates. With the formulas from Ref.[5], this enables us to determine the Laplace transform of the stationary distribution.

In this article we make extensive use of matrix notation; we now list the main notational conventions that we adopt. Bold symbols will denote column vectors and in particular, $\mathbf{1}$ and $\mathbf{0}$ are the vectors of 1's and 0's respectively; $\boldsymbol{e}_i$ is a vector with ith element being 1 and all others being 0. The identity and the null matrices are denoted by $\mathbb{I}$ and $\mathbb{O}$, respectively, whereas $\Delta_{\boldsymbol{x}} = \mathrm{diag}[x_1, \ldots, x_n]$ represents the diagonal matrix whose diagonal elements are those of the vector $\boldsymbol{x}$. Given a column vector $\boldsymbol{x}$ we denote by $\boldsymbol{x}^{\mathrm{T}}$ the transposed row vector.

The article is organized as follows. In Section 2 the model is introduced and preliminaries are given. Section 3 reviews results from the literature on matrix polynomials, which is used in Section 4 to analyze the matrix exponent. Section 5 determines the upper and lower loss-rate vectors, which enable the computation of the Laplace transform of the stationary

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

workload. Finally Section 6 provides an example that illustrates the developed methodology.

2. MODEL AND PRELIMINARIES

Markov modulated Brownian motion (MMBM) is a bivariate continuous-time Markov process, denoted by $(X(t), J(t))$, defined as follows. Let $J(\cdot)$ be an irreducible continuous-time Markov chain with finite state space $E = \{1, \ldots, N\}$, transition rate matrix $Q = (q_{ij})$ and a (unique) stationary distribution $\boldsymbol{\pi}$. In addition, let $B(t)$ be a standard Brownian motion (independent of $X(t)$) and denote by $\boldsymbol{a}$ and $\boldsymbol{\sigma} \geq \mathbf{0}$ the drift and diffusion vectors of size N. We now define the MMBM $X(t)$ as the process starting at zero and evolving according to the stochastic differential equation

$$dX(t) = a_{J(t)}dt + \sigma_{J(t)}dB(t).$$

The MMBM $(X(t), J(t))$ can alternatively be characterized as the subclass of Markov additive processes with a.s. continuous sample paths in the component $X(t)$. In particular it can be proved that its moment generating function is well defined for any $\alpha \in \mathbb{C}$ and is given by

$$\mathbb{E}_i[e^{\alpha X(t)} 1\{J(t) = j\}] = (e^{F(\alpha)t})_{ij}, \tag{1}$$

cf.[4] (Prop. XI.2.2); here $\mathbb{E}_i(\cdot)$ denotes expectation given that $J(0) = i$, and the matrix term $F(\alpha)$ is a second order matrix polynomial defined by

$$F(\alpha) = \frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\alpha^2 + \Delta_{\boldsymbol{a}}\alpha + Q. \tag{2}$$

It is noted that $F(\alpha)$ is usually referred to as the *matrix exponent* of $(X(t), J(t))$, see also Ref.[8], and it is essentially the multidimensional analog of the Laplace exponent in the Lévy setting. An important quantity associated to the process $(X(t), J(t))$ is the *asymptotic drift*:

$$\kappa = \sum_{j=1}^{N} \pi_j a_j = \lim_{t\to\infty} \frac{X(t)}{t}, \quad \mathbb{P}_i\text{-a.s.}, \tag{3}$$

which does not depend on the initial state i of $J(t)$, cf.[4] (Cor. XI.2.7); here $\mathbb{P}_i$ denotes probability conditional on $J(0) = i$.

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

2.1. The Two-Sided Skorokhod Reflection

The two-sided reflection of an MMBM $(X(t), J(t))$, at the barriers 0 and b (where $b > 0$) is defined as the Markov process $(W(t), J(t))$, with

$$W(t) = X(t) + L(t) - U(t) \in [0, b] \quad \forall t \geq 0,$$

where $L(t)$ and $U(t)$ (called the *lower-regulator* and *upper-regulator* processes, respectively) are two non-negative, non-decreasing, continuous processes that can increase only when $W(t) = 0$ and $W(t) = b$, respectively. This last property usually is expressed by the integral relations

$$\int_0^\infty W(s)dL(s) = 0 \quad \text{and} \quad \int_0^\infty (b - W(s))dU(s) = 0.$$

It is known that such a reflected process $W(t)$ exists and is unique, see e.g., Refs.[14,19,26]. The literature on finite buffer queues is extensive; references can be found in[4] (Ch. XIV). For a literature overview for the specific case of fluid queues, we refer to Ref.[1].

Let (W, J) refer to the (unique) stationary distribution of $(W(t), J(t))$. It is shown in Ref.[5] that

$$\mathbb{E}\left[e^{\alpha W}\boldsymbol{e}_J^{\mathrm{T}}\right] \cdot F(\alpha) = \alpha(e^{\alpha b}\boldsymbol{u}^{\mathrm{T}} - \boldsymbol{\ell}^{\mathrm{T}}), \tag{4}$$

where $\boldsymbol{u}$ and $\boldsymbol{\ell}$ are column vectors of constants having the following sample-path interpretation:

$$\boldsymbol{\ell} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \boldsymbol{e}_{J(s)}L(ds) \quad \text{and} \quad \boldsymbol{u} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \boldsymbol{e}_{J(s)}U(ds). \tag{5}$$

The vectors $\boldsymbol{u}$ and $\boldsymbol{\ell}$ were not identified in Ref.[5]; in Section 5 we show that it is always possible to compute them by solving a system of linear equations. This knowledge is then used to determine the stationary distribution of (W, J), in terms of its Laplace transform, through Eq. (4). This result completes the analysis of finite-buffer MMBM-driven queues contained in[5] (Section 9) and extends the previous works (Refs.[16,17,24]).

2.2. First Passage Process

The study of the doubly-reflected process is closely related with the analysis of the exit times for the process $X(t)$ from an interval, see for example[4] (Prop. XIV.3.7). We therefore define, for $x \geq 0$, the first passage times above the level $+x$ (resp. below level $-x$) for the process

$X(t)$ as

$$\tau^{\pm}(x) = \inf\{t > 0 : \pm X(t) > x\}, \tag{6}$$

where we follow the convention that $\inf\emptyset = \infty$.

By the strong Markov property, on $\{\tau^{\pm}(x) < \infty, J(\tau^{\pm}(x)) = i\}$, the process $\{(X(t+\tau^{\pm}(x)) - X(\tau^{\pm}(x)), J(t+\tau^{\pm}(x))), t \geq 0\}$ is independent from $\{(X(t), J(t)), t \in [0, \tau^{\pm}(x)]\}$ and has the same law as the original process under $\mathbb{P}_i$ (i.e., the law of the process $(X(t), J(t))$ given that $J(0) = i$). It follows that the time-changed processes $J(\tau^{\pm}(x))$, $x \geq 0$ are time-homogeneous Markov chains.

According to the values of the diffusion and the drift coefficients, the process $X(t)$ is allowed to reach a new minimum or a new maximum only when $J(t)$ belongs to some subset of the state space E. We denote by E_+ (E_-) the set of states where the process $X(t)$ can increase (decrease, respectively), and by $E_{\uparrow}$ ($E_{\downarrow}$) the states where the process $X(t)$ has monotone non-decreasing (non-increasing, respectively) paths. In other words,

$$E_+ = \{j \in E : \sigma_j > 0 \text{ or } a_j > 0\}, \quad E_{\uparrow} = \{j \in E : \sigma_j = 0 \text{ and } a_j \geq 0\},$$
$$E_- = \{j \in E : \sigma_j > 0 \text{ or } a_j < 0\}, \quad E_{\downarrow} = \{j \in E : \sigma_j = 0 \text{ and } a_j \leq 0\}.$$

We also denote the number of states in $E_{\pm}$ by $N_{\pm}$.

Let $\{\partial\}$ be an absorbing state corresponding to $J(\infty)$. We note that $\{J(\tau^{\pm}(x)),\ x \geq 0\}$ lives on $E_{\pm} \cup \{\partial\}$, because $X(t)$ cannot hit a new maximum (resp. minimum) when $J(t)$ is in a state belonging to $E_{\downarrow}$ (resp. $E_{\uparrow}$), see also Ref.[21]. Let $\Lambda^{\pm}$ be the $N_{\pm} \times N_{\pm}$ dimensional transition rate matrices of $J(\tau^{\pm}(x))$ restricted to $E_{\pm}$. In addition define the matrices of initial distributions $\Pi^{\pm}$ as the $N \times N_{\pm}$ matrices with

$$\Pi^{\pm}_{ij} = \mathbb{P}_i(J(\tau^{\pm}(0)) = j), \quad \text{where } i \in E \text{ and } j \in E_{\pm}. \tag{7}$$

The restriction of the matrices $\Pi^{\pm}$ to the rows in $E_{\pm}$ are the identity matrices, as we know that $\mathbb{P}_i(\tau^{\pm}(x) = 0) = 1$ when $i \in E_{\pm}$ (see Ref.[20], Theorem 6.5).

In the following, for a given matrix A whose rows are indexed by the set E, we denote by $A_{\pm}$ the submatrices obtained from the rows with indices in $E_{\pm}$. In addition let $\mathscr{M}$ be a set of all irreducible transition rate matrices. We partition $\mathscr{M}$ into two sets $\mathscr{M}_0$ and $\mathscr{M}_1$, where $\mathscr{M}_1 = \{M \in \mathscr{M} : M\mathbf{1} = \mathbf{0}\}$. Note that $\mathscr{M}_0$ corresponds to the case when an absorbing state occurs. In other words, for any $M \in \mathscr{M}_0$ it holds that $M\mathbf{1} \leq 0$ with at least one inequality being strict. It is important to note that if $\kappa > 0$, then we have that $\Lambda^+ \in \mathscr{M}_1$ and $\Lambda^- \in \mathscr{M}_0$; if $\kappa < 0$, then we have that $\Lambda^+ \in \mathscr{M}_0$ and $\Lambda^- \in \mathscr{M}_1$; and if $\kappa = 0$, then we have that $\Lambda^{\pm} \in \mathscr{M}_1$. This fact follows from[4] (Prop. XI.2.10),

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

and in the following we are going to denote it by the shorter expression $\Lambda^{\pm} \in \mathscr{M}_{1\{\pm\kappa\geq 0\}}$.

Noticing that the process $(-X(t), J(t))$ is an MMBM with diffusion vector $\boldsymbol{\sigma}$ and drift vector $-\boldsymbol{a}$, the following result holds (see Ref.[8], (Corollary. 3 and Remark. 1)).

Proposition 1. *If $(X(t), J(t))$ is an MMBM, then $(\Pi^{\pm}, \Lambda^{\pm})$ are the unique pairs (P, M), with $P_{\pm} = \mathbb{I}^{\pm}$ and $M \in \mathscr{M}_{1\{\pm\kappa\geq 0\}}$, which satisfy the matrix equations*

$$\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 PM^2 \mp \Delta_{\boldsymbol{a}} PM + QP = \mathbb{O}^{\pm}. \tag{8}$$

The above result has appeared in different degrees of generality in various previous works, such as Refs.[4,7,18,24]. As an aside, we mention here that the uniqueness part is not required for the present work; in fact, this uniqueness follows from the analysis presented in Section 4.

3. MATRIX POLYNOMIALS

In this section we review some basic facts from the theory of matrix polynomials; for a more complete treatment we suggest the reader the book.[12] In the following a *λ-Jordan block* refers to a square matrix with λ on the main diagonal and 1 on the first upper diagonal, and the remaining elements being 0. A *Jordan matrix* is a block diagonal matrix whose diagonal elements are all given by Jordan blocks.

Let $A(z)$ be a complex matrix polynomial ($N \times N$ dimensional) of order ℓ, i.e.,

$$A(z) = \sum_{j=0}^{\ell} A_j z^j$$

and associate to it the complex function $\alpha(z) = \det(A(z))$ that we assume is not identically equal to 0 for $z \in \mathbb{C}$. A complex number λ is an *eigenvalue* of $A(z)$ of algebraic multiplicity n_λ if λ is a zero of $\alpha(z)$ of multiplicity n_λ. Given a sequence of vectors $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}$, with $\boldsymbol{v}_0 \neq \boldsymbol{0}$, we say that it is a *Jordan chain* of size $n > 0$ associated to the eigenvalue λ if

$$\sum_{j=0}^{k} \frac{1}{j!} A^{(j)}(\lambda) \boldsymbol{v}_{k-j} = \boldsymbol{0}, \quad k = 0, \ldots, n-1. \tag{9}$$

It can be shown that n can not exceed n_λ. Observe that $\boldsymbol{v}_0$ is a regular eigenvector associated to the eigenvalue 0 of the matrix $A(\lambda)$, while the vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}$ are generally referred to as *generalized eigenvectors.*

Define the matrix $V = [\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}]$ and denote by Γ the λ-Jordan block of size n. It is known that Eq. (9) is equivalent to the following matrix relation (see Ref.[12] Prop. 1.10) and the comments thereafter):

$$\sum_{j=0}^{\ell} A_j V \Gamma^j = 0. \tag{10}$$

Let m_λ be the geometric multiplicity of the eigenvalue λ, i.e., the dimension of the kernel of $A(\lambda)$. Then there exist m_λ Jordan chains corresponding to the eigenvalue λ, such that

(i) the first vectors of these chains form a basis of the kernel of $A(\lambda)$,
(ii) the total number of vectors is n_λ.

Letting (V_i, Γ_i) be the corresponding pairs of matrices as defined above, we put $V = [V_1, \ldots, V_{m_\lambda}]$ and $\Gamma = \mathrm{diag}[\Gamma_1, \ldots, \Gamma_{m_\lambda}]$. The pair (V, Γ) is called a *Jordan pair* corresponding to the eigenvalue λ. Observe that this pair also satisfies Eq. (10). It is noted that Jordan pairs are not unique. However, they are *similar*, in the sense stated in the following proposition.

Proposition 2. *Let (V', Γ') and (V'', Γ'') be Jordan pairs corresponding to some eigenvalue λ. Then there exists an invertible $n_\lambda \times n_\lambda$ matrix S such that*

$$V'' = V'S \quad \textit{and} \quad \Gamma'' = S^{-1}\Gamma' S. \tag{11}$$

Let us make some additional remarks concerning the structure of a Jordan pair. Firstly, Jordan pairs of the polynomial $z\mathbb{I} - M$ are the Jordan pairs of the matrix M in the classical sense: $MV = V\Gamma$. Secondly, properties (i) and (ii) above are equivalent to requiring that the rank of the column space $\mathrm{col}[V\Gamma^j]_{j=0}^{\ell-1}$ be n_λ. If the order of the polynomial, say ℓ, is 1, as in the classical case, then V must have independent columns. This is not the case in general.

By collecting all the Jordan pairs $(V_{\lambda(i)}, \Gamma_{\lambda(i)})$ corresponding to every distinct eigenvalue $\lambda(i)$ of $A(z)$, we extend the concept of a Jordan pair to the whole matrix polynomial. More specifically, we define $V = [V_{\lambda(i)}]_i$ and $\Gamma = \mathrm{diag}[\Gamma_{\lambda(i)}]_i$. This pair (V, Γ) is generally known as a *finite Jordan pair* of the matrix polynomial $A(z)$ and it can be shown that V and Γ have sizes respectively $N \times n_A$ and $n_A \times n_A$, where n_A is the number of zeros (counting multiplicities) of $\alpha(z)$. The pair (V, Γ) satisfies Eq. (10) and in addition it has the characterizing property that the rank of the column space $\mathrm{col}[V\Gamma^j]_{j=0}^{\ell-1}$ is equal to n_A.

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

Let $f(z)$ be an entire function. Then for an arbitrary square matrix M one defines

$$f(M) = \sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(0) M^j.$$

It is well-known that if Γ is a Jordan block of size n with eigenvalue λ, then

$$f(\Gamma) = \begin{pmatrix} f(\lambda) & f'(\lambda)/1! & \dots & f^{(n-1)}(\lambda)/(n-1)! \\ 0 & f(\lambda) & \dots & f^{(n-2)}(\lambda)/(n-2)! \\ \dots & & & \\ 0 & 0 & \dots & f(\lambda) \end{pmatrix}. \tag{12}$$

Finally we state the following technical result that will be used in the next sections.

Proposition 3. *Assume that $\boldsymbol{c}^{\mathrm{T}}(z)A(z) = \sum_{i=1}^{K} f_i(z)\boldsymbol{u}_i^{\mathrm{T}}$, where $f_i(z)$ are entire functions and $\boldsymbol{u}_i$ are vectors in $\mathbb{C}^N$. Then it holds that*

$$\sum_{i=1}^{K} \boldsymbol{u}_i^{\mathrm{T}} V f_i(\Gamma) = \boldsymbol{0}^{\mathrm{T}}. \tag{13}$$

Proof. It is enough to show that Eq. (13) holds for an arbitrary Jordan chain $\boldsymbol{v}_0, \dots, \boldsymbol{v}_{n-1}$ of $A(z)$ corresponding to some eigenvalue λ. Letting $\boldsymbol{b}^{\mathrm{T}}(z) = \sum_{i=1}^{K} f_i(z)\boldsymbol{u}_i^{\mathrm{T}}$ we have

$$\sum_{j=0}^{k} \frac{1}{j!} \boldsymbol{b}^{\mathrm{T}(j)}(\lambda)\boldsymbol{v}_{k-j} = \boldsymbol{0}^{\mathrm{T}} \quad k = 0, \dots, n-1,$$

see, e.g., Ref.[13] It only remains to observe that the columns of $Vf_i(\Gamma)$ are given by

$$\sum_{j=0}^{k} \frac{1}{j!} f_i^{(j)}(\lambda)\boldsymbol{v}_{k-j} \quad k = 0, \dots, n-1,$$

according to Eq. (12). $\square$

4. THE MATRIX EXPONENT

Recall that the matrix exponent $F(\alpha)$, as defined in Eq. (2), is a second order matrix polynomial. The primary objective of this section is to show that its finite Jordan pair has a very special structure. The following exposition heavily relies on Proposition 1. Consider a Jordan decomposition of the matrices $\Lambda^{\pm}$:

$$\Lambda^{+} = -R^{+}\Gamma^{+}(R^{+})^{-1} \quad \text{and} \quad \Lambda^{-} = +R^{-}\Gamma^{-}(R^{-})^{-1}, \tag{14}$$

and define $V^{\pm} = \Pi^{\pm} R^{\pm}$. Observe that $V_{+}^{+} = R^{+}$ and hence V_{+}^{+} is invertible; the same holds for $V_{-}^{-} = R^{-}$. Proposition 1 implies that

$$\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^{2} V^{\pm}(\Gamma^{\pm})^{2} + \Delta_{\boldsymbol{a}} V^{\pm}\Gamma^{\pm} + QV^{\pm} = 0.$$

In other words, (V^{+}, Γ^{+}) and (V^{-}, Γ^{-}) are composed from Jordan chains of $F(\alpha)$, see Eq. (10). These Jordan chains are possibly incomplete, that is, they may not form Jordan pairs as the total number of vectors associated to a given eigenvalue λ may be strictly smaller then the corresponding multiplicity n_{λ}.

Observe that exactly $N_{+} - 1\{\kappa \geq 0\}$ (resp. $N_{-} - 1\{\kappa \leq 0\}$) eigenvalues (counting multiplicities) of Λ^{+} (resp. Λ^{-}) have a negative real part and that if λ is an eigenvalue of Λ^{+} (resp. Λ^{-}) then $-\lambda$ (resp. $+\lambda$) is also an eigenvalue of $F(\alpha)$. Hence $F(\alpha)$ has at least $N_{+} - 1\{\kappa \geq 0\}$ eigenvalues with positive real parts and at least $N_{-} - 1\{\kappa \leq 0\}$ eigenvalues with negative real parts. As the total number of zeros of $\det(F(\alpha))$ is $N_{+} + N_{-}$ and one of them is $\alpha = 0$. The analysis of zeros is complete when $\kappa \neq 0$. In this case $([V^{+}, V^{-}], \operatorname{diag}[\Gamma^{+}, \Gamma^{-}])$ is a finite Jordan pair of $F(\alpha)$. The case $\kappa = 0$, however, is substantially more delicate.

Lemma 4. *If* $\kappa = 0$, *then* 0 *is a root of* $\det(F(\alpha))$ *with multiplicity* 2.

Proof. If $\kappa = 0$ then $F(\alpha)$ has at least $N_{+} + N_{-} - 2$ eigenvalues with non-zero real parts. But the number of zeros of $\det(F(\alpha))$ is $N_{+} + N_{-}$. So it remains to prove that the multiplicity of 0 is at least 2. Note that $F(0) = Q$, whose kernel is of dimension 1 by the irreducibility of Q. Hence we need to show that there is a Jordan chain corresponding to 0 of length 2. Let us show that there exists a vector $\boldsymbol{h}$ such that $(\boldsymbol{1}, \boldsymbol{h})$ is such a Jordan chain. According to (9) the following equation has to be satisfied:

$$F(0)\boldsymbol{h} + F'(0)\boldsymbol{1} = Q\boldsymbol{h} + \boldsymbol{a} = \boldsymbol{0}. \tag{15}$$

Such vector $\boldsymbol{h}$ exists if $[Q, \boldsymbol{a}]$ has rank $N-1$, as $\boldsymbol{\pi}^{\mathrm{T}}[Q, \boldsymbol{a}] = \boldsymbol{0}$ and $\boldsymbol{\pi} > \boldsymbol{0}$. $\square$

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

Let (V_0, Γ_0) be a Jordan pair associated with the null eigenvalue, given by

$$V_0 = \begin{cases} \mathbf{1} & \text{if } \kappa \neq 0 \\ [\mathbf{1}, \boldsymbol{h}] & \text{if } \kappa = 0 \end{cases} \quad \text{and} \quad \Gamma_0 = \begin{cases} (0) & \text{if } \kappa \neq 0 \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{if } \kappa = 0 \end{cases}, \tag{16}$$

where $\boldsymbol{h}$ satisfies (15). Exclude the eigenvalue 0 from the pairs $(V^{\pm}, \Gamma^{\pm})$ and denote these new pairs through $(\hat{V}^{\pm}, \widehat{\Gamma}^{\pm})$. Then $([V_0, \widehat{V}^{+}, \widehat{V}^{-}], \mathrm{diag}[\Gamma_0, \widehat{\Gamma}^{+}, \widehat{\Gamma}^{-}])$ is a finite Jordan pair of $F(\alpha)$, irrespective of the sign of κ.

Let us now solve the reverse problem. We start by considering an arbitrary finite Jordan pair (V, Γ) of $F(\alpha)$, where the Jordan pair (V_0, Γ_0) corresponding to the null eigenvalue is taken as in Eq. (16). Let also $(V^{\pm}, \Gamma^{\pm})$ be formed from all the Jordan pairs corresponding to eigenvalues $\lambda(i)$ with $\pm\mathrm{Re}(\lambda(i)) > 0$ and in addition $(\mathbf{1}, 0)$ if $\pm\kappa \geq 0$.

Proposition 5. *The matrices V_+^+ and V_-^- are invertible, and the following holds:*

$$\Lambda^+ = -V_+^+ \Gamma^+ (V_+^+)^{-1}, \quad \Pi^+ = V^+ (V_+^+)^{-1},$$
$$\Lambda^- = +V_-^- \Gamma^- (V_-^-)^{-1}, \quad \Pi^- = V^- (V_-^-)^{-1}.$$

Proof. Observe that similarity relation between Jordan pairs, see Proposition 1, can be extended to the pairs $(V^{\pm}, \Gamma^{\pm})$. The result follows immediately from representation (14). □

5. THE UPPER AND LOWER LOSS-RATE VECTORS

In this section we construct a system of linear equations, which uniquely determines the loss vectors $\boldsymbol{u}$ and $\boldsymbol{\ell}$; recalling (4), this also yields the Laplace transform of the stationary workload (jointly with the state of the background process). These equations are stated in terms of an arbitrary finite Jordan pair (V, Γ) of $F(\alpha)$. Without loss of generality it is assumed that the Jordan pair (V_0, Γ_0) associated to the null eigenvalue is taken as in Eq. (16). It is also assumed that (V_0, Γ_0) is the first Jordan pair in representation (V, Γ).

Theorem 6. *It holds that $\boldsymbol{u}_{\downarrow} = \mathbf{0}$ and $\boldsymbol{\ell}_{\uparrow} = \mathbf{0}$, whereas the vectors $\boldsymbol{u}_+$ and $\boldsymbol{\ell}_-$ are the unique solutions of the system of linear equations*

$$(\boldsymbol{u}_+^{\mathrm{T}}, \boldsymbol{\ell}_-^{\mathrm{T}}) \begin{pmatrix} V_+ e^{b\Gamma} \\ -V_- \end{pmatrix} = (\boldsymbol{k}^{\mathrm{T}}, 0, \ldots, 0), \tag{17}$$

with

$$\boldsymbol{k}^{\mathrm{T}} = \boldsymbol{\pi}^{\mathrm{T}}(\Delta_{\boldsymbol{a}}, \frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2)\begin{pmatrix} V_0 \\ V_0\,\Gamma_0 \end{pmatrix}. \tag{18}$$

Remark 7. Since $\boldsymbol{\pi}^{\mathrm{T}}\Delta_{\boldsymbol{a}}\mathbf{1}$ equals the asymptotic drift κ, (18) reduces to

$$\boldsymbol{k}^{\mathrm{T}} = \begin{cases} \kappa, & \text{if } \kappa \neq 0 \\ (0, \boldsymbol{\pi}^{\mathrm{T}}(\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\mathbf{1} + \Delta_{\boldsymbol{a}}\boldsymbol{h})), & \text{if } \kappa = 0. \end{cases} \tag{19}$$

Let us first present a result concerning $\boldsymbol{k}^{\mathrm{T}}$ in the case $\kappa = 0$.

Lemma 8. *If* $\kappa = 0$, *then*

$$b\boldsymbol{u}^{\mathrm{T}}\mathbf{1} + (\boldsymbol{u}^{\mathrm{T}} - \boldsymbol{\ell}^{\mathrm{T}})\boldsymbol{h} = \boldsymbol{\pi}^{\mathrm{T}}\left(\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\mathbf{1} + \Delta_{\boldsymbol{a}}\boldsymbol{h}\right) \neq 0. \tag{20}$$

Proof. Differentiating (4) at 0 and right-multiplying by $\boldsymbol{h}$, we obtain the identity

$$(\boldsymbol{u}^{\mathrm{T}} - \boldsymbol{\ell}^{\mathrm{T}})\boldsymbol{h} = \mathbb{E}[W\boldsymbol{e}_J^{\mathrm{T}}]Q\boldsymbol{h} + \mathbb{E}[\boldsymbol{e}_J^{\mathrm{T}}]\Delta_{\boldsymbol{a}}\boldsymbol{h} = -\mathbb{E}[W\boldsymbol{e}_J^{\mathrm{T}}]\Delta_{\boldsymbol{a}}\mathbf{1} + \mathbb{E}[\boldsymbol{e}_J^{\mathrm{T}}]\Delta_{\boldsymbol{a}}\boldsymbol{h}, \tag{21}$$

where the second equality follows from Eq. (15). Differentiating Eq. (4) twice at 0 and right-multiplying by $\mathbf{1}$, we find

$$b\boldsymbol{u}^{\mathrm{T}}\mathbf{1} = \mathbb{E}[W\boldsymbol{e}_J^{\mathrm{T}}]\Delta_{\boldsymbol{a}}\mathbf{1} + \mathbb{E}[\boldsymbol{e}_J^{\mathrm{T}}]\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\mathbf{1}, \tag{22}$$

which summed with the previous equation gives Eq. (20). We conclude the proof by showing that the resulting expression cannot be 0. Assume that it is 0. Recall that $\boldsymbol{\pi}^{\mathrm{T}}Q = 0$ and the rank of Q is $N-1$, so that the vector $\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\mathbf{1} + \Delta_{\boldsymbol{a}}\boldsymbol{h}$ is in the column space of Q. In other words, there exists a vector $\boldsymbol{v}$ such that

$$\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2\mathbf{1} + \Delta_{\boldsymbol{a}}\boldsymbol{h} - Q\boldsymbol{v} = \mathbf{0}.$$

This implies that $(\mathbf{1}, \boldsymbol{h}, -\boldsymbol{v})$ is a Jordan chain associated to the null eigenvalue, which has multiplicity $2 < 3$. This is a contradiction. □

Proof of Theorem 6. The proofs of the facts that $\boldsymbol{u}_{\downarrow} = \mathbf{0}$ and $\boldsymbol{\ell}_{\uparrow} = \mathbf{0}$ and, moreover, $(\boldsymbol{u}^{\mathrm{T}} - \boldsymbol{\ell}^{\mathrm{T}})\mathbf{1} = \boldsymbol{\pi}^{\mathrm{T}}\Delta_{\boldsymbol{a}}\mathbf{1} = \kappa$ were already given in[5] and follow directly from (5). The rest of the proof is split into two steps. First we show that $(\boldsymbol{u}_+^{\mathrm{T}}, \boldsymbol{\ell}_-^{\mathrm{T}})$ solves (17), and then we show that the solution is unique.

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

Step I: Apply Proposition 3 to (4) to obtain

$$\boldsymbol{u}^{\mathrm{T}} V e^{b\Gamma}\Gamma - \boldsymbol{\ell}^{\mathrm{T}} V\Gamma = \mathbf{0}^{\mathrm{T}}. \tag{23}$$

Let $\widehat{\Gamma}$ be the matrix Γ with Jordan block Γ_0 replaced with $\mathbb{I}$. Suppose first that $\kappa \neq 0$. Then $(V_0, \Gamma_0) = ([\mathbf{1}], (0))$, and as a result Eq. (23) can be rewritten as

$$\boldsymbol{u}^{\mathrm{T}} V e^{b\Gamma}\widehat{\Gamma} - \boldsymbol{\ell}^{\mathrm{T}} V\widehat{\Gamma} = ((\boldsymbol{u}^{\mathrm{T}} - \boldsymbol{\ell}^{\mathrm{T}})\mathbf{1}, 0, \ldots, 0). \tag{24}$$

Multiply by $\widehat{\Gamma}^{-1}$ from the right to obtain (17). Now consider the case $\kappa = 0$. Adding

$$\hat{\boldsymbol{k}}^{\mathrm{T}} = \boldsymbol{u}^{\mathrm{T}} V_0 e^{b\Gamma_0}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} - \boldsymbol{\ell}^{\mathrm{T}} V_0 \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \tag{25}$$

to the first two elements of the vectors appearing on the both sides of Eq. (23), we obtain

$$\boldsymbol{u}^{\mathrm{T}} V e^{b\Gamma}\widehat{\Gamma} - \boldsymbol{\ell}^{\mathrm{T}} V\widehat{\Gamma} = (\hat{\boldsymbol{k}}^{\mathrm{T}}, 0, \ldots, 0). \tag{26}$$

To complete Step 1 it is enough to show that $\hat{\boldsymbol{k}} = \boldsymbol{k}$. A simple computation reveals that $\hat{\boldsymbol{k}}^{\mathrm{T}} = \boldsymbol{u}^{\mathrm{T}}(\mathbf{1}, (b-1)\mathbf{1} + \boldsymbol{h}) - \boldsymbol{\ell}^{\mathrm{T}}(\mathbf{1}, -\mathbf{1} + \boldsymbol{h}) = (0, b\boldsymbol{u}^{\mathrm{T}}\mathbf{1} + (\boldsymbol{u}^{\mathrm{T}} - \boldsymbol{\ell}^{\mathrm{T}})\boldsymbol{h})$, where we used that $(\boldsymbol{u}^{\mathrm{T}} - \boldsymbol{\ell}^{\mathrm{T}})\mathbf{1} = \kappa = 0$. Use Lemma 8 and (19) to see that $\hat{\boldsymbol{k}} = \boldsymbol{k}$.

Step II (Uniqueness): Without loss of generality we assume that $\kappa \geq 0$. Recall the definition of $(V^{\pm}, \Gamma^{\pm})$ as given in Section 4. It is easy to see that $(\boldsymbol{u}_+^{\mathrm{T}}, \boldsymbol{\ell}_-^{\mathrm{T}})$ solves (17) if and only if

$$(\boldsymbol{u}_+^{\mathrm{T}}, -\boldsymbol{\ell}_-^{\mathrm{T}})\begin{pmatrix} V_+^+ e^{b\Gamma^+} & V_+^- e^{b\Gamma^-} \\ V_-^+ & V_-^- \end{pmatrix} = \kappa \boldsymbol{e}_1^{\mathrm{T}} \tag{27}$$

and, in addition in case $\kappa = 0$ we have that Eq. (20) holds true because of the construction of the matrices $V^{\pm}$ and $\Gamma^{\pm}$. Note also that we can right-multiply both sides of the above display by the same matrix to obtain

$$(\boldsymbol{u}_+^{\mathrm{T}}, -\boldsymbol{\ell}_-^{\mathrm{T}})\begin{pmatrix} V_+^+ & V_+^- e^{b\Gamma^-} \\ V_-^+ e^{-b\Gamma^+} & V_-^- \end{pmatrix} = \kappa \boldsymbol{e}_1^{\mathrm{T}}\begin{pmatrix} e^{-b\Gamma^+} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{pmatrix}. \tag{28}$$

According to Proposition 5 we have

$$\begin{aligned} V_-^+ &= \Pi_-^+ V_+^+, \quad & V_+^+ e^{-b\Gamma^+} &= e^{b\Lambda^+} V_+^+, \\ V_+^- &= \Pi_+^- V_-^-, \quad & V_-^- e^{+b\Gamma^-} &= e^{b\Lambda^-} V_-^-, \end{aligned} \tag{29}$$

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

and thus

$$\begin{pmatrix} V_+^+ & V_+^- e^{b\Gamma^-} \\ V_-^+ e^{-b\Gamma^+} & V_-^- \end{pmatrix} = \begin{pmatrix} \mathbb{I} & \Pi_+^- e^{b\Lambda^-} \\ \Pi_-^+ e^{b\Lambda^+} & \mathbb{I} \end{pmatrix} \begin{pmatrix} V_+^+ & \mathbb{O} \\ \mathbb{O} & V_-^- \end{pmatrix}. \tag{30}$$

Observe that $\Pi_-^+ e^{b\Lambda^+}$ and $\Pi_+^- e^{b\Lambda^-}$ are irreducible transition probability matrices, so that the first matrix on the right hand side of Eq. (30), call it M, is an irreducible non-negative matrix, which is non-strictly diagonally dominant. If $\kappa > 0$, then $\Pi_+^- e^{b\Lambda^-}$ is transient, which implies that M is irreducibly diagonally dominant and hence invertible.[15] If $\kappa = 0$, then the Perron-Frobenius theory implies that M has a simple eigenvalue at 0, so

$$(\boldsymbol{u}_+^{\mathrm{T}}, -\boldsymbol{\ell}_-^{\mathrm{T}}) \begin{pmatrix} \mathbb{I} & \Pi_+^- e^{b\Lambda^-} \\ \Pi_-^+ e^{b\Lambda^+} & \mathbb{I} \end{pmatrix} = \mathbf{0}^{\mathrm{T}} \tag{31}$$

determines the vector $(\boldsymbol{u}_+^{\mathrm{T}}, -\boldsymbol{\ell}_-^{\mathrm{T}})$ up to a scalar, which is then identified using (20):

$$(\boldsymbol{u}_+^{\mathrm{T}}, -\boldsymbol{\ell}_-^{\mathrm{T}}) \begin{pmatrix} b\mathbf{1} + \boldsymbol{h} \\ \boldsymbol{h} \end{pmatrix} = \boldsymbol{\pi}^{\mathrm{T}}(\frac{1}{2}\Delta_{\boldsymbol{\sigma}}^2 \mathbf{1} + \Delta_{\boldsymbol{a}} \boldsymbol{h}), \tag{32}$$

and which is non-zero by Lemma 8. □

Finally, we state a corollary, which identifies (in the case of non-zero asymptotic drift) the vectors $\boldsymbol{u}_+$ and $\boldsymbol{\ell}_-$ in terms of matrices $\Lambda^\pm$ and $\Pi^\pm$. We believe that there should exist a direct probabilistic argument underlying this identity, but we have not succeeding in finding this so far. Moreover, an interesting topic for future research would be to investigate whether such a result holds in the case of a countably infinite state space E.

Corollary 9. *It holds that*

$$(\boldsymbol{u}_+^{\mathrm{T}}, -\boldsymbol{\ell}_-^{\mathrm{T}}) \begin{pmatrix} \mathbb{I} & \Pi_+^- e^{b\Lambda^-} \\ \Pi_-^+ e^{b\Lambda^+} & \mathbb{I} \end{pmatrix} = \begin{cases} \kappa(\boldsymbol{\pi}_{\Lambda^+}^{\mathrm{T}}, \mathbf{0}_-^{\mathrm{T}}), & \text{if } \kappa > 0 \\ (\mathbf{0}_+^{\mathrm{T}}, \mathbf{0}_-^{\mathrm{T}}), & \text{if } \kappa = 0 \\ \kappa(\mathbf{0}_+^{\mathrm{T}}, \boldsymbol{\pi}_{\Lambda^-}^{\mathrm{T}}), & \text{if } \kappa < 0 \end{cases}, \tag{33}$$

where $\boldsymbol{\pi}_{\Lambda^\pm}$ is the unique stationary distribution of $\Lambda^\pm$, which is well-defined if $\kappa \neq 0$.

Proof. Assume that $\kappa > 0$. From the above proof we know that

$$(\boldsymbol{u}_+^{\mathrm{T}}, -\boldsymbol{\ell}_-^{\mathrm{T}}) \begin{pmatrix} \mathbb{I} & \Pi_+^- e^{b\Lambda^-} \\ \Pi_-^+ e^{b\Lambda^+} & \mathbb{I} \end{pmatrix} \begin{pmatrix} V_+^+ & \mathbb{O} \\ \mathbb{O} & V_-^- \end{pmatrix} = \kappa(\boldsymbol{e}_1^{\mathrm{T}}, \mathbf{0}_-^{\mathrm{T}}). \tag{34}$$

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

Hence it is enough to check that $\boldsymbol{\pi}_{\Lambda^+}^{\mathrm{T}} = \boldsymbol{e}_1^{\mathrm{T}}(V_+^+)^{-1}$. Indeed, $\boldsymbol{e}_1^{\mathrm{T}}(V_+^+)^{-1}\Lambda^+ = \mathbf{0}$ according to Proposition 5. Moreover, it is easy to see that the components of this vector sum up to 1. □

6. THE NOISY M/E$_2$/1 QUEUE

One of the main contributions of our work is that it does not require the spectrum of $F(\alpha)$ to be semi-simple, i.e., that the algebraic multiplicities of all of its eigenvalues are not larger than their geometric multiplicities. The aim of this section is to show that a non-semi-simple spectrum may arise naturally. We do so by constructing non-trivial examples (that is, with $\kappa \neq 0$).

Consider the M/E$_2$/1 queue, that is a queuing system whose arrival process is given by a Poisson process with parameter λ and whose service times are Erlang(2) distributed with mean $2/\mu$. The virtual workload of this queue can be analyzed by a Markovian arrival process having three states, with diffusion vector $\boldsymbol{\sigma} = \mathbf{0}$, drift vector $\boldsymbol{a} = (-1, 1, 1)$ and background transition rate matrix

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -\mu & \mu \\ \mu & 0 & -\mu \end{pmatrix}.$$

When in state 1 the MAP-queue coincides with the workload process of the M/E$_2$/1 queue where the workload drains at rate -1 when it is positive. After an exponential time with mean $1/\lambda$, a new customer arrives and the queue jumps to state 2 where the workload is incremented by an independent exponential distributed amount with parameter μ corresponding to the first stage of the Erlang distribution (where the queue content cannot exceed b). After this, the environment jumps to state 3 where an equivalent and independent exponential amount of work is added corresponding to the second stage of the Erlang distribution (again truncated at b). Finally the system goes back to the state 1 where work is served at rate 1, etc.

We now slightly modify this example by adding a diffusion component when in state 1. In other words, we include a diffusion, with coefficient vector $\boldsymbol{\sigma} = (\sigma, 0, 0)$. We call the resulting system a 'noisy-M/E$_2$/1 queue'. In particular we consider the special case when $\sigma = 1$, $\lambda = 3/100$ and $\mu = 3/5$, so that the matrix polynomial is given by

$$F(\alpha) = \begin{pmatrix} a^2 - a - \frac{3}{100} & \frac{3}{100} & 0 \\ 0 & a - \frac{3}{5} & \frac{3}{5} \\ \mu & 0 & a - \frac{3}{5} \end{pmatrix}, \tag{35}$$

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

the asymptotic drift is negative, namely $\kappa = -9/11$, and the stationary distribution of the environment is given by the vector $\boldsymbol{\pi} = (10/11, 1/22, 1/22)^{\mathrm{T}}$. We have that $E^+ = \{1, 2, 3\}$ and $E^- = \{1\}$, and the roots of $\det(F(a))$ are given by $\{0, 2/5, 9/10, 9/10\}$, with $9/10$ having algebraic multiplicity equal to 2 and geometric multiplicity equal to 1.

Trivially we have that $\Lambda^- = (0)$, $\Pi^- = (1, 1, 1)^{\mathrm{T}}$, $\boldsymbol{\pi}_{\Lambda^-} = (1)$ and, by Proposition 5, the matrix $V^- = \Pi^-$. Using (8) it is easy to verify that $\Pi^+ = \mathbb{I}$ and

$$\Lambda^+ = \begin{pmatrix} -1 & \frac{1}{20} & \frac{1}{20} \\ 0 & -\frac{3}{5} & \frac{3}{5} \\ \frac{3}{5} & 0 & -\frac{3}{5} \end{pmatrix}; \quad V^+ = \begin{pmatrix} \frac{1}{3} & -\frac{1}{2} & -\frac{5}{3} \\ 3 & -2 & \frac{20}{3} \\ 1 & 1 & 0 \end{pmatrix}; \quad \Gamma^+ = \begin{pmatrix} \frac{2}{5} & 0 & 0 \\ 0 & \frac{9}{10} & 1 \\ 0 & 0 & \frac{9}{10} \end{pmatrix}, \tag{36}$$

with $\Lambda^+ = -V_+^+ \Gamma^+ (V_+^+)^{-1}$. From (36) one can see that $[(-1/2, -2, 1)^{\mathrm{T}}, (-5/3, 20/3, 0)^{\mathrm{T}}]$ is a Jordan chain associated to the eigenvalue $9/10$. It follows that a complete Jordan pair (V, Γ) of $F(\alpha)$ is then given by

$$V = \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{2} & -\frac{5}{3} \\ 1 & 3 & -2 & \frac{20}{3} \\ 1 & 1 & 1 & 0 \end{pmatrix}; \quad \Gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & 0 \\ 0 & 0 & \frac{9}{10} & 1 \\ 0 & 0 & 0 & \frac{9}{10} \end{pmatrix}. \tag{37}$$

Applying Theorem 6 we get $\ell_2, \ell_3 = 0$ and

$$(u_1, u_2, u_3, \ell_1) \begin{pmatrix} 1 & \frac{1}{3}e^{2b/5} & -\frac{1}{2}e^{9b/10} & -\frac{5}{3}e^{9b/10} - \frac{1}{2}be^{9b/10} \\ 1 & 3e^{2b/5} & -2e^{9b/10} & \frac{20}{3}e^{9b/10} - 2be^{9b/10} \\ 1 & e^{2b/5} & e^{9b/10} & be^{9b/10} \\ -1 & -\frac{1}{3} & \frac{1}{2} & \frac{5}{3} \end{pmatrix}$$
$$= \left(-\frac{9}{11}, 0, 0, 0\right). \tag{38}$$

Solving the system (38), we get the following loss rates as functions of the barrier level b:

$$u_1(b) = \frac{9\left(42 - 9b + 8e^{b/2}\right)}{11\left(-32 + 9b - 18e^{b/2} + 50e^{9b/10}\right)},$$

$$l_1(b) = \frac{450}{11\left(50 + e^{-9b/10}\left(-32 + 9b - 18e^{b/2}\right)\right)},$$

$$u_2(b) = \frac{9\left(-4 + 3b + 4e^{b/2}\right)}{22\left(-32 + 9b - 18e^{b/2} + 50e^{9b/10}\right)}, \quad l_2(b) = 0,$$

$$u_3(b) = \frac{9\left(-3b + 16\left(-1 + e^{b/2}\right)\right)}{22\left(-32 + 9b - 18e^{b/2} + 50e^{9b/10}\right)}, \quad l_3(b) = 0.$$

One can prove that for any $\mu > 1/2$ it is possible to obtain one root with algebraic multiplicity exceeding the geometric multiplicity by choosing

$$\lambda(\mu) = \frac{44\mu^2 - 16\mu - 1 + \sqrt{(10\mu - 1)^3(2\mu - 1)}}{8} \tag{39}$$

and that in general such a condition exists, for some region of the parameter space, for any value of $\sigma > 0$. This situation can be dealt with in the way demonstrated above.

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013

ACKNOWLEDGMENTS

The first author is partially supported by the Spanish Ministry of Education and Science Grants MTM2007-63140, SEJ2007-64500 and MTM2010-16519. Part of his research was done when he was visiting the Hebrew University of Jerusalem by partial support of Madrid University Carlos III Grant for Young Researchers' Mobility. The third author is partially supported by grant 964/06 from the Israel Science Foundation and the Vigevani Chair in Statistics.

REFERENCES

1. Ahn, S.; Badescu, A.L.; Ramaswami, V. Time dependent analysis of finite buffer fluid flows and risk models with a dividend barrier. Queueing Systems **2007**, *55*, 207–222.
2. Anick, D.; Mitra, D.; Sondhi, M. Stochastic theory of data-handling system with multiple sources. Bell Systems Technology Journal **1982**, *61*, 1871–1894.
3. Asmussen, S. Stationary distributions for fluid flow models with or without Brownian noise. Stochastic Models **1995**, *11*, 21–49.
4. Asmussen, S. *Applied Probability and Queues*. 2nd ed.; Springer-Verlag: New York, 2003.
5. Asmussen, S.; Kella, O. A multi-dimensional martingale for Markov additive processes and its applications. Advances in Applied Probability **2000**, *32*, 376–393.
6. Asmussen, S.; Philsgrd, M. Loss rates for Lévy processes with two reflecting barriers. Mathamatical Operations Research **2007**, *32*, 308–321.
7. Breuer, L. First passage times for Markov additive processes with positive jumps of phase type. Journal of Applied Probability **2008**, *45*, 779–799.
8. D'Auria, B.; Ivanovs, J.; Kella, O.; Mandjes, M. First passage of a Markov additive process and generalized Jordan chains. Journal of Applied Probability **2010**, *47*, 1048–1057.
9. Dukhovny, A. Multiple roots in some equations of queueing theory. Stochastic Models **1994**, *10*, 519–524.
10. Elwalid, A.I.; Mitra, D. Analysis and design of rate-based congestion control of high speed networks, I: stochastic fluid models, access regulation. Queueing System **1991**, *9*, 29–64.
11. Gaver, D.; Lehoczky, J. Channels that cooperatively service a data stream and voice messages. IEEE Transaction and Communication **1982**, *30*, 1153–1161.
12. Gohberg, I.; Lancaster, P.; Rodman, L. *Matrix Polynomials*; Academic Press: New York, 1982.
13. Gohberg, I.; Rodman, L. Analytic matrix functions with prescribed local data. Journal d'Analyse Mathématique **1981**, *40*, 90–128.

14. Harrison, J.M. *Brownian Motion and Stochastic Flow Systems*; Wiley: New York, 1985.
15. Horn, R.; Johnson, C. *Matrix Analysis*; Cambridge University Press: New York, 1990.
16. Karandikar, R.; Kulkarni, V. Second-order fluid flow models: reflected Brownian motion in a random environment. Operations Research **1995**, *43*, 77–88.
17. Kella, O.; Stadje, W. A Brownian motion with two reflecting barriers and Markov-modulated speed. Jornal of Applied Probability **2004**, *41*, 1237–1242.
18. Kennedy, J.; Williams, D. Probabilistic factorization of a quadratic matrix polynomial. Mathamatical Proceedings of the Cambridge Philosophical Society **1990**, *107*, 591–600.
19. Kruk, L.; Lehoczky, J.; Ramanan, K.; Shreve, S. An explicit formula for the Skorokhod map on $[0, A]$. Annals of Probability **2007**, *35*, 1740–1768.
20. Kyprianou, A.E. *Introductory Lectures on Fluctuations of Lévy Processes with Applications*; Springer-Verlag: Berlin, Heidelberg, 2006.
21. Kyprianou, A.E.; Palmowski, Z. Fluctuations of spectrally negative Markov additive processes. In *Séminaire de Probabilités XL*; Vol. 1934 of *Lecture Notes in Mathematics*; Springer: Berlin, 2008; 121–135.
22. Mitra, D. Stochastic theory of a fluid model of producers and customers coupled by a buffer. Advances in Applied Probability **1988**, *20*, 646–676.
23. Pacheco, A.; Prabhu, N.U. A Markovian storage model. Annals of Applied Probability **1996**, *6*, 76–91.
24. Rogers, L. Fluid models in queueing theory and Wiener–Hopf factorization of Markov chains. Annals of Applied Probability **1994**, *4*, 390–413.
25. Stern, T.; Elwalid, A. Analysis of separable Markov-modulated rate models for information-handling systems. Advances in Applied Probability **1991**, *23*, 105–139.
26. Whitt, W. *Stochastic-Process Limits. An Introduction to Stochastic-Process Limits and Their Applications to Queues*; Springer: New York, 2002.

Downloaded by [UVA Universiteitsbibliotheek SZ] at 06:29 22 January 2013