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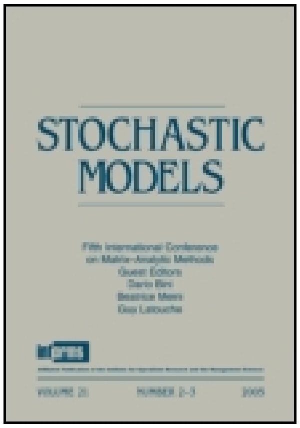

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QUASI-STATIONARY WORKLOAD IN A LÉVY-DRIVEN STORAGE SYSTEM

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□ *This article we analyzes the quasi-stationary workload of a Lévy-driven storage system. More precisely, assuming the system is in stationarity, we study its behavior conditional on the event that the busy period T in which time 0 is contained has not ended before time t, as $t \to \infty$. We do so by first identifying the double Laplace transform associated with the workloads at time 0 and time t, on the event $\{T > t\}$. This transform can be explicitly computed for the case of spectrally one-sided jumps. Then asymptotic techniques for Laplace inversion are relied upon to find the corresponding behavior in the limiting regime that $t \to \infty$. Several examples are treated; for instance in the case of Brownian input, we conclude that the workload distribution at time 0 and t are both Erlang(2).*

Keywords Fluctuation theory; Heaviside principle; Laplace transforms; Lévy processes; Quasi-stationary distribution; Storage systems.

Mathematics Subject Classification Primary 60G51, 60G50, 60K25; Secondary 60J99, 93E20.

1. INTRODUCTION

Consider a storage system with Lévy netput, i.e., the process $(Q(t))_t$ that evolves as a Lévy process $X(t)$ that is reflected at 0. In mathematical terms, this means that the workload in the storage system at time t is given by

$$Q(t) := x + X(t) - \inf_{s \le t}(x + X(s))^-,$$

where $a^- = \min\{a, 0\}$. Assuming $\mathbb{E}X(1) < 0$, there exists a stationary distribution π of $Q(t)$; it is known that the stationary workload is

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distributed as the all-time supremum:

$$\pi(x) = \mathbb{P}\left(\sup_{t\geq 0} X(t) \leq x\right).$$

In the sequel we add the subscript π to the probability measure $\mathbb{P}$ and the associated expectation $\mathbb{E}$ when we wish to indicate that $Q(0)$ is distributed according to this stationary distribution.

Now let T denote the *busy period*, that is $T = \inf\{t \geq 0 : Q(t) = 0\}$. In this article the object of our interest concerns the existence and characterization of the joint conditional distribution

$$\lim_{t\to\infty} \mathbb{P}_\pi(Q(0) \in \mathrm{d}x, Q(t) \in \mathrm{d}y \mid T > t) =: \mu(\mathrm{d}x, \mathrm{d}y),$$

where the convergence is to be understood in the weak sense. We study this so-called *quasi-stationary distribution* μ in detail; special attention is paid to the marginal distributions $\mu(\cdot \times \mathbb{R}) = \mu_L^{\mathrm{QS}}(\cdot)$ and $\mu(\mathbb{R} \times \cdot) = \mu_R^{\mathrm{QS}}(\cdot)$. As an aside we mention that sometimes $\mu(\mathbb{R}_+ \times \mathrm{d}y)$ is called quasi-stationary distribution; here we do not follow that convention.

A substantial body of work has been devoted to the analysis of quasi-stationary distributions. Over the past decades, various settings were considered; we here give a brief (non-exhaustive) overview. Seneta and Vere-Jones[15], Tweedie[16], Jacka and Roberts[7] focus on a Markov chain setting, Iglehart[6] addresses a random walk setup, Kyprianou[8] considers the M/G/1 queue (i.e., a storage system with compound Poisson input), whereas Martinez and San Martin[11] treat the case of Brownian motion with drift. We also mention the contribution by Kyprianou and Palmowski[10], who found the quasi-stationary distribution associated with a general light-tailed Lévy process. Recently, Rivero[14] (after appropriate scaling) found the quasi-stationary distribution for the specific situation that the Lévy process under study has a jump measure with a regularly varying tail.

The contribution of this article is twofold. In the first place, a general formula for the double Laplace transform $(Q(0), Q(t))$ on the event $\{T > t\}$, that is,

$$\int_0^\infty e^{-\vartheta t}\mathbb{E}_\pi[e^{-\alpha Q(0) - \beta Q(t)}, T > t]\,\mathrm{d}t \tag{1}$$

is given (to which we refer to as the *master formula*). The derivation is based on the Wiener-Hopf factorization, and can be evaluated explicitly when all jumps are either all positive (the so-called spectrally-positive case) or all jumps are negative (spectrally-negative case). These formulae allow us to identify the quasi-stationary measures for the spectrally one-sided cases (relying on Tauberian-type theorems), which can be regarded as the second major contribution.

The article provides interesting insights into the distribution of the workload conditional on a long busy period. The distributions found tend to be stochastically larger than the normal, stationary distribution. For instance in the case of regulated standard Brownian motion (with drift -1), both $\mu_L^{QS}(\cdot)$ and $\mu_R^{QS}(\cdot)$ correspond to Erlang(2) distributions (with mean 2), whereas the stationary distribution is exponential (with mean $\frac{1}{2}$). This type of insights can potentially be used when setting up efficient importance sampling algorithms[4] (as in those algorithms a change of measure is looked for that mimics the distribution conditional on the rare event under consideration).

This article is organized as follows. In Section 2 preliminaries are given: (i) we first recapitulate a set of main results on fluctuation theory for Lévy processes, and (ii) then present Tauberian theorems that are useful in the context of this article (which can be used to identify the tail behavior of a random variable from its Laplace transform). The main objective of Section 3 concerns the derivation of the master formula, i.e., an expression for (1) in terms of the Wiener-Hopf factorization (with explicit results for the spectrally one-sided cases). Our findings on the quasi-stationary distribution are then given in Section 4. The last section treats a number of examples.

2. PRELIMINARIES

2.1. Lévy Processes

Here we follow Ref.[9] for definitions, notations and basic facts on Lévy processes. Let in the sequel $X \equiv (X(t))_t$ be a Lévy process which is defined on the filtered space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with the natural filtration that satisfies the usual assumptions of right continuity and completion. Later if we write $\mathbb{P}_x$, it means that $\mathbb{P}_x(X(0) = x) = 1$ and $\mathbb{P}_0 = \mathbb{P}$; similarly, $\mathbb{E}_x$ is expectation with respect to $\mathbb{P}_x$. We denote by $\Pi(\cdot)$ the jump measure of X. Later we will focus on asymmetric Lévy processes, which are either spectrally negative (having non-positive jumps) or spectrally positive Lévy processes (having non-negative jumps).

First Passage Times.

For any Lévy process we can define its Laplace exponent $\psi(\eta)$ by

$$\mathbb{E}e^{\eta X(t)} = e^{t\psi(\eta)}, \tag{2}$$

for $\eta \in \Theta$ such that the left-hand side of (2) is well-defined. Later, we also need the first passage time

$$\tau(x) := \min\{t : X(t) \geq x\},$$

which for a spectrally negative process X with positive drift (i.e., $\mathbb{E}X(1) > 0$) has Laplace transform

$$\mathbb{E}e^{-s\tau(x)} = e^{-\Phi(s)x}, \tag{3}$$

where $\Phi(s) := \inf\{\eta \geq 0 : \psi(\eta) > s\}$ is the right inverse of ψ (see Ref.[9] for details).

Exponential Change of Measure.
For $\eta \in \Theta$ we define a new probability measure $\mathbb{P}_x^\eta$ by the relation

$$\left.\frac{d\mathbb{P}_x^\eta}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = e^{\eta(X(t)-x)-\psi(\eta)t};$$

we say that we have performed an exponential change of measure. Under $\mathbb{P}^\eta$, the process X is still a Lévy process, but now with Laplace exponent

$$\psi_\eta(\beta) := \psi(\eta+\beta) - \psi(\eta). \tag{4}$$

We will use the subscript η to indicate that the quantity under consideration relates to $\mathbb{P}^\eta$.

Dual Process.
We will also consider the so-called *dual process* $\widehat{X}_t = -X_t$ with jump measure $\widehat{\Pi}\left(0, y\right) = \Pi\left(-y, 0\right)$. Characteristics of $\widehat{X}$ will be indicated by using the same symbols as for X, but with a '^' added.

Ladder Heights.
For the process X we define the associated $(L^{-1}(t), H(t))_t$:

$$L^{-1}(t) := \begin{cases} \inf\{s > 0 : L(s) > t\} & \text{if } t < L(\infty), \\ \infty & \text{otherwise,} \end{cases}.$$

and

$$H(t) := \begin{cases} X_{L^{-1}(t)} & \text{if } t < L(\infty), \\ \infty & \text{otherwise,} \end{cases}.$$

where $L \equiv (L(t))_t$ is the local time at the maximum[9] (p. 140) . Recall that (L^{-1}, H) is a bivariate subordinator with the Laplace exponent

$$\kappa(\varphi, \beta) := -\frac{1}{t}\log\mathbb{E}\left(e^{-\varphi L^{-1}(t)-\beta H(t)}\mathbf{1}_{\{t \leq L(\infty)\}}\right)$$

and with the jump measure Π_H. In addition to this, we define the *descending ladder height process* $(\widehat{L}^{-1}(t), \widehat{H}(t))_{t\geq 0}$ with the Laplace exponent $\hat{\kappa}(\vartheta, \beta)$ constructed from the dual process $\widehat{X}$. Recall that under the stability assumption $\mathbb{E}X(1) < 0$, the random variable $L(\infty)$ has an exponential distribution with parameter $\kappa(0, 0)$. Moreover, for a spectrally negative Lévy process the Wiener-Hopf factorization implies that

$$\kappa(\varphi, \beta) = \Phi(\varphi) + \beta, \quad \hat{\kappa}(\varphi, \beta) = \frac{\varphi - \psi(\beta)}{\Phi(\varphi) - \beta}; \tag{5}$$

see Ref.[9] (pp. 169–170). It follows that $\kappa(0, 0) = \Phi(0)$.

We introduce a *potential measure* $\mathcal{U}$ defined by

$$\mathcal{U}(\mathrm{d}x, \mathrm{d}s) = \int_{t=0}^{\infty} \mathbb{P}\left(L^{-1}(t) \in \mathrm{d}s, H(t) \in \mathrm{d}x\right) \mathrm{d}t$$

with the Laplace transform $\int_{[0,\infty)^2} e^{-\varphi s - \beta x}\mathcal{U}(\mathrm{d}x, \mathrm{d}s) = 1/\kappa(\varphi, \beta)$ and renewal function

$$V(\mathrm{d}x) = \int_{s=0}^{\infty} \mathcal{U}(\mathrm{d}x, \mathrm{d}s) = \mathbb{E}\left(\int_{t=0}^{\infty} \mathbf{1}_{\{H(t)\in \mathrm{d}x\}}\mathrm{d}t\right).$$

In particular,

$$\int_0^{\infty} e^{-\beta x} V(x)\mathrm{d}x = \frac{1}{\beta\kappa(0, \beta)}. \tag{6}$$

For a spectrally negative Lévy process, the upward ladder height process is just a linear drift, and hence the renewal measure corresponds to the Lebesgue measure:

$$V(\mathrm{d}x) = \mathrm{d}x. \tag{7}$$

From (5) and Ref.[9] (p. 195), we have that

$$\int_0^{\infty} e^{-\varphi x}\widehat{V}(\mathrm{d}x) = \frac{\varphi}{\psi(\varphi)}.$$

According to our convention $V_\eta, \widehat{V}_\eta, \kappa_\eta, \hat{\kappa}_\eta$ are quantities computed under $\mathbb{P}^\eta$.

2.2. Tauberian-Type Results

Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = 0$ for $x < 0$. Let $\tilde{f}(z) := \int_0^{\infty} e^{-zx} f(x)\, \mathrm{d}x$ for $z \in \mathbb{R}$ be its Laplace transform. Consider singularities of $\tilde{f}(z)$; among these, let $\vartheta^\star < 0$ the one with the largest real part. Notice that this yields the integrability of $\int_0^\infty |f(x)|\,\mathrm{d}x$. The inversion formula reads

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \tilde{f}(z) e^{zx}\,\mathrm{d}z$$

for some (and then any) $a > \vartheta^\star$.

We now focus on a class of theorems that infer the tail behavior of a function from its Laplace transform, commonly referred to as *Tauberian theorems*. Importantly, the behavior of the Laplace transform around the singularity $\vartheta^\star$ plays a crucial role here. The following heuristic principle given in Ref.[1] is often relied upon. Suppose that for $\vartheta^\star$, some constants K and C, and a non-integer $s > 0$,

$$\tilde{f}(\vartheta) = K - C(\vartheta - \vartheta^\star)^s + o((\vartheta - \vartheta^\star)^s), \quad \text{as } \vartheta \downarrow \vartheta^\star.$$

Then

$$f(x) = \frac{C}{\Gamma(-s)} x^{-s-1} e^{\vartheta^\star x}(1 + o(1)), \quad \text{as } x \to \infty,$$

where $\Gamma(s)$ is the gamma function. Below we specify conditions under which this relation can be rigorously proven. Later in our article we apply it for the specific case that $s = 1/2$; recall that $\Gamma(-1/2) = -2\sqrt{\pi}$.

A formal justification of the above relation can be found in Doetsch[2] (Theorem 37.1). Following Miyazawa and Rolski[12], we consider the following specific form. For this we first recall the concept of the $\mathfrak{W}$-contour with an half-angle of opening $\pi/2 < \psi \leq \pi$, as depicted in Ref.[2] (Fig. 30, p. 240); also, $\mathcal{G}_\alpha(\psi)$ is the region between the contour $\mathfrak{W}$ and the line $\Re(z) = 0$. More precisely,

$$\mathcal{G}_\alpha(\psi) \equiv \{z \in \mathbb{C}; \Re(z) < 0, z \neq \alpha, |\arg(z - \alpha)| < \psi\},$$

where $\arg z$ is the principal part of the argument of the complex number z. In the following theorem, conditions are identified such that the above principle holds; we refer to this as the *Heaviside's operational principle*, or simply *Heaviside principle*.

Theorem 1 (Heaviside Principle). *Suppose that for $\tilde{f} : \mathbb{C} \to \mathbb{C}$ and $\zeta^\star < 0$ the following three conditions hold:*

(A1) $\tilde{f}(\cdot)$ *is analytic in a region $\mathcal{G}_{\zeta^\star}(\psi)$ for some $\pi/2 < \psi \leq \pi$;*
(A2) $\tilde{f}(z) \to 0$ *as $|z| \to \infty$ for $z \in \mathcal{G}_{\zeta^\star}(\psi)$;*

(A3) *for some constants K and C, and a non-integer $s > 0$,*

$$\tilde{f}(z) = K\mathbf{1}_{\{s>0\}} - C(z-\zeta^\star)^s + o((z-\zeta^\star)^s), \tag{8}$$

where $\mathcal{G}_{\zeta^\star}(\psi) \ni z \to \zeta^\star$.

Then

$$f(x) = \frac{C}{\Gamma(-s)} x^{-s-1} e^{\zeta^\star x}(1+o(1)),$$

as $x \to \infty$, where $K := \tilde{f}(\zeta^\star)$ if $s > 0$.

We now discuss when assumption (A1) is satisfied. To check that the Laplace transform $\tilde{f}(\cdot)$ is analytic in the region $\mathcal{G}_{\zeta^\star}(\psi)$, we can use the concept of semiexponentiality of f (see Ref.[5]).

Definition 2 (Semiexponentiality). f is said to be *semiexponential* if for some $0 < \phi \leq \pi/2$, there exists finite and strictly negative $\gamma(\vartheta)$, defined as the infimum of all such a such that

$$\left|f(e^{i\vartheta} r)\right| < e^{ar}$$

for all sufficiently large r; here $-\phi \leq \vartheta \leq \phi$ and $\sup \gamma(\vartheta) < 0$.

Relying on this concept, the following sufficient condition for (A1) applies.

Proposition 3[5] (Thm. 10.9f). *Suppose that f is semiexponential with $\gamma(\vartheta)$ fulfilling the following conditions: (i) $\gamma = \gamma(0) < 0$, (ii) $\gamma(\vartheta) \geq \gamma(0)$ in a neighborhood of $\vartheta = 0$, and (iii) it is smooth. Then* (A1) *is satisfied.*

3. MASTER FORMULA

The objective of this section is to derive a general formula for the double Laplace–Stieltjes transforms

$$L_x(\vartheta; \alpha, \beta) := \int_0^\infty e^{-\vartheta t} \mathbb{E}_x[e^{-\alpha x - \beta Q(t)}, T > t] \mathrm{d}t$$

and

$$\begin{aligned} L(\vartheta; \alpha, \beta) &:= \int_0^\infty L_x(\vartheta; \alpha, \beta) \mathrm{d}\mathbb{P}(Q(0) \leq x) \\ &= \int_0^\infty e^{-\vartheta t} \mathbb{E}_\pi[e^{-\alpha Q(0) - \beta Q(t)}, T > t] \mathrm{d}t. \end{aligned}$$

Let e_ϑ be an exponentially distributed random variable with parameter $\vartheta > 0$, independent of the process X. Denote

$$\overline{X}(t) := \sup_{s\le t} X(s), \quad \underline{X}(t) := \inf_{s\le t} X(s).$$

From Eq. (10) in Ref.[10] we can get the the following result.

Theorem 4. *For $\alpha, \beta, \vartheta > 0$ and $x \geq 0$,*

$$L_x(\vartheta;\alpha,\beta) = \frac{1}{\vartheta}\frac{\kappa(\vartheta,0)}{\kappa(\vartheta,\beta)} e^{-(\alpha+\beta)x}\left(\int_0^x e^{\beta z}\widehat{\mathbb{P}}(\overline{X}(e_\vartheta) \in \mathrm{d}z)\right).$$

We now evaluate $L(\vartheta;\alpha,\beta)$ for the spectrally one-sided cases.

Proposition 5. *If X is a spectrally positive Lévy process, then*

$$L(\vartheta;\alpha,\beta) = \frac{\hat{\psi}'(0+)}{\vartheta - \hat{\psi}(\beta)}\left(\frac{\alpha+\beta}{\hat{\psi}(\alpha+\beta)} - \frac{\alpha+\widehat{\Phi}(\vartheta)}{\hat{\psi}(\alpha+\widehat{\Phi}(\vartheta))}\right). \tag{9}$$

Proof. Note that

$$\widehat{\mathbb{P}}(\overline{X}(e_\vartheta) \geq x) = \widehat{\mathbb{P}}(\tau(x) \leq e_\vartheta) = \widehat{\mathbb{E}}e^{-\vartheta\tau(x)} = e^{-\widehat{\Phi}(\vartheta)x}.$$

Integration by parts yields

$$\int_0^x e^{\beta z}\,\widehat{\mathbb{P}}(\overline{X}(e_\vartheta) \in \mathrm{d}z) = \frac{\widehat{\Phi}(\vartheta)}{\beta - \widehat{\Phi}(\vartheta)}\left(e^{(\beta-\widehat{\Phi}(\vartheta))x} - 1\right).$$

The Pollaczek–Khintchine formula[9] (Eq. (4.14), p. 101) states that

$$\tilde{\pi}(s) := \int_0^\infty e^{-sx}\,\pi(\mathrm{d}x) = \frac{\hat{\psi}'(0+)s}{\hat{\psi}(s)}. \tag{10}$$

It now follows that

$$\begin{aligned}
L(\vartheta;\alpha,\beta) &= \frac{1}{\vartheta}\frac{\kappa(\vartheta,0)}{\kappa(\vartheta,\beta)}\int_0^\infty e^{-(\alpha+\beta)x}\int_0^x e^{\beta z}\widehat{\mathbb{P}}(\overline{X}(e_\vartheta) \in \mathrm{d}z)\,\pi(\mathrm{d}x)\\
&= \frac{1}{\vartheta}\frac{\kappa(\vartheta,0)}{\kappa(\vartheta,\beta)}\frac{\widehat{\Phi}(\vartheta)}{\beta-\widehat{\Phi}(\vartheta)}\int_0^\infty \left(e^{-(\alpha+\widehat{\Phi}(\vartheta))x} - e^{-(\alpha+\beta)x}\right)\pi(\mathrm{d}x)\\
&= \frac{\hat{\psi}'(0+)}{\vartheta}\frac{\kappa(\vartheta,0)}{\kappa(\vartheta,\beta)}\frac{\widehat{\Phi}(\vartheta)}{\beta-\widehat{\Phi}(\vartheta)}\left(\frac{\alpha+\widehat{\Phi}(\vartheta)}{\hat{\psi}(\alpha+\widehat{\Phi}(\vartheta))} - \frac{\alpha+\beta}{\hat{\psi}(\alpha+\beta)}\right).
\end{aligned}$$

The Wiener–Hopf factorization[9] (Section 6.5.2) and (5) complete the proof. □

A similar result can be derived for the spectrally negative case.

Proposition 6. *If X is a spectrally negative Lévy process, then*

$$L(\vartheta;\alpha,\beta) = \frac{\Phi(\vartheta)-\alpha-\Phi(0)}{\Phi(\vartheta)+\beta}\frac{\Phi(0)}{\alpha+\beta+\Phi(0)}\frac{1}{\vartheta-\psi(\alpha+\Phi(0))}. \tag{11}$$

Proof. Recall the well-known fact that $\pi(\mathrm{d}x) = \Phi(0)e^{-\Phi(0)x}\,\mathrm{d}x$ by (3). Applying Theorem 4 and interchanging the order of integration,

$$\begin{aligned} L(\vartheta;\alpha,\beta) &= \frac{\Phi(0)}{\vartheta}\frac{\kappa(\vartheta,0)}{\kappa(\vartheta,\beta)}\int_0^\infty e^{-(\alpha+\beta+\Phi(0))x}\left(\int_0^x e^{\beta z}\widehat{\mathbb{P}}(\overline{X}(e_\vartheta)\in \mathrm{d}z)\right)\mathrm{d}x \\ &= \frac{1}{\vartheta}\frac{\kappa(\vartheta,0)}{\kappa(\vartheta,\beta)}\frac{\Phi(0)}{\alpha+\beta+\Phi(0)}\widehat{\mathbb{E}}e^{-(\alpha+\Phi(0))\overline{X}(e_\vartheta)} \\ &= \frac{1}{\vartheta}\frac{\kappa(\vartheta,0)}{\kappa(\vartheta,\beta)}\frac{\Phi(0)}{\alpha+\beta+\Phi(0)}\mathbb{E}e^{-(\alpha+\Phi(0))\underline{X}(e_\vartheta)}. \end{aligned}$$

This gives by Eqs. (5) and[9] (Th. 6.16(ii)), in conjunction with the fact that $\vartheta = \kappa(\vartheta,0)\hat{\kappa}(\vartheta,0)$,

$$\begin{aligned} L(\vartheta;\alpha,\beta) &= \frac{1}{\vartheta}\frac{\kappa(\vartheta,0)}{\kappa(\vartheta,\beta)}\frac{\Phi(0)}{\alpha+\beta+\Phi(0)}\frac{\hat{\kappa}(0,0)}{\hat{\kappa}(\vartheta,\alpha+\Phi(0))} \\ &= \frac{\Phi(\vartheta)-\alpha-\Phi(0)}{\Phi(\vartheta)+\beta}\frac{\Phi(0)}{\alpha+\beta+\Phi(0)}\frac{1}{\vartheta-\psi(\alpha+\Phi(0))}. \end{aligned}$$

This completes the proof. □

4. QUASI-STATIONARY DISTRIBUTION

In this section we use the Laplace transforms given in (9) and (11) to identify the quasi-stationary distribution $\mu(\mathrm{d}x,\mathrm{d}y)$ for the spectrally one-sided cases.

4.1. Spectrally Positive Lévy Process

We impose the following additional assumptions:

[SP1] There exists $\vartheta_+ < 0$ such that

- $\hat{\psi}(\vartheta) < \infty$ for $\vartheta_+ < \vartheta$,

- $\hat{\psi}(\vartheta)$ attains its strictly negative minimum at $\vartheta^\star < 0$, where $\vartheta_+ < \vartheta^\star < 0$ (and hence $\hat{\psi}'(\vartheta^\star) = 0$).

Denote $\zeta^\star := \hat{\psi}(\vartheta^\star) < 0$. Note that the function $\hat{\Phi}$ can be considered in the complex domain. It is clearly analytic for $\Re(\vartheta) > \zeta^\star$. But, as it will turn out, more is required to obtain the quasi-stationary distribution.

[SP2] One can extend analytically $L(\vartheta; \alpha, \beta)$ into $\mathcal{G}_{\zeta^\star}(\psi)$ for some $\pi/2 < \psi \leq \pi$.

Example 7. Since $\hat{\Phi}(\vartheta)$ is the Laplace exponent of a subordinator (viz. a first passage time process), we have the following spectral representation:

$$\widehat{\Phi}(\vartheta) = d_+\vartheta + \int_0^\infty (1 - e^{-\vartheta x})\Pi_+(\mathrm{d}x), \tag{12}$$

and $\int_0^\infty (x \wedge 1)\Pi_+(\mathrm{d}x) < \infty$[9] (Exercise 2.11) . From its definition we see that $\zeta^\star$ must be a singular point of $\hat{\Phi}(\vartheta)$. Moreover, if there exists a density of Π_+ which is of semiexponential type, then from Proposition 3 it follows then that $\widehat{\Phi}$ is analytic in $\mathcal{G}_{\zeta^\star}(\phi)$ and assumption [SP2] is satisfied. In particular, assumption [SP2] is for example satisfied for

$$\Pi_+(\mathrm{d}x) = e^{\zeta^\star x} x^\alpha \, \mathrm{d}x \, \mathbf{1}_{\{x>0\}}, \tag{13}$$

for $\alpha > -2$. Clearly, then $\gamma(\vartheta) = \zeta^\star \cos\vartheta$. Note that (13) is satisfied for a linear Brownian motion $X(t) = B(t) - t$. Indeed, in this case $\hat{\Phi}(\vartheta)$ is the Laplace exponent of the inverse Gaussian process with Lévy measure $\Pi_+(\mathrm{d}x) = (2\pi)^{-1/2} x^{-3/2} \exp\{-\frac{x}{2}\} \, \mathrm{d}x$.

Theorem 8. *If X is a spectrally positive Lévy process satisfying conditions* [SP1–SP2], *then*

$$\mu(\mathrm{d}x, \mathrm{d}y) = Q_+ \left(-\hat{\psi}'(0+) \frac{\hat{\psi}(\vartheta) - \vartheta\hat{\psi}'(\vartheta)}{(\hat{\psi}(\vartheta))^2} \right) e^{\vartheta^\star (y-x)} x V_{-\vartheta^\star}(y) \, \pi(\mathrm{d}x)\mathrm{d}y \mathbf{1}_{\{x \geq 0, y \geq 0\}},$$

where $Q_+ := (\int_0^\infty e^{\vartheta^\star z} V_{-\vartheta^\star}(z)\mathrm{d}z)^{-1}$.

Corollary 9. *We have*

$$\mu_L^{\mathrm{QS}}(\mathrm{d}x) = \left(-\hat{\psi}'(0+) \frac{\hat{\psi}(\vartheta) - \vartheta\hat{\psi}'(\vartheta)}{(\hat{\psi}(\vartheta))^2} \right) e^{-\vartheta^\star x} x \, \pi(\mathrm{d}x) \mathbf{1}_{\{x \geq 0\}}$$

and

$$\mu_R^{\mathrm{QS}}(\mathrm{d}y) = Q_+ e^{\vartheta^\star y} V_{-\vartheta^\star}(y) \, \mathrm{d}y \mathbf{1}_{\{y \geq 0\}}.$$

Before we prove Theorem 8, we first present a few facts. Let $k^\star := \sqrt{2/\hat{\psi}''(\vartheta^\star)}$.

Lemma 10. *Under* [SP1–SP2],

$$\widehat{\Phi}(\vartheta) = \vartheta^\star + k^\star(\vartheta - \zeta^*)^{1/2} + o((\vartheta - \zeta^*)^{1/2})$$

as $\vartheta \downarrow \zeta^\star$.

Proof. From a Taylor series expansion and the condition that $\hat{\psi}'(\vartheta^\star) = 0$, we have

$$\hat{\psi}(\vartheta) - \hat{\psi}(\vartheta^\star) = \frac{(\vartheta - \vartheta^\star)^2}{2}\hat{\psi}''(\vartheta^\star) + o((\vartheta - \vartheta^\star)^2).$$

After some rearranging, it is obtained that

$$\vartheta - \vartheta^\star = \sqrt{\frac{2}{\hat{\psi}''(\vartheta^\star)}}\sqrt{\hat{\psi}(\vartheta) - \hat{\psi}(\vartheta^\star)} + o((\vartheta - \vartheta^\star)^2).$$

We now substitute $\vartheta = \widehat{\Phi}(s)$, and use $\hat{\psi}(\widehat{\Phi}(s)) = s$ to complete the proof. □

Proposition 11. *Under* [SP1–SP2] *we have*

$$\begin{aligned}&\lim_{t\to\infty} \mathbb{E}_\pi[e^{-\alpha Q(0)-\beta Q(t)} \mid T > t] \\ &\quad = \left(\zeta^\star \cdot \frac{\hat{\psi}(\alpha + \vartheta^\star) - (\alpha + \vartheta^\star)\hat{\psi}'(\alpha + \vartheta^\star)}{\hat{\psi}^2(\alpha + \vartheta^\star)}\right)\left(\frac{\zeta^\star}{\zeta^\star - \hat{\psi}(\beta)}\right). \qquad (14)\end{aligned}$$

Proof. By [SP1–SP2] and Proposition 5 $L(\vartheta, \alpha, \beta)$, as given in (9) as a function of ϑ, is analytic in $\mathcal{G}_{\zeta^\star}(\phi)$ for $\pi/2 < \phi \le \pi$ when $\hat{\psi}(\alpha + \widehat{\Phi}(z))$ is analytical there. Recall that $\widehat{\Phi}$ is analytic in this region and note that $\hat{\psi}(\alpha + \widehat{\Phi}(z))$ is analytical there since $\hat{\psi}(\widehat{\Phi}(z)) = z$ is analytical in this region. Thus condition (A1) of Theorem 1 is satisfied.

To check that condition (A2) of Thm. 1 holds for $L(\vartheta, \alpha, \beta)$, it suffices to prove that

$$\left|\frac{1}{z}\frac{\alpha + \widehat{\Phi}(z)}{\hat{\psi}(\alpha + \widehat{\Phi}(z))}\right| \qquad (15)$$

tends to 0 for $z \in \mathcal{G}_{\zeta^\star}(\phi)$ tending to ∞ two-dimensionally, that is in particular for $\Im z \to \pm\infty$. Denoting $\widehat{\Phi}(z) := a + bi$ note that

$$\Im \frac{\hat{\psi}(a+b\mathrm{i})}{|a+b\mathrm{i}|} = \frac{1}{\sqrt{a^2+b^2}}\left(\hat{\psi}'(0)b + ab\sigma^2 + \int_{-\infty}^{0}(\sin bx - b1_{\{|x|\le 1\}}x)\widehat{\Pi}(\mathrm{d}x)\right)$$

cannot be zero. Taking into account the term $1/z$ in (15), this completes the verification of condition (A2). We will check now that also condition (A3) of Theorem 1 is satisfied. Now using Lemma 10 we write

$$\begin{aligned}\hat{\psi}(\alpha + \widehat{\Phi}(\vartheta)) &= \hat{\psi}(\alpha + \vartheta^\star + k^\star(\vartheta - \zeta^\star)^{1/2} + o((\vartheta-\zeta^\star)^{1/2})\\ &= \hat{\psi}(\alpha+\vartheta^\star) + \hat{\psi}'(\alpha+\vartheta^\star)k^\star(\vartheta-\zeta^\star)^{1/2} + o((\vartheta-\zeta^\star)^{1/2}).\end{aligned}$$

Hence by Proposition 5 (for $\vartheta \downarrow \zeta^\star$) we have, for some $\overline{K}$,

$$\begin{aligned}&L(\vartheta;\alpha,\beta)\\ &= \frac{\hat{\psi}'(0+)}{\vartheta - \hat{\psi}(\beta)}\left(\frac{\alpha+\beta}{\hat{\psi}(\alpha+\beta)} - \frac{\alpha+\vartheta^\star + k^\star(\vartheta-\zeta^\star)^{1/2} + o((\vartheta-\zeta^\star)^{1/2})}{\hat{\psi}(\alpha+\vartheta^\star+\gamma(\vartheta-\zeta^\star)^{1/2}+o((\vartheta-\zeta^\star)^{1/2})}\right)\\ &= \overline{K} - \frac{\hat{\psi}'(0+)}{\vartheta-\hat{\psi}(\beta)}\,\frac{\alpha+\vartheta^\star+k^\star(\vartheta-\zeta^\star)^{1/2}+o((\vartheta-\zeta^\star)^{1/2}}{\hat{\psi}(\alpha+\vartheta^\star)+\hat{\psi}'(\alpha+\vartheta^\star)k^\star(\vartheta-\zeta^\star)^{1/2}+o((\vartheta-\zeta^\star)^{1/2})}\\ &= \overline{K} - \frac{\hat{\psi}'(0+)}{\vartheta-\hat{\psi}(\beta)}\,\frac{\alpha+\vartheta^\star+k^\star(\vartheta-\zeta^\star)^{1/2}+o((\vartheta-\zeta^\star)^{1/2}}{\hat{\psi}(\alpha+\vartheta^\star)+\hat{\psi}'(\alpha+\vartheta^\star)k^\star(\vartheta-\zeta^\star)^{1/2}+o((\vartheta-\zeta^\star)^{1/2})}\\ &\quad\times\frac{\hat{\psi}(\alpha+\vartheta^\star)-\hat{\psi}'(\alpha+\vartheta^\star)k^\star(\vartheta-\zeta^\star)^{1/2}+o((\vartheta-\zeta^\star)^{1/2})}{\hat{\psi}(\alpha+\vartheta^\star)-\hat{\psi}'(\alpha+\vartheta^\star)k^\star(\vartheta-\zeta^\star)^{1/2}+o((\vartheta-\zeta^\star)^{1/2})}.\end{aligned}$$

Thus we obtain that, for some K,

$$\begin{aligned}L(\vartheta;\alpha,\beta) = K &- \frac{\hat{\psi}'(0+)k^\star}{\zeta^\star-\hat{\psi}(\beta)}\frac{\hat{\psi}(\alpha+\vartheta^\star)-(\alpha+\vartheta^\star)\hat{\psi}'(\alpha+\vartheta^\star)}{\hat{\psi}^2(\alpha+\vartheta^\star)}(\vartheta-\zeta^\star)^{1/2}\\ &+ o((\vartheta-\zeta^\star)^{1/2}).\end{aligned}$$

Conclude by invoking 'Heaviside' that

$$\begin{aligned}&\mathbb{E}_\pi[e^{-\alpha Q(0)-\beta Q(t)}, T > t]\\ &\quad= \frac{\hat{\psi}'(0+)k^\star}{\zeta^\star-\hat{\psi}(\beta)}\frac{\hat{\psi}(\alpha+\vartheta^\star)-(\alpha+\vartheta^\star)\hat{\psi}'(\alpha+\vartheta^\star)}{\hat{\psi}^2(\alpha+\vartheta^\star)}\frac{t^{-3/2}}{\Gamma(-1/2)}e^{\zeta^\star t}(1+o(1)).\end{aligned}$$

By setting $\alpha = \beta = 0$ we have

$$\mathbb{P}_\pi(T > t) = \frac{\hat{\psi}'(0+)k^\star}{(\zeta^\star)^2}\frac{t^{-3/2}}{\Gamma(-1/2)}e^{\zeta^\star t}(1 + o(1)).$$

It is now seen that Eq. (14) holds and the proof is completed. □

Proof of Theorem 8. From Proposition 11 it follows that

$$\tilde{\mu}(\alpha,\beta) = \int_0^\infty \int_0^\infty e^{-\alpha x}e^{-\beta y}\,\mu(\mathrm{d}x,\mathrm{d}y) = \widetilde{A}_+(\alpha)\widetilde{B}_+(\beta),$$

where

$$\widetilde{A}_+(\alpha) := \zeta^\star \cdot \frac{\hat{\psi}(\alpha+\vartheta^\star) - (\alpha+\vartheta^\star)\hat{\psi}'(\alpha+\vartheta^\star)}{\hat{\psi}^2(\alpha+\vartheta^\star)}, \quad \widetilde{B}_+(\beta) := \frac{\zeta^\star}{\zeta^\star - \hat{\psi}(\beta)}.$$

- By the Pollaczek–Khintchine formula (10) and[9] ((4.14), p. 101) applied for the dual, we derive

$$\int_0^\infty e^{-(\alpha+\vartheta^\star)x}x\,\pi(\mathrm{d}x) = -\tilde{\pi}'(\vartheta)\big|_{\vartheta=\alpha+\vartheta^\star}$$
$$= -\hat{\psi}'(0+)\frac{\hat{\psi}(\alpha+\vartheta^\star) - (\alpha+\vartheta^\star)\hat{\psi}'(\alpha+\vartheta^\star)}{\hat{\psi}^2(\alpha+\vartheta^\star)}.$$

  Hence, $\mu_L^{\mathrm{QS}}(\cdot)$ has the desired form.
- The dual process $\widehat{X}$ is spectrally negative, so that $\widehat{V}_{-\vartheta^\star}(y) = y$ and $\hat{\kappa}_{-\vartheta^\star}(0,\vartheta) = \vartheta$ for all $\vartheta \geq 0$. The Wiener–Hopf factorization gives (up to a multiplicative constant k that relates to the normalization of the local time) that under $\mathbb{P}^{-\vartheta^\star}$ for all $\vartheta \in \mathbb{R}$ we have

$$\psi_{-\vartheta^\star}(\vartheta) = -k\vartheta\kappa_{-\vartheta^\star}(0,-\vartheta) \tag{16}$$

  for all $\vartheta \leq -\vartheta_+$. From (4) and (6) we have that

$$\int_0^\infty e^{-\beta y}e^{\vartheta^\star y}V_{-\vartheta^\star}(y)\,\mathrm{d}y = \frac{1}{(\beta-\vartheta^\star)\kappa_{-\vartheta^\star}(0,\beta-\vartheta^\star)}$$
$$= \frac{k}{\psi_{-\vartheta^\star}(-\beta+\vartheta^\star)}$$
$$= \frac{k}{\psi(-\beta)-\psi(-\vartheta^\star)} = \frac{k}{\hat{\psi}(\beta)-\zeta^\star}.$$

Conclude that $\mu_R^{\mathrm{QS}}(\cdot)$ has the desired form, which completes the proof. □

Remark 12. The transform $\widetilde{A}_+(\cdot)$ can be used to interpret the quasi-stationary distributions. Because of Pollaczek–Khinchine,

$$\frac{\alpha+\vartheta^\star}{\hat{\psi}(\alpha+\vartheta^\star)}\frac{\hat{\phi}(\vartheta^\star)}{\vartheta^\star}$$

is a Laplace transform (i.e., corresponding to an exponentially twisted version of the steady-state workload). In addition, by virtue of[3] (Lemma 3.5),

$$\frac{2}{\hat{\psi}'(0)}\frac{\alpha\hat{\psi}'(\alpha)-\hat{\psi}(\alpha)}{\alpha^2}$$

is a Laplace transform, and therefore also its $\vartheta^\star$-twisted version

$$\frac{(\alpha+\vartheta^\star)\hat{\psi}'(\alpha+\vartheta^\star)-\hat{\psi}(\alpha+\vartheta^\star)}{(\alpha+\vartheta^\star)^2}\frac{(\vartheta^\star)^2}{\vartheta^\star\hat{\psi}'(\vartheta^\star)-\hat{\psi}(\vartheta^\star)}. \tag{17}$$

This reasoning indicates that, conditional on a long busy period, $Q(0)$ is distributed as the sum of three independent random variables. Two of these are distributed as the $\vartheta^\star$-twisted version of the steady-state workload, while a third has transform (17).

4.2. Spectrally Negative Lévy Process

Like for the spectrally positive case, also in the spectrally negative case we need to impose additional assumptions to find the quasi-stationary distribution.

[SN1] There exists $\vartheta_- > 0$ such that

- $\psi(\vartheta) < \infty$ for $0 < \vartheta < \vartheta_-$,
- $\psi(\vartheta)$ attains its strictly negative minimum at $\vartheta^\star > 0$, where $0 < \vartheta^\star < \vartheta_-$ (and hence $\psi'(\vartheta^\star) = 0$).

[SN2] Φ is analytical in $\mathscr{G}_{\zeta^\star}(\phi)$ for $\pi/2 < \phi \leq \pi$, where $\zeta^\star := \psi(\vartheta^\star) < 0$.

Example 13. Since for spectrally negative Lévy process $\Phi(\vartheta)$ is the Laplace exponent of a subordinator (viz. a first passage time process), the spectral representation

$$\Phi(\vartheta) = d_-\vartheta + \int_0^\infty (1-e^{-\vartheta x})\Pi_-(\mathrm{d}x), \tag{18}$$

applies, with $\int_0^\infty (x \wedge 1)\Pi_-(\mathrm{d}x) < \infty$; cf. (12). This means that if there exists a density of Π_- which is of semixponential type, then Proposition 3 entails that Φ is analytic in $\mathcal{G}_{\zeta^\star}(\phi)$ and hence assumption [SN2] is satisfied.

Theorem 14. *If X is a spectrally negative Lévy process satisfying conditions* [SN1–SN2], *then*

$$\mu(\mathrm{d}x, \mathrm{d}y) = Q_-(\vartheta^\star)^2 y e^{-\vartheta^\star(x+y)} e^{-\Phi(0)x} \widehat{V}_{\vartheta^\star}(x)\mathrm{d}x\,\mathrm{d}y\, \mathbf{1}_{\{x\geq 0, y\geq 0\}},$$

where $Q_- := (\int_0^\infty e^{-(\Phi(0)+\vartheta^\star)z} V_{\vartheta^\star}(z)\mathrm{d}z)^{-1}$.

Corollary 15. *We have*

$$\mu_L^{\mathrm{QS}}(\mathrm{d}y) = Q_- e^{-(\Phi(0)+\vartheta^\star)x} \widehat{V}_{\vartheta^\star}(x)\mathrm{d}x \mathbf{1}_{\{x\geq 0\}}$$

and

$$\mu_R^{\mathrm{QS}}(\mathrm{d}y) = (\vartheta^\star)^2 y e^{-\vartheta^\star y}\mathrm{d}y \mathbf{1}_{\{y\geq 0\}}.$$

Observe that $\mu_R^{\mathrm{QS}}(\cdot)$ corresponds to an Erlang(2) distribution. The proof of these results is based on the following lemma, which is proven as Lemma 10.

Lemma 16. *Under* [SN1–SN2],

$$\Phi(\vartheta) = \vartheta^\star + k^\star(\vartheta - \zeta^*)^{1/2} + o((\vartheta - \zeta^*)^{1/2})$$

as $\vartheta \downarrow \zeta^\star$, *where* $k^\star := \sqrt{2/\psi''(\vartheta^\star)}$.

Proof of Theorem 14. Note that all assumptions of Theorem 1 are satisfied by Proposition 6. In particular, as $|z|$ tends to infinity in $\mathcal{G}_{\zeta^\star}(\phi)$ function $|\Phi(z)|$ is either bounded or tends to infinity. In both cases condition (A2) is satisfied. Moreover, for some K,

$$L(\alpha, \beta; \vartheta) = K - \frac{\Phi(0)k^\star}{(\vartheta^\star + \beta)^2} \frac{1}{\psi(\alpha + \Phi(0)) - \zeta^\star} (\vartheta - \zeta^\star)^{1/2} + o((\vartheta - \zeta^\star)^{1/2})$$

as $\vartheta \downarrow \zeta^\star$, and we can conclude by 'Heaviside' that

$$\mathbb{E}_\pi[e^{-\alpha Q(0) - \beta Q(t)}, T > t] = \frac{\Phi(0)k^\star}{(\vartheta^\star + \beta)^2} \frac{1}{\psi(\alpha + \Phi(0)) - \zeta^\star} \frac{t^{-3/2}}{\Gamma(-1/2)} e^{-\zeta^\star t}(1 + o(1))$$

as $t \to \infty$. Therefore,

$$\tilde{\mu}(\alpha, \beta) = \lim_{t\to\infty} \mathbb{E}_\pi[e^{-\alpha Q(0) - \beta Q(t)} \mid T > t] = \widetilde{A}_-(\alpha)\widetilde{B}_-(\beta),$$

where

$$\widetilde{A}_{-}(\alpha) := \frac{-\zeta^{\star}}{\psi(\alpha+\Phi(0))-\zeta^{\star}}, \quad \widetilde{B}_{-}(\beta) := \frac{(\vartheta^{\star})^2}{(\vartheta^{\star}+\beta)^2}.$$

It is not hard to see that the proposed density indeed corresponds with this transform. $\square$

5. EXAMPLES

In this section we illustrate our theory by means of a number of examples. We indicate for which Lévy processes our assumptions are fulfilled, and for a few of those processes we perform the computations.

According to Vigon's theory of *philanthropy*[17], a (killed) subordinator is called a philanthropist if its Lévy measure has a decreasing density on $\mathbb{R}_{+}$. Moreover, given any two subordinators H_1 and H_2 which are philanthropists, providing that at least one of them is not killed, there exists a Lévy process X such that H_1 and H_2 have the same law as the ascending and descending ladder height processes of X, respectively. Suppose we denote the killing rate, drift coefficient and Lévy measures of H_1 and H_2 by the respective triples (b, δ, Π_{H_1}) and $(\hat{b}, \hat{\delta}, \Pi_{H_2})$. Then[17] shows that the Lévy measure of X satisfies the following identity

$$\Pi(x,\infty) = \int_0^{\infty} \Pi_{H_2}(u,\infty)\,\Pi_{H_1}(x+\mathrm{d}u) + \hat{\delta}\pi_{H_1}(x) + \hat{b}\Pi_{H_1}(x,\infty), \quad x>0,$$

where $\pi_{H_1}(x)$ is the density corresponding to Π_{H_1}. By symmetry, an obvious analogue of the above equation holds for the negative tail $\Pi(-\infty, x)$, with $x<0$.

Choosing then e.g. $H_2(t) = t$ and $H_1(t) = \tau(t)$ with Laplace exponent $\widehat{\Phi}$ and jump measure Π_{+} being semiexponential, then, using the above construction, we can easily give examples of spectrally positive Lévy processes satisfying conditions [SP1–SP2]. Similarly, using the above method we can construct spectrally negative Lévy processes satisfying [SN1–SN2].

Usually these conditions can be verified in a straightforward manner, as we did in the examples below.

Example 17 (M/M/1 Queue). In this case

$$X(t) = \sum_{i=1}^{N(t)} \sigma_i - t, \tag{19}$$

where σ_i (where $i = 1, 2, \ldots$) are i.i.d. service times that have an exponential distribution with mean $1/\nu$. The arrival process is a homogeneous Poisson process $N(t)$ with rate λ; it is assumed that $\varrho := \lambda/v < 1$. We apply the theory of Section 4.1.

We have

$$\hat{\psi}(\eta) = \eta - \lambda\left(1 - \frac{v}{\eta + v}\right) = \eta - \frac{\lambda\eta}{\eta + v},$$

yielding $\vartheta^\star = \sqrt{\lambda v} - v$, and $\zeta^\star = -(\sqrt{v} - \sqrt{\lambda})^2$. Furthermore

$$\widehat{\Phi}(\eta) = \frac{\eta + \lambda - v + \sqrt{(\eta + \lambda - v)^2 + 4\theta v}}{2}$$

and hence assumptions [SP1–SP2] are satisfied.

Let us first concentrate on $\mu_L^{\mathrm{QS}}(\cdot)$. We use Remark 12. Using that

$$\eta\hat{\psi}'(\eta) - \hat{\psi}(\eta) = \frac{\lambda\eta^2}{(\eta + v)^2},$$

we obtain

$$\frac{(\alpha + \vartheta^\star)\hat{\psi}'(\alpha + \eta^\star) - \hat{\psi}(\alpha + \vartheta^\star)}{(\alpha + \vartheta^\star)^2} \frac{(\vartheta^\star)^2}{\vartheta^\star\hat{\psi}'(\vartheta^\star) - \hat{\psi}(\vartheta^\star)} = \left(\frac{\sqrt{\lambda v}}{\alpha + \sqrt{\lambda v}}\right)^2.$$

This corresponds to the sum of two $\mathrm{Exp}(\sqrt{\lambda v})$ random variables. Also,

$$\begin{aligned}\frac{\alpha + \vartheta^\star}{\hat{\psi}(\alpha + \vartheta^\star)} \frac{\hat{\psi}(\vartheta^\star)}{\vartheta^\star} &= (1 - \sqrt{\varrho})\frac{\alpha + \sqrt{\lambda v}}{\alpha + \sqrt{\lambda v} - \lambda} \\ &= (1 - \sqrt{\varrho})\sum_{n=0}^{\infty}(\sqrt{\varrho})^n\left(\frac{\sqrt{\lambda v}}{\alpha + \sqrt{\lambda v}}\right)^n,\end{aligned}$$

corresponding with a $\mathrm{Geom}(\sqrt{\varrho})$-distributed number of $\mathrm{Exp}(\sqrt{\lambda v})$ random variables. Conclude that $\mu_L^{\mathrm{QS}}(\cdot)$ corresponds to the sum of M independent $\mathrm{Exp}(\sqrt{\lambda v})$ random variables, where

$$\mathbb{P}(M = m) = (m - 1)(\sqrt{\varrho})^{m-2}(1 - \sqrt{\varrho})^2,$$

i.e., M has a shifted negative binomial distribution with parameters 2 and $\sqrt{\varrho}$. A similar form is found for the general light-tailed M/G/1 case.

Let us now study $\mu_R^{\mathrm{QS}}(\cdot)$. It is a matter of straightforward calculus to find that

$$\widetilde{B}_+(\beta) = (v + \beta)\left(\frac{\sqrt{v} - \sqrt{\lambda}}{\beta + \sqrt{v}(\sqrt{v} - \sqrt{\lambda})}\right)^2.$$

A partial fraction expansion argument gives that this equals

$$(1-\sqrt{\varrho})\frac{\nu-\sqrt{\lambda\nu}}{\nu-\sqrt{\lambda\nu}+\beta}+\sqrt{\varrho}\left(\frac{\nu-\sqrt{\lambda\nu}}{\nu-\sqrt{\lambda\nu}+\beta}\right)^2.$$

In other words, the Quasi-stationary distribution at time t (for t large) equals a mixture of an exponential and an Erlang(2) distribution.

Example 18 (Linear Brownian Motion). In this case $X(t)=\sigma B(t)-t$, where $\sigma>0$ and $B(t)$ is a standard Brownian motion. Remark that this process is spectrally positive *and* spectrally negative, so we can use both Theorems 8 and 14.

Let us first see what the spectrally positive results would give. It is not hard to check that

$$\hat{\psi}(\vartheta)=\vartheta+\frac{\sigma^2\vartheta^2}{2},$$

so that, in the setting of Section 4.1, $\vartheta^\star=-1/\sigma^2$ and $\zeta^\star=-1/(2\sigma^2)$. It is a matter of straightforward computations now to obtain that

$$\widetilde{A}_+(\alpha)=\left(\frac{1/\sigma^2}{1/\sigma^2+\alpha}\right)^2,\quad \widetilde{B}_+(\beta)=\left(\frac{1/\sigma^2}{1/\sigma^2+\beta}\right)^2.$$

Conclude that the quasi-stationary distributions of $Q(0)$ and $Q(t)$ (t large) are both Erlang(2) with mean $2/\sigma^2$, whereas the stationary workload has an exponential distribution with mean $1/(2\sigma^2)$. (In the decomposition of Remark 12, the first two random variables have exponential distributions with mean $1/\sigma^2$, the third is equal to 0). Interestingly, the relation with the Erlang(2) distribution has also been observed in, Refs.[6,11,13].

The same result can be obtained by using the results from Section 4.2. Now $\vartheta^\star=1/\sigma^2$ and $\zeta^\star=-1/(2\sigma^2)$. It is easily checked that $\Phi(0)=2/\sigma^2$. As expected, we obtain $\widetilde{A}_-(\alpha)=\widetilde{A}_+(\alpha)$ and $\widetilde{B}_-(\beta)=\widetilde{B}_+(\beta)$.

In fact, in this case the quasi-stationarity distributions can be found in an explicit manner. For simplicity we restrict ourselves to studying just $\mu_L^{\rm QS}(\cdot)$; we do so by investigating the density

$$\frac{\mathrm{d}}{\mathrm{d}q}\mathbb{P}_\pi(Q(0)\le q\,|\,T>t)=:f_t(q).$$

We rely on the standard equality

$$\mathbb{P}_q(T>t)=\Psi_{\rm N}\left(\frac{t-q}{\sqrt{t}}\right)-e^{2q}\Phi_{\rm N}\left(\frac{-t-q}{\sqrt{t}}\right),$$

and the fact that $Q(0)$ (unconditioned) has an exponential distribution with mean $\frac{1}{2}$; here, $\Phi_N(\cdot)$ denotes the distribution function of a standard Normal random variable, where $\Psi_N(x) := 1 - \Phi_N(x)$ is its tail. It is known that, as $x \to \infty$,

$$\Psi_N(x) \sim \left(\frac{1}{x} - \frac{1}{x^3} + \frac{3}{x^5}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}. \tag{20}$$

Let us first determine $\mathbb{P}_\pi(T > t)$ (where $Q(0)$ has an exponential distribution with mean $\frac{1}{2}$), which can evidently be rewritten as

$$1 - \int_0^\infty 2e^{-2q} \Phi_N\left(\frac{t-q}{\sqrt{t}}\right) \mathrm{d}q - 2\int_0^\infty \Phi_N\left(\frac{-t-q}{\sqrt{t}}\right) \mathrm{d}q.$$

Consider the first integral of the previous display. It can be evaluated as

$$\begin{aligned}\int_0^\infty \int_{-\infty}^{\frac{t-q}{\sqrt{t}}} 2e^{-2q} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \mathrm{d}y\mathrm{d}q &= \int_{-\infty}^{\sqrt{t}} \int_0^{t-y\sqrt{t}} 2e^{-2q} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \mathrm{d}q\mathrm{d}y \\ &= \int_{-\infty}^{\sqrt{t}} \left(1 - e^{-2(t-y\sqrt{t})}\right) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \mathrm{d}y \\ &= \Phi_N(\sqrt{t}) - \Phi_N(-\sqrt{t}).\end{aligned}$$

Likewise, the second integral can be rewritten as

$$\begin{aligned}\int_0^\infty \int_{-\infty}^{\frac{-t-q}{\sqrt{t}}} 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \mathrm{d}y\mathrm{d}q &= \int_{-\infty}^{-\sqrt{t}} \int_0^{-t-y\sqrt{t}} 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \mathrm{d}q\mathrm{d}y \\ &= \int_{-\infty}^{-\sqrt{t}} (-2t - 2y\sqrt{t}) \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \mathrm{d}y \\ &= -2t\Phi_N(-\sqrt{t}) + 2\sqrt{t} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t}.\end{aligned}$$

We arrive at

$$\mathbb{P}_\pi(T > t) = \Psi_N(\sqrt{t}) + \Phi_N(-\sqrt{t}) + 2t\Phi_N(-\sqrt{t}) - 2\sqrt{t} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t}. \tag{21}$$

Using (20) it is readily verified that, for t large,

$$\mathbb{P}_\pi(T > t) \sim \frac{4}{t\sqrt{t}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t}.$$

Also, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}q} \mathbb{P}_\pi(Q_0 \le q, T > t) = 2e^{-2q} \Psi_N\left(\frac{t-q}{\sqrt{t}}\right) - 2\Phi_N\left(\frac{-t-q}{\sqrt{t}}\right), \tag{22}$$

so that we now have an explicit expression for $f_t(q)$, viz. the ratio of (22) and (21). Due to the asymptotic equivalence (20), Expression (22) behaves for t large as

$$\frac{2}{\sqrt{2\pi}}\sqrt{t}\left(\frac{1}{t-q}-\frac{1}{t+q}\right)\exp\left(-\frac{1}{2}\frac{(q+t)^2}{t}\right)\sim 4\left(\frac{1}{t\sqrt{t}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t}\right)qe^{-q}.$$

We conclude that we again find that the quasi-stationary distribution of $Q(0)$ is Erlang(2) with expected value 2.

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