# Inducing good behavior

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# F. Proofs of Propositions "Keeping out Trojan Horses"

*Proof of Proposition 5.1.* Let $\tilde{u}(\theta,\tilde{\theta})$be the utility of bidder 1 with type $\theta$ who bids as if having type $\tilde{\theta}$ "close" to $\theta$ while the other two bidders bid according to the same strictly increasing bidding function $B$ with $B(\theta) < 2\theta$. Then,

$$\begin{aligned}\tilde{u}(\theta,\tilde{\theta}) &= \int_0^{\tilde{\theta}}\int_0^{\tilde{\theta}} \max\left\{\theta+\theta_2+\theta_3-B(\tilde{\theta}),0\right\} d\frac{\theta_2}{100}d\frac{\theta_3}{100} - \frac{1}{20,000}c\left[B(\tilde{\theta})-\theta\right]^2 \\ &= \frac{1}{60,000}\left[\theta+2\tilde{\theta}-B(\tilde{\theta})\right]^3 - \frac{1}{30,000}\left[\theta+\tilde{\theta}-B(\tilde{\theta})\right]^3 - \frac{1}{20,000}c\left[B(\tilde{\theta})-\theta\right]^2.\end{aligned}$$

The first [second] term on the right-hand side in the first line refers to situations in which bidder 1 does not go [goes] bankrupt. The first-order condition of the equilibrium is given by

$$\left.\frac{\partial\tilde{u}(\theta,\tilde{\theta})}{\partial\tilde{\theta}}\right|_{\tilde{\theta}=\theta} = \frac{\frac{1}{2}\left[3\theta-B(\theta)\right]^2\left[2-B'(\theta)\right]-\left[2\theta-B(\theta)\right]^2\left[1-B'(\theta)\right]-cB'(\theta)\left[B(\theta)-\theta\right]}{10,000}=0$$

from which differential equation (5.10) follows.

□

*Proof of Proposition 5.2.* Let $B$ be the equilibrium bid function. According to the ranking lemma (see e.g., Milgrom, 2004), the proposition holds true if $B(0)=0$and if $B(\theta)=\frac{5}{3}\theta$ implies that $B'(\theta)<\frac{5}{3}$. It is standard that $B(0)=0$ must hold in a symmetric equilibrium. Moreover, suppose that bidders 2 and 3 bid according to $B$ and that bidder 1 with signal $\theta$ bids as if having signal $\tilde{\theta}$. Bidder 1's utility equals

$$\tilde{u}(\theta,\tilde{\theta}) = \int_0^{\tilde{\theta}}\int_0^{\tilde{\theta}} u(\theta+\theta_2+\theta_3-B(\tilde{\theta}))d\frac{\theta_2}{100}d\frac{\theta_3}{100}.$$

The first-order condition of the equilibrium implies that if $B(\theta) = \frac{5}{3}\theta$,

$$\begin{aligned}
0 &= 10,000 * \tilde{u}_2(\theta, \theta) \\
&= 2\int_0^\theta u(2\theta + \theta_2 - B(\theta))d\theta_2 - B'(\theta)\int_0^\theta\int_0^\theta u'(\theta + \theta_2 + \theta_3 - B(\theta))d\theta_2 d\theta_3 \\
&= 2\int_0^\theta u\left(\frac{1}{3}\theta + \theta_2\right)d\theta_2 - B'(\theta)\int_0^\theta\left[u\left(\frac{1}{3}\theta + \theta_2\right) - u\left(\theta_2 - \frac{2}{3}\theta\right)\right]d\theta_2 \Rightarrow \\
B'(\theta) &= \frac{2\int_0^\theta u\left(\frac{1}{3}\theta + \theta_2\right)d\theta_2}{\int_0^\theta\left[u\left(\frac{1}{3}\theta + \theta_2\right) - u\left(\theta_2 - \frac{2}{3}\theta\right)\right]d\theta_2} < \frac{5}{3}.
\end{aligned}$$

The third equality follows by direct integration and by substituting $B(\theta) = \frac{5}{3}\theta$. The inequality follows because the strict concavity of implies that

$$\int_0^\theta\left[u\left(\frac{1}{3}\theta + \theta_2\right) + 5u\left(\theta_2 - \frac{2}{3}\theta\right)\right]d\theta_2 < u'(0)\int_0^\theta\left[\left(\frac{1}{3}\theta + \theta_2\right) + 5\left(\theta_2 - \frac{2}{3}\theta\right)\right]d\theta_2 = 0.$$

□

*Proof of Corollary 5.1.* The expected winning bid equals

$$\mathbb{E}\left\{\min\left(\frac{\delta_n\theta_m}{1-\delta_n}, \delta_n\theta_n\right) + \theta_k\right\} \le \mathbb{E}\left\{\delta_n\theta_n + \theta_k\right\} \le \mathbb{E}\left\{\theta_n + \theta_k\right\} \le \mathbb{E}\left\{\theta^{(1)} + \theta^{(2)}\right\} = 125 = R_E^\infty,$$

from which the result immediately follows.

□

*Proof of Proposition 5.4.* Suppose both opponents of bidder 1 bid according to (5.19). Bidder 1 wishes to step out of the auction at a price equal to her (perceived) expected value. If both of her opponents step out at the same price $p$, bidder 1 knows that both have signal

$$\theta = \frac{p - 100\chi}{3 - 2\chi}.$$

She steps out at price $p$ equal to her perceived expected value, i.e.,

$$v = \theta_1 + 2(1-\chi)\theta + 100\chi = \theta_1 + 2(1-\chi)\frac{p - 100\chi}{3 - 2\chi} + 100\chi = p.$$

It is readily verified that $B_E^{1,\chi}$ in (5.19) is a solution. Similarly, $B_E^{2,\chi}$ follows by taking into account that bidder 1 updates her beliefs about the signal of the lowest bidder with probability $1-\chi$.

□

*Proof of Proposition 5.5.* Let $\tilde{u}\left(\theta, \tilde{\theta}\right)$ be the perceived utility of bidder 1 with type $\theta$ who bids as if having type $\tilde{\theta}$ while the other two bidders bid according to the same strictly increasing bidding function $B$. Then,

$$\tilde{u}(\theta, \tilde{\theta}) = \tilde{\theta}^2 \left[ (1-\chi) \left( \theta + \tilde{\theta} \right) + \chi (\theta + 100) - B(\tilde{\theta}) \right].$$

The first-order condition of the equilibrium is given by

$$\left. \frac{\partial \tilde{u}(\theta, \tilde{\theta})}{\partial \tilde{\theta}} \right|_{\tilde{\theta}=\theta} = 2\theta \left[ 2\theta (1-\chi) + \chi (\theta + 100) - B(\theta) \right] + \theta^2 \left[ (1-\chi) - B'(\theta) \right] = 0.$$

It is readily verified that (5.20) is a solution.

□

*Proof of Proposition 5.6.* Bidder 1 steps out at price $p$ equal to her perceived expected value of winning given that her two opponents bid according to equilibrium. Because bidder 1 is fully cursed, she assumes that the other two bidders' signals are uniformly distributed on $[0, 100]$ regardless of her winning the auction and regardless of the price at which an opponent steps out. Therefore, she indeed steps out at a price $p$ which solves $\tilde{U}(p, \theta) = 0$.

□

*Proof of Proposition 5.7.* Let $\tilde{u}(\theta, \tilde{\theta})$ be the utility of bidder 1 with type $\theta$ who bids as if having type $\tilde{\theta}$ while the other two bidders bid according to the same strictly increasing bidding function $B$. Then

$$\tilde{u}(\theta, \tilde{\theta}) = G(\tilde{\theta}) \tilde{U}(B(\tilde{\theta}), \theta)$$

where

$$G(\theta) \equiv \frac{\theta^2}{10,000}$$

is the distribution function of the higher of two draws from $U[0, 100]$. Equation (5.27) follows immediately from the first-order condition of the equilibrium:

$$\left. \frac{\partial \tilde{u}(\theta, \tilde{\theta})}{\partial \tilde{\theta}} \right|_{\tilde{\theta}=\theta} = G'(\theta) \tilde{U}(B(\theta), \theta) + G(\theta) \tilde{U}_1(B(\theta), \theta) B'(\theta) = 0.$$

□

*Proof of Corollary 5.3.* (The proof proceeds along the same lines as Maskin and Riley's (1984) proof of their Theorem 4.) Conditional on a bidder with type $\theta$ winning, the expected winning bid in EN is given by

$$R_E(\theta) \quad = \quad \int_0^\theta \frac{b_E^{\chi=1}(t)}{G(\theta)} dG(t)$$

where $G$ is the distribution function of the higher of two draws from $U[0,100]$. Consequently,

$$R_E'(\theta) \quad = \quad \left[b_E^{\chi=1}(\theta) - R_E(\theta)\right] \frac{G'(\theta)}{G(\theta)}.$$

The winning bid in FP equals $R_F(\theta) = b_F^{\chi=1}(\theta)$. Therefore,

$$R_F'(\theta) \quad = \quad b_F^{\chi=1\prime}(\theta) = -\frac{\tilde{U}(b_F^{\chi=1}(\theta), \theta)}{\tilde{U}_1(b_F^{\chi=1}(\theta), \theta)} \frac{G'(\theta)}{G(\theta)}.$$

Because $b_E(0) = b_F(0)$, it follows that $R_E(0) = R_F(0)$. According to the ranking lemma (see e.g., Milgrom (2004)), the proposition follows if $R_E(\theta) = R_F(\theta) \Rightarrow R_E'(\theta) > R_F'(\theta)$, which is equivalent to

$$b_E^{\chi=1}(\theta) - b_F^{\chi=1}(\theta) \quad > \quad -\frac{\tilde{U}(b_F^{\chi=1}(\theta), \theta)}{\tilde{U}_1(b_F^{\chi=1}(\theta), \theta)}.$$

Consider the left- and right-hand sides as functions of $b_F$. For $b_F = b_E$, both sides vanish. The derivative of the right-hand side is equal to $-1 + \frac{\tilde{U}\tilde{U}_{11}}{(\tilde{U}_1)^2} < -1$ whereas the derivative of the left-hand side equals -1. Therefore, because $b_F^{\chi=1}(\theta) < b_E^{\chi=1}(\theta)$, we conclude that the inequality is satisfied.

$\square$