Inducing good behavior

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F. Proofs of Propositions “Keeping out Trojan Horses”

Proof of Proposition 5.1. Let \( \tilde{u}(\theta, \tilde{\theta}) \) be the utility of bidder 1 with type \( \theta \) who bids as if having type \( \tilde{\theta} \) “close” to \( \theta \) while the other two bidders bid according to the same strictly increasing bidding function \( B \) with \( B(\theta) < 2\theta \). Then,

\[
\tilde{u}(\theta, \tilde{\theta}) = \int_0^\tilde{\theta} \int_0^\theta \max \left\{ \theta + \theta_2 + \theta_3 - B(\tilde{\theta}), 0 \right\} d\theta_2 \frac{d\theta_3}{100} - \frac{1}{20,000} c \left[ B(\tilde{\theta}) - \theta \right]^2
\]

The first term on the right-hand side in the first line refers to situations in which bidder 1 does not go bankrupt. The first-order condition of the equilibrium is given by

\[
\left. \frac{\partial \tilde{u}(\theta, \tilde{\theta})}{\partial \theta} \right|_{\tilde{\theta}=\theta} = \frac{1}{60,000} \left[ \theta + 2\tilde{\theta} - B(\tilde{\theta}) \right]^3 - \frac{1}{30,000} \left[ \theta + \tilde{\theta} - B(\tilde{\theta}) \right]^3 - \frac{1}{20,000} c \left[ B(\tilde{\theta}) - \theta \right]^2
\]

from which differential equation (5.10) follows.

\[ \square \]

Proof of Proposition 5.2. Let \( B \) be the equilibrium bid function. According to the ranking lemma (see e.g., Milgrom 2004), the proposition holds true if \( B(0) = 0 \) and if \( B(\theta) = \frac{5}{3}\theta \) implies that \( B'(\theta) < \frac{5}{3} \). It is standard that \( B(0) = 0 \) must hold in a symmetric equilibrium. Moreover, suppose that bidders 2 and 3 bid according to \( B \) and that bidder 1 with signal \( \theta \) bids as if having signal \( \tilde{\theta} \). Bidder 1’s utility equals

\[
\tilde{u}(\theta, \tilde{\theta}) = \int_0^\tilde{\theta} \int_0^\theta u(\theta + \theta_2 + \theta_3 - B(\tilde{\theta})) d\theta_2 \frac{d\theta_3}{100}.
\]
The first-order condition of the equilibrium implies that if \( B(\theta) = \frac{5}{3} \theta \),

\[
0 = 10,000 + \tilde{u}_2(\theta, \theta)
\]

\[
= 2 \int_0^\theta u(2\theta + \theta_2 - B(\theta))d\theta_2 - B'(\theta) \int_0^\theta \int_0^\theta u'(\theta + \theta_2 + \theta_3 - B(\theta))d\theta_2d\theta_3
\]

\[
= 2 \int_0^\theta u \left( \frac{1}{3}\theta + \theta_2 \right) d\theta_2 - B'(\theta) \int_0^\theta \left[ u \left( \frac{1}{3}\theta + \theta_2 \right) - u \left( \theta_2 - \frac{2}{3}\theta \right) \right] d\theta_2 \Rightarrow
\]

\[
B'(\theta) = \frac{2 \int_0^\theta u \left( \frac{1}{3}\theta + \theta_2 \right) d\theta_2}{\int_0^\theta [u \left( \frac{1}{3}\theta + \theta_2 \right) - u \left( \theta_2 - \frac{2}{3}\theta \right)] d\theta_2} < \frac{5}{3}.
\]

The third equality follows by direct integration and by substituting \( B(\theta) = \frac{5}{3} \theta \). The inequality follows because the strict concavity of implies that

\[
\int_0^\theta \left[ u \left( \frac{1}{3}\theta + \theta_2 \right) + 5u \left( \theta_2 - \frac{2}{3}\theta \right) \right] d\theta_2 < u'(0) \int_0^\theta \left[ \left( \frac{1}{3}\theta + \theta_2 \right) + 5 \left( \theta_2 - \frac{2}{3}\theta \right) \right] d\theta_2 = 0.
\]

\[\square\]

**Proof of Corollary 5.4.** The expected winning bid equals

\[
\mathbb{E} \left\{ \min \left( \delta_n \theta_n, \delta_n \theta_n \right) + \theta_k \right\} \leq \mathbb{E} \left\{ \theta_n + \theta_k \right\} \leq \mathbb{E} \left\{ \theta^{(1)} + \theta^{(2)} \right\} = 125 = R^\infty_E,
\]

from which the result immediately follows.

\[\square\]

**Proof of Proposition 5.4.** Suppose both opponents of bidder 1 bid according to (5.19). Bidder 1 wishes to step out of the auction at a price equal to her (perceived) expected value. If both of her opponents step out at the same price \( p \), bidder 1 knows that both have signal

\[
\theta = \frac{p - 100\chi}{3 - 2\chi}.
\]

She steps out at price \( p \) equal to her perceived expected value, i.e.,

\[
u = \theta_1 + 2(1 - \chi)\theta + 100\chi = \theta_1 + 2(1 - \chi) \frac{p - 100\chi}{3 - 2\chi} + 100\chi = p.
\]

It is readily verified that \( B_1^{\chi,\chi} \) in (5.19) is a solution. Similarly, \( B_2^{\chi,\chi} \) follows by taking into account that bidder 1 updates her beliefs about the signal of the lowest bidder with probability \( 1 - \chi \).

\[\square\]

**Proof of Proposition 5.5.** Let \( \tilde{u} \left( \theta, \hat{\theta} \right) \) be the perceived utility of bidder 1 with type \( \theta \) who bids
as if having type $\tilde{\theta}$ while the other two bidders bid according to the same strictly increasing bidding function $B$. Then,

$$\tilde{u}(\theta, \tilde{\theta}) = \tilde{\theta}^2 \left[ (1 - \chi) \left( \theta + \tilde{\theta} \right) + \chi (\theta + 100) - B(\tilde{\theta}) \right].$$

The first-order condition of the equilibrium is given by

$$\left. \frac{\partial \tilde{u}(\theta, \tilde{\theta})}{\partial \tilde{\theta}} \right|_{\tilde{\theta} = \theta} = 2\theta \left[ 2\theta (1 - \chi) + \chi (\theta + 100) - B(\theta) \right] + \theta^2 \left[ (1 - \chi) - B'(\theta) \right] = 0.$$

It is readily verified that (5.20) is a solution.

**Proof of Proposition 5.6.** Bidder 1 steps out at price $p$ equal to her perceived expected value of winning given that her two opponents bid according to equilibrium. Because bidder 1 is fully cursed, she assumes that the other two bidders’ signals are uniformly distributed on $[0, 100]$ regardless of her winning the auction and regardless of the price at which an opponent steps out. Therefore, she indeed steps out at a price $p$ which solves $\tilde{U}(p, \theta) = 0$.

**Proof of Proposition 5.7.** Let $\tilde{u}(\theta, \tilde{\theta})$ be the utility of bidder 1 with type $\theta$ who bids as if having type $\tilde{\theta}$ while the other two bidders bid according to the same strictly increasing bidding function $B$. Then

$$\tilde{u}(\theta, \tilde{\theta}) = G(\tilde{\theta})\tilde{U}(B(\tilde{\theta}), \theta)$$

where

$$G(\theta) \equiv \frac{\theta^2}{10,000}$$

is the distribution function of the higher of two draws from $U[0, 100]$. Equation (5.27) follows immediately from the first-order condition of the equilibrium:

$$\left. \frac{\partial \tilde{u}(\theta, \tilde{\theta})}{\partial \tilde{\theta}} \right|_{\tilde{\theta} = \theta} = G'(\theta)\tilde{U}(B(\theta), \theta) + G(\theta)\tilde{U}_1(B(\theta), \theta)B'(\theta) = 0.$$

**Proof of Corollary 5.3.** (The proof proceeds along the same lines as Maskin and Riley’s [1984] proof of their Theorem 4.) Conditional on a bidder with type $\theta$ winning, the expected winning
The winning bid in EN is given by

\[ R_E(\theta) = \frac{\int_0^\theta b_{E}^{(\theta)}(t) dG(t)}{G(\theta)} \]

where \( G \) is the distribution function of the higher of two draws from \( U[0,100] \). Consequently,

\[ R'_E(\theta) = \left[ b_{E}^{(\theta)}(\theta) - R_E(\theta) \right] \frac{G'(\theta)}{G(\theta)} \]

The winning bid in FP equals \( R_F(\theta) = b_{F}^{(\theta)}(\theta) \). Therefore,

\[ R'_F(\theta) = b_{F}^{(\theta)}(\theta) = -\frac{\bar{U}(b_{F}^{(\theta)}(\theta), \theta) G'(\theta)}{U_1(b_{F}^{(\theta)}(\theta), \theta) G(\theta)} \]

Because \( b_E(0) = b_F(0) \), it follows that \( R_E(0) = R_F(0) \). According to the ranking lemma (see e.g., Milgrom (2004)), the proposition follows if \( R_E(\theta) = R_F(\theta) \Rightarrow R'_E(\theta) > R'_F(\theta) \), which is equivalent to

\[ b_{E}^{(\theta)}(\theta) - b_{F}^{(\theta)}(\theta) > -\frac{\bar{U}(b_{F}^{(\theta)}(\theta), \theta)}{U_1(b_{F}^{(\theta)}(\theta), \theta)}. \]

Consider the left- and right-hand sides as functions of \( b_F \). For \( b_F = b_E \), both sides vanish. The derivative of the right-hand side is equal to \(-1 + \frac{\bar{U}(U_1)}{(U_1)^2} < -1 \) whereas the derivative of the left-hand side equals -1. Therefore, because \( b_{F}^{(\theta)}(\theta) < b_{E}^{(\theta)}(\theta) \), we conclude that the inequality is satisfied.

\( \square \)

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