Inducing good behavior

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F. Proofs of Propositions “Keeping out Trojan Horses”

Proof of Proposition 5.1. Let \( \tilde{u}(\theta, \tilde{\theta}) \) be the utility of bidder 1 with type \( \theta \) who bids as if having type \( \tilde{\theta} \) “close” to \( \theta \) while the other two bidders bid according to the same strictly increasing bidding function \( B \) with \( B(\theta) < 2\theta \). Then,

\[
\tilde{u}(\theta, \tilde{\theta}) = \int_0^{\tilde{\theta}} \int_0^\theta \max\left\{ \theta + \theta_2 + \theta_3 - B(\tilde{\theta}), 0 \right\} d\frac{\theta_2}{100} d\frac{\theta_3}{100} - \frac{1}{20,000} c \left[ B(\tilde{\theta}) - \theta \right]^2
\]

The first [second] term on the right-hand side in the first line refers to situations in which bidder 1 does not go [goes] bankrupt. The first-order condition of the equilibrium is given by

\[
\left. \frac{\partial \tilde{u}(\theta, \tilde{\theta})}{\partial \theta} \right|_{\tilde{\theta} = \theta} = \frac{1}{100} \left[ 3\theta - B(\theta) \right] \left[ \frac{2 - B'(\theta)}{2} - \frac{B'(\theta) - B(\theta)}{2} \right] - \frac{1}{20,000} c \left[ B(\tilde{\theta}) - \theta \right]^2. 
\]

from which differential equation (5.10) follows.

Proof of Proposition 5.2. Let \( B \) be the equilibrium bid function. According to the ranking lemma (see e.g., Milgrom 2004), the proposition holds true if \( B(0) = 0 \) and if \( B(\theta) = \frac{5}{3} \theta \) implies that \( B'(\theta) < \frac{5}{3} \). It is standard that \( B(0) = 0 \) must hold in a symmetric equilibrium. Moreover, suppose that bidders 2 and 3 bid according to \( B \) and that bidder 1 with signal \( \theta \) bids as if having signal \( \tilde{\theta} \). Bidder 1’s utility equals

\[
\tilde{u}(\theta, \tilde{\theta}) = \int_0^{\tilde{\theta}} \int_0^\theta u(\theta + \theta_2 + \theta_3 - B(\tilde{\theta})) d\frac{\theta_2}{100} d\frac{\theta_3}{100}.
\]
The first-order condition of the equilibrium implies that if $B(\theta) = \frac{5}{3} \theta$,

\[
0 = 10,000 + \tilde{u}_2(\theta, \theta)
\]

\[
= 2 \int_0^\theta u(2\theta + \theta_2 - B(\theta))d\theta_2 - B'(\theta) \int_0^\theta \int_0^\theta u'(\theta + \theta_2 - \theta_3 - B(\theta))d\theta_2d\theta_3
\]

\[
= 2 \int_0^\theta u \left( \frac{1}{3} \theta + \theta_2 \right) d\theta_2 - B'(\theta) \int_0^\theta \left[ u \left( \frac{1}{3} \theta + \theta_2 \right) - u \left( \theta_2 - \frac{2}{3} \theta \right) \right] d\theta_2 \Rightarrow
\]

\[
B'(\theta) = \frac{2 \int_0^\theta u \left( \frac{1}{3} \theta + \theta_2 \right) d\theta_2}{\int_0^\theta \left[ u \left( \frac{1}{3} \theta + \theta_2 \right) - u \left( \theta_2 - \frac{2}{3} \theta \right) \right] d\theta_2} < \frac{5}{3}.
\]

The third equality follows by direct integration and by substituting $B(\theta) = \frac{5}{3} \theta$. The inequality follows because the strict concavity of $\tilde{u}_2$ implies that

\[
\int_0^\theta \left[ u \left( \frac{1}{3} \theta + \theta_2 \right) + 5u \left( \theta_2 - \frac{2}{3} \theta \right) \right] d\theta_2 < u'(0) \int_0^\theta \left[ \left( \frac{1}{3} \theta + \theta_2 \right) + 5 \left( \theta_2 - \frac{2}{3} \theta \right) \right] d\theta_2 = 0.
\]

**Proof of Corollary 5.1.** The expected winning bid equals

\[
\mathbb{E} \left\{ \min \left( \frac{\delta_n \theta_1}{\delta_n}, \delta_n \theta_n \right) + \theta_k \right\} \leq \mathbb{E} \left\{ \delta_n \theta_n + \theta_k \right\} \leq \mathbb{E} \left\{ \theta_n + \theta_k \right\} \leq \mathbb{E} \left\{ \theta^{(1)} + \theta^{(2)} \right\} = 125 = R_E^\infty,
\]

from which the result immediately follows.

**Proof of Proposition 5.4.** Suppose both opponents of bidder 1 bid according to (5.19). Bidder 1 wishes to step out of the auction at a price equal to her (perceived) expected value. If both of her opponents step out at the same price $p$, bidder 1 knows that both have signal

\[
\theta = \frac{p - 100 \chi}{3 - 2 \chi}.
\]

She steps out at price $p$ equal to her perceived expected value, i.e.,

\[
v = \theta_1 + 2(1 - \chi)\theta + 100\chi = \theta_1 + 2(1 - \chi) \frac{p - 100 \chi}{3 - 2 \chi} + 100 \chi = p.
\]

It is readily verified that $B_E^{1, \chi}$ in (5.19) is a solution. Similarly, $B_E^{2, \chi}$ follows by taking into account that bidder 1 updates her beliefs about the signal of the lowest bidder with probability $1 - \chi$.

**Proof of Proposition 5.5.** Let $\tilde{u}(\theta, \tilde{\theta})$ be the perceived utility of bidder 1 with type $\theta$ who bids
as if having type \( \tilde{\theta} \) while the other two bidders bid according to the same strictly increasing bidding function \( B \). Then,

\[
\tilde{u}(\theta, \tilde{\theta}) = \tilde{\theta}^2 \left[ (1 - \chi)(\theta + \tilde{\theta}) + \chi(\theta + 100) - B(\tilde{\theta}) \right].
\]

The first-order condition of the equilibrium is given by

\[
\frac{\partial \tilde{u}(\theta, \tilde{\theta})}{\partial \tilde{\theta}} \bigg|_{\tilde{\theta} = \theta} = 2\theta \left[ 2\theta (1 - \chi) + \chi(\theta + 100) - B(\theta) \right] + \theta^2 [(1 - \chi) - B'(\theta)] = 0.
\]

It is readily verified that (5.20) is a solution.

**Proof of Proposition 5.6.** Bidder 1 steps out at price \( p \) equal to her perceived expected value of winning given that her two opponents bid according to equilibrium. Because bidder 1 is fully cursed, she assumes that the other two bidders’ signals are uniformly distributed on \([0, 100]\) regardless of her winning the auction and regardless of the price at which an opponent steps out. Therefore, she indeed steps out at a price \( p \) which solves \( \tilde{U}(p, \theta) = 0 \).

**Proof of Proposition 5.7.** Let \( \tilde{u}(\theta, \tilde{\theta}) \) be the utility of bidder 1 with type \( \theta \) who bids as if having type \( \tilde{\theta} \) while the other two bidders bid according to the same strictly increasing bidding function \( B \). Then

\[
\tilde{u}(\theta, \tilde{\theta}) = G(\tilde{\theta}) \tilde{U}(B(\tilde{\theta}), \theta)
\]

where

\[
G(\theta) \equiv \frac{\theta^2}{10,000}
\]

is the distribution function of the higher of two draws from \( U[0, 100] \). Equation (5.27) follows immediately from the first-order condition of the equilibrium:

\[
\frac{\partial \tilde{u}(\theta, \tilde{\theta})}{\partial \tilde{\theta}} \bigg|_{\tilde{\theta} = \theta} = G'(\theta) \tilde{U}(B(\theta), \theta) + G(\theta) \tilde{U}_1(B(\theta), \theta) B'(\theta) = 0.
\]

**Proof of Corollary 5.3.** (The proof proceeds along the same lines as Maskin and Riley’s (1984) proof of their Theorem 4.) Conditional on a bidder with type \( \theta \) winning, the expected winning
The winning bid in EN is given by

$$R_E(\theta) = \int_0^\theta \frac{b^{\chi=1}(t)}{G(\theta)} dG(t)$$

where $G$ is the distribution function of the higher of two draws from $U[0, 100]$. Consequently,

$$R'_E(\theta) = \left[ b^{\chi=1}(\theta) - R_E(\theta) \right] \frac{G'(\theta)}{G(\theta)}$$

The winning bid in FP equals $R_F(\theta) = b^{\chi=1}(\theta)$. Therefore,

$$R'_F(\theta) = b^{\chi=1'}(\theta) = -\frac{\tilde{U}(b^{\chi=1}(\theta), \theta)}{U_1(b^{\chi=1}(\theta), \theta)} G(\theta).$$

Because $b_E(0) = b_F(0)$, it follows that $R_E(0) = R_F(0)$. According to the ranking lemma (see e.g., Milgrom (2004)), the proposition follows if $R_E(\theta) = R_F(\theta) \Rightarrow R'_E(\theta) > R'_F(\theta)$, which is equivalent to

$$b^{\chi=1}(\theta) - b^{\chi=1}(\theta) > -\frac{\tilde{U}(b^{\chi=1}(\theta), \theta)}{U_1(b^{\chi=1}(\theta), \theta)}.$$ 

Consider the left- and right-hand sides as functions of $b_F$. For $b_F = b_E$, both sides vanish. The derivative of the right-hand side is equal to $-1 + \frac{\tilde{U}(b^{\chi=1}(\theta), \theta)}{(b^{\chi=1}(\theta))^2} < -1$ whereas the derivative of the left-hand side equals -1. Therefore, because $b^{\chi=1}_E(\theta) < b^{\chi=1}_E(\theta)$, we conclude that the inequality is satisfied.

\[\square\]