Essays in pension economics and intergenerational risk sharing

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Chapter 3

Sharing of Demographic Risks in a General Equilibrium Model with funded Pensions

In general, pension arrangements will affect the risks faced by both workers and retirees. One type of risk that one typically thinks of in this context is financial market risk. This risk is relatively large for funded pension systems in which workers accumulate assets for future retirement via their pension fund. Another important source of risk that is present in pension systems is demographic risk. Expected and unexpected developments in both survival probabilities and fertility rates have a direct impact on the distribution of burdens and benefits through pension systems. Taking them explicitly into account in the design of the pension arrangement may therefore lead to an improved allocation of these types of risks over the different generations of participants in the pension system.

In the literature, the impact of longevity or mortality risk has received a lot of attention. The paper by Andersen (2005) studies the impact of of longevity risk on PAYG systems, while Cocco and Gomes (2009) studies the effects on retirement savings. Also, the structural effect of increasing longevity and decreasing fertility (population aging) has been studied extensively, see for example Brsch-Supan et al. (2006) and Bovenberg and Knaap (2005).

In this chapter, the effects of the presence of demographic and financial shocks on optimal pension system design are investigated. Closest to our set-up are the papers by and Bovenberg and H.Uhlig (2006). Bohn (1999b) discusses the impact of demographic uncertainty on PAYG arrangements, while Bovenberg and H.Uhlig (2006) investigate the impact of demographic shocks in a model with capital and endogenous growth. Their set-up is an infinite horizon model with 2 overlappig generations, yielding generational conditions for optimal intergeneration risk sharing.

In this chapter, to keep the set-up as simple as possible, the model used is a two-period model with two generations whose lifes overlap only in the second period. This set-up allows for a very detailed numerical analysis of the effects of demographic shocks on the optimal set-up of the pension system. The young generation is born in the second period after all shocks have materialized. Hence, under laissez-faire – i.e., in the absence of a pension system – the young generation is unable to participate in financial markets and thus unable to share risks with the old generation. The pension systems under consideration consist of a pay-as-you-go (PAYG) first tier and a fully-funded second tier. Benefits in the second tier can be organized in different ways, as a defined-contribution (DC) or a defined-benefit (DB) scheme. Defined benefits can be defined either in real terms or relative to wages.

The two types of demographic shocks affect optimal defined-benefit pension arrangement in a highly nonlinear nature and are integral to both the total size of the economy and the distribution of resources over different generations. Specifically, fertility risk is important for both the wage level and the returns to capital (and thus aggregate production), and also for the relative size of transfers within the pension system. Mortality risk is important both for the returns on the funded part of the pension system and for the relative sizes of transfers within the pension system.

Optimal pension arrangements are explored under the assumption that the parameters of the arrangement are constant. In particular, they are not contingent on the stochastic shocks. Allowing the pension system parameters to be shock dependent would allow for more flexibility in distributing the various shocks over the pension system participants. However, frequent changes in the pension parameters may undermine the confidence in the pension system, while, moreover, each time that the system's parameter combinations are to be reset, an intergenerational conflict may be re-opened on how the parameters should exactly be set. Finally, it may be hard to fine-tune the parameters in response to shocks.

It is well known (see Beetsma and Bovenberg (2009)) that in the absence of demographic risks the decentralized pension system is able to replicate the social planner's solution by determining optimal redistribution via the first pillar and optimal risk sharing through the second pillar. When demographic risk is added to the model, this result disappears. Even if there is only one source of demographic risk (either fertility risk of mortality risk), it is no longer possible to replicate the social planner's solution anymore due to the complicated and non-linear effects each demographic shock has on the economy. Specifically, fertility risk affects both the wage level and the returns to capital (and thus aggregate production), as well as the relative size of transfers within the pension system. Mortality risk is important both for the returns on the funded part of the pension system and for the relative sizes of transfers within the pension system. Finding a solution that is as close as possible to the social planner's solution requires abandoning the separation of redistribution and risk sharing between the first and second pillar. Instead, in the presence of demographic risk, both pillars are involved in achieving an amount of risk sharing that is as close as possible to that under the social planner's solution.

With all four sources of risk present, it is socially optimal for the pension system to feature a funded second pillar of the defined-benefit type where the benefit is linked to the aggregate wage level. This setup, combined with a PAYG first pillar consisting of a lump-sum transfer and a transfer linked to individual wages, allows for an allocation of all sources of risks such that the resulting welfare gains (measured in terms of the percent increase in consumption in all periods and all states of the world) are very close to those obtained under the social planner's solution. Even with highly volatile demographic shocks, the short-fall under the wage-linked DB scheme from the social planner's solution is only about 0.01%, compared to welfare gains of over 5% relative to the laissez-faire economy. Further, the numerical analysis suggests that welfare gains from a defined real-benefit scheme are close to those under a scheme with defined wage-linked benefits. Hence, the precise format in which benefits are defined in a DB scheme may be less important than the scheme being defined benefit rather than defined contribution – the latter corresponds to the laissez-faire economy and implies that substantial gains from risk sharing may be foregone.

The setup of the remainder of this chapter is as follows. Section 2 presents the command economy and the social planner's solution. The decentralized economy in combination with the different pension schedules is introduced in Section 3. Section 4 presents the optimal pension scheme in the case of perfect foresight. Section 5 discusses the numerical procedure and the calibration, while Section 6 presents the numerical results. Section 7 investigates the robustness of the results for variations in the degree of risk aversion. Finally, Section 8 concludes the main text of the chapter.

3.1 The command economy

The model is deliberately kept as simple as possible, in order to describe the intuitions as clearly as possible.

3.1.1 Individuals and preferences

The model represents a closed economy. It incorporates two periods (0 and 1) and two generations. In period 0, a generation of mass N_0^o is born. This generation is referred to as the "old generation". The old generation consumes only in period 1 when it has become old. Therefore, this generation's utility function is given by:

$$\mathrm{E}_0 \left[\psi u \left(c_o \right) \right], \tag{3.1}$$

where c_o represents consumption when the agent is old and ψ is the probability that an individual old person survives period 0 and enters period 1. Further, $\mathrm{E}_0 [.]$ is the expectations operator conditional on information before any of the shocks have occurred (see below). It is assumed that function $u(.)$ is twice continuously differentiable, with $u' > 0$ and $u'' < 0$.

In period 1 a new generation of size N_1^y is born. This generation is referred to as the "young generation". The representative individual of the young generation features utility

$$u \left(c_y \right), \tag{3.2}$$

which is defined over per-young person consumption c_y in period 1.

3.1.2 Demographics

There are two sources of demographic risk in the model. The first source of risk is uncertainty about the survival probability ψ of the old generation. In the remainder of the chapter this type of risk is referred to as "mortality risk". The size of the old generation in period 0 is denoted by N_0^o and is nonstochastic. The size of the old generation in period 1 is denoted N_1^o and depends on the realization of ψ, i.e. $N_1^o = \psi N_0^o$. The second source of risk is fertility risk. The size of the young generation in period 1, N_1^y, is also uncertain.

3.1.3 Production

In period 0 each old generation member receives an exogenous non-storable initial endowment η. Production is endogenous only in period 1, when the two generations co-exist. Labour supply is exogenous in the model and amounts to 1 unit of labour per young person. Hence, production in period 1 is given by

$$Y = AF\left(K, N_1^y\right), \tag{3.3}$$

where A denotes (stochastic) total factor productivity, K represents the aggregate capital stock and N_1^y is the aggregate labor input. The production function exhibits constant returns to scale. In the closed economy setting in this chapter, the capital stock K in period 1 is the result of investment in the previous period, period 0.

3.1.4 Resource constraints

The resource constraints in periods 0 and 1 are given by, respectively,

$$\begin{aligned} N_0^o \eta &= K, && (3.4) \\ AF\left(K, N_1^y\right) + (1-\delta) K &= N_1^y c_y + N_1^o c_o, && (3.5) \end{aligned}$$

where $0 \leq \delta \leq 1$ is the stochastic depreciation rate of the capital stock. The left-hand sides of (3.4) and (3.5) are the available resources in the two periods. Specifically, the left-hand side of (3.5) equals total production plus what is left over of the capital stock after taking depreciation into account. The right-hand sides of (3.4) and (3.5) describe the use of these resources. The entire endowment in period 0 is spent on investment in physical capital (equation (3.4)), while the right-hand side of (3.5) is total consumption in the economy.

3.1.5 The social planner's solution

The vector of the stochastic shocks hitting the command economy is $\xi \equiv \{A, \delta, N_1^y, \psi\}$. It is unknown in period 0, but becomes known before period 1 variables are determined. A (utilitarian) social planner who aims at maximising the sum of the discounted expected utilities of all individuals serves as a benchmark. In period 0, the planner commits to an optimal state-contingent plan. Hence, the consumption levels are functions of the shocks, so that $c_o = c_o(\xi)$ and $c_y = c_y(\xi)$. The planner's problem can be written as:

$$\pounds = \int \left[\begin{array}{c} N_1^o \left[u\left(c_o\left(\xi\right)\right)\right] + N_1^y \left[u\left(c_y\left(\xi\right)\right)\right] + \\ \lambda\left(\xi\right)\left[AF\left(K, N_1^y\right) + \left(1-\delta\right)K - N_1^y c_y\left(\xi\right) - N_1^o c_o\left(\xi\right)\right] \end{array} \right] f\left(\xi\right) d\xi. \tag{3.6}$$

$$\tag{3.7}$$

Here, $f(\xi)$ stands for the probability density function of the vector of stochastic shocks ξ. The Lagrange multiplier on the resource constraint in period 1 is denoted by λ. Maximization of the planner's program with respect to $c_y(\xi)$ and $c_o(\xi)$ for all ξ yields the following first-order conditions:

$$\frac{\partial \pounds}{\partial c_y} = 0 \Rightarrow u'\left(c_y\left(\xi\right)\right) = \lambda\left(\xi\right), \forall \xi, \tag{3.8}$$

$$\frac{\partial \pounds}{\partial c_o} = 0 \Rightarrow u'\left(c_o\left(\xi\right)\right) = \lambda\left(\xi\right), \forall \xi. \tag{3.9}$$

By eliminating the Lagrange multiplier from these first-order conditons, the following is obtained:

$$u'(c_y) = u'(c_o), \forall \xi, \tag{3.10}$$

which, owing to the identical utility functions reduces to:

$$c_y = c_o, \forall\ \xi, \tag{3.11}$$

Condition (3.10) equalises the marginal utilities of the two generations which implies that with identical utility functions for the two generations all shocks are perfectly spread over all individuals. If a decentralised equilibrium is to replicate the planner's solution, conditions (3.8) and (3.9) need to be met in addition to the resource constraints (3.4) and (3.5). This is also the set of necessary conditions. This case differs from Beetsma and Bovenberg (2009) in that the latter include a trade off between period 0 and period 1 consumption for the old generation. Hence, in their paper there is an extra optimality condition that needs to be fulfilled to replicate the social planner solution. However, the intertemporal consumption trade off of the old generation is dropped in this chapter as it does not substantially add to the insights that can be gained from the analysis below.

3.2 The decentralized economy

3.2.1 The pension arrangements

The pension system in the decentralized economy consists of a PAYG first pillar and a funded second pillar. The first pillar in the pension system is composed of a lump-sum part and a wage-indexed part. Each old person receives a (possibly negative) amount θ^p and a fraction θ^w of wage income of an individual young. Therefore, the systematic transfer per surviving old member via the first pillar is $\theta^p + \theta^w w$.

The second pillar of the pension system consists of a pension fund that collects contributions θ^f per old-generation member in period 0, invests these contributions and pays out benefits to the fraction ψ of surviving old-generation members in period 1. Denoting by r^f the rate of return that a surviving member of the old generation receives on his contributions, the payout to this individual is $(1+r^f)\theta^f$. Depending on whether the pension fund is of the DC, DRB or DWB type, the format of the pension benefit will be different (see below).

The pension fund can invest the old generation's contribution in price-indexed bonds or in physical capital:

$$N_0^o\theta^f = B^f + K^f, \tag{3.12}$$

where B^f and K^f are the fund's aggregate investments in price-indexed bonds and physical capital, respectively. The bonds provide a non-stochastic return of r and physical capital provides the stochastic net-of-depreciation rate of return $r^{kn} \equiv AF_K - \delta$, where F_K is the marginal product of capital. The average return on the assets held by the fund is denoted by r^a. Therefore, the total value of the fund before the payout is

$$(1+r^a)N_0^o\theta^f = (1+r)\,B^f + \left(1+r^{kn}\right)K^f.$$

Depending on the pension scheme and the fund's investment scheme, a difference may arise between the value of the fund and the payout equal to $\left[(1+r^a)\,N_0^o - \left(1+r^f\right)N_1^o\right]\theta^f$, where $\left(1+r^f\right)N_1^o\theta^f$ is the total pension benefit payout to the old generation. The young are the residual claimants of the fund and receive this difference. Denoting the generational account *per old person in period* 1 by G:

$$G = \theta^p + \theta^w w + \left[\left(1+r^f\right) - \frac{N_0^o}{N_1^o}\left(1+r^a\right)\right]\theta^f.$$

Defined contribution (DC)

The DC scheme is discussed purely as a benchmark in which risk sharing between the two cohorts is completely absent. For this reason, it is not discussed in the formal analysis below. In a DC system the pension fund invests the contributions of the old generation in period 0 and pays out whatever these investments turn out to be worth in period 1:

$$\left(1+r^f\right)\theta^f = \frac{N_0^o}{N_1^o}(1+r^a)\theta^f,$$

which can be rewritten as:

$$r^f = \frac{N_0^o}{N_1^o}(1+r^a) - 1. \tag{3.13}$$

Therefore, the payout and the value of the pension fund coincide in every state of the world and there is no residual claim left for the young generation. This implies that the generational account per old person in period 1 under a DC system simplifies to:

$$G = \theta^p + \theta^w w. \tag{3.14}$$

Note that the pension assets of the non-surviving members of the generation go to the survivors.

Defined real benefit (DRB)

Under a DRB system, the pension fund promises to provide each retired individual with a given real benefit of size p:

$$p = \left(1+r^f\right)\theta^f, \tag{3.15}$$

where r^f is the return to the pension fund contribution which will be determined by the so-called "actuarial fairness condition" to be discussed below.

In period 1 the pension fund will have accumulated assets equal to $(1+r)\,B^f + \left(1+r^{kn}\right)K^f$. Hence, the difference between the total benefit payout and the assets of the pension fund equals

$$\left(1+r^f\right)N_1^o\theta^f - (1+r)\,B^f - \left(1+r^{kn}\right)K^f.$$

The generational account per old person in period 1 is equal to:

$$G = \theta^p + \theta^w w + \left(1+r^f\right)\theta^f - \frac{1}{N_1^o}\left[(1+r)\,B^f + \left(1+r^{kn}\right)K^f\right]. \tag{3.16}$$

Defined wage-indexed benefit (DWB)

Under a DWB system, the pension fund benefit is indexed to the aggregate wage sum:

$$\left(1 + r^f\right) \theta^f N_1^o = \theta^{dwb} N_1^y w, \tag{3.17}$$

where $N_1^y w$ is the total wage bill and parameter θ^{dwb} links the pension payout to the total wage bill. This parameter is also determined by the actuarial fairness condition discussed below.

The return to the old generation on their pension contribution is stochastic, because the size of the young generation is unknown when the contribution is made, while the wage rate is only determined by market forces after the shocks have materialised and depends on the a priori unknown size of the young generation and level of productivity. Hence, if the realised total wage bill exceeds the expected total wage bill, the payout of the pension fund is higher than expected. This structure is intended to capture indexation of occupational pension benefits to aggregate wage growth, which exists in some funded pension schemes in Germany, the Netherlands, Norway, Slovenia and Sweden (OECD, 2011).

The difference between the total pension benefit payout and total assets of the pension fund is:

$$\left[\left(1 + r^f\right) N_1^o - \left(1 + r^a\right) N_0^o\right] \theta^f = \theta^{dwb} N_1^y w - \left[\left(1 + r\right) B^f + \left(1 + r^{kn}\right) K^f\right]. \tag{3.18}$$

Hence, in this case the generational account per old person in period 1 is:

$$G = \theta^p + \theta^w w + \frac{N_1^y}{N_1^o} \theta^{dwb} w - \frac{1}{N_1^o} \left[\left(1 + r\right) B^f + \left(1 + r^{kn}\right) K^f\right]. \tag{3.19}$$

3.2.2 Individual budget constraints and generational accounts

In period 0, each individual old person receives his deterministic endowment η, pays the mandatory pension fund contribution θ^f, and invests the remainder of his income in bonds, $\frac{B^p}{N_0^o}$, and physical capital, $\frac{K^p}{N_0^o}$:

$$\eta = \frac{B^p}{N_0^o} + \frac{K^p}{N_0^o} + \theta^f, \tag{3.20}$$

where B^p and K^p are the aggregate private investments in real bonds and physical capital, respectively.

In period 1, the surviving members of the old generation receive the returns on their private investments plus their pension benefits, while the members of the young generation receive their wage plus the residual (potentially negative) value of the pension fund. In addition, the non-pension assets of those old generation members that do not survive until period 1 are collected by the government and distributed evenly across all individuals alive in period 1. Thus, in period 1 each individual alive receives a lumpsum "bequest" τ, such that

$$c_o = (1+r)\frac{B^p}{N_0^o} + \left(1+r^{kn}\right)\frac{K^p}{N_0^o} + \frac{N_0^o}{N_1^o}(1+r^a)\,\theta^f + \tau + G, \tag{3.21}$$

$$c_y = w + \tau - \frac{N_1^o}{N_1^y}G, \tag{3.22}$$

where τ is given by:

$$\tau = \frac{N_0^o - N_1^o}{N_1^o + N_1^y}\left[(1+r)\,B^p + \left(1+r^{kn}\right)K^p\right].$$

3.2.3 Individual and firm optimization

The model is solved by backward induction. A continuum of perfectly competitive representative firms, with mass normalized to unity, produce according to (3.3) and maximize period-1 profits $AF(K,L) - wL - r^k K$ over labour L and capital K, taking the wage rate and rental rate on capital as given. The first-order conditions are:

$$AF_L = w, \tag{3.23}$$

$$AF_K = r^k, \tag{3.24}$$

where the subscripts on F denote its first-order partial derivatives.

In period 0, the old generation decides on the allocation of its savings over the two assets. Subject to (3.20) they maximize (3.1) over $\frac{B^p}{N_0^o}$ and $\frac{K^p}{N_0^o}$, where c_o is given by (3.21). The first-order condition is:

$$(1+r)\,\mathrm{E}_0\left[\psi u'(c_o)\right] = \mathrm{E}_0\left[\psi\left(1+r^{kn}\right)u'(c_o)\right]. \tag{3.25}$$

Finally, the so-called "actuarial fairness condition" of the pension fund states that, in expectation, the pension fund should not redistribute resources between generations. This condition is based on the work of Yaari (1965) that shows that in a model with lifetime uncertainty, the actuarially fair rate of return on an annuity is higher than the market interest rate. The rate of return on an annuity contains a mark-up for mortality risk. The reason for this is the zero-profit condition for insurance companies offering the annuities. An insurance company that would try to sell annuities with a rate of return below the actuarial fair one would be driven out of the market by other insurance companies offering a rate of return closer to the actuarial fair rate.

In the setting in this chapter, the pension fund effectively provides an annuity, because the contribution paid to the pension fund is not part of the bequest if an individual dies and can be used by the pension fund to increase the benefits to surviving participants. Analoguous to Yaari's setup, the actuarial fairness condition is written as:

$$\theta^f\ \mathrm{E}_0\left[\left(1+r^f\right)N_1^o\right] = \theta^f\mathrm{E}_0\left[\left(1+r^a\right)N_0^o\right], \tag{3.26}$$

where the left-hand side represents the expected total pension fund payout and the right-hand side represents the value of the assets in the pension fund expected at the moment that the fund makes its investments.

For the DRB pension system, $p = \left(1+r^f\right)\theta^f$, which is deterministic, so that the actuarial fairness condition can be rewritten as:

$$\begin{aligned} & p\ \mathrm{E}_0\left[N_1^o\right] = N_0^o\theta^f\ \mathrm{E}_0\left(1+r^a\right), \\ \Rightarrow\quad & p = \theta^f\frac{N_0^o}{\mathrm{E}_0\left[N_1^o\right]}\ \mathrm{E}_0\left(1+r^a\right). \end{aligned} \tag{3.27}$$

For the DWB system, using (3.17) the actuarial fairness condition (3.26) can be written as:

$$\begin{aligned} & \mathrm{E}_0\left[\left(1+r^f\right) N_1^o\right]=N_0^o \mathrm{E}_0\left(1+r^a\right) \\ \Rightarrow \quad & \theta^{dwb}=\theta^f \frac{N_0^o}{\mathrm{E}_0\left[N_1^y w\right]} \mathrm{E}_0\left(1+r^a\right). \end{aligned} \tag{3.28}$$

3.2.4 Market equilibrium conditions

The goods market equilibrium conditions in periods 0 and 1 are (3.4) and (3.5), respectively. Equilibrium in the labour market requires that the amount of labour L demanded by firms is equal to N_1^y. Equilibrium in the capital market requires that

$$K=K^p+K^f, \tag{3.29}$$

while, with zero net outstanding debt, equilibrium in the market for risk-free debt in period 0 requires that:

$$B^p+B^f=0. \tag{3.30}$$

Finally, the factor prices are determined as $w=AF_L\left(K,N_1^y\right)$ and $r^k=AF_K\left(K,N_1^y\right)$.

3.3 Optimal pension policy

This section investigates the optimal pension parameter settings and checks whether the pension system designer is able to mimick the social planner's solution. The assumed objective of the pension system designer (for example, the government) is a utilitarian maximisation of the welfare of the pension system participants. Hence, the pension system designer solves the following problem

$$\max_{\Pi}\left\{W \equiv \mathrm{E}_0\left[N_1^o u\left(c_o\right)+N_1^y u\left(c_y\right)\right]\right\}, \tag{3.31}$$

where Π is the set of instruments (pension system parameters) available to the pension system designer, subject to the resource constraints ((3.4) and (3.5)), the individual's and firm's optimality conditions ((3.23), (3.24) and (3.25)), the actuarial fairness condition (3.26), and the market equilibrium conditions (3.29) and (3.30). In its choice of the optimal pension parameters, the pension designer internalizes how individuals react to changes in those parameters (see Chamley, 1986; Chari et al., 1994, for a detailed treatment of this method). The set Π under a DRB, respectively a DWB system, consists of the following parameters:

$$\begin{aligned}\Pi^{drb} &= \left\{\theta^p, \theta^w, K^f, p\right\},\\ \Pi^{dwb} &= \left\{\theta^p, \theta^w, K^f, \theta^{dwb}\right\}.\end{aligned}$$

Specific realisations of the shocks A and N_1^y and the solution for K yield w and r^k. Using the same specific realisations for A and N_1^y and specific realisations for δ and N_1^o, realisations of c_o and c_y are obtained via (3.21) and (3.22). Using all probability-weighted possible realisations of the shock vector ξ and the associated outcomes for c_o and c_y, W can be computed for the given set of pension system parameters. The set that generates the highest value for W yields the optimal pension system in this procedure.

3.3.1 The optimum under perfect demographic foresight

Under perfect foresight about the demographic parameters N_1^y and N_1^o of the young and elderly in period 1 the social planner solution can be achieved, as the following proposition states:

Proposition 3.1. *(i) Under a DRB system, the social planner solution can be replicated and only be replicated by setting the pension parameters as:*

$$\theta^p + p = (1+r)\,\frac{N_0^o + N_1^y}{N_0^o\left(N_1^y + N_1^o\right)}B^f,$$
$$K^f = \frac{N_1^y N_0^o}{N_1^y + N_0^o}\eta,\ \ \theta^w = \frac{N_1^y}{N_1^y + N_1^o}.$$

(ii) Under a DWB system, the social planner solution can be replicated and only be

replicated by setting the pension parameters as:

$$\theta^p = (1+r)\,\frac{N_0^o + N_1^y}{N_0^o\,(N_1^y + N_1^o)} B^f,\ \ K^f = \frac{N_1^y N_0^o}{N_1^y + N_0^o}\eta,$$
$$\text{and } \theta^w + \frac{N_1^y}{N_1^o}\theta^{dwb} = \frac{N_1^y}{N_1^y + N_1^o}.$$

Proof. See Appendix 3.B for the proof. □

In the special case in which demographic risks are absent, the pension parameters are known with certainty at the moment that the old generation takes its investment decisions. This is the case discussed in Beetsma and Bovenberg (2009). The intuition for the results closely follows that in Beetsma and Bovenberg (2009). However, there is a major difference in that their "full-funding condition" is replaced by the actuarial fairness condition. The full-funding condition says that the expected return on an additional euro contributed to the pension fund should be the same as the expected return on an additional euro invested in any asset traded in the financial market. Here, this condition is dropped in favour of the actuarial fairness condition, because the probability of prematurely dying before retirement results in an above-market return on the contribution to the pension fund made by those who survive into retirement.

The intuition for the parameter settings in Proposition 1 is as follows. Let us focus on the case of the DRB system. Risks associated with the pension fund's asset portfolio are effectively shifted to the young generation, because the old generation gets a fixed benefit, while the residual risk is carried by the young. Hence, each young generation member has an individual capital exposure of $\left(1 + r^{kn}\right)\left(K^f/N_1^y\right)$, while each old generation member retains an individual capital exposure of $\left(1 + r^{kn}\right)(K^p/N_0^o)$. Here, the capital exposure implicit in the accidental bequest τ is ignored, because the bequest is identical for young and old individuals. Identical capital exposures among all individuals are achieved by setting $K^f/N_1^y = K^p/N_0^o$, which can be written as $K^f = K^p\,(N_1^y/N_0^o)$. Hence, if $N_1^y > N_0^o$, then $K^f > K^p$ is needed in order to bestow the same amount of capital risk on young and old generation individuals. Using $K^p + K^f = K$, the equation $K^f = K^p\,(N_1^y/N_0^o)$ can be rewritten as $K^f = [N_1^y/\,(N_1^y + N_0^o)]\,K$, which can be written as the expression in the proposition by using the resource constraint in period 0.

The intuition for the expression for θ^w follows from the fact that for condition (3.11) to be fulfilled, exposure of all individuals to the realisation of the wage rate must be identical. Since each old person receives a fixed transfer of size $\theta^w w$, each young person has to pay a transfer of the size $\frac{N_1^o}{N_1^y}\theta^w w$. Identical exposure of each individual to wages then requires that $\left(1-\frac{N_1^o}{N_1^y}\theta^w\right)w = \theta^w w$, which is equivalent to $\theta^w = \frac{N_1^y}{N_1^y+N_1^o}$. Thus, the combination $\theta^w = \frac{N_1^y}{N_1^y+N_1^o}$ and $K^f = \frac{N_1^y N_0^o}{N_1^y+N_0^o}\eta$ is necessary to simultaneously ensure identical exposures to w and r^{kn}.

Finally, after having ensured that both generations' consumption responds identically to movements in wages and the net return on capital, the expression for $\theta^p + p$ ensures that the lump-sum first-pillar transfer and the benefit from the funded pillar are set such that the period-1 consumption levels of the two generations are identical. Obviously, this expression depends on the sizes of both generations.

3.3.2 Demographic uncertainty

In the absence of perfect foresight about the future sizes of the two cohorts, it is not possible to design a pension arrangement in period 0 that reproduces the social planner's solution. While multiple combinations of the pension system parameters can reproduce the social planner's solution under perfect foresight, all these combinations require the realisations of N_1^y and N_1^o to be known in period 0. If this is not the case, then the social planner's solution is not attained.

It is no longer possible to solve analytically for the optimal pension system arrangement in the presence of demographic risk. In the remainder of the chapter a numerical analysis is performed to investigate how well the different pension systems perform in the presence of demographic uncertainty. As already motivated above, only arrangements that do not depend on the shocks are considered. This means that only arrangements for which the pension parameters are known in period 0 are considered.

3.4 Calibration

In the numerical analysis, the pension parameter combination that yields the highest welfare level W under the assumption that all elements of Π are independent of the shocks is determined. Under this constraint the optimal solution will in general not replicate the social planner's solution. Before turning to the analysis, functional forms for the utility and production functions need to be specified, as well as assumptions on the distributions of the various shocks in the model.

For the utility and production functions, commonly-used specifications in macroeconomic analysis are adopted. The utility function is of the constant relative risk aversion (CRRA) format:

$$u(c_i) = \frac{c_i^{1-\varphi}}{1-\varphi}, \quad i = y, o,$$

where φ is the relative risk aversion parameter. The baseline value is $\varphi = 2.5$, which is close to the median of the values found in the literature (see, for example Gertner, 1993; Beetsma and P.C.Schotman, 2001). However, later φ is varied to see how the results are affected by changes in risk attitude.

Production is given by the Cobb-Douglas function

$$AF(K, N_1^y) = AK^{\alpha}(N_1^y)^{1-\alpha},$$

where α is the share of national income accruing to the equity providers. The value assumed is $\alpha = 0.3$, a value commonly used in the literature. Hence, 70% of national income accrues to workers.

For each of the four shocks, a 2-point distribution is assumed, with probability $\frac{1}{2}$ for each of the two values that the shock can attain. Therefore, the mean of each stochastic variable is always the mid-point between the two possible values of that variable. The amount of uncertainty associated with a shock is measured by absolute value of the distance between the mean of the shock and its possible realisations. This measure is denoted by Δ. It can differ over the various types of shocks.

The mean size of the young generation N_1^y is normalized to 1, such that in expectation both generations are equally large at the moment of birth. The mean of the

Table 3.1: Parameters

parameter	mean	Δ
α	0.3	
η	1	
φ	2.5	
A	3	0.3
δ	0.5	0.1
N_1^y	1	0.2
N_1^o	0.8	0.1

mortality shock ψ is 0.8, so that on average 80% of the old generation members survive until period 1.[1] The mean of the depreciation shock δ is 0.5, which corresponds to an annual depreciation rate of around 2% over a 30-year period. The remaining parameters to be calibrated are η and A. The choice of η is only relevant in relation to the average value of A. These two parameters determine the scale of the economy and one of them can be fixed to unity. The choice made is to set $\eta = 1$ and set the mean of the productivity shock A to 3. This mean is chosen so that the numerical simulations yield a realistic net-of-depreciation return on capital and a realistic return on bonds. In the numerical experiments, mean-preserving increases in the variances of the shocks will be considered. The base calibration is summarised in Table 3.1, including the base values of Δ for each of the shocks.

3.5 Numerical results

3.5.1 Measures for welfare comparison

Two welfare comparisons are conducted. First, welfare is compared under the optimal pension fund arrangement to welfare under the laissez-faire situation. Second, welfare

[1] According to the 2010 tables of the Dutch Association of Actuarial Scientists (AG, 2010), the probability for a newborn to survive until the current mandatory retirement age of 65 is approximately 0.87, while the probability of surviving until 67 is roughly 0.84. Ideally, the calibration of ψ would serve two purposes. One is to capture the probability of surviving until retirement date, while the other is to replicate the old-age dependency ratio, defined as the number of retired divided by the number of people of working age. The choice for the value of ψ is made to replicate closely the former objective. The stylised two generation objective employed prevents the replication of both objectives simultaneously.

under the social planner solution is compared to welfare under the optimal pension fund arrangement. The welfare consequences of introducing a pension arrangement are measured by calculating the constant perunage increase Ω^{pf} in consumption of both generations in all possible states of the world that is required to raise social welfare under the laissez-faire economy to that under the economy with the optimal pension arrangement. The welfare difference between the economy with the optimal pension arrangement and the social-planner economy is measured in an analogous way by Ω^{sp}.

$$\begin{aligned}
\mathrm{E}_0 \left\{ u \left[\left(1+\Omega^{pf}\right) c_o^{lf} \right] + u \left[\left(1+\Omega^{pf}\right) c_y^{lf} \right] \right\} &= \mathrm{E}_0 \left[u \left(c_o^{pf} \right) + u \left(c_y^{pf} \right) \right], \\
\mathrm{E}_0 \left\{ u \left[\left(1+\Omega^{sp}\right) c_o^{pf} \right] + u \left[\left(1+\Omega^{sp}\right) c_y^{pf} \right] \right\} &= \mathrm{E}_0 \left[u \left(c_o^{sp} \right) + u \left(c_y^{sp} \right) \right],
\end{aligned}$$

where the superscripts lf, pf and sp on consumption indicate consumption in the laissez-faire economy, consumption under the optimal pension arrangement and consumption under the social planner solution, respectively.

3.5.2 No demographic uncertainty

To establish some intuition for the numerical results, first the case in which there is no demographic uncertainty is considered. In this case, the values for the demographic shocks are fixed at their means, $N_1^y = 1$, $N_1^o = 0.8$.

Table 3.2 reports the instrument settings for optimal pension arrangements. For the case of a DRB system the optimal arrangement with $p = 0$ is selected, while for the case of the DWB system the optimal arrangement with $\theta^w = 0$ is chosen. Obviously, for $p = 0$ to be consistent with the actuarial fairness condition, it is necessary that either $\theta^f = 0$ or $\mathrm{E}_0 (1 + r^a) = 0$. If θ^f is not zero, then $\mathrm{E}_0 (1 + r^a) = 0$. Table 2 presents a solution with $\mathrm{E}_0 (1 + r^a) = 0$. This solution allows the fund to take a short position in real debt that is slightly larger in absolute value than its long position in capital, the reason being that the expected net return on capital $\mathrm{E}_0 \left[r^{kn} \right]$ is slightly larger than r. Because K^f and r^{kn} are tied down by demographic variables and period-0 resources, equation $\mathrm{E}_0 (1 + r^a) = 0$ and the arbitrage equation (3.25) determine together B^f and

Table 3.2: Different pension systems, no demographic uncertainty

	DRB	DWB
θ^p	-0.76	0.24
θ^w	0.56	0
θ^{dwb}	-	0.45
p	0	-
θ^f	-0.01	0.65
K^f	0.5	0.5
K^p	0.5	0.5
K	1	1
B^f	-0.51	0.15
B^p	0.51	-0.15
r	0.37	0.37
$E\left(r^{kn}\right)$	0.40	0.40
$E\left(r^a\right)$	0	0.39
$E\left(w\right)$	2.10	2.10
$E\left(c_o\right)$	1.94	1.94
$E\left(c_y\right)$	1.94	1.94
Ω^{pf}	0.067	0.067
Ω^{sp}	0	0

r. Equation (3.12) then ties down θ^f. Individuals from both generations have an identical exposure to capital income, which implies that $K^p/N_0^o = K^f/N_1^y = \frac{1}{2}$, and to labour income in period 1. Hence, they have an identical exposure to both remaining sources of risk, productivity and depreciation, which are perfectly shared. Because the two generations are of different size in period 1, the pension system parameters correct for this to ensure identical exposures at the individual level. Hence, conform Proposition 1, $\theta^w = 1/1.8 \simeq 0.56$ and $\theta^{dwb} = 0.8/1.8 \simeq 0.44$.

3.5.3 Deterministic variation in demographic variables

Table 3.3 shows how under perfect foresight about the future demography the parameters of the optimal pension arrangement change when either the fertility rate N_1^y or the survival probability ψ differs from its mean value, while keeping the other demographic parameter at its mean. The outcomes can be partly obtained by applying Proposition 1. To compare them to the outcomes in Table 3.2, those optimal arrangements are selected in which $p = 0$ in the DRB system and $\theta^w = 0$ in the DWB system.

Table 3.3: Varying values for demographic shocks

	$N_1^y = 0.8$	$\psi = 0.8$	$N_1^y = 1.2$	$\psi = 0.8$	$N_1^y = 1$	$\psi = 0.6$	$N_1^y = 1$	$\psi = 1$
	DRB	DWB	DRB	DWB	DRB	DWB	DRB	DWB
θ^p	-0.64	0.37	-0.92	0.13	-0.88	0.10	-0.70	0.34
θ^w	0.50	0	0.60	0	0.63	0	0.50	0.00
θ^{dwb}	0	0.5	0	0.40	0	0.38	0	0.50
p	0	0	0	0	0	0	0	0
θ^f	-0.01	0.71	-0.01	0.62	-0.01	0.56	-0.01	0.75
K^f	0.44	0.44	0.55	0.55	0.50	0.50	0.50	0.50
K^p	0.56	0.56	0.45	0.45	0.50	0.50	0.50	0.50
K	1	1	1	1	1	1	1	1
B^f	-0.45	0.26	-0.56	0.08	-0.51	0.06	-0.51	0.25
B^p	0.45	-0.26	0.56	-0.08	0.51	-0.06	0.51	-0.25
r	0.25	0.25	0.49	0.49	0.37	0.37	0.37	0.37
$E\left(r^{kn}\right)$	0.27	0.27	0.52	0.52	0.40	0.40	0.40	0.40
$E\left(r^a\right)$	0	0.26	0	0.52	0	0.40	0	0.39
$E\left(w\right)$	2.25	2.25	1.99	1.99	2.10	2.10	2.10	2.10
$E\left(c_o\right)$	1.92	1.92	1.95	1.95	1.75	1.75	2.19	2.19
$E\left(c_y\right)$	1.92	1.92	1.95	1.95	1.75	1.75	2.19	2.19
Ω^{lf}	0.089	0.089	0.045	0.045	0.08	0.08	0.054	0.054
Ω^{sp}	0	0	0	0	0	0	0	0

Because the total capital stock K is fixed by the period-0 resource constraint, a higher fertility rate N_1^y produces a higher expected net return r^{kn} on capital and a lower expected wage rate w. Owing to the trade-off that individuals make between investing in capital and investing in real debt, the higher expected net return on capital requires a rise in the risk-free interest rate in order for individuals to be willing to invest in the risk-free asset. The increase in N_1^y also produces an increase in K^f. The young generation carries the residual risk associated with the pension fund's asset portfolio. To evenly distribute the exposure to capital risk between the two generations, K^f needs to make up a larger fraction of the total capital stock if the relative size of the young generation increases. Given that the volume of benefit payments is kept fixed, the increase in K^f requires a fall in B^f. Further, an increase in N_1^y implies a higher value of θ^w. For a given value of θ^w, an increase in N_1^y implies that, per young person, a smaller fraction of its wage is taken away by the government and given to the old generation, since θ^w is defined per old-generation member. Hence, for a fixed θ^w, the young's exposure to wage risk increases. Therefore, to ensure indentical exposure to wage risk for all individuals, an increase in θ^w is needed, thereby raising the exposure of the old generation members to wage risk. However, the increase in θ^w raises the average amount of period-1 resources transfered to the old. To compensate for this systematic increase in transfers to the old, θ^p is reduced.

The consequences of the increase in N_1^y for the optimal pension arrangement under DWB are qualitatively the same as under DRB. However, because θ^w is fixed to zero under DWB, the identical exposures to wage risk now have to be restored through a reduction in θ^{dwb}, which is defined as a share of the total wage sum. Finally, although a priori θ^f could go both up or down, for this specific parameter combination the effect is that it goes down.

Now, the effect of a foreseen increase in the survival probability of the old generation is discussed. As old generation members do not work in period 1, factor prices and the return on debt are unaffected. The increase in the survival probability implies an increase in the old-age dependency ratio. That is, it implies an increase in the number of retirees N_1^o over the number of people of working age N_1^y. As a result, under DRB and for given θ^w, a larger fraction of the young generation members wage income is transfered to the old via the PAYG pillar of the pension arrangement. To restore their exposure to wage risk, θ^w needs to go down. However, this implies a smaller average transfer from the young to the old. To undo this, θ^p increases.

Analogously, under the DWB system θ^{dwb} needs to increase and θ^p needs to decrease. Further, under this system θ^f rises. The reason is that there are more old generation members, so that the "survivor-dividend" of the pension fund is smaller. Hence, θ^f needs to rise to compensate for this decrease in the survivor dividend. Indeed, for the actuarial fairness condition to continue to hold in the face of an increase in θ^{dwb} and a fall in $\mathrm{E}_0\left(1+r^a\right)$, while the term $N_0^o/\mathrm{E}_0\left[N_1^y w\right]$ remains constant, θ^f needs to rise. The average return on the pension fund portfolio falls, because the increase in θ^f leads to a higher investment in the risk-free asset, while the investment in equity and the factor prices remain unaltered.

3.5.4 Introducing demographic uncertainty

In this section optimal pension arrangements in the presence of demographic risk are explored. This is done by introducing each one of the sources of demographic risk separately, while shutting the other source off. In the final step, both sources are introduced simultaneously. Two important observations need to be made.

First, in general the social planner solution can no longer be achieved. Second, the multiplicity of solutions that were obtained under foresight about the demography will vanish. All four pension system parameters have to be deployed to maximise social welfare.

Tables 3.4-3.11 display the results of introducing uncertainty about N_1^y and N_1^o. This is done by keeping the mean values of N_1^y and N_1^o at 1 and 0.8, respectively, but increasing the spread between the possible outcomes of the two-point distributions as indicated in the tables by ΔN_1^y and ΔN_1^o, respectively. When considering only one source of demographic risk, the limiting case in which the spread of the outcomes is set to zero is also reported. This corresponds to the situation without demographic uncertainty. One of the solutions in this situation is displayed in the column under $\Delta N_1^y = 0$ ($\Delta N_1^o = 0$). Specifically, in the case of the DRB system when varying ΔN_1^y, the solution is chosen such that $p_{|\Delta N_1^y=0} = p_{|\Delta N_1^y=0.1}$, while in the case of the DWB system the solution is chosen such that $\theta^{dwb}_{|\Delta N_1^y=0} = \theta^{dwb}_{|\Delta N_1^y=0.1}$. In other words, those solutions under demographic certainty are chosen that are "close" to the solutions when a minimum level of uncertainty is introduced.

Obviously, under uncertainty about N_1^y and N_1^o, the solutions for the pension system parameters differ from those in Proposition 3.1. To obtain additional intuition for the solutions under demographic uncertainty, the deviations of the solutions under uncertainty from those when $\Delta N_1^y = \Delta N_1^o = 0$ are decomposed into two components. The first component is the change in the average value of the pension system parameters under each of the two possible realisations of the demographic parameter under consideration (N_1^y or N_1^o) assuming that these realisations are foreseen with certainty when the pension parameters are decided. This component arises purely from the non-linearity of the solutions in Proposition 3.1. As an example, consider the case of uncertainty about N_1^y. Then, formally, for each pension system parameter $\pi \in \Pi^{\mathbf{drb}}$ or $\pi \in \Pi^{\mathbf{dwb}}$, the following is computed:

$$d_1 = \frac{\pi_{|N_1^y=\bar{N}_1^y-\Delta N_1^y} + \pi_{|N_1^y=\bar{N}_1^y+\Delta N_1^y}}{2} - \pi_{|N_1^y=\bar{N}_1^y},$$

where $\pi_{|N_1^y=\bar{N}_1^y}$, $\pi_{|N_1^y=\bar{N}_1^y-\Delta N_1^y}$ and $\pi_{|N_1^y=\bar{N}_1^y+\Delta N_1^y}$ are the solutions for parameter π under certainty (computed using Proposition 3.1) for the indicated value of N_1^y. Here, $\bar{N}_1^y$ denotes the mean of N_1^y.

The second component measures the change in the solution that is the result purely from the demographic uncertainty. Formally, for the case of uncertainty about N_1^y it is given by

$$d_2 = \pi_{|N_1^y \sim U\left\{\bar{N}_1^y-\Delta N_1^y, \bar{N}_1^y+\Delta N_1^y\right\}} - \frac{\pi_{|N_1^y=\bar{N}_1^y-\Delta N_1^y} + \pi_{|N_1^y=\bar{N}_1^y+\Delta N_1^y}}{2},$$

where $\pi_{|N_1^y \sim U\left\{\bar{N}_1^y-\Delta N_1^y, \bar{N}_1^y+\Delta N_1^y\right\}}$ is the solution under uncertainty about N_1^y given by the discrete uniform distribution with support elements $\bar{N}_1^y - \Delta N_1^y$ and $\bar{N}_1^y + \Delta N_1^y$.

Analytical insight into the signs of d_1 for the various pension parameters is obtained by differentiating the solutions in Proposition 3.1 with respect to N_1^y or N_1^o. The results are reported in Appendix B. For DRB, the values of the first- and second-order derivatives for $\theta^p + p$ are ambiguous. For K^f, the derivative with respect to N_1^y is positive and the second derivative is negative. This means that K^f is a concave function of N_1^y and, by Jensen's inequality, an increase in the spread of the possible values of N_1^y leads to a decrease in the expected value of K^f. The reason is the correlation between the realisation for N_1^y and the return on capital r^{kn}. If N_1^y is low, there are few young and the return on capital is low, implying a shortfall in the pension fund which has to be paid for by relatively few young individuals, which is costly in utility terms. Utility costs in the low state become increasingly higher as uncertainty goes up (there are fewer young individuals in the low state), so K^f decreases to make sure the utility loss in the low state does not become too large. The derivative of K^f with respect to N_1^o is zero, since only the size of the old generation in the initial period, N_0^o, matters for K^f.

Further, the first derivative of θ^w with respect to N_1^y is positive and the second derivative is negative, implying that θ^w is also a concave function of N_1^y and that its expected value falls when the spread in N_1^y rises. A low realisation of N_1^y implies relatively many old individuals each one of them receiving $\theta^w w$. Since the aggregate amount of resources transfered to the old generation has to be paid for by relatively few young individuals, this becomes increasingly costly in terms of young individuals' utility as the realisation of N_1^y in the low state becomes lower. Hence, θ^w decreases to compensate for the bad low state as uncertainty goes up.

Conversely, the first derivative of θ^w with respect to N_1^o is negative, while the second derivative is positive. At a higher realisation of N_1^o the optimal value of θ^w decreases by less for a given increase in N_1^o. The reason is that precisely because of the increased size of the old generation at a higher value for N_1^o a given reduction in θ^w becomes more effective in reducing the transfer per young individual. Thus, the average of the optimal values of θ^w goes up as N_1^o increases.

Analogous results are obtained for the case of DWB. The signs of the first- and second-order derivatives of θ^p with respect to both sources of demographic risk are ambiguous, while K^f is a concave function of N_1^y and does not depend on N_1^o. Finally, the expression from Proposition 3.1 involving the wage-related parameters, $\theta^w + \frac{N_1^y}{N_1^o}\theta^{dwb}$ is a concave function of N_1^y and a convex function of N_1^o.

Table 3.4: Varying fertility risk, no mortality risk

System	DRB				DWB			
ΔN_1^y	0	0.1	0.2	0.3	0	0.1	0.2	0.3
θ^p	-1.1468	-1.1376	-1.0851	-1.0011	-0.6207	-0.6160	-0.6043	-0.5841
θ^w	0.5556	0.5498	0.5326	0.5043	0.4776	0.4738	0.4640	0.4470
θ^{dwb}	-	-	-	-	0.0623	0.0623	0.0609	0.0583
p	3.3210	3.3210	3.0937	2.7319	-	-	-	-
θ^f	1.9246	1.9266	1.8005	1.5989	0.0933	0.0932	0.0910	0.0867
K^f	0.5000	0.4951	0.4806	0.4567	0.5000	0.4970	0.4877	0.4714
K^p	0.5000	0.5049	0.5194	0.5433	0.5000	0.5030	0.5123	0.5286
K	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
B^f	1.4246	1.4316	1.3199	1.1421	-0.4067	-0.4038	-0.3968	-0.3847
B^p	-1.4246	-1.4316	-1.3199	-1.1421	0.4067	0.4038	0.3968	0.3847
r	0.3736	0.3719	0.3667	0.3572	0.3736	0.3719	0.3667	0.3571
$E\left(r^{kn}\right)$	0.4000	0.3991	0.3962	0.3913	0.4000	0.3991	0.3962	0.3913
$E\left(w\right)$	2.1000	2.1041	2.1168	2.1391	2.1	2.1041	2.1168	2.1391
$E\left(r^f\right)$	0.7256	0.7238	0.7182	0.7086	0.7542	0.7534	0.7512	0.7481
$E\left(c_o\right)$	1.9444	1.9421	1.9351	1.9226	1.9444	1.9422	1.9352	1.9229
$E\left(c_y\right)$	1.9444	1.9422	1.9353	1.9231	1.9444	1.9422	1.9353	1.9232
Ω^{pf}	0.0450	0.0465	0.0512	0.0589	0.0450	0.0466	0.0512	0.0589
Ω^{sp}	0	$3.92*10^{-6}$	$1.57*10^{-5}$	$3.55*10^{-5}$	0	$2.41*10^{-6}$	$9.91*10^{-6}$	$2.34*10^{-5}$

3.5.5 Fertility risk

Table 3.4 displays the results for the introduction of uncertainty about N_1^y, i.e. fertility risk. For a given DRB pension arrangement, a change in N_1^y has a variety of effects. First, if N_1^y is low, then w is high and r^{kn} is low, which is beneficial for the young. Second, the wage-linked transfer in the first pillar has to be paid for by fewer young, increasing the amount to be transfered per young. Third , the lumpsum transfer θ^p from the old to the young is divided over less young individuals, increasing the per young transfer as well. Finally, if N_1^y is low, then the amount in the pension fund is not sufficient to pay out the defined real benefit p, because of the lower-than-expected r^{kn}. This is bad news for the few young people, because of the guarantee they have given to the old generation.

As a result of these effects, it can be observed that as uncertainty about N_1^y goes up, θ^f, p, K^f and B^f all go down. This means that the size of the funded pillar becomes smaller and that the "leverage" in the system for young participants goes down. The reason is that as the uncertainty about N_1^y becomes larger, the outcomes for the young generation when it turns out to be small (N_1^y is low) can be particularly unfortunate, and, hence, utility can be particularly low, because of the cost of providing the guarantee p to each old generation member. Specifically, this is the case for a low realisation of productivity and a high realisation of the depreciation rate. Since the expected cost to the young generation of providing the guarantee to the old generation rises as the uncertainty about N_1^y becomes larger, the utilitarian pension planner lowers p, which also lowers θ^f because of the actuarial link between the two pension parameters. Since the guarantee that the young generation provides becomes more costly in utility terms as the uncertainty surrounding N_1^y goes up, the size of the guarantee goes down. This can be seen from the fact that r^f goes down for increasing uncertainty about N_1^y (remember that for the DRB case this is defined as $\frac{p}{\theta^f} - 1$ and is thus the guaranteed return per old generation member). This is corroborated by the results from Table 3.5, where for the DRB system it can be observed that d_2 ranges from -0.0037 to -0.0317 for the solution for θ^f and from -0.0036 to -0.0318 for the solution for K^f.

Moreover, for the first pillar, θ^w falls as uncertainty about N_1^y rises. This compensates for the nonlinearity in the movement in w. If N_1^y is low, the wage rate is very high, while for at high N_1^y the wage rate does not go decrease quite as fast when N_1^y rises due to the Cobb-Douglas specification of the production function. To prevent the wage-linked transfer from becoming very burdensome for the young agents alive when N_1^y is low, θ^w goes down slightly. Decreasing θ^w in this fashion causes redistribution from old to young, because the average wage linked transfer goes down. To offset this redistribution, θ^p rises, which implies that the lump-sum transfer from old to young goes down on average.

For the DWB system, an almost identical response to the introduction of N_1^y risk can be observed from Tables 3.4 and 3.5: the second-pillar parameters θ^{dwb}, θ^f and K^f all fall to prevent the occurrence of truly bad outcomes for the young generation. Parameter θ^w goes down, and to prevent systematic redistribution θ^p needs to be raised. Finally, both systems can handle the presence of fertility risk very well; the welfare gains compared to the laissez-faire situation continue to be sizable (an increase in the certainty-equivalent consumption of 4,5% to 6%), while the welfare loss compared to the social planner's solution remains very small (on the order of magnitude of 0,001%).

3.5.6 Mortality risk

The results for the case with mortality risk, ψ, as the only source of demographic uncertainty are displayed in Tables 3.6 and 3.7. For the DRB system, all mortality risk falls on the young generation, since both the first- and second-pillar benefits are fixed per old generation member. To ensure that the income of the young generation members does not fluctuate too much, the mortality risk the young generation runs through the second pillar is reversed. This is done by setting the pension fund contribution θ^f and the pension fund benefit p at negative values. To prevent distortions of risk sharing of the financial shocks, this is achieved by a large negative pension fund position in bonds. The move in the bond portfolio means that the old generation now borrows money from the pension fund, invests this privately in bonds and pays back a per-surviving old amount of p to the fund in period 1. This means that the risk the young generation runs is now reversed: if fewer than expected old generation members survive until period 1, there is a funding shortfall in the pension fund. This contrasts with the mortality risk transmitted through the first pillar, where the parameters only

Table 3.5: Decomposed solutions, fertility risk, no mortality risk

ΔN_1^y	0.1		0.2		0.3	
	d_1	d_2	d_1	d_2	d_1	d_2
DRB						
$\theta^p + p$	0.0087	0.0005	-0.1674	0.0018	-0.4470	0.0036
θ^w	-0.0014	-0.0044	-0.0056	-0.0174	-0.0127	-0.0386
θ^f	0.0057	-0.0037	-0.1098	-0.0143	-0.2940	-0.0317
K^f	-0.0013	-0.0036	-0.0051	-0.0143	-0.0115	-0.0318
DWB						
θ^p	0.0050	-0.0003	0.0174	-0.0010	0.0390	-0.0024
$\theta^w + \frac{N_1^y}{N_1^o}\theta^{dwb}$	-0.0013	-0.0025	-0.0055	-0.0099	-0.0126	-0.0230
θ^f	0.0016	-0.0017	0.0048	-0.0071	0.0105	-0.0171
K^f	-0.0013	-0.0017	-0.0051	-0.0072	-0.0115	-0.0171

Table 3.6: Varying mortality risk, no fertility risk

System	DRB					DWB				
ΔN_1^o	0	0.05	0.1	0.15	0.2	0	0.05	0.1	0.15	0.2
θ^p	-0.7333	-0.7262	-0.7055	-0.6733	-0.6323	-0.5144	-0.5119	-0.5084	-0.5036	-0.4985
θ^w	0.5556	0.5528	0.5450	0.5333	0.5193	0.4195	0.4179	0.4156	0.4122	0.4083
θ^{dwb}	-	-	-	-	-	0.1089	0.1089	0.1072	0.1044	0.1005
p	-0.4007	-0.4007	-0.4057	-0.4142	-0.4267	-	-	-	-	-
θ^f	-0.2430	-0.2420	-0.2421	-0.2422	-0.2423	0.1629	0.1629	0.1604	0.1562	0.1505
K^f	0.5000	0.4993	0.4975	0.4951	0.4926	0.5000	0.4997	0.4989	0.4977	0.4965
K^p	0.5000	0.5007	0.5025	0.5049	0.5074	0.5000	0.5003	0.5011	0.5023	0.5035
K	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
B^f	-0.7430	-0.7414	-0.7396	-0.7373	-0.7350	-0.3371	-0.3368	-0.3385	-0.3415	-0.3460
B^p	0.7430	0.7414	0.7396	0.7373	0.7350	0.3371	0.3368	0.3385	0.3415	0.3460
r	0.3736	0.3736	0.3737	0.3738	0.3739	0.3736	0.3736	0.3736	0.3737	0.3737
$E\left(r^{kn}\right)$	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
$E\left(w\right)$	2.1	2.1	2.1	2.1	2.1	2.1	2.1	2.1	2.1	2.1
$E\left(r^f\right)$	0.6490	0.6558	0.6758	0.7102	0.7611	0.7542	0.7611	0.7820	0.8180	0.8710
$E\left(c_o\right)$	1.9444	1.9458	1.9498	1.9567	1.9665	1.9444	1.9459	1.9503	1.9578	1.9683
$E\left(c_y\right)$	1.9444	1.9460	1.9508	1.9587	1.9698	1.9444	1.9460	1.9505	1.9582	1.9690
Ω^{pf}	0.0450	0.0452	0.0456	0.0462	0.0471	0.0450	0.0452	0.0456	0.0462	0.0472
Ω^{sp}	0	$2.61*10^{-6}$	$9.80*10^{-6}$	$1.99*10^{-5}$	$3.08*10^{-5}$	0	$5.35*10^{-7}$	$2.04*10^{-6}$	$4.24*10^{-6}$	$6.78*10^{-6}$

change to compensate for the nonlinearity in wage income w. Here, a smaller-than-expected old generation implies a lower aggregate transfers. Hence, the first and second pillars hedge each other implying a more even distribution of mortality risk across the generations. The first-pillar settings of θ^w and θ^p move into the same direction as in the case of fertility risk: θ^w falls as uncertainty increases and θ^p becomes less negative to compensate for the redistribution effect of the change in θ^w.

Under the DWB system, mortality risk is more evenly distributed across generations through the set-up of the pension fund. In the first pillar, the young generation still bears all the mortality risk. However, in the second pillar, the total payout of the fund depends on the aggregate wage sum, which does not depend on the size of the old generation. Hence, the mortality risk associated with the second pillar falls on the old generation. Thus, by construction the DWB system distributes mortality risk over the different generations and there is no need – as there is in the DRB system – to hedge mortality risk by making second-pillar pension savings negative. Therefore, the pension designer can afford to pay more attention to the distribution of the financial shocks.

As in the case of fertility risk alone, θ^p becomes less negative and θ^w becomes smaller when uncertainty increases. Similarly, in order to limit the impact on consumption in the bad state in the second pillar, θ^f and θ^{dwb} decline as uncertainty rises. In terms of welfare, the DWB system deals slightly better with mortality risk than the DRB system, since its set-up is more amenable to evenly distributing the effects of a mortality shock. However, compared to the welfare gains that are obtained relative to the laissez-faire solution (between 4.5% to 4.7%), the difference between the two systems is very small.

3.5.7 Simultaneous presence of both types of demographic risk

Varying fertility risk with constant mortality risk

Finally, the combined effect of the presence of both sources of demographic uncertainty are analyzed. In Table 3.8 the results when varying fertility risk, while mortality risk is

Table 3.7: Decomposed solutions, mortality risk, no fertility risk

ΔN_1^o	0.05		0.1		0.15		0.2	
	d_1	d_2	d_1	d_2	d_1	d_2	d_1	d_2
DRB								
$\theta^p + p$	0.0016	0.0055	0.0016	0.0212	0.0007	0.0458	-0.0020	0.0770
θ^w	0.0004	-0.0032	0.0017	-0.0123	0.0038	-0.0261	0.0069	-0.0432
θ^f	0.0016	-0.0006	0.0034	-0.0025	0.0057	-0.0049	0.0080	-0.0073
K^f	0.0000	-0.0007	0.0000	-0.0025	0.0000	-0.0049	0.0000	-0.0074
DWB								
θ^p	0.0000	0.0025	-0.0038	0.0098	-0.0105	0.0213	-0.0203	0.0362
$\theta^w + \frac{N_1^y}{N_1^o}\theta^{dwb}$	0.0004	-0.0014	0.0017	-0.0055	0.0039	-0.0120	0.0069	-0.0202
θ^f	0.0003	-0.0003	-0.0014	-0.0011	-0.0044	-0.0023	-0.0089	-0.0035
K^f	0.0000	-0.0003	0.0000	-0.0011	0.0000	-0.0023	0.0000	-0.0035

Table 3.8: Varying fertility risk, mortality risk constant at $\Delta N_1^o = 0.1$

System	DRB				DWB			
ΔN_1^o	0	0.1	0.2	0.3	0	0.1	0.2	0.3
θ^p	-0.7055	-0.5933	-0.5105	-0.4661	-0.5084	-0.5855	-0.5918	-0.5736
θ^w	0.5450	0.4889	0.4446	0.4162	0.4156	0.4544	0.4539	0.4380
θ^{dwb}	-	-	-	-	0.1072	0.0847	0.0776	0.0731
p	-0.4057	-0.3963	-0.3998	-0.4153	-	-	-	-
θ^f	-0.2421	-0.2357	-0.2381	-0.2494	0.1604	0.1266	0.1158	0.1087
K^f	0.4975	0.4162	0.3583	0.3302	0.4989	0.5194	0.5154	0.5001
K^p	0.5025	0.5838	0.6417	0.6698	0.5011	0.4806	0.4846	0.4999
K	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
B^f	-0.7396	-0.6519	-0.5964	-0.5796	-0.3385	-0.3928	-0.3997	-0.3915
B^p	0.7396	0.6519	0.5964	0.5796	0.3385	0.3928	0.3997	0.3915
r	0.3737	0.3719	0.3666	0.3567	0.3736	0.3719	0.3664	0.3563
$E\left(r^{kn}\right)$	0.4000	0.3991	0.3962	0.3913	0.4000	0.3991	0.3962	0.3913
$E\left(w\right)$	2.1000	2.1041	2.1168	2.1391	2.1000	2.1041	2.1168	2.1391
$E\left(r^f\right)$	0.6758	0.6814	0.6791	0.6652	0.7820	0.7820	0.7801	0.7772
$E\left(c_o\right)$	1.9498	1.9468	1.9395	1.9275	1.9503	1.9465	1.9388	1.9258
$E\left(c_y\right)$	1.9508	1.9485	1.9417	1.9298	1.9505	1.9495	1.9436	1.9330
Ω^{pf}	0.0456	0.0471	0.0517	0.0596	0.0456	0.0471	0.0518	0.0597
Ω^{sp}	$9.80 * 10^{-6}$	$6.45 * 10^{-5}$	$1.07 * 10^{-4}$	$1.33 * 10^{-4}$	$2.04 * 10^{-6}$	$2.03 * 10^{-5}$	$5.05 * 10^{-5}$	$1.02 * 10^{-4}$

kept constant at $\Delta N_1^o = 0.1$ are displayed. For the DRB arrangement, mortality risk is shared in the same way as in the case when mortality risk was the only source of risk, by decreasing θ^f and p to -0.25 and -0.44 respectively. Since this prohibits sharing of fertility risk the way this was done previously (through increases in θ^f, B^f and p), this has to be done in another way. Fertility risk is compensated for by having both θ^w and K^f going down. The decrease in θ^w ensures that in case of a bad fertility shock – the young generation turns out to be small – the burden on individual young generation members does not become too high. However, this also shifts more productivity risk to the young generation. To mitigate this effect, K^f is reduced and, thus, K^p is increased. This shifts some of the productivity as well as some of the depreciation risk to the old generation. As the variance of the fertility shock rises, these effects become more and more pronounced. To compensate, θ^w and K^f have to fall by more.

The DWB system turns out to be more robust to the presence of two types of demographic risks. Since for both the fertility shock and the mortality shock the changes in the pension fund parameters were qualitatively the same – a decrease in the size of the second pillar and an increase in the size of the wage-linked transfer in the first pillar, and the lump-sum first pillar transfer becoming negative to offset the redistributive impact of the increase in θ^w – the parameters are simply set as an average of the parameter settings under the separate demographic risks.

Not surprisingly, given that the DWB system is better suited for handling the presence of both types of demographic risks simultaneously, the welfare gains under the DWB system are slightly higher than under the DRB system. The presence of two sources of demographic risk causes the resulting allocation of consumption to both generations to be more remote from the social planner's solution than when only one source of demographic risk is present. Therefore, the shortfall from the welfare gain under the social planner has become slightly higher than in the cases with only one source of demographic risk.

Varying mortality risk with constant fertility risk

In Table 3.10 fertility risk is kept constant at $\Delta N_1^y = 0.2$, while the uncertainty surrounding the mortality shock is increased progressively. The same patterns are observed

Table 3.9: Decomposed solutions, varying fertility risk, constant mortality risk at $\Delta N_1^o = 0.1$

ΔN_1^y	0.1		0.2		0.3	
	d_1	d_2	d_1	d_2	d_1	d_2
DRB						
$\theta^p + p$	0.1141	0.0075	0.2015	-0.0006	0.2320	-0.0022
θ^w	0.0109	-0.0670	0.0067	-0.1071	-0.0004	-0.1284
θ^f	0.0889	-0.0825	0.1406	-0.1366	0.1510	-0.1583
K^f	0.0012	-0.0825	-0.0026	-0.1366	-0.0090	-0.1583
DWB						
θ^p	-0.0924	0.0153	-0.1011	0.0177	-0.0853	0.0201
$\theta^w + \frac{N_1^y}{N_1^o}\theta^{dwb}$	0.0042	0.0061	0.0000	0.0007	-0.0071	-0.0138
θ^f	-0.0545	0.0207	-0.0652	0.0206	-0.0634	0.0117
K^f	-0.0002	0.0207	-0.0040	0.0205	-0.0104	0.0116

Table 3.10: Varying mortality risk, fertility risk constant at $\Delta N_1^y = 0.2$

System	DRB					DWB				
ΔN_1^o	0	0.05	0.1	0.15	0.2	0	0.05	0.1	0.15	0.2
θ^p	-1.0851	-0.5417	-0.5105	-0.4806	-0.4442	-0.6043	-0.6018	-0.5918	-0.5736	-0.5497
θ^w	0.5326	0.4525	0.4446	0.4353	0.4243	0.4640	0.4614	0.4539	0.4424	0.4284
θ^{dwb}	-	-	-	-	-	0.0609	0.0670	0.0776	0.0846	0.0871
p	3.0937	-0.3069	-0.3998	-0.4250	-0.4434	-	-	-	-	-
θ^f	1.8005	-0.1868	-0.2381	-0.2479	-0.2513	0.0910	0.1001	0.1158	0.1262	0.1299
K^f	0.4806	0.3612	0.3583	0.3579	0.3579	0.4877	0.4984	0.5154	0.5237	0.5221
K^p	0.5194	0.6388	0.6417	0.6421	0.6421	0.5123	0.5016	0.4846	0.4763	0.4779
K	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
B^f	1.3199	-0.5480	-0.5964	-0.6058	-0.6092	-0.3968	-0.3983	-0.3997	-0.3976	-0.3922
B^p	-1.3199	0.5480	0.5964	0.6058	0.6092	0.3968	0.3983	0.3997	0.3976	0.3922
r	0.3667	0.3667	0.3666	0.3664	0.3661	0.3667	0.3666	0.3664	0.3660	0.3656
$E\left(r^{kn}\right)$	0.3962	0.3962	0.3962	0.3962	0.3962	0.3962	0.3962	0.3962	0.3962	0.3962
$E\left(w\right)$	2.1168	2.1168	2.1168	2.1168	2.1168	2.1168	2.1168	2.1168	2.1168	2.1168
$E\left(r^f\right)$	0.7182	0.6429	0.6791	0.7144	0.7644	0.7512	0.7585	0.7801	0.8166	0.8699
$E\left(c_o\right)$	1.9351	1.9351	1.9395	1.9469	1.9575	1.9352	1.9359	1.9388	1.9450	1.9547
$E\left(c_y\right)$	1.9353	1.9368	1.9417	1.9498	1.9613	1.9353	1.9376	1.9436	1.9528	1.9650
Ω^{pf}	0.0512	0.0512	0.0517	0.0525	0.0536	0.0512	0.0513	0.0518	0.0526	0.0536
Ω^{sp}	$1.57*10^{-5}$	$9.66*10^{-5}$	$1.07*10^{-4}$	$1.20*10^{-4}$	$1.33*10^{-4}$	$9.91*10^{-6}$	$2.23*10^{-5}$	$5.05*10^{-5}$	$8.30*10^{-5}$	$1.14*10^{-4}$

as in the previous cases that included mortality risk. Under the DRB system, the first two columns provide an illustration of the reaction of the pension system parameters to the introduction of mortality risk. In the absence of mortality risk (the first column) the pension system parameters are set to distribute the fertility shock equally across generations as in Table 3.4. When mortality risk is introduced (the second column, where the realization of N_1^o is either 0.05 above or below its mean), the funded pillar is used as a hedge against mortality risk that the young generation runs through the first pillar by setting θ^f and p at negative values (-0.19 and -0.31, respectively). However, with these settings, young generation members run more fertility risk than the old generation members. The solution then is to shift some fertility risk towards the old generation by increasing their exposure to the equity return. This is accomplished by raising K^p and reducing K^f. However, the downside of this is that it also shifts more productivity and depreciation risk to the old generation. The various consequences are balanced by raising K^p up to the point at which the additional benefits of increased sharing of fertility risk no longer outweigh the utility costs of decreased risk sharing of productivity and depreciation risk. The larger the uncertainty about fertility risk becomes, the higher the benefits of sharing this risk and the larger (lower) is the optimal value of K^p (K^f). The presence of two sources of demographic risk increases the shortfall in welfare terms relative to that under the social planner. However, the increase in the shortfall is still very small compared to the gains as measured in terms of certainty-equivalent consumption relative to the laissez-faire solution.

For the DWB system, the same result is obtained as in the previous subsection: because the pension system parameters react in the same way to fertility and mortality risk, the pension system parameters simply are an average of the parameter settings that resulted in the presence of each type of demographic risk separately. As before, the welfare gains under DWB are slightly higher than under DRB, because the optimal instrument settings under DRB are forced to deviate more from those under the social planner. Nonetheless, both types of pension arrangement constitute a substantial improvement over the laissez-faire solution but lead to only small deviations from the social planner solution.

Table 3.11: Decomposed solutions, varying mortality risk, constant fertility risk at $\Delta N_1^y = 0.2$

ΔN_1^o	0.05		0.1		0.15		0.2	
	d_1	d_2	d_1	d_2	d_1	d_2	d_1	d_2
DRB								
$\theta^p + p$	-2.8425	-0.0147	-2.9183	-0.0006	-2.9362	0.0220	-2.9465	0.0503
θ^w	0.0178	-0.0979	0.0191	-0.1071	0.0213	-0.1186	0.0244	-0.1327
θ^f	-1.8536	-0.1337	-1.9020	-0.1366	-1.9114	-0.1370	-1.9148	-0.1370
K^f	0.0143	-0.1337	0.0143	-0.1366	0.0143	-0.1370	0.0143	-0.1370
DWB								
θ^p	-0.0017	0.0042	-0.0052	0.0177	-0.0043	0.0350	0.0007	0.0539
$\theta^w + \frac{N_1^y}{N_1^o}\theta^{dwb}$	0.0103	-0.0050	0.0116	0.0007	0.0138	-0.0019	0.0169	-0.0125
θ^f	0.0056	0.0035	0.0042	0.0206	0.0063	0.0289	0.0117	0.0272
K^f	0.0072	0.0035	0.0072	0.0205	0.0072	0.0288	0.0072	0.0272

3.6 Robustness: varying the degree of risk aversion

In the previous section, the effect of changes in the volatility of the shocks on the optimal pension system parameters was investigated. In this section, the robustness of the results for different values of the CRRA parameter ϕ (which was assumed to be equal to 2.5 so far) is checked. Both a lower degree of risk aversion $\phi = 1$ (i.e., log-utility preferences) and a higher degree of risk aversion, $\phi = 5$, are considered. Results are presented for the three baseline scenarios considered in the previous section: only fertility risk ($\Delta N_1^y = 0.2$), only mortality risk ($\Delta N_1^o = 0.1$), fertility and mortality risk at the same time ($\Delta N_1^y = 0.2$ and $\Delta N_1^y = 0.1$). They are found in Table 3.12.

The top panel of the table shows the case with only fertility risk. For both DRB and DWB, in general the impact of varying the degree of risk aversion on the values of the parameters is quite limited, indicating that the chosen degree of risk aversion is immaterial for the results reported so far. As risk aversion becomes lower (i.e., $\phi = 1$), slightly more wage risk is shifted to the old generation by increasing in θ^w and θ^{dwb}, while at the same time slightly more financial risk is shifted to the young generation by increasing K^f somewhat. The intuition can be extracted from Table 3.5. Here, it was seen for $\phi = 2.5$ that the solutions for both θ^w, θ^{dwb} and K^f under a pension scheme were slightly lower than the corresponding social planner's solutions under perfect foresight. This way young individuals were protected against low income and utility in the case of a bad combination of shocks. However, when risk aversion is lower, the penalty in terms of utility when income is low is less severe and, hence, the optimal pension parameters shift in the direction of the social planner's solution. The opposite happens when risk aversion rises. The risk-free interest rate is quite sensitive to the degree of risk aversion. As individuals become less risk averse, they require a return on the risk-free bonds that is closer to the expected return on capital, so that the risk-free rate goes up. Finally, the size of the welfare gains from introducing a pension arrangement depend quite strongly on the risk aversion parameter: as risk aversion falls from $\phi = 2.5$ to $\phi = 1$, the welfare gain relative to laissez-faire falls from around 5% to slightly less than 2% for both DRB and DWB, while raising risk aversion to $\phi = 5$ increases the welfare gain to almost 10.5%. This sensitivity is not surprising, because the advantage of having a pension arrangement is tightly linked to the risk-sharing it provides. If risk aversion becomes smaller the value of having a pension arrangement that allows for risk sharing goes down. For both lower and higher risk aversion, the welfare differences between the social planner and the decentralised solutions with pension arrangements are small. Moreover, the differences become smaller as risk aversion falls.

The results for mortality risk only and the simultaneous presence of both types of demographic risk are reported in the middle and bottom panels of Table 3.12. In these cases varying the degree of risk aversion has a rather small impact on the outcomes, comparable to the effects for the case of only fertility risk. The directions of the changes in the parameters is the same as in the case of fertility risk only. Again the quantitatively largest effects are observed for the risk-free rate and the welfare gains of introducing a defined-benefit pension arrangement.

3.7 Conclusion

In this chapter intergenerational risk sharing within two-tier pension systems in the presence of productivity, financial market and demographic risks has been explored. Compared to the laissez-faire situation, relatively large welfare gains from the presence of a two-tier pension system with a defined-benefit second pillar are found. The first, PAYG pillar takes care of appropriate redistribution between the young and the old generation while also allowing for some sharing of wage risks and demographic risks. The fully-funded second pillar allows for further risk sharing between the two generations. This is accomplished by the mismatch between the assets and the liabilities of the pension fund, where the young are the residual claimants to the pension fund. Obviously, the size of the welfare gains associated with the pension arrangements under consideration should not be over-emphasized, as the underlying economic model is necessarily very stylised.

The simulation results suggest that while having a defined-benefit second pension pillar yields large welfare gains, the exact form in which the benefits are defined is less relevant, as the welfare differences between the defined-benefit systems under con-

Table 3.12: Varying risk aversion

System	DRB			DWB		
ϕ	1	2.5	5	1	2.5	5
$\Delta N_1^y = 0.2,\ \Delta N_1^o = 0$						
θ^p	-1.0911	-1.0851	-1.0776	-0.6117	-0.6043	-0.5924
θ^w	0.5331	0.5326	0.5320	0.4639	0.4640	0.4640
θ^{dwb}	-	-	-	0.0614	0.0609	0.0603
p	3.1398	3.0937	3.0353	-	-	-
θ^f	1.8104	1.8005	1.7931	0.0918	0.091	0.0898
K^f	0.4808	0.4806	0.4808	0.4884	0.4877	0.4868
B^f	1.3296	1.3199	1.3124	-0.3966	-0.3968	-0.3971
r	0.3843	0.3667	0.3388	0.3843	0.3667	0.3387
Ω^{pf}	0.0197	0.0512	0.1043	0.0197	0.0512	0.1043
Ω^{sp}	$6.37 * 10^{-6}$	$1.57 * 10^{-5}$	$2.98 * 10^{-5}$	$3.93 * 10^{-6}$	$9.91 * 10^{-6}$	$1.98 * 10^{-5}$
$\Delta N_1^y = 0,\ \Delta N_1^o = 0.1$						
θ^p	-0.7077	-0.7055	-0.7023	-0.5153	-0.5084	-0.4977
θ^w	0.5472	0.5450	0.5417	0.4162	0.4156	0.4147
θ^{dwb}	-	-	-	0.1075	0.1072	0.1067
p	-0.4219	-0.4057	-0.3814	-	-	-
θ^f	-0.2429	-0.2421	-0.2416	0.1611	0.1604	0.1594
K^f	0.4973	0.4975	0.4981	0.4988	0.4989	0.4991
B^f	-0.7402	-0.7396	-0.7397	-0.3376	-0.3385	-0.3397
r	0.3894	0.3737	0.3487	0.3894	0.3736	0.3487
Ω^{pf}	0.0176	0.0456	0.0913	0.0176	0.0456	0.0913
Ω^{sp}	$4.20 * 10^{-6}$	$9.80 * 10^{-6}$	$1.58 * 10^{-5}$	$8.70 * 10^{-7}$	$2.04 * 10^{-6}$	$3.31 * 10^{-6}$
$\Delta N_1^y = 0.2,\ \Delta N_1^o = 0.1$						
θ^p	-0.5140	-0.5105	-0.5026	-0.6040	-0.5918	-0.5730
θ^w	0.4472	0.4446	0.4393	0.4565	0.4539	0.4501
θ^{dwb}	-	-	-	0.0776	0.0776	0.0775
p	-0.4107	-0.3998	-0.3828	-	-	-
θ^f	-0.2367	-0.2381	-0.241	0.1161	0.1158	0.1151
K^f	0.3567	0.3583	0.3611	0.5173	0.5154	0.5121
B^f	-0.5934	-0.5964	-0.6021	-0.4013	-0.3997	-0.397
r	0.3843	0.3666	0.3380	0.3842	0.3664	0.3377
Ω^{pf}	0.0197	0.0517	0.1065	0.0197	0.0518	0.1066
Ω^{sp}	$4.29 * 10^{-5}$	$1.07 * 10^{-4}$	$2.09 * 10^{-4}$	$2.06 * 10^{-5}$	$5.05 * 10^{-5}$	$9.52 * 10^{-5}$

sideration are very small, irrespective of the degree of uncertainty and the degree of risk aversion. Moreover, while it is not possible to perfectly replicate the social planner's solution in the presence of demographic risks, the defined-benefit arrangements under consideration yield welfare gains compared to the situation without a pension arrangement that are very close to the welfare gains realised under the social planner's solution. The DWB system performs slightly better in this regard.

An avenue for further research concerns the design of pension arrangements that are as robust as possible to foreseen and unforeseen changes in the environment. While changes in the underlying socio-economic environment tend to make adjustments to pension arrangements desirable, making those changes is often a difficult process because of the unavoidable conflicting interests. This has been made painfully clear by the arduous process of switching from the old to a new pension contract in the Netherlands. While it is widely recognised that the funded pension promises given under the old contract are unsustainable in view of past and expected future increases in life expectancy and the poor performance of the pension funds' assets during the current crisis, it has proven to be extremely difficult to agree on the terms of the new contract. In many instances when reform is contemplated the interests are split along generational lines. The difficulties in changing the arrangements increase the desirability of automatic adjustment rules, such as an automatic link between life expectancy and the retirement age or an explicit loss sharing rule in case the financial position of the pension fund deteriorates. However, while the game-theoretic aspects of resetting pension arrangements are highly relevant, they have been underexplored so far.

APPENDICES

3.A Description of solution of model

Given the set of pension system parameters Π^{drb} or Π^{dwb}, we solve the model as follows. The pension contribution θ^f is tied down by the actuarial fairness condition (3.27) for the DRB fund and (3.28) for the DWB fund. In particular, in the case of a DRB system for each combination θ^p and p that fulfills the condition in the proposition, an outcome for θ^f is found that fulfills the actuarial fairness condition. Similarly, for the DWB system for each combination θ^w and θ^{dwb} that fulfills the condition in the proposition, an outcome for θ^f is found that fulfills the actuarial fairness condition. Then, equation (3.12) pins down $B^f = N_0^o \theta^f - K^f$. Next, via (3.30) we obtain $B^p = -B^f$. Using the solutions for θ^f and B^p in (3.20) we obtain K^p and, hence, we also obtain the total capital stock $K = K^p + K^f$. Finally, substituting the relevant expression for G into (3.21) and substituting the resulting expression into (3.25), we are left with an equation that we can solve for the remaining unknown, r. In solving this equation, we make use of the firm's first-order conditions (3.23) and (3.24).

3.B Proof of Proposition 1

3.B.1 Part (i)

A necessary condition for replicating the social planner solution is $u'(c_o) = u'(c_y)$, which simplifies to $c_o = c_y$. The general expressions for consumption of the young and of the old are given by equations (3.21) and (3.22), respectively.

Under the DRB system, the generational account is given by equation (3.16). Substituting this into the consumption functions (3.21) and (3.22) yields:

$$\begin{aligned} c_o &= (1+r)\frac{B^p}{N_0^o} + \left(1+r^{kn}\right)\frac{K^p}{N_0^o} + \frac{(1+r)B^f + \left(1+r^{kn}\right)K^f}{N_1^o} + \tau + \theta^p + \theta^w w \\ &+ p - \frac{(1+r)B^f + \left(1+r^{kn}\right)K^f}{N_1^o} \\ &= \left(1+r^{kn}\right)\frac{K^p}{N_0^o} + (1+r)\frac{B^p}{N_0^o} + \tau + \theta^p + \theta^w w + p, \\ c_y &= w + \tau - \frac{N_1^o}{N_1^y}\left[\theta^p + \theta^w w + p - \frac{(1+r)B^f + \left(1+r^{kn}\right)K^f}{N_1^o}\right]. \end{aligned}$$

Setting $c_o = c_y$, using that $K^p = K - K^f$, and rearranging, we obtain:

$$\begin{aligned} &\left(1+r^{kn}\right)\left(\frac{K-K^f}{N_0^o} - \frac{K^f}{N_1^y}\right) + (1+r)\frac{B^p}{N_0^o} + \left(\frac{N_1^y + N_1^o}{N_1^y}\right)\left(\theta^p + \theta^w w + p\right) \\ &= w + (1+r)\frac{B^f}{N_1^y} \end{aligned}$$

By setting $\theta^w = \frac{N_1^y}{N_1^y + N_1^o}$ we obtain identical exposure to w on both sides of the equation. The equation then reduces to:

$$\begin{aligned} &\left(1+r^{kn}\right)\left(\frac{K-K^f}{N_0^o} - \frac{K^f}{N_1^y}\right) + (1+r)\frac{B^p}{N_0^o} + \left(\frac{N_1^y + N_1^o}{N_1^y}\right)\left(\theta^p + p\right) \\ &= (1+r)\frac{B^f}{N_1^y} \end{aligned}$$

By setting $\frac{K-K^f}{N_0^o} - \frac{K^f}{N_1^y} = 0$, hence $K^f = \frac{N_1^y}{N_1^y + N_0^o}K = \frac{N_1^y N_0^o}{N_1^y + N_0^o}\eta$, where we have used (3.4), we ensure that consumption of the young and the old responds identically to r^{kn}. Note that the combination $\theta^w = \frac{N_1^y}{N_1^y + N_1^o}$ and $K^f = \frac{N_1^y}{N_1^y + N_0^o}K$ is also necessary to simultaneously ensure identical exposures to w and r^{kn}. The vector of factor prices is a function of the productivity and depreciation shocks. Effectively, we have used the two instruments θ^w and K^f to force identical exposures of all period-1 individuals to these fundamental sources of shocks. This leaves us with the equation

$$\begin{aligned}
& (1+r)\frac{B^p}{N_0^o} + \left(\frac{N_1^y + N_1^o}{N_1^y}\right)(\theta^p + p) = (1+r)\frac{B^f}{N_1^y} \\
\Leftrightarrow\ & \left(\frac{N_1^y + N_1^o}{N_1^y}\right)(\theta^p + p) = (1+r)\left(\frac{B^f}{N_1^y} + \frac{B^f}{N_0^o}\right) \\
\Leftrightarrow\ & \theta^p + p = (1+r)\frac{N_0^o + N_1^y}{N_0^o\left(N_1^o + N_1^y\right)}B^f
\end{aligned}$$

where we have used the bonds market equilibrium condition $B^p = -B^f$ to obtain the second line. This completes the proof of part (i).

3.B.2 Part (ii)

Under the DWB system the generational account is given by equation (3.19). Substituting this into the consumption functions (3.21) and(3.22) yields:

$$\begin{aligned}
c_o &= (1+r)\frac{B^p}{N_0^o} + \left(1+r^{kn}\right)\frac{K^p}{N_0^o} + \frac{(1+r)B^f + \left(1+r^{kn}\right)K^f}{N_1^o} + \tau + \theta^p + \theta^w w \\
&+ \frac{N_1^y}{N_1^o}\theta^{dwb}w - \frac{1}{N_1^o}\left[(1+r)B^f + \left(1+r^{kn}\right)K^f\right] \\
&= (1+r)\frac{B^p}{N_0^o} + \left(1+r^{kn}\right)\frac{K^p}{N_0^o} + \tau + \theta^p + \theta^w w + \frac{N_1^y}{N_1^o}\theta^{dwb}w, \\
c_y &= w + \tau - \frac{N_1^o}{N_1^y}\left\{\theta^p + \theta^w w + \frac{N_1^y}{N_1^o}\theta^{dwb}w - \frac{1}{N_1^o}\left[(1+r)B^f + \left(1+r^{kn}\right)K^f\right]\right\}.
\end{aligned}$$

Again, a necessary condition for replicating the social planner solution is to set $c_o = c_y$. Doing so, using that $K^p = K - K^f$, and rearranging, we obtain:

$$\begin{aligned}
& (1+r)\frac{B^p}{N_0^o} + \left(1+r^{kn}\right)\frac{K-K^f}{N_0^o} + \left(\frac{N_1^y + N_1^o}{N_1^y}\right)\left[\theta^p + \theta^w w + \frac{N_1^y}{N_1^o}\theta^{dwb}w\right] \\
=\ & w + \frac{1}{N_1^y}\left[(1+r)B^f + \left(1+r^{kn}\right)K^f\right].
\end{aligned}$$

Setting $\theta^w + \frac{N_1^y}{N_1^o}\theta^{dwb} = \frac{N_1^y}{N_1^y + N_1^o}$ makes the exposure to wages on both sides of the equation identical. Further, setting $K^f = \frac{N_1^y}{N_1^y + N_0^o}K = \frac{N_1^y N_0^o}{N_1^y + N_0^o}\eta$ results in identical exposure to

r^{kn} for individuals from both generations. We are then left with:

$$(1+r)\,\frac{B^p}{N_0^o} + \left(\frac{N_1^y + N_1^o}{N_1^y}\right)\theta^p = (1+r)\,\frac{B^f}{N_1^y}.$$

Using that $B^p = -B^f$, we rewrite this as:

$$\begin{aligned}
\frac{N_1^y + N_1^o}{N_1^y}\theta^p &= (1+r)\left[\frac{B^f}{N_1^y} + \frac{B^f}{N_0^o}\right] \\
\Rightarrow \theta^p &= (1+r)\,\frac{N_0^o + N_1^y}{N_0^o\left(N_1^y + N_1^o\right)}B^f.
\end{aligned}$$

This concludes the proof of part (ii).

3.C Derivatives of expressions in Proposition 1

In this part of the appendix we provide the derivatives of the expressions in Proposition 1.

3.C.1 DRB

$$
\begin{aligned}
\frac{\partial\left(\theta^{p}+p\right)}{\partial N_{1}^{y}} &= (1+r)\,B^{f}\frac{N_{0}^{o}\left(N_{1}^{y}+N_{1}^{o}\right)-\left(N_{0}^{o}+N_{1}^{y}\right)N_{0}^{o}}{\left[N_{0}^{o}\left(N_{1}^{y}+N_{1}^{o}\right)\right]^{2}} \\
&= (1+r)\,B^{f}\frac{N_{1}^{o}-N_{0}^{o}}{N_{0}^{o}\left(N_{1}^{y}+N_{1}^{o}\right)^{2}} \lesseqgtr 0, \\
\frac{\partial^{2}\left(\theta^{p}+p\right)}{\partial^{2}N_{1}^{y}} &= -2\,(1+r)\,B^{f}\frac{N_{1}^{o}-N_{0}^{o}}{N_{0}^{o}\left(N_{1}^{y}+N_{1}^{o}\right)^{3}} \lesseqgtr 0, \\
\frac{\partial\left(\theta^{p}+p\right)}{\partial N_{1}^{o}} &= -\,(1+r)\,B^{f}\frac{N_{0}^{o}+N_{1}^{y}}{N_{0}^{o}\left(N_{1}^{y}+N_{1}^{o}\right)^{2}} \lesseqgtr 0, \\
\frac{\partial^{2}\left(\theta^{p}+p\right)}{\partial^{2}N_{1}^{o}} &= 2\,(1+r)\,B^{f}\frac{N_{0}^{o}+N_{1}^{y}}{N_{0}^{o}\left(N_{1}^{y}+N_{1}^{o}\right)^{3}} \lesseqgtr 0, \\
\frac{\partial K^{f}}{\partial N_{1}^{y}} &= \eta\frac{\left(N_{1}^{y}+N_{0}^{o}\right)N_{0}^{o}-N_{1}^{y}N_{0}^{o}}{\left(N_{1}^{y}+N_{0}^{o}\right)^{2}} = \eta\left[\frac{N_{0}^{o}}{N_{1}^{y}+N_{0}^{o}}\right]^{2} > 0, \\
\frac{\partial^{2}K^{f}}{\partial^{2}N_{1}^{y}} &= -2\eta\frac{\left(N_{0}^{o}\right)^{2}}{\left(N_{1}^{y}+N_{0}^{o}\right)^{3}} < 0, \\
\frac{\partial K^{f}}{\partial N_{1}^{o}} &= 0, \\
\frac{\partial^{2}K^{f}}{\partial^{2}N_{1}^{o}} &= 0, \\
\frac{\partial\theta^{w}}{\partial N_{1}^{y}} &= \frac{N_{1}^{y}+N_{1}^{o}-N_{1}^{y}}{\left(N_{1}^{y}+N_{1}^{o}\right)^{2}} = \frac{N_{1}^{o}}{\left(N_{1}^{y}+N_{1}^{o}\right)^{2}} > 0, \\
\frac{\partial^{2}\theta^{w}}{\partial^{2}N_{1}^{y}} &= -2\frac{N_{1}^{o}}{\left(N_{1}^{y}+N_{1}^{o}\right)^{3}} < 0, \\
\frac{\partial\theta^{w}}{\partial N_{1}^{o}} &= -\frac{N_{1}^{y}}{\left(N_{1}^{y}+N_{1}^{o}\right)^{2}} < 0, \\
\frac{\partial^{2}\theta^{w}}{\partial^{2}N_{1}^{o}} &= 2\frac{N_{1}^{y}}{\left(N_{1}^{y}+N_{1}^{o}\right)^{3}} > 0.
\end{aligned}
$$

3.C.2 DWB

$$
\begin{aligned}
\frac{\partial \theta^p}{\partial N_1^y} &= (1+r)\, B^f \frac{N_0^o\left(N_1^y+N_1^o\right)-N_0^o\left(N_1^y+N_0^o\right)}{\left[N_0^o\left(N_1^y+N_1^o\right)\right]^2}, \\
&= (1+r)\, B^f \frac{N_1^o-N_0^o}{N_0^o\left(N_1^y+N_1^o\right)^2} \lesseqgtr 0, \\
\frac{\partial^2 \theta^p}{\partial^2 N_1^y} &= -2\,(1+r)\, B^f \frac{N_1^o-N_0^o}{N_0^o\left(N_1^y+N_1^o\right)^3} \lesseqgtr 0, \\
\frac{\partial \theta^p}{\partial N_1^o} &= -\,(1+r)\, B^f \frac{N_0^o+N_1^y}{N_0^o\left(N_1^y+N_1^o\right)^2} \lesseqgtr 0, \\
\frac{\partial \theta^p}{\partial N_1^o} &= 2\,(1+r)\, B^f \frac{N_0^o+N_1^y}{N_0^o\left(N_1^y+N_1^o\right)^3} \lesseqgtr 0, \\
\frac{\partial K^f}{\partial N_1^y} &= \eta \frac{\left(N_1^y+N_0^o\right)N_0^o-N_1^y N_0^o}{\left(N_1^y+N_0^o\right)^2} = \eta\left[\frac{N_0^o}{N_1^y+N_0^o}\right]^2 > 0, \\
\frac{\partial^2 K^f}{\partial^2 N_1^y} &= -2\eta \frac{\left(N_0^o\right)^2}{\left(N_1^y+N_0^o\right)^3} < 0, \\
\frac{\partial K^f}{\partial N_1^o} &= 0, \\
\frac{\partial^2 K^f}{\partial^2 N_1^o} &= 0, \\
\frac{\partial\left[\theta^w+\left(N_1^y/N_1^o\right)\theta^{dwb}\right]}{\partial N_1^y} &= \frac{N_1^o}{\left(N_1^y+N_1^o\right)^2} > 0, \\
\frac{\partial^2\left[\theta^w+\left(N_1^y/N_1^o\right)\theta^{dwb}\right]}{\partial^2 N_1^y} &= -2\frac{N_1^o}{\left(N_1^y+N_1^o\right)^3} < 0, \\
\frac{\partial\left[\theta^w+\left(N_1^y/N_1^o\right)\theta^{dwb}\right]}{\partial N_1^o} &= -\frac{N_1^y}{\left(N_1^y+N_1^o\right)^2} < 0, \\
\frac{\partial^2\left[\theta^w+\left(N_1^y/N_1^o\right)\theta^{dwb}\right]}{\partial^2 N_1^o} &= 2\frac{N_1^y}{\left(N_1^y+N_1^o\right)^3} > 0.
\end{aligned}
$$