Essays in pension economics and intergenerational risk sharing

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Chapter 4

Voluntary Participation and Intergenerational Risk Sharing in a Funded Pension System

4.1 Introduction

All over the world countries introduce funded pension systems in anticipation of the projected rise in aging costs. Often those systems are of the defined-contribution (DC) type, which means that workers save through a pension fund and at retirement receive whatever they have accumulated in their own account. Hence, under such an arrangement much of the potential benefit from intergenerational risk sharing, in particular risk sharing between workers and retirees, will be lost. However, this need not be the case if the funded pension scheme is appropriately designed. For example, in the Netherlands, a country with a large pillar of sector and company pension funds that pay out benefits defined in nominal terms, risks are shared among generations of workers and retirees through changes in the contributions paid by the workers and changes in the indexation of pension benefits and pension rights to wage or price inflation. In the case of underfunding (low buffers), reduced indexation implies a reduction in the purchasing power of all generations' benefits or rights, while, in addition, contributions paid by working generations may be raised. The opposite tends to happen when the ratio of assets to liabilities becomes very large. Participation of employees in the system

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is mandatory.[2] This is important, because a risk-sharing arrangement that is not legally enforced is only viable when there exists no ex-post incentive to quit the arrangement whenever a transfer has to be made. In a situation of underfunding of the pension fund, the young would be tempted to leave the system instead of guaranteeing the pensions of the retired through higher contributions. If the buffers become too large, the old will be tempted to close the fund. If participation in a funded pension system is voluntary, the ex-ante benefits of intergenerational risk sharing may be forgone.

In this chapter, the feasibility and welfare consequences of a funded pension system with voluntary participation and intergenerational risk sharing is expored. So far, there hardly exists any formal analysis addressing this issue. Such an analysis is of substantial policy relevance, because the countries that currently increase or plan to increase the funded component of their pension systems need to take informed decisions about its design. In particular they need to decide whether to introduce certain defined-benefit elements and whether to make participation obligatory.

Ongoing discussions or even reforms take place in many countries. Of particular relevance for this chapter is the discussion in the U.K. The Department for Work and Pensions (DWP, 2012) is preparing a revision of the Pensions Act 2011 with the aim of making 'auto-enrolment' of employees in pension schemes provided by the employer obligatory. At the same time, the European Commission is revising the Directive for Institutions of Occupational Retirement Provision (IORPs) and plans to have draft legislation before the summer of 2013 (Barnier, 2012). The current IORP directive provides minimum standards for supervision and governance of occupational pension plans. Whether the revision will support or discourage moving towards funded second-pillar pensions in EU member states will depend to a large extent on the exact details of the proposal. In the Netherlands, pressure on mandatory pension fund participation has been growing. The current generosity of funded occupational pensions is unsustainable due to the ageing of the population, while the financial crisis has left many pension funds underfunded. Young generations see their contributions rise, while they are concerned that insufficient assets will remain in the pension funds to secure

[2]More precisely, participation is mandatory when there is a collective labour agreement between the social partners (trade unions and firms or employers' organizations). Except for the self-employed this is the case for most of the workers.

their own future pension benefits. Hence, their reluctance to participate in the system grows. Therefore, social partners, supported by the government, have been working on redesigning current pension contracts, which should preserve mandatory participation and the risk-sharing benefits that this brings to all generations involved.

Our analysis takes place in the context of a simple infinite horizon model with two overlapping generations. Voluntary participation is modelled through a participation constraint that requires the expected utility from participation to be at least as large as expected utility under autarky, which is defined as a situation without any form of pension system and, hence, without any risk sharing between generations. When the return on the pension fund portfolio is low, the current young make a transfer to the current retirees. Individuals always have the option to not participate after the current shock has materialised. Participation yields benefits in terms of intergenerational risk sharing. However, numerical results show that the ex-post option to not participate in the system renders a funded system unfeasible for low degrees of risk aversion and when portfolio returns have low variance. In those circumstances the risk-sharing benefits will be lost. Increases in risk aversion and uncertainty about the returns raise the likelihood of a sustainable funded system based on voluntary participation. However, risk sharing would still be less than under the optimal arrangement with obligatory participation. Raising risk-aversion and volatility further, we find that the optimal solutions under voluntary and obligatory participation coincide.

This chapter draws from several strands of the literature. First, there is a literature on intergenerational risk sharing in pension systems. Bohn (1999b, 2003) analyses such risk sharing in PAYG pension systems. Krueger and Kubler (2006) find that PAYG pension systems can only be welfare improving when markets are incomplete, while, moreover, their welfare effects depend on the degrees of risk aversion and intertemporal substitution. Gottardi and Kubler (2011) extend the analysis of Krueger and Kubler (2006) and Ball and Mankiw (2007) compare risk sharing of an overlapping generations economy under autarky and in the presence of a full set of state-contingent assets. De Menil and Sheshinski (2003), Hassler and Lindbeck (1997) and Matsen and Thøgersen (2004) explore the trade off between a PAYG and a funded pension system. The optimal relative sizes of the two components of the pension system depend on the preferences and the characteristics of the stochastic processes driving wages and returns. Beetsma and Bovenberg (2009) and Beetsma et al. (2011) explore risk sharing through a combination of a PAYG and a fully funded system. Finally, Cui et al. (2011) compare risk sharing in various types of funded pension systems such as an individual DC scheme, a collective DC scheme and a collective defined-benefit (DB) scheme.

Second, this chapter is linked to the literature on participation constraints. Kocherlakota (1996) and Thomas and Worrall (1988) explore risk-sharing between two types of infinitely-lived agents in which transfers can go both ways between the agents, who always have the possibility to walk away from the risk-sharing agreement. Krueger and Perri (2011) allow for a continuum of infinitely-lived agents and explore participation with limited enforcement when public insurance is introduced into a market in which private insurance contracts are not fully enforcable. They show that public insurance can crowd out private insurance, because agents who break the private contract have the option of joining the public insurance contract. Our setup differs from these other contributions in that our agents only live for two periods and transfers always go from the young to the old. The young are prepared to make such transfers because they expect the future young to also honour the risk-sharing arrangement.

The third strand of the literature to which this chapter is related is that on discontinuity risk in pension systems. Demange and G.Laroque (2001) study risk-sharing in a PAYG pension system with voluntary contributions, while Demange (2009) explores political sustainability risk in PAYG systems. In Teulings and De Vries (2006) the young may be exposed to equity risk already before they enter the labour market, implying that they may have accumulated losses even before they start working. Under those circumstances young individuals may prefer not to participate in the pension fund. Bovenberg et al. (2007) also discuss the problem of negative buffers that make it unattractive for new young generations to enter a pension fund. Other articles exploring discontinuity risk in funded pension systems are van Ewijk et al. (2009), Gollier (2008), Westerhout (2009) and Molenaar et al. (2011). The latter two articles are closest to our analysis. Westerhout (2009) employs an infinite horizon model with two overlapping generations. He quantifies the feasible amount of risk-sharing under the assumption that the old are bound by their pension contract and the young are free to choose whether they will join the fund. They will not join when the financial position of the fund is weak. Hence, risk sharing is possible only for a limited set of states. An important assumption is that the return on the pension funds' assets exceeds that on private savings. This makes it relatively attractive to join a pension fund. We do not need to make such an assumption. Molenaar et al. (2011) explore the break-even ratio of pension fund assets over liabilities at which it is optimal for a participant to quit the pension fund. They model a Dutch DB pension fund and in this and other respects their model differs substantially from our model. Moreover, in contrast to our analysis they only optimise the static asset allocation under the alternative of no participation, while there is no fully-fledged analysis of the existence and characterisation of equilibria with voluntary participation.

The remainder of the chapter is structured as follows. In Section 2, the model is set up and solved for the autarky solution. Section 3 introduces a pension fund and solves for the optimal transfer rule in the absence of a participation constraint. In Section 4, we introduce the participation constraint and characterise the various equilibria. This section also solves for the optimal pension fund rule. Section 5 works out a numerical example. Finally, Section 6 concludes the main text of the chapter.

4.2 Model and autarky solution

We set up an infinite-horizon overlapping generations model. In each period, a new, young generation is born. The generation born in period t will be referred to as the "period-t generation". Each generation consists of identical individuals and lives for two periods. In the first period of its life, the generation works, consumes and saves for its old age. When old in the second period of its life, the generation is retired and consumes all of its savings. Savings can be invested in a single risky asset. We assume that subsequent returns on the asset are identically and independently distributed. All generations are of the same size, which we normalise to unity.

The preferences of an individual born in period t are given by:

$$U_t = u\left(c_{t,t}\right) + \beta E_t\left[u\left(c_{t+1,t}\right)\right], \tag{4.1}$$

where $c_{t,t}$ denotes consumption in the first period of its life and $c_{t+1,t}$ denotes consumption in the second period of its life. Hence, the first subscript denotes the period in which consumption takes place and the second subscript denotes the period in which the individual is born. Function $u(.)$ is increasing and concave on $[0, \infty)$ and twice differentiable.

In the absence of any form of pension system, that is, under "autarky", consumption of the period-t generation in the two periods of its life is:

$$c_{t,t} = w_t - s_t, \tag{4.2}$$

$$c_{t+1,t} = (1 + r_{t+1}) s_t. \tag{4.3}$$

where w_t is the exogenous wage rate, s_t are private savings under autarky and r_{t+1} is the return on savings. The individual solves the intertemporal consumption allocation problem:

$$\max_{s_t} \quad u(w_t - s_t) + \beta E_t \{u[(1 + r_{t+1}) s_t]\}, \tag{4.4}$$

which implies the first-order condition:

$$u'\left(c^a_{t,t}\right) = \beta E_t \left[(1 + r_{t+1}) u'\left(c^a_{t+1,t}\right)\right]. \tag{4.5}$$

Utility under autarky is given by:

$$U^a_t \equiv u\left(c^a_{t,t}\right) + \beta E_t \left[u\left(c^a_{t+1,t}\right)\right], \tag{4.6}$$

where $c^a_{t,t}$ and $c^a_{t+1,t}$ are the optimal consumption levels under autarky.

4.3 Introduction of a pension fund

We introduce a pension system with a simple funding rule. This introduces the possibility of intergenerational risk sharing. A young person privately saves an amount s_t and contributes an amount θ to the pension fund. The fund invests this contribution in financial assets and pays out the gross return as a pension benefit one period later. In addition, if the return on the fund's assets in period t is too low to provide a "decent" pension benefit, each individual of the period-t generation pays an additional re-funding contribution τ_t, which is used to supplement the benefit to the current old generation. The transfer τ_t is constrained to be non-negative.

4.3.1 Individuals

The consumption levels of a period-t generation member are:

$$c_{t,t} = w_t - s_t - \theta - \tau_t, \tag{4.7}$$

$$c_{t+1,t} = (1 + r_{t+1})(s_t + \theta) + \tau_{t+1}. \tag{4.8}$$

The individual takes as given the known policy function set by the pension fund and now maximises (4.1) subject to (4.7) and (4.8). This implies the following first-order condition for his intertemporal consumption trade-off:

$$u'\left(c^p_{t,t}\right) = \beta E_t\left[(1 + r_{t+1})\, u'\left(c^p_{t+1,t}\right)\right], \tag{4.9}$$

where $c^p_{t,t}$ and $c^p_{t+1,t}$ are the optimal consumption levels under participation in period t.

4.3.2 The pension fund

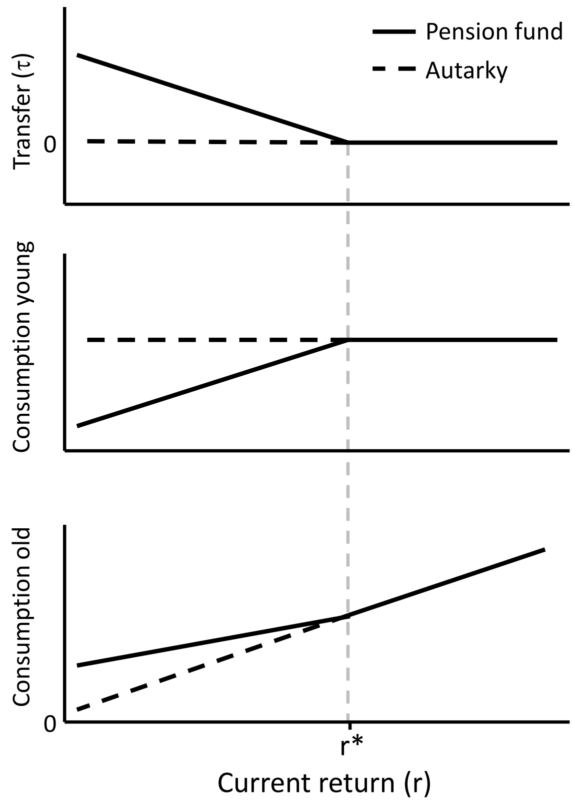
We assume that the pension fund applies the very simple funding rule given by:

$$\tau_t = \begin{cases} (r^* - r_t)\,\theta & \forall\, r_t < r^*, \\ 0 & \forall\, r_t \geq r^*. \end{cases} \tag{4.10}$$

We choose this rule, because it is appealing from a practical point of view, as it is intuitive and it would be easy to implement. It features two parameters to be set by the pension fund, r^* and θ. In the remainder of the chapter, we assume that θ is given and that the pension fund uses r^* as its policy instrument.[3] Figure 4.1 shows the size of

[3] An alternative would be to vary θ and keep r^* fixed. The analysis is similar to the analysis presented here. Because the contribution rate is easier to observe and, hence, to calibrate we choose to vary r^*. If we also allow for different values of θ, we get a two dimensional feasible region in r^*

Figure 4.1: Transfer and consumption of young and old



the transfer τ_t as a function of the current asset return and the effect that this rule has on consumption of the young (in the middle panel) and the old (in the bottom panel). The transfer is rising in the short-fall of the actual market return from r^* (i.e., as we move to the left along the horizontal axis in Figure 1). This dampens the sensitivity of the old's consumption to fluctuations in financial market returns when compared to the autarky situation (the dashed line).

The objective of the pension fund is to maximise from an ex-ante point of view the

and θ and the optimal contract would be a combination of these two pension parameters for which welfare is maximised given that these parameters lie within this feasible region.

sum of the utilities of the current and all future generations:

$$\begin{aligned} V &= E_0 \left\{ u\left(c_{1,0}\right) + \left[u\left(c_{1,1}\right) + \beta u\left(c_{2,1}\right)\right] + \beta\left[u\left(c_{2,2}\right) + \beta u\left(c_{3,2}\right)\right] + \ldots \right\} \\ &= E_0 u\left(c_{1,0}\right) + \sum_{t=1}^{\infty} \beta^{t-1} E_0\left[u\left(c_{t,t}\right) + \beta u\left(c_{t+1,t}\right)\right] \\ &= E_0 u\left(c_{1,0}\right) + \sum_{t=1}^{\infty} \beta^{t-1} E_0\left[U_t^p\right], \end{aligned} \tag{4.11}$$

where E_0 denotes the expectation taken at the start of period 1, before r_1 has materialised, and $U_t^p \equiv u\left(c_{t,t}\right) + \beta u\left(c_{t+1,t}\right)$. Notice that the transfer is not a function of the savings of the previous young generation, but of the time-invariant pension fund rule parameters r^* and θ. This implies that all young generations are affected in an identical way by the funding rule and in expectation the decisions of all young generations will be identical. Hence, $E_0 U_1^p = E_0 U_2^p = \cdots \equiv E_0 U^p$. Using this, we can rewrite (4.11) as:

$$V = E_0 u\left(c_{1,0}\right) + \frac{1}{1-\beta} E_0\left[U^p\right].$$

Aggregate welfare is maximised by setting r^* such that the derivative of the objective function with respect to r^* equals 0:

$$\frac{\partial V}{\partial r^*} = E_0\left[\frac{\partial u\left(c_{1,0}\right)}{\partial r^*}\right] + \frac{1}{1-\beta} E_0\left[\frac{\partial U^p}{\partial r^*}\right] = 0.$$

We call the value for r^* implied by this first-order condition $r^{*,opt}$. Hence, $r^{*,opt}$ yields the optimal transfer rule in the absence of a participation constraint.

4.4 The participation constraint

In the previous section we optimised welfare assuming that participation is mandatory. In this section we relax this assumption. This gives rise to a participation constraint that needs to be fulfilled to sustain the collective pension scheme. We characterise the equilibria that arise in the presence of this constraint. Finally, it solves for the optimal value for r^* under this participation constraint.

For any period t, we denote the set of possible realisations of the state of the world by $R \equiv [-1, \infty)$. The particular state that actually materialises in period t is exhaustively described by the asset return r_t in that period, with the continuously differentiable density function $p(r_t)$ on R.

We define participation or lack of participation in the pension fund as:

Definition 1: A young generation in period t is said to *participate* in the pension fund if it chooses to follow the funding rule $\tau = \tau(r_t)$ for the current realization r_t. It does *not participate* if it does not follow this funding rule.

Before we continue the analysis we make the following assumptions for an arbitrary period $t \geq 1$:

Assumption 1: All preceding young generations as of period 1 have participated in the pension fund. Each period-t young individual is free to choose whether or not to participate in the pension system after the state r_t has materialised.

Assumption 2: If the period-t young decide not to participate, the economy shifts to autarky and remains in autarky forever after. Hence, none of the future young generations will ever participate.

In the remainder of this chapter we assume that Assumptions 1 and 2 hold. Using these assumptions we can write the participation constraint for the period-t young individual as:

$$u\left(c^{p}_{t,t}\right) + \beta \left\{ P^{in}_t E_t \left[u\left(c^{in}_{t+1,t} | \text{in}\right)\right] + P^{out}_t E_t \left[u\left(c^{out}_{t+1,t} | \text{out}\right)\right]\right\} \geq U^a_t, \quad \forall\, r_t \in R, \quad (4.12)$$

with $P^{out}_t = 1 - P^{in}_t$ and $E_t(\cdot|\text{in/out})$ the expected value given that the next generation participates (in) or not (out). For the sake of readability in our notation we have dropped the dependence of consumption levels on the current return r_t. Next period's consumption of the current young depends on whether the next young generation participates or not. If the next young generation participates, the current young receive $c^{in}_{t+1,t}$ in the next period and $c^{out}_{t+1,t}$, otherwise. P^{in}_t and P^{out}_t denote the current young's perceived probabilities that the next young will participate, respectively opt out.

4.4.1 Recursive formulation of the participation constraint

Since subsequent generations are identical, and assuming that the economy has not shifted to autarky in the past, the participation constraint for each state of the world is identical in any period. This means that we can drop the time subscripts and write (4.12) in terms of current and next-period variables (which are denoted by a prime) as:

$$u\left(c^{p}\right)+\beta\left\{P^{in}E\left[u\left(c^{in\,\prime}|\text{in}\right)\right]+P^{out}E\left[u\left(c^{out\,\prime}|\text{out}\right)\right]\right\}\geq U^{a},\quad\forall\, r\in\ R,\qquad(4.13)$$

where a prime denotes next-period variables.

Hence, in our OLG setting, there is a participation constraint analogous to that in the literature that deals with participation constraints in settings with two types of infinitely-lived agents, such as Thomas and Worrall (1988) and Kocherlakota (1996). However, because in our setting the old have no incentive to walk away from the contract, the pension fund does not need to ensure that it delivers at least the utility promised in the previous period. Instead, the fund should ensure that it does not violate the promised funding rule, so that the insurance it promises the current young for the next period remains credible. Otherwise, the economy falls into autarky (Assumption 2). In addition, in contrast to models with infinitely-lived agents in which transfers can go both ways, in our model transfers always go from young to old individuals. While they may have to make a transfer to the current old, the current young depend on the next period's young for their own insurance. That implies that we do not have a repeated game between agents who can punish each other when one of them deviates from the rule. Hence, the beliefs about the willingness of the next period's young to make a transfer are crucial for the current young's willingness to pay a transfer, as will become clear below.

The next period's young will decide not to participate if the transfer they have to pay is relatively large, i.e. if $r' < \tilde{r}'$, where $\tilde{r}'$ denotes the threshold value for the next-period return r' that makes next period's young indifferent between participating and not participating. Using Bayes' rule we obtain

$$P^{in}p(r'|\text{in}) = P(\text{in}|r')p(r') = \begin{cases} 0 & \text{for } r' < \tilde{r}' \\ p(r') & \text{for } r' \geq \tilde{r}' \end{cases},$$

so we can rewrite the participation constraint (4.13) as:

$$\begin{aligned} U^a \leq U^p(r, \tilde{r}') \equiv u\left(w - s^p(r) - \theta - \tau(r)\right) \\ + \beta \left\{ \begin{array}{c} \int_{-1}^{\tilde{r}'} p(r')\, u\left[(1+r')\left(s^p(r)+\theta\right)\right] dr' + \\ \int_{\tilde{r}'}^{r^*} p(r')\, u\left[(1+r')\, s^p(r) + (1+r^*)\,\theta\right] dr' + \\ \int_{r^*}^{\infty} p(r')\, u\left[(1+r')\left(s^p(r)+\theta\right)\right] dr' \end{array} \right\}, \quad (4.14) \end{aligned}$$

where the right-hand side is the utility to the young when the current return is r and the next period's cut-off return for participating is $\tilde{r}'$.

4.4.2 Equilibrium definition

We explore equilibria that are defined as follows:

Definition 2: A recursive equilibrium with participation is an autarky savings decision s^a, an autarky value U^a, a pension funding rule $\{\tau(r)\}$, a set of savings decisions under participation $\{s^p(r)\}$, values under participation $\{U^p(r, \tilde{r}')\}$, a current cut-off return $\tilde{r}$ and expectations about the next period's cut-off return $\tilde{r}'$, such that

1. For any r, given the funding rule and expectations about future participation, the savings decisions $\{s^p(r)\}$ solve the young generation's optimization problem.

2. For $r < \tilde{r}$, $U^p(r, \tilde{r}') < U^a$, while for $r \geq \tilde{r}$, $U^p(r, \tilde{r}') \geq U^a$.

3. The cut-off return for the current young, $\tilde{r}$, computed given the expectation about the future cut-off return $\tilde{r}'$, equals the cut-off return for the next period's young, $\tilde{r}'$.

4. For at least one element $r \in R$, the funding rule sets $\tau(r) > 0$ and has $U^p(r, \tilde{r}') \geq U^a$.

4.4.3 Solutions for $\tilde{r}$

The cut-off return $\tilde{r}$ depends on the size of the current transfer prescribed by the funding rule and the current belief about the cut-off return of the young in the next period. If $r \geq r^*$, the current young would certainly want to participate, so the cut-off return cannot exceed r^*. Because subsequent young generations are identical in all respects under the assumption that all preceding young generations have participated in the pension fund, the cut-off return must be the same for every young generation that still has the option to participate. This implies that $\tilde{r} = \tilde{r}' = (\tilde{r}')' = \ldots$.

An equilibrium solution for $\tilde{r}$ requires that if the current young believe that the next young generation has the same threshold, then these current young want to participate if confronted with a return higher than this threshold and want to opt out if confronted with a lower return. That is, $\tilde{r}$ is such that

$$\begin{array}{l} U^p(r, \tilde{r}' = r) < U^a \text{ for all } -1 \leq r < \tilde{r} \\ U^p(r, \tilde{r}' = r) \geq U^a \text{ for all } \tilde{r} \leq r \leq r^* \end{array}. \tag{4.15}$$

Here, $U^p(r, \tilde{r}' = r)$ is the utility from participation under the assumption that the next generation uses the current portfolio return as the cut-off return.

As the Appendix shows, the derivative of $U^p(r, \tilde{r}' = r)$ with respect to r is the sum of a positive and a negative term and its sign is generally indeterminate. The negative term measures the utility cost of the extra transfer the current generation has to make this period. The positive term measures the benefit from extra insurance, because the future young have a higher threshold. This indeterminate derivative complicates our analysis. However, as the Appendix also shows, $U^p(r, \tilde{r}' = r)$ is equal to U^a and upward sloping in $r = r^*$. That is, it approaches U^a from below as r approaches r^*. In the examples below, we will work with a constant relative risk aversion utility specification. Plots of the resulting function $U^p(r, \tilde{r}' = r)$ show that it is always convex on $r \in [-1, r^*]$. Henceforth, we restrict ourselves to the case in which $U^p(r, r' = \tilde{r})$ is convex in r on this interval. Under this assumption, depending on whether $U^p(r, \tilde{r}' = r)$ is larger or smaller than U^a at $r = -1$, there are three possibilities as illustrated by Figure 4.2, which as a function of r plots the function $\Delta^p(r) \equiv U^p(r, \tilde{r}' = r) - U^a$:

Situation 1: If $\Delta^p(r) < 0$ at $r = -1$, then $U^p(r, \tilde{r}' = r) < U^a$ for all $-1 \leq r < r^*$. This is the situation depicted by the lowest curve in Figure 4.2. In this case, $\tilde{r} = r^*$ is the only threshold that is consistent with our equilibrium definition. This threshold implies that the pension fund collapses as soon as $r < r^*$, so there can effectively be no risk sharing via a pension fund.

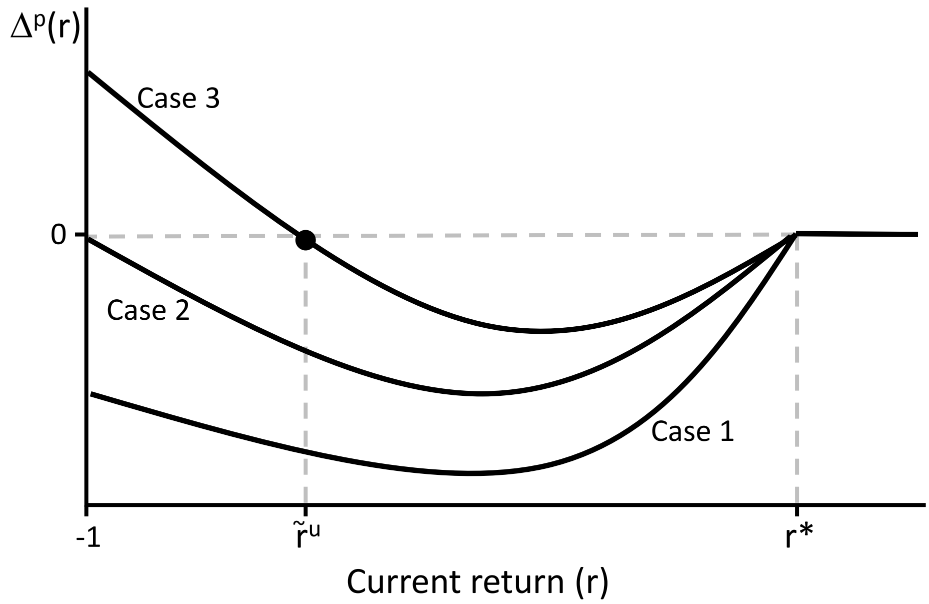
Situation 2: If $\Delta^p(r) = 0$ at $r = -1$, then $U^p(r, \tilde{r}' = r) < U^a$ for all $-1 < r < r^*$. In this case, depicted by the middle curve in Figure 4.2, equation (4.15) has exactly two solutions for $\tilde{r}$, namely $\tilde{r} = -1$ and $\tilde{r} = r^*$. This first solution implies that if the current young believe that the next period's young will participate in the pension system with certainty, then in the worst scenario today (i.e. $r = -1$), the current young are indifferent between participating and not participating. Hence, the pension system continues to exist forever.

Situation 3: If $\Delta^p(r) > 0$ at $r = -1$, as depicted by the upper curve in Figure 4.2, $\tilde{r} = -1$ is a corner solution, because under the belief that the next period's young will always participate the current young have higher utility from participating in the pension system than under autarky, even if they have to pay the highest possible transfer. Again, $\tilde{r} = r^*$ is also a solution. Finally, there exists a third solution $-1 < \tilde{r} < r^*$, such that $U^p(r, \tilde{r}' = r) = U^a$.

4.4.4 Properties of the solutions for $\tilde{r}$

We shall now further analyse Situation 3, because Situation 1 effectively excludes the possibility of risk sharing via a pension fund, while Situation 2 corresponds to a very specific parameter combination. In Situation 3, if the current young, given their belief that the next period's young will participate in every state ($\tilde{r}' = -1$), are also willing to participate, there exist three solutions, namely $\tilde{r} = -1$, $\tilde{r} = r^*$ and a solution $-1 < \tilde{r} < r^*$. The first two solutions are stable, while the third is a knife-edge solution that will never be realised unless the current young start with an initial belief at exactly

Figure 4.2: Net benefit from participation



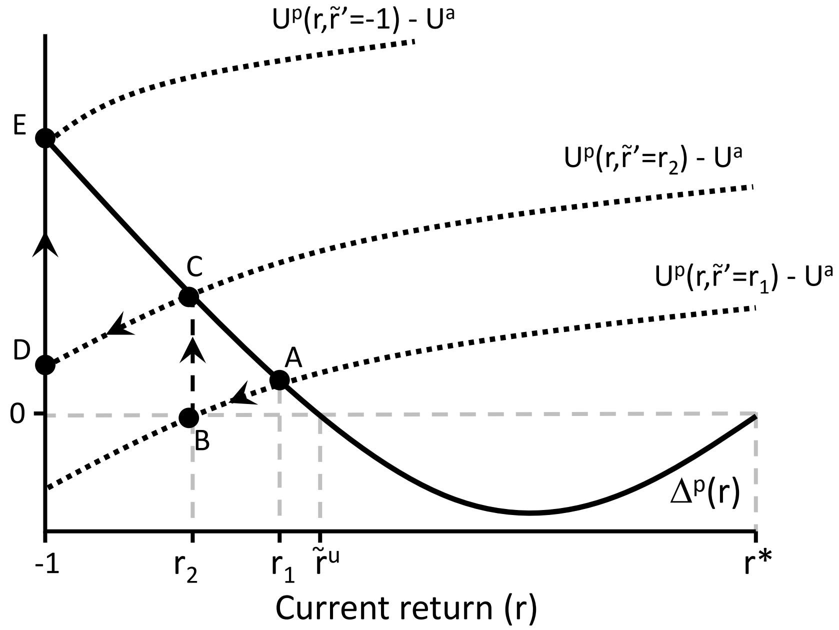
that point. Let us denote this solution by $\tilde{r}^u$. Which one of the two stable equilibria is reached depends on the initial belief of the current young.

Initial belief lies between -1 and unstable solution

If the current young initially believe that the cut-off return $\tilde{r}'$ of the next young generation lies between -1 and $\tilde{r}^u$, then the economy will end up at the equilibrium with $\tilde{r} = -1$. We illustrate this case using Figure 4.3, in which we plot the solid curve $\Delta^p(r)$ as a function of r and the dashed curves $U^p(r, \tilde{r}') - U^a$ for three different values $(r_1 > r_2 > -1)$ of $\tilde{r}'$. Note that there exist an infinite number of such curves, one for each possible belief about next-period young's cut-off value $\tilde{r}'$. Since U^a does not depend on $\tilde{r}'$ and the insurance value of participation is lower as the next generation's threshold is higher, $U(r, \tilde{r}' = r_1) < U(r, \tilde{r}' = r_2) < U(r, \tilde{r}' = -1)$ on the whole interval $r \in [-1, r^*]$.

Take an arbitrary starting value r_1 between -1 and $\tilde{r}^u$ for $\tilde{r}$ and assume that the next generation uses the same threshold. This corresponds to point (A) in Figure 4.3 where the lower dashed line crosses the solid line. This return r_1 cannot be the cut-off return $\tilde{r}$ of the current young since there are lower returns for which the current young

Figure 4.3: Stable solution with participation



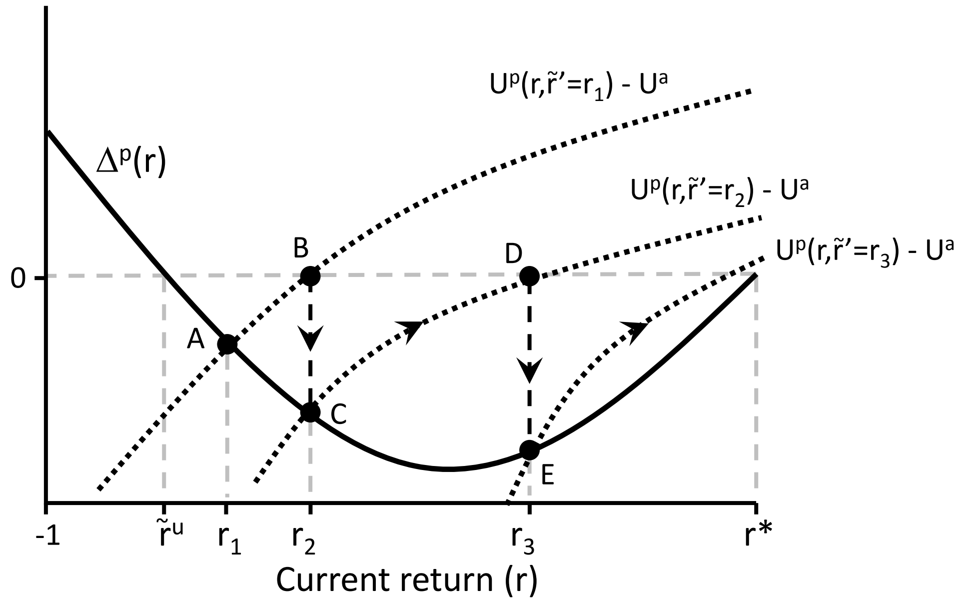
are also willing to participate given their beliefs about the future's young generation's threshold. While keeping $\tilde{r}'$ fixed at r_1, we decrease r, thereby raising the required transfer and reducing expected utility from participation $U^p(r, \tilde{r}' = r_1)$ until we reach the value of r such that $U^p(r, \tilde{r}' = \tilde{r}_1) = U^a$ at $r = r_2$ in point (B). This defines the cut-off return $\tilde{r}_2$ of the current young given their initial belief about future participation. But, $\tilde{r}_2 < r_1$, implying that the belief $\tilde{r}' = r_1$ cannot be an equilibrium belief, as in equilibrium the cut-off returns of all subsequent young generations must be identical. Hence, the current young update their belief about the next young's cut-off return to $\tilde{r}' = r_2 < r_1$ to make their threshold consistent with their own (point C). However, at $r = \tilde{r}_2$, $U^p(r, \tilde{r}' = r_2)$ again exceeds U^a. Lowering r again, we move along the next-to-lowest dashed line to the left until we hit the vertical axis. This yields the cut-off return $\tilde{r} = -1$ of the current young which implies that they will never want to opt out. If opting out is never optimal for the current young, then it is also never optimal for future generations, so the threshold consistent with our equilibrium definition is the stable equilibrium $\tilde{r} = \tilde{r}' = -1$.

Initial belief lies between unstable solution and r^*

We illustrate this case using Figure 4.4, in which we again plot the solid curve $\Delta^p(r)$ as a function of r and the dashed curves $U^p(r, \tilde{r}') - U^a$ for three different values of $\tilde{r}'$. As before, there exist an infinite number of these curves, one for each possible value of $\tilde{r}'$. All these curves are upward sloping since the required transfer decreases and the insurance value stays constant along each line as $\tilde{r}'$ is constant. By definition the curve $U^p(r, \tilde{r}') - U^a$ crosses $\Delta^p(r)$ at $r = \tilde{r}'$. Finally, these curves are positive (or zero) at $r = r^*$ since $U^p(r, r^*) - U^a = 0$ for $r = r^*$ and all curves with $\tilde{r}' < r^*$ lie above this curve.

To show that every initial belief about the next generation's threshold between $\tilde{r}^u$ and r^* results in the autarky solution r^* start in point A in Figure 4.4. At $r = r_1$, we have $U^p(r, \tilde{r}' = r_1) < U^a$ so that the current young do not want to participate. However, there are also higher returns r for which they do not want to participate, but according to their initial belief $\tilde{r}' = r_1$ the next generation does. So, r_1 cannot be the cut-off return $\tilde{r}$ of the current young. While keeping $\tilde{r}'$ fixed at r_1, we raise r, thereby reducing the required transfer and raising expected utility from participation $U^p(r, \tilde{r}' = r_1)$ until we reach the value of r such that $U^p(r, \tilde{r}' = r_1) = U^a$ in point B. This defines the cut-off return r_2 of the current young given their initial belief about future participation. However, $r_2 > r_1$, implying that the belief $\tilde{r}' = r_1$ cannot be an equilibrium belief, as in equilibrium the cut-off returns of all subsequent young generations must be identical. Hence, the current young update their belief about the next young's cut-off return to $\tilde{r}' = r_2$ (point C). However, at $r = r_2 = \tilde{r}'$, one has $U^p(r, \tilde{r}' = r_2) < U^a$. Raising r, we move along the middle dashed line to the right until we hit the horizontal dashed line again. This yields the new cut-off return $r = r_3$ of the current young. Because r_3 exceeds the next generation's threshold r_2, the latter cannot be an equilibrium cut-off return. We repeat the updating procedure until the current generation's cut-off and the beliefs about the next generation's cut-off converge to the stable equilibrium value $\tilde{r} = \tilde{r}' = r^*$.

Figure 4.4: Stable solution without participation



4.4.5 Assumption about initial beliefs

The analysis thus far has shown that the only stable equilibrium can be one in which $\tilde{r} = \tilde{r}' = -1$ or $\tilde{r} = \tilde{r}' = r^*$. The latter case is effectively ruled out by requirement (iv) of Definition 2 of an equilibrium. Hence, in the sequel we limit ourselves to the former case and we assume that each young generation starts with the belief that the next-period young will participate in all states of the world, i.e $\tilde{r}' = -1$. If, given this belief, the current young are prepared to participate for $r = -1$, i.e. when the transfer they have to make is at its largest, we have a sustainable pension fund that continues to operate in all periods under any state of the world. If for this initial belief the current young are unwilling to participate for $r = -1$, then, as the analysis in the previous subsection has shown, there exists no equilibrium with participation and the pension system breaks down.

4.4.6 Feasible pension fund rules

Until now we have fixed the contract at an arbitrary r^* and varied the cut-off value $\tilde{r}$ for young generations. Next, we will vary r^* to find the set of feasible funding rules by the pension fund. This part of the analysis closely follows the analysis by Ljunqvist and Sargent (2004, pp. 724-726) and Kocherlakota (1996).

We evaluate U^p for the worst state of the world ($r = -1$) under the assumption that the next period's young will participate irrespective of the next period's state of the world (i.e., $\tilde{r}' = -1$):

$$\begin{aligned} U^p(-1,-1) &= u\left(w - s^p - \theta - (1+r^*)\theta\right) + \\ &\quad \beta \left\{ \begin{array}{c} \int_{-1}^{r^*} p(r')\, u\left((1+r')\, s^p + (1+r^*)\,\theta\right) dr' \\ + \int_{r^*}^{\infty} p(r')\, u\left((1+r')\left(s^p+\theta\right)\right) dr' \end{array} \right\}. \end{aligned}$$

We can check for all possible values of r^* whether the current young are indeed prepared to participate or not. Start with $r^* = -1$. In this case, the young will never make a transfer, hence their consumption is constant across all states of the world and the pension fund effectively operates as an individual DC system. To see what happens when we increase r^*, we differentiate $U^p(-1,-1)$ with respect to r^*. Applying Leibniz' integral rule, this yields,

$$\begin{aligned} \frac{dU^p(-1,-1)}{dr^*} &= -\theta u'\left(w - s^p - \theta - (1+r^*)\theta\right) + \\ &\quad \beta\theta \int_{-1}^{r^*} p(r')\, u'\left((1+r')\, s^p + (1+r^*)\,\theta\right) dr'. \end{aligned} \tag{4.16}$$

The first term on the right-hand side of (4.16) is negative: raising r^* from -1 means that the young individual has to pay a transfer with certainty because $r = -1$. The second term on the right-hand side is positive, except at $r^* = -1$. It arises from the fact that the increase in the transfer from the future young for given r' raises future consumption and, hence, future utility to the current young. At $r^* = -1$, this term is exactly zero, because the probability of receiving the higher future transfer is zero in this case. In differentiating $U^p(-1,-1)$ two terms have cancelled out. A marginal increase in r^* expands the interval $[-1, r^*)$ over which a transfer is received in the future. However, this positive contribution to $U^p(-1,-1)$ cancels out against the reduction of utility from the shrinkage of the interval $[r^*, \infty)$ over which no transfer is received.

Figure 4.5 plots the gain from participating $\Delta^p(-1) = U^p(-1,-1) - U^a$ for various constant relative aversion utility functions. As shown above, $\Delta^p(-1)$ initially always falls below 0. Further from $r^* = -1$, the utility loss from the initial transfer is still low, but the probability of receiving income protection in the worst case scenarios in the second period grows. For some parameter combinations (as in cases A and B), this second effect, measured by the term on the right-hand side of (4.16) starts to dominate and the function increases. For high values of r^*, increasing r^* further always has a negative effect on the gain from participating as the insurance value does not offset the relatively large initial transfer. For values of r^* close to $w/\theta - 1$, participation is never attractive, because the initial transfer is so large that positive consumption in the first period may lead to bankruptcy in the second. To ensure positive consumption this value of r^* requires that $s^p + \theta < 0$, so the young are privately heavily indebted. But, if $r' \geq r^*$ then the payout from the pension fund is insufficient to pay the principal plus interest rate on this debt. In our numerical analysis below, the gain from participating always shows the same rotated S-pattern.

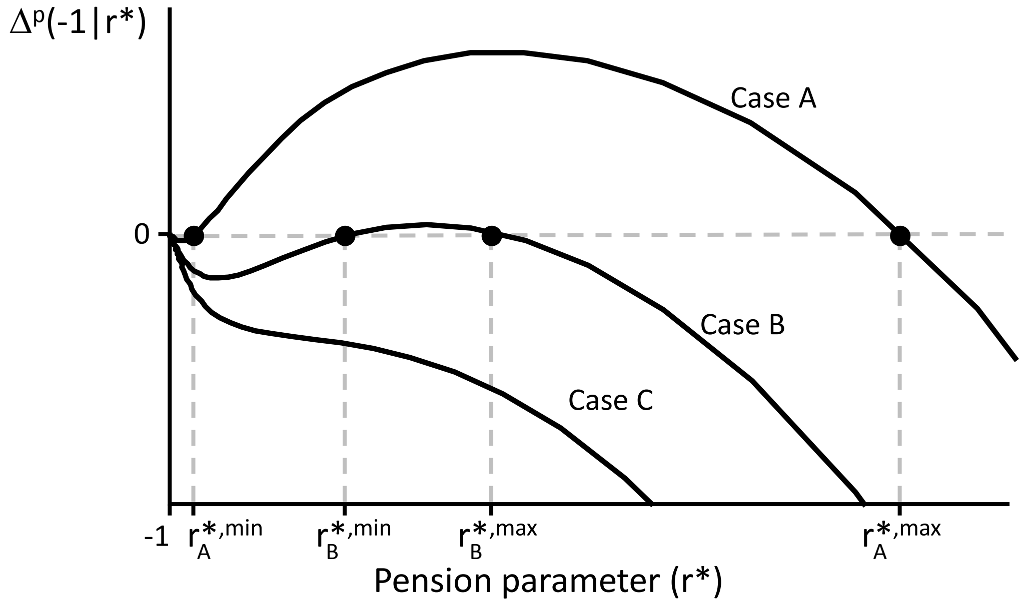
Figure 4.5 shows that there are intervals for r^* on which the gain from participating is positive. In this case, we define $r^{*,\max}$ as the maximum value for r^* for which a pension fund is sustainable and $r^{*,\min}$ as the largest value of $r^* < r^{*,\max}$ for which $\Delta^p(-1) = 0$. However, there are also instances (as in Case C in Figure 4.5) for which no such interval exists.

4.4.7 The optimal pension fund rule

We continue to focus on the stable equilibrium with participation in Situation 3 and assume that the funding rule is in place at the start of period 1. Writing out the terms in the pension fund objective function, equation (4.11), we obtain

$$\begin{aligned} E_0 u\left(c_{1,0}\right) &= \int_{-1}^{r^*} p\left(r\right) u\left[\left(1+r\right) s^p + \left(1+r^*\right)\theta\right] dr \\ &+ \int_{r^*}^{\infty} p\left(r\right) u\left[\left(1+r\right)\left(s^p+\theta\right)\right] dr, \end{aligned}$$

Figure 4.5: Feasible pension schemes when varying r^*



and

$$
\begin{aligned}
E_0U^p &= \int_{-1}^{r^*} p(r)\,u\left[w - s^p - \theta - (r^* - r)\,\theta\right]dr \\
&+ \int_{r^*}^{\infty} p(r)\,u\left[w - s^p - \theta\right]dr \\
&+ \beta\left\{\begin{array}{c}\int_{-1}^{r^*} p(r)\,u\left[(1+r)\,s^p + (1+r^*)\,\theta\right]dr \\ +\int_{r^*}^{\infty} p(r)\,u\left[(1+r)\,(s^p+\theta)\right]dr\end{array}\right\}.
\end{aligned}
$$

Substituting these terms into (4.11), it is straightforward to write the pension fund objective as:

$$
\begin{aligned}
V(r^*) &= \frac{1}{1-\beta}\left\{\begin{array}{c}\int_{-1}^{r^*} p(r)\,u\left[w - s^p - \theta - (r^* - r)\,\theta\right]dr \\ +\int_{r^*}^{\infty} p(r)\,u\left[w - s^p - \theta\right]dr\end{array}\right\} \\
&+ \frac{1}{1-\beta}\left\{\begin{array}{c}\int_{-1}^{r^*} p(r)\,u\left[(1+r)\,s^p + (1+r^*)\,\theta\right]dr \\ +\int_{r^*}^{\infty} p(r)\,u\left[(1+r)\,(s^p+\theta)\right]dr\end{array}\right\}.
\end{aligned}
\tag{4.17}
$$

If an interval $\left[r^{*,\min}, r^{*,\max}\right]$ exists, we denote the value for $r^* \in \left[r^{*,\min}, r^{*,\max}\right]$ that maximises (4.17) subject to the participation constraint by $r^{*,pc}$.

4.5 A numerical example

In this section we work out a simple numerical example of the analysis we have presented in the previous sections. The purpose of this section is to explore under what circumstances a funded pension arrangement with voluntary participation exists and, conditional on this existence, to investigate under what circumstances the participation constraint leads to welfare losses because it is binding.

We assume that period utility is given by,

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where parameter γ captures the constant degree of relative risk-aversion. The baseline values of the model parameters are set as follows. Consistent with standard calibrations in the macro-economic literature, we set $\gamma = 5$. Further, we assume that $\beta = 0.5$. With one generation spanning 30 years, this corresponds to an annual discount factor of 0.977. The wage rate is simply a parameter that determines the scale of the economy and we fix it at $w = 100$. Given the wage rate w, we set $\theta = 10$. Hence, this amounts to a realistic pension contribution rate of 10%. We assume that portfolio returns are lognormally distributed and independent over time. In setting the parameters for the portfolio returns, we make use of Campbell et al. (2003) (see Appendix). Specifically, the average annual portfolio return is set at $\mu = 0.089$, which over a period of thirty years translates into an average return of 2.14. The standard deviation of the annual equity return in Campbell et al. (2003) is $\sigma = 0.182$, which implies a standard deviation of 0.91 over a thirty-year period. We will consider a wide range of standard deviations that comfortably covers this value. Given that the degree of risk aversion is somewhat on the high side, the conditions for a recursive equilibrium with participation are relatively favourable.

First, we solve for $r^{*,opt}$, the optimal choice of the pension fund for r^* in the absence of participation constraints. Subsequently, we solve the autarky problem, which yields expected utility U^a. Given U^a, we can then solve for the minimum and maximum values $r^{*,\min}$ and $r^{*,\max}$ of r^* for which our stable equilibrium in the presence of participation constraints still exists as well as the optimum value $r^{*,pc}$ for r^* in the interval $\left[r^{*,\min}, r^{*,\max}\right]$. If there is no interval $\left[r^{*,\min}, r^{*,\max}\right]$ for the specific parameter constellation under consideration, then there is no sustainable pension contract under voluntary participation as indicated by the non-existence of $r^{*,\max}$ and $r^{*,pc}$ in the tables. In these instances we indicate $r^{*,\max}$ and $r^{*,pc}$ in the tables by "–". Obviously, if $r^{*,\max}$ exists, then also $r^{*,\min}$ and $r^{*,pc}$ exist. However, we do not separately report $r^{*,\min}$, because numerically it always lies extremely close to -1. Finally, we compute for the constrained optimum the ex-ante welfare gains Ω^{init} for the initial old generation and Ω for the current young and all future generations. These ex-ante welfare gains are defined as follows:

$$\begin{aligned} E_0 u\left(c_{1,0}^p\right) &= E_0 u\left[\left(1+\Omega^{init}\right) c_{1,0}^a\right], \\ E_0 U^p &= u\left[\left(1+\Omega\right) c_{t,t}^a\right] + \beta E_0 u\left[\left(1+\Omega\right) c_{t+1,t}^a\right]. \end{aligned}$$

We report the results in Table 4.1 for different values of the risk aversion parameter γ, the *annual* standard deviation of the portfolio return σ and the pension contribution θ. The results show that the higher is the degree of relative risk aversion and the higher is the standard deviation of the asset returns, the more likely it becomes that participation in the pension fund is beneficial because of the risk-sharing gains that it provides.

Consider first the case when the pension contribution is at its baseline of $\theta = 10$. Then, for a relatively low degree of risk aversion $\gamma = 3$ the pension fund is viable for high volatilities of 0.25 and larger, while for a relatively high degree of risk aversion $\gamma = 7.5$ it is viable already for a moderate volatility of 0.15. For a given degree of risk aversion, the benefit from future risk sharing and thus the attractiveness of participation increases with the uncertainty about the future asset returns. Vice versa, for a given variance of the asset returns, future risk sharing and thus participation becomes more attractive when risk aversion rises. For $\gamma = 7.5$, when $\sigma = 0.20$ or larger, the young are even prepared to make a transfer that is larger than under the optimum without the participation constraint, i.e. $r^{*,\max} > r^{*,opt}$. In this case, the participation constraint is not binding at the unconstrained optimum and, hence, the constrained optimum coincides with the unconstrained optimum, i.e. $r^{*,pc} = r^{*,opt} > r^{*,\min}$. For

$\gamma = 3$ the pension fund is viable for $\sigma = 0.25$ and for $\gamma = 5$ it is viable for $\sigma = 0.20$, although in these cases it produces less risk sharing than under the unconstrained optimum. In this case, $r^{*,pc} = r^{*,\max} < r^{*,opt}$.[4]

In Table 4.1 we also vary the size of the pension contribution. If $\theta = 20$, then for two risk aversion – volatility combinations, $(\gamma = 3, \sigma = 0.25)$ and $(\gamma = 7.5, \sigma = 0.15)$, the participation constraint now prohibits any risk sharing, while it did not do so before when $\theta = 10$. We also see that whenever there is voluntary participation under $\theta = 20$, the values for $r^{*,\max}$ and $r^{*,pc}$ have dropped relative to the corresponding ones under $\theta = 10$, hence the scope for risk sharing has become smaller. The intuition for this is the following. With a transfer by the current young of the format $(r^* - r_t)\,\theta$, an increase in θ implies a larger transfer for given $r_t < r^*$ and, hence, to avoid violation of the participation constraint for given future transfers, $r^{*,\max}$ needs to fall. However, this reduces the number of states in the future for which the current young will receive a transfer back and, hence, the insurance value of the pension falls. For some parameter combinations under consideration the insurance value falls so much that participation by the young is no longer beneficial to them. A reduction in θ from $\theta = 10$ to $\theta = 5$ yields the opposite effects and the scope for risk sharing increases.

The degree to which the participation constraint is binding is reflected in the welfare gain associated with participation. If voluntary participation is not optimal then, obviously, there can be no welfare gain, while once participation has become beneficial, further increases in risk aversion and the variability of the returns raise the welfare gains from participation. The welfare gain of the initial old is always higher than that of all subsequent generations, because they reap the benefits from the potential transfer, without ever having to pay a transfer themselves.

We also vary the discount factor. A higher discount factor corresponds to a lower time preference rate, implying that future events become relatively more important. Hence, voluntary participation becomes more attractive to the young. After all, any benefit of participation in the form of a transfer only materialises in the future. This

[4] We have also done a more elaborate numerical analysis in which we varied σ by small steps of 0.02 and extended the range beyond the one reported in the Table 4.1. In all instances the results were qualitatively identical to those reported here, while for volatilities larger than 0.3 we always find that $r^{*,pc}$ and $r^{*,opt}$ are equal. These results are available upon request.

Table 4.1: Results for 100% equity

	γ	3					5					7.5				
θ	σ	0.1	0.15	0.2	0.25	0.3	0.1	0.15	0.2	0.25	0.3	0.1	0.15	0.2	0.25	0.3
5	$r^{*,opt}$	0.057	0.057	0.061	0.066	0.070	0.063	0.063	0.067	0.071	0.073	0.066	0.066	0.069	0.073	0.075
	$r^{*,max}$	-	-	-	0.068	0.090	-	-	0.074	0.095	0.103	-	0.058	0.094	0.103	0.105
	$r^{*,pc}$	-	-	-	0.066	0.070	-	-	0.067	0.071	0.073	-	0.058	0.069	0.073	0.075
	Ω^{init}	0	0	0	0.018	0.036	0	0	0.019	0.047	0.094	0	0.011	0.043	0.103	0.192
	Ω	0	0	0	0.009	0.022	0	0	0.012	0.036	0.080	0	0.009	0.035	0.093	0.181
10	$r^{*,opt}$	0.053	0.049	0.048	0.049	0.050	0.057	0.052	0.050	0.050	0.051	0.057	0.051	0.050	0.050	0.051
	$r^{*,max}$	-	-	-	0.037	0.064	-	-	0.045	0.070	0.077	-	0.020	0.069	0.078	0.079
	$r^{*,pc}$	-	-	-	0.037	0.050	-	-	0.045	0.050	0.051	-	0.020	0.050	0.050	0.051
	Ω^{init}	0	0	0	0.015	0.036	0	0	0.018	0.047	0.093	0	0.007	0.043	0.103	0.192
	Ω	0	0	0	0.009	0.022	0	0	0.013	0.037	0.081	0	0.006	0.036	0.094	0.181
20	$r^{*,opt}$	0.046	0.039	0.035	0.032	0.031	0.049	0.039	0.034	0.030	0.029	0.047	0.037	0.031	0.029	0.028
	$r^{*,max}$	-	-	-	-	0.037	-	-	0.012	0.044	0.053	-	-	0.043	0.053	0.055
	$r^{*,pc}$	-	-	-	-	0.031	-	-	0.012	0.031	0.029	-	-	0.031	0.029	0.028
	Ω^{init}	0	0	0	0	0.036	0	0	0.013	0.046	0.092	0	0	0.042	0.102	0.190
	Ω	0	0	0	0	0.022	0	0	0.010	0.036	0.080	0	0	0.036	0.094	0.180

is borne out by the results in Table 4.2, in which we vary the discount factor and, for convenience, repeat the middle panel with $\beta = 0.5$. Discount factors of $\beta = 0.3$ and $\beta = 0.7$ correspond to annual time preference rates of roughly 4% and 1%, respectively. In all instances in which the participation constraint is binding, we see that a higher discount factor raises $r^{*,pc}$ and $r^{*,\max}$. The qualitative effects of an increase in risk aversion or in the standard deviation of the portfolio return that we described earlier are confirmed for different values of the discount factor β.

So far, the benchmark level of riskiness of the pension and individual investment portfolios was based on the implicit assumption of a 100% stake in equity. We will now make the implicit assumption that both the individual and the pension portfolio are invested half in equity and half in a risk-free asset. This implies that both the standard deviation and expected value of the overall portfolio return fall. The reason why we consider this case is that pension funds often invest in a mixture of risk-free assets and risk-bearing assets (mostly equity) and that we want to see what is the scope for risk sharing under voluntary participation in this case. The 50-50 division between the two categories is quite realistic for pension funds in the Netherlands. Obviously, the optimal individual savings rate in our model takes account of the new situation. Following Campbell et al. (2003), we calibrate the risk-free rate at 2.1% per year (see Appendix). Table 4.3 reports the numerical outcomes, where $r^{*,opt}$, $r^{*,\max}$ and $r^{*,pc}$ all refer to returns on the total portfolio. For relatively low aversion ($\gamma = 3$), participation under the constraint is only attractive when the volatility of the equity returns is at its highest value $\sigma = 0.3$. Again, for higher degrees of risk aversion participation can be attractive at lower levels of volatility. Hence, for relatively high risk aversion and medium and high equity volatility, the pension fund is still viable under voluntary participation, although, not surprisingly, the welfare gains relative to autarky have dropped due to the reduced risk-sharing benefits that the fund provides.

Table 4.2: Results for 100% equity, $\theta = 10$ and varying β

	γ	3					5					7.5				
β	σ	0.1	0.15	0.2	0.25	0.3	0.1	0.15	0.2	0.25	0.3	0.1	0.15	0.2	0.25	0.3
	$r^{*,opt}$	0.055	0.051	0.050	0.051	0.052	0.058	0.053	0.047	0.051	0.052	0.058	0.052	0.051	0.051	0.051
	$r^{*,max}$	-	-	-	0.032	0.061	-	-	0.043	0.069	0.076	-	0.016	0.068	0.078	0.079
$\beta = 0.3$	$r^{*,pc}$	-	-	-	0.032	0.052	-	-	0.043	0.051	0.052	-	0.016	0.051	0.051	0.051
	Ω^{init}	0	0	0	0.014	0.036	0	0	0.012	0.046	0.092	0	0.007	0.043	0.102	0.191
	Ω	0	0	0	0.008	0.020	0	0	0.017	0.035	0.078	0	0.006	0.035	0.092	0.179
	$r^{*,opt}$	0.053	0.049	0.048	0.049	0.050	0.057	0.052	0.050	0.050	0.051	0.057	0.051	0.050	0.050	0.051
	$r^{*,max}$	-	-	-	0.037	0.064	-	-	0.045	0.070	0.077	-	0.020	0.069	0.078	0.079
$\beta = 0.5$	$r^{*,pc}$	-	-	-	0.037	0.050	-	-	0.045	0.050	0.051	-	0.020	0.050	0.050	0.051
	Ω^{init}	0	0	0	0.015	0.036	0	0	0.018	0.047	0.093	0	0.007	0.043	0.103	0.192
	Ω	0	0	0	0.009	0.022	0	0	0.013	0.037	0.081	0	0.006	0.036	0.094	0.181
	$r^{*,opt}$	0.052	0.048	0.046	0.047	0.049	0.056	0.051	0.049	0.049	0.050	0.057	0.051	0.049	0.050	0.051
	$r^{*,max}$	-	-	-	0.040	0.065	-	-	0.047	0.070	0.078	-	0.022	0.069	0.078	0.079
$\beta = 0.7$	$r^{*,pc}$	-	-	-	0.040	0.049	-	-	0.047	0.049	0.050	-	0.022	0.049	0.050	0.051
	Ω^{init}	0	0	0	0.016	0.036	0	0	0.019	0.047	0.094	0	0.008	0.043	0.104	0.192
	Ω	0	0	0	0.010	0.024	0	0	0.013	0.038	0.082	0	0.007	0.037	0.095	0.182

Table 4.3: Results for 50% invested in risk free and 50% in equity

γ	3					5					7.5				
σ	0.1	0.15	0.2	0.25	0.3	0.1	0.15	0.2	0.25	0.3	0.1	0.15	0.2	0.25	0.3
$r^{*,opt}$	0.038	0.039	0.041	0.044	0.047	0.042	0.042	0.045	0.047	0.049	0.044	0.044	0.046	0.048	0.050
$r^{*,max}$	-	-	-	-	0.051	-	-	-	0.059	0.071	-	-	0.057	0.072	0.077
$r^{*,pc}$	-	-	-	-	0.047	-	-	-	0.047	0.049	-	-	0.046	0.048	0.050
Ω^{init}	0	0	0	0	0.020	0	0	0	0.024	0.044	0	0	0.022	0.047	0.087
Ω	0	0	0	0	0.010	0	0	0	0.016	0.034	0	0	0.016	0.039	0.078

4.6 Conclusion

In this chapter we have explored the feasibility and welfare consequences of a funded pension system with voluntary participation and intergenerational risk sharing. Our numerical results showed that the ex-post option to not participate obviates the existence of such a system when both risk aversion and financial market uncertainty are relatively low. Under those circumstances mandatory participation in the system is needed to reap the benefits of intergenerational risk sharing. Increases in these parameters raise the benefits from risk sharing and make the existence of an equilibrium with voluntary participation more likely. For intermediate values of these parameters, risk sharing would still be less than under the optimal arrangement with obligatory participation. However, raising risk-aversion and volatility further, the optimal solutions under voluntary and obligatory participation coincide.

Many countries that are currently trying to expand the funded pillar of their pension system need to decide on its design. In particular, will it be a defined-contribution system or will it contain certain defined-benefit elements that allow for intergenerational risk sharing? In the latter case, the question whether participation is obligatory becomes important. Giving individuals too much freedom in deciding whether to participate or not may lead to a breakdown of the system and, hence, to a loss of the risk-sharing benefits. The Dutch funded pillar has long and successfully operated under mandatory participation by the employees and employers and by collectively sharing risks among all the fund participants. However, the recent shrinkage of the pension buffers has undermined the confidence in the system, especially among the younger participants. Once large groups start losing their confidence in the system, the pressure to abolish mandatory fund participation will intensify. To prevent a collapse of the system and a loss of the benefits from intergenerational risk sharing, it is important that its planned overhaul leaves young participants with sufficient guarantee that they will also receive a benefit when they are old.

APPENDICES

4.A Details on first-order condition pension fund without participation constraint

The first-order condition is given by

$$\frac{\partial V}{\partial r^*} = E_0\left[\frac{\partial u\left(c_{1,0}\right)}{\partial r^*}\right] + \frac{1}{1-\beta}E_0\left[\frac{\partial U^p}{\partial r^*}\right] = 0,$$

where, using Leibniz' integral rule,

$$\begin{aligned} E_0\left[\frac{\partial u\left(c_{1,0}\right)}{\partial r^*}\right] &= p\left(r^*\right)u\left[\left(1+r^*\right)\left(s_0^p+\theta\right)\right] - p\left(r^*\right)u\left[\left(1+r^*\right)\left(s_0^p+\theta\right)\right] \\ &\quad + \int_{-1}^{r^*} p\left(r\right)u'\left[\left(1+r\right)s_0^p + \left(1+r^*\right)\theta\right]\theta dr \\ &= \int_{-1}^{r^*} p\left(r\right)u'\left[\left(1+r\right)s_0^p + \left(1+r^*\right)\theta\right]\theta dr, \end{aligned}$$

and

$$\begin{aligned} E_0\left[\frac{\partial U^p}{\partial r^*}\right] &= -\int_{-1}^{r^*} p\left(r\right)u'\left[w - s^p - \theta - \left(r^* - r\right)\theta\right]\left(\frac{\partial s^p}{\partial r^*} + \theta\right)dr \\ &\quad - \int_{r^*}^{\infty} p\left(r\right)u'\left[w - s^p - \theta\right]\frac{\partial s^p}{\partial r^*}dr \\ &\quad + \beta\int_{-1}^{r^*} p\left(r\right)u'\left[\left(1+r\right)s^p + \left(1+r^*\right)\theta\right]\left[\left(1+r\right)\frac{\partial s^p}{\partial r^*} + \theta\right]dr \\ &\quad + \beta\int_{r^*}^{\infty} p\left(r\right)u'\left[\left(1+r\right)\left(s^p+\theta\right)\right]\left(1+r\right)\frac{\partial s^p}{\partial r^*}dr. \end{aligned}$$

4.B Details on $U^p\left(r, \tilde{r}' = r\right)$

4.B.1 $U^p\left(r^*, \tilde{r}' = r^*\right)$ approaches U^a from below as $r \uparrow r^*$

Recall that $U^p\left(r, \tilde{r}' = r\right) = U^a$. Since $\Delta(r)$ was defined as $U^p\left(r, \tilde{r}' = r\right) - U^a$, all we need to show is that Δ^p approaches 0 from below. Differentiating $\Delta^p(r)$, with respect to r, and using Leibniz' integral rule, we obtain:

$$\begin{aligned}\frac{d\Delta(r)}{dr} &= u'\left[w - s^p(r) - \theta - (r^* - r)\theta\right]\left(\theta - \frac{\partial s^p(r)}{\partial r}\right) \\ &+ \beta p(r) u\left[(1+r)\left(s^p(r)+\theta\right)\right] \\ &+ \beta \int_{-1}^{r} p(r')\, u'\left[(1+r')\left(s^p(r)+\theta\right)\right](1+r')\frac{\partial s^p(r)}{\partial r} dr' \\ &- \beta p(r) u\left[(1+r)s^p(r) + (1+r^*)\theta\right] \\ &+ \beta \int_{r}^{r^*} p(r')\, u'\left[(1+r')s^p(r) + (1+r^*)\theta\right](1+r')\frac{\partial s^p(r)}{\partial r} dr' \\ &+ \beta \int_{r^*}^{\infty} p(r')\, u'\left[(1+r')\left(s^p(r)+\theta\right)\right](1+r')\frac{\partial s^p(r)}{\partial r} dr',\end{aligned}$$

which we can write as:

$$\begin{aligned}\frac{d\Delta(r)}{dr} &= u'\left[w - s^p(r) - \theta - (r^* - r)\theta\right]\left(\theta - \frac{\partial s^p(r)}{\partial r}\right) \\ &+ \beta \frac{\partial s^p(r)}{\partial r}\left\{\begin{array}{c} \int_{-1}^{r} p(r')\, u'\left[(1+r')\left(s^p(r)+\theta\right)\right](1+r')dr' \\ + \int_{r}^{r^*} p(r')\, u'\left[(1+r')s^p(r) + (1+r^*)\theta\right](1+r')dr' \\ + \int_{r^*}^{\infty} p(r')\, u'\left[(1+r')\left(s^p(r)+\theta\right)\right](1+r')dr' \end{array}\right\} \\ &+ \beta p(r)\left\{u\left[(1+r)\left(s^p(r)+\theta\right)\right] - u\left[(1+r)s^p(r) + (1+r^*)\theta\right]\right\}\end{aligned}$$

where the term between big curly brackets measures $E[(1+r')u'\left(c^p_{t+1,t}\right)]$. According to the Euler equation (4.9) this equals $u'\left(c^p_{t,t}\right)$, so all terms involving $\frac{\partial s^p(r)}{\partial r}$ cancel out and we are left with:

$$\begin{aligned}\frac{d\Delta(r)}{dr} &= u'\left[w - s^p(r) - \theta - (r^* - r)\theta\right]\theta \\ &+ \beta p(r)\left\{u\left[(1+r)\left(s^p(r)+\theta\right)\right] - u\left[(1+r)s^p(r) + (1+r^*)\theta\right]\right\} \quad (4.18)\end{aligned}$$

Evaluated in $r = r^*$ this equals:

$$\frac{d\Delta(r)}{dr} = u'\left[w - s^p(r^*) - \theta\right]\theta > 0.$$

4.B.2 Second-order derivative of $U^p\left(r, \tilde{r}' = r\right)$

The second-order derivative of $U^p\left(r, \tilde{r}' = r\right)$ equals the second derivative of $\Delta(r)$ since the difference is the constant U^a. Differentiating (4.18) once more gives:

$$\begin{aligned}
\frac{d^2\Delta}{d^2r} &= u''\left[w - s^p(r) - \theta - (r^* - r)\theta\right]\theta\left(\theta - \frac{\partial s^p(r)}{\partial r}\right) \\
&\quad + \beta p(r)\left\{\begin{array}{c} u'\left[(1+r)\left(s^p(r)+\theta\right)\right]\left[s^p(r) + \theta + (1+r)\frac{\partial s^p(r)}{\partial r}\right] \\ -u'\left[(1+r)s^p(r) + (1+r^*)\theta\right]\left[s^p(r) + (1+r)\frac{\partial s^p(r)}{\partial r}\right] \end{array}\right\} \\
&\quad + \beta p'\left(r\right)\left\{u\left[(1+r)\left(s^p\left(r\right)+\theta\right)\right] - u\left[(1+r)\,s^p\left(r\right) + (1+r^*)\,\theta\right]\right\}.
\end{aligned}$$

The first line is of ambiguous sign, since $u''\left(\cdot\right)$ is negative and θ and $\partial s^p\left(r\right)/\partial r$ are both positive (see below). The second line is positive or zero since $p\left(r\right) \geq 0$ and since $u'\left[(1+r)\left(s^p\left(r\right)+\theta\right)\right] > u'\left[(1+r)\,s^p\left(r\right) + (1+r^*)\,\theta\right]$, while the sign of the third line is ambiguous, since we do not know the sign of $p'\left(r\right)$. This means that we can not make any general statements about the shape of $U^p\left(r, \tilde{r}' = r\right)$.

The sign of $\partial s^p\left(r\right)/\partial r > 0$

Define

$$EU(r, s^p) \equiv u'\left[w - s^p - \theta - (r^* - r)\theta\right] - \beta\left\{\begin{array}{c} \int_{-1}^{r} p\left(r'\right)u'\left[(1+r')\left(s^p+\theta\right)\right](1+r')dr' \\ + \int_{r}^{r^*} p\left(r'\right)u'\left[(1+r')\,s^p + (1+r^*)\theta\right](1+r')dr' \\ + \int_{r^*}^{\infty} p\left(r'\right)u'\left[(1+r')\left(s^p+\theta\right)\right](1+r')dr' \end{array}\right\}, \quad (4.19)$$

so we can write the Euler equation as $EU(r, s^p) = 0$. Implicit differentiation gives:

$$\frac{\partial s^p\left(r\right)}{\partial r} = -\frac{\partial EU/\partial r}{\partial EU/\partial s^p}. \quad (4.20)$$

Working out, we have:

$$\frac{\partial EU}{\partial r} = -u''\left[w - s^p - \theta - (r^* - r)\,\theta\right]\theta$$
$$+\,\beta p(r)(1+r)\left\{u'\left[(1+r)\,(s^p+\theta)\right] - u'\left[(1+r)s^p + (1+r^*)\theta\right]\right\},$$

The first line of this derivative is positive, since $u''(\cdot)$ is negative, while the second line is greater than or equal to zero, because $r \leq r^*$ on the relevant domain for $\Delta(r)$. Further,

$$\frac{\partial EU}{\partial s^p} = u''\left[w - s^p(r) - \theta - (r^* - r)\,\theta\right] +$$
$$\beta\left\{\begin{array}{c} \int_{-1}^{r} p\,(r')\,u''\left[(1+r')\,(s^p+\theta)\right](1+r')^2\,dr' \\ +\int_{r}^{r^*} p\,(r')\,u''\left[(1+r')\,s^p + (1+r^*)\theta\right](1+r')^2\,dr' \\ +\int_{r^*}^{\infty} p\,(r')\,u''\left[(1+r')\,(s^p+\theta)\right](1+r')^2\,dr' \end{array}\right\} < 0.$$

Both lines of this derivative are negative, since $u''(.)$ is negative. Hence, by (4.20), we have $\partial s^p(r)/\partial r > 0$.

4.C Calibration of the returns process

The portfolio returns follow a lognormal process and are independently and identically distributed over time. A period in our model corresponds to one generation, which we take here to span 30 years. We use the Campbell *et al.* (2003) estimates of annual returns on stocks and bonds to construct corresponding 30-year figures. Their sample period covers annual returns over the period 1890 - 1998. We assume the return on T-bills to correspond to that on a risk-free investment. The average annual real return on T-bills is 2.101%. Hence, this is the calibrated value for the risk-free rate of return in our model. Campbell *et al.* (2003) calculate an average annual equity risk premium of 6.797%. Hence, the average annual real equity return is $2.101\% + 6.797\% = 8.898\%$. Further, it has a standard deviation of 18.192%. Hence, the variance σ_{30}^2 of lognormally distributed thirty-year equity returns is calculated as $\sigma_{30}^2 = 30\ln\left(1 + \frac{\text{var}(r)}{[E(r)]^2}\right) = 30\ln\left(1 + \frac{0.18192^2}{1.08898^2}\right) = 0.027525 * 30 = 0.8258$. The corresponding standard deviation is $\sqrt{0.825756} = 0.908711$. Next, the mean return over a period of thirty years is calculated as $\ln\left\{[E(r)]^{30}\right\} - \frac{1}{2}\sigma_{30}^2 = \ln(12.9002) - \frac{1}{2} * 0.8258 = 2.1443$. To conclude, we calibrate the logarithm of the equity return to follow a normal process with mean 2.1443 and variance 0.8258 over a thirty-year period.