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Methods

Technical Note—Dual Approach for Two-Stage Robust Nonlinear Optimization

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Abstract. Adjustable robust minimization problems where the objective or constraints depend in a convex way on the adjustable variables are generally difficult to solve. In this paper, we reformulate the original adjustable robust nonlinear problem with a polyhedral uncertainty set into an equivalent adjustable robust linear problem, for which all existing approaches for adjustable robust linear problems can be used. The reformulation is obtained by first dualizing over the adjustable variables and then over the uncertain parameters. The polyhedral structure of the uncertainty set then appears in the linear constraints of the dualized problem, and the nonlinear functions of the adjustable variables in the original problem appear in the uncertainty set of the dualized problem. We show how to recover linear decision rules to the original primal problem and how to generate bounds on its optimal objective value.

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1. Introduction

1.1. Problem Formulation

We consider the following general two-stage robust nonlinear minimization problem:

$$\inf_{x\in\mathcal{X}} \sup_{\zeta\in\mathcal{U}} \inf_{y} \{ f_0(x) + g_0(y) \mid \zeta^\top F_{i\cdot}(x) + f_i(x) + g_i(y) \le 0,\ i = 1,\dots,m_1,\ A(x)\zeta + By = b(x)\}. \quad (1)$$

Here, $\mathcal{X} \subseteq \mathbb{R}^{n_x}$, the functions $f_i : \mathbb{R}^{n_x} \to \mathbb{R}$, $g_i : \mathbb{R}^{n_y} \to \mathbb{R}$ are convex for all $i = 0,\dots,m_1$, $F_{i\cdot}(x) = (F_{i1}(x), \dots, F_{in_\zeta}(x))$, and $F_{ij} : \mathbb{R}^{n_x} \to \mathbb{R}$ are real-valued functions for all $i = 1,\dots,m_1$ and $j = 1,\dots,n_\zeta$. The matrices $A(x) \in \mathbb{R}^{m_2\times n_\zeta}$ and the vector $b(x) \in \mathbb{R}^{m_2}$ depend on $x \in \mathbb{R}^{n_x}$ in an affine way:

$$A(x) = A^0 + \sum_{l=1}^{n_x} A^l x_l, \qquad b(x) = b^0 + \sum_{l=1}^{n_x} b^l x_l, \quad (2)$$

with $A^l \in \mathbb{R}^{m_2\times n_\zeta}$ and $b^l \in \mathbb{R}^{m_2}$ for all $l = 0,\dots,n_\zeta$. Note that Problem (1) has *fixed recourse* because the functions g_i, $i = 0,\dots,m_1$, and the matrix B do not depend on ζ. Therefore, there are no direct interaction terms between ζ and y, such as products $\zeta^\top y$. Throughout this paper, we focus on nonempty polyhedral uncertainty sets:

$$\mathcal{U} = \{\zeta \ge 0 : D\zeta = d\}, \quad (3)$$

where $D \in \mathbb{R}^{p\times n_\zeta}$ and $d \in \mathbb{R}^p$.

1.2. Literature Review

Problem (1) is generally intractable even if all the objective and constraint functions are linear. Adjustable robust optimization techniques in the literature, such as nonlinear decision rules, Benders decomposition, the column and constraint generation method (Zeng and Zhao 2013), the copositive approach (Hanasusanto and Kuhn 2018, Xu and Burer 2018), and Fourier–Motzkin elimination (Zhen et al. 2017), are developed for linear adjustable problems and are not applicable for (1). Furthermore, even if we impose linear decision rules

$$y(\zeta) = y_0 + \sum_{j=1}^{n_\zeta} y_j \zeta_j, \quad (4)$$

where $y_0,\dots,y_{n_\zeta} \in \mathbb{R}^{n_y}$, to the wait and see decision variables, the resulting conservative approximation of (1),

$$\inf_{\substack{x\in\mathcal{X}\\ y_0,y_j}} \sup_{\zeta_0\in\mathcal{U}} \left\{ f_0(x) + g_0(y(\zeta_0)) \,\middle|\, \forall \zeta \in \mathcal{U} : \begin{array}{l} \zeta^\top F_{i\cdot}(x) + f_i(x) + g_i(y(\zeta)) \le 0,\ i = 1,\dots,m_1 \\ A(x)\zeta + By(\zeta) = b(x) \end{array} \right\}, \quad (5)$$

is still difficult to solve. This difficulty is because of the fact that the objective and constraint functions contain terms $g_i(y_0 + \sum_{j=1}^{n_\zeta} y_j\zeta_j)$, $i = 0,\dots,m_1$, which are convex in the uncertain parameters if g_i is nonlinear and convex. The inner maximization problem in (5) tries to maximize a convex function over a polyhedron, which is in general NP hard.

There are only a few papers on adjustable robust nonlinear optimization known to the authors. Pinar and Tütüncü (2005) study a two-period adjustable robust portfolio problem to identify robust arbitrage opportunities when the uncertainty is ellipsoidal. They derive optimal decision rules from exploiting the explicit structure of their formulation, but it is unclear how this can be generalized to problems with more constraints, other uncertainty sets, or other model formulations. Takeda et al. (2008) consider an adjustable robust nonlinear model with a polyhedral uncertainty set, similar to the models considered in this paper. They solve a sampled model while enumerating all vertices of the polytope uncertainty set. This quickly becomes unviable for even medium-sized problems as the number of extreme points of the uncertainty set is exponential in the dimension of the uncertain parameter. Boni and Ben-Tal (2008) consider adjustable robust optimization models with conic quadratic constraints with ellipsoidal uncertainty sets. They approximate the model with linear decision rules and finally end up with a semidefinite optimization model.

Our paper significantly extends the approach of Bertsimas and de Ruiter (2016), where only linear problems are considered. Note that in the linear case, the original adjustable robust optimization models could already be solved with techniques, such as Fourier–Motzkin elimination, linear and nonlinear decision rules, Benders decomposition, and the column and constraint generation method of Zeng and Zhao (2013). This is not the case (at least not directly) for the nonlinear problems, where the original formulation cannot be solved with these techniques. However, in this paper, we show that the dual of the nonlinear problem is linear in the adjustable variables. For this dual problem, the mentioned well-known adjustable linear robust optimization techniques can be used.

1.3. Contributions

This paper uses the consecutive dualization scheme in Bertsimas and de Ruiter (2016) for linear problems and extends it to two-stage robust *nonlinear* problems that have a polyhedral uncertainty set. The major contributions of this paper are follows.

1. We extend the consecutive dualization approach of Bertsimas and de Ruiter (2016) to reformulate two-stage robust nonlinear problems that have fixed recourse and a polyhedral uncertainty set. The reformulation can again be interpreted as a two-stage robust problem but with adjustable variables that appear linearly. We show that this linear reformulation is equivalent to the original one (i.e., the optimal objective value and the feasible region of the here and now decisions of both formulations coincide).

2. We apply a new relaxation technique to establish a close relation between linear decision rules for the original nonlinear problem and its equivalent dual (linear) reformulation.

3. We provide lower bounds on the optimal objective value. Furthermore, we show how binding scenarios from the original uncertainty set can be obtained from binding scenarios in the dual formulation. This new technique also considerably improves the lower bounds proposed in Bertsimas and de Ruiter (2016) for the linear case.

We show that we can use our method to efficiently solve two numerical problems. First, we solve a distribution problem on a network with nonlinear commitments. Second, we find the equilibrium of a system with several springs.

1.4. Paper Organization and Notation

The rest of this paper is organized as follows. In Section 2, we present our framework and derive our dualized formulation and linear decision rule model. We recover the linear decision rule for the original primal problem in Section 3. In Section 4, we explain how we obtain lower bounds on the optimal objective value to assess the quality of our solutions. Our numerical examples are presented in Sections 3 and 4, respectively, of the e-companion.

The function g^* is the convex conjugate of the function $g: \mathbb{R}^{n_\nu} \to \mathbb{R}$ and is defined by

$$g^*(z) = \sup_{\nu \in \mathrm{dom}(g)} \{\nu^\top z - g(\nu)\},$$

where $\mathrm{dom}(g)$ is the domain of the function g. The perspective $h: \mathbb{R}^{n_\nu} \times \mathbb{R}_+ \to \mathbb{R}$ of a real-valued convex function $f: \mathbb{R}^{n_\nu} \to \mathbb{R}$ is defined for all $\nu \in \mathbb{R}^{n_\nu}$ and $t \in \mathbb{R}_+$ as $h(\nu, t) = tf(\nu/t)$ if $t > 0$, and $h(\nu, 0) = \liminf_{(\nu', t') \to (\nu, 0)} t' f(\nu'/t')$ (Rockafellar 1970, p. 67). For ease of exposition, we use $tf(\nu/t)$ to denote the perspective function $h(\nu, t)$ in the rest of this paper.

2. The Dual Formulation

We first use the consecutive dualization approach of Bertsimas and de Ruiter (2016) to derive an equivalent linear reformulation of (1). Linear decision rules are then applied to the linear reformulation to obtain a conservative approximation. This constitutes a convex program that can be efficiently solved using off-the-shelf solvers. To this end, we first assume that (1) has a *relatively complete recourse*.

Assumption 1 (Relatively Complete Recourse). *For all $x \in \mathcal{X}$ and all $\zeta \in \mathcal{U}$, there exists a $y \in \mathbb{R}^{n_y}$, such that*

$$\begin{cases} \zeta^\top F_{i\cdot}(x) + f_i(x) + g_i(y) \le 0 & i = 1, \ldots, m_1 \\ A(x)\zeta + By = b(x), \end{cases}$$

and for all $i = 1, \ldots, m_1$ for which g_i is nonlinear, we have $\zeta^\top F_{i\cdot}(x) + f_i(x) + g_i(y) < 0$.

This assumption implies that each here and now decision is strictly feasible for all nonlinear constraints that contain the wait and see decision variables. It seems to be restrictive from a modeling perspective at first. However, in practice, models can be cast in such a way that undesirable here and now decisions x will result in very high second-stage costs $g_0(y)$. Also, the slightly weaker condition of relatively complete recourse (that does not require strict feasibility) is common in two-stage stochastic and robust linear optimization; see Birge and Louveaux (2011). In the following theorem, we introduce a two-stage robust linear reformulation of (1).

Theorem 1 (Dual Formulation). *Let $\mathcal{U}$ be a polyhedral set as in* (3) *and assume that Assumption* 1 *holds. The here and now decision x is feasible for* (1) *if and only if x is feasible for the following dualized model*:

$$\inf_{x\in\mathcal{X}} \sup_{(u,v,w,z)\in\mathcal{V}} \inf_{\lambda} \left\{ \sum_{i=0}^{m_1} v_i f_i(x) + d^\top\lambda - w^\top b(x) - \sum_{i=0}^{m_1} z_i \;\middle|\; \sum_{k=1}^{p} D_{kj}\lambda_k \geq w^\top A_{\cdot j}(x) + \sum_{i=1}^{m_1} v_i F_{ij}(x),\ j=1,\dots,n_\zeta \right\}, \tag{6}$$

where $u=(u_0,\dots,u_{m_1})\in\mathbb{R}^{(m_1+1)n_y}$, $u_i\in\mathbb{R}^{n_y}$ *for* $i=0,\dots,m_1$ *and*

$$\mathcal{V}=\left\{(u,v,w,z):\ v\geq 0,\ v_0=1,\ v_i(g_i)^*\left(\frac{u_i}{v_i}\right)\leq z_i,\ i=0,\dots,m_1,\ \sum_{i=0}^{m_1} u_i = -B^\top w\right\}.$$

Moreover, the infimum of (1) *coincides with that of* (6).

Proof. See Section 1 of the e-companion. □

Note that the linear structure of the uncertainty set appears in the constraints of the dual formulation (6) and that the convex structure of the adjustable variables is in the new uncertainty set $\mathcal{V}$. When $F_{i\cdot}(x)$, $f_i(x)$, and $g_i(y)$ are affine functions, Theorem 1 coincides with the result in theorem 1 of Bertsimas and de Ruiter (2016).

The obtained two-stage robust linear reformulation (6) can be conservatively approximated via linear decision rules. We impose the following linear decision rules to the wait and see variable λ,

$$\lambda(u,v,w,z)=\sum_{i=0}^{m_1}\Psi_i^\top u_i+\sum_{i=0}^{m_1} t_i v_i+\Phi^\top w+\sum_{i=0}^{m_1}\eta_i z_i,$$

where $\Psi_i\in\mathbb{R}^{n_y\times p}$, $t_i,\eta_i\in\mathbb{R}^p$ for all $i=0,\dots,m_1$ and $\Phi\in\mathbb{R}^{m_2\times p}$. The resulting conservative approximation of (6) constitutes a robust optimization problem:

$$\begin{aligned}\inf_{\substack{x\in\mathcal{X},t_i\\ \Psi_i,\Phi,\eta_i}} \sup_{(u,v,w,z)\in\mathcal{V}} \ & v^\top F_0(x)+d^\top\lambda(u,v,w,z) - w^\top b(x)-\sum_{i=0}^{m_1} z_i\\ \text{s.t. } & \forall (u,v,w,z)\in\mathcal{V}: D_{\cdot j}^\top\lambda(u,v,w,z)\geq w^\top A_{\cdot j}(x) + \sum_{i=1}^{m_1} v_i F_{ij}(x) \qquad j=1,\dots,n_\zeta,\end{aligned} \tag{7}$$

where $F_0(x)=(f_0(x),\dots f_{m_1}(x))^\top\in\mathbb{R}^{m_1+1}$, $F_{\cdot j}(x)\in\mathbb{R}^{m_1}$ and $D_{\cdot j}\in\mathbb{R}^p$ are the jth column vectors of $F(x)$ and D, respectively. It follows from Theorem 1 that (7) constitutes a conservative approximation of (1). Because the uncertain parameters appear linearly in (7) and $\mathcal{V}$ is convex, one can use standard robust optimization techniques to obtain the following tractable reformulation:

$$\begin{aligned}\inf_{\substack{x\in\mathcal{X}\\ y_0,y_j,\Psi_i\\ \gamma\geq 0,t,\eta,\Phi}} \quad & f_0(x)+\gamma_{00}g_0\left(\frac{y_0+\Psi_0 d}{\gamma_{00}}\right)+d^\top t_0 \\ \text{s.t.} \quad & \gamma_{0j}g_0\left(\frac{y_j-\Psi_0 D_{\cdot j}}{\gamma_{0j}}\right)\leq D_{\cdot j}^\top t_0 && j=1,\dots,n_\zeta\\ & f_i(x)+\gamma_{i0}g_i\left(\frac{y_0+\Psi_i d}{\gamma_{i0}}\right)+d^\top t_i\leq 0 && i=1,\dots,m_1\\ & F_{ij}(x)+\gamma_{ij}g_i\left(\frac{y_j-\Psi_i D_{\cdot j}}{\gamma_{ij}}\right)\leq D_{\cdot j}^\top t_i && i=1,\dots,m_1\\ & && j=1,\dots,n_\zeta\\ & \gamma_{i0}+d^\top\eta_i=1 && i=0,\dots,m_1\\ & \gamma_{ij}=D_{\cdot j}^\top\eta_i && i=0,\dots,m_1\\ & && j=1,\dots,n_\zeta\\ & By_0+\Phi d=b(x)\\ & A_{\cdot j}(x)+By_j=\Phi D_{\cdot j} && j=1,\dots,n_\zeta,\end{aligned} \tag{8}$$

where $y_j\in\mathbb{R}^{n_y}$ for all $j=0,\dots,n_\zeta$, $\Psi_i\in\mathbb{R}^{n_y\times p}$, $t_i,\eta_i\in\mathbb{R}^p$ for all $i=0,\dots,m_1$, $\gamma\in\mathbb{R}^{(m_1+1)\times(n_\zeta+1)}$ and $\Phi\in\mathbb{R}^{m_2\times p}$.

Because of the introduction of the additional optimization variables (i.e., $y_j\in\mathbb{R}^{n_y}$ for all $j=1,\dots,n_\zeta$, $\Psi_i\in\mathbb{R}^{n_y\times p}$, $t_i,\eta_i\in\mathbb{R}^p$ for all $i=0,\dots,m_1$, $\gamma\in\mathbb{R}^{(m_1+1)\times(n_\zeta+1)}$ and $\Phi\in\mathbb{R}^{m_2\times p}$), there are significantly more optimization variables in (8) than in the original Problem (1). The original problem has only (n_x+n_y) variables but is intractable because of its nature. The tractability of (8) relies on the functions f_0, g_0, f_i, F_{ij}, and g_i for all $i=1,\dots,m_1$ and $j=1,\dots,n_\zeta$. For example, if all these functions are conic quadratic functions, then (8) simply constitutes a conic quadratic program. More generally, the perspective function of a conically representable function can be represented in the same cone (Roos et al. 2018, theorem 8). Therefore, the perspective functions do not lift Model (8) to a higher complexity class if the original functions admit a conic representation.

Finally, we remark that if the uncertainty set $\mathcal{U}$ in (1) is nonpolyhedral, one can outer approximate $\mathcal{U}$ by a polyhedral set before applying the approach developed in this section. For instance, if $\mathcal{U}$ is a conic representable set, one can use the developed scheme in Ben-Tal and Nemirovski (2001) to outer approximate $\mathcal{U}$ efficiently via a bounded polyhedron.

3. Recover a Primal Linear Decision Rule

We show that we can obtain a feasible linear decision rule for the primal Model (1) from the linear decision rule model used to solve the dual Model (8). For the proof, we need a novel perspective relaxation, which can be seen as an extended version of Jensen's inequality.

Lemma 1 (Perspective Relaxation). *If the function $f:\mathbb{R}^{n_x}\to\mathbb{R}$ is convex, then for any $x_1,\dots,x_N\in\mathbb{R}^{n_x}$, $\alpha\in\mathbb{R}^N_+$ and $\gamma\in\mathbb{R}^N_+$ such that $\sum_{i=1}^N\alpha_i\gamma_i=1$, we have*

$$f\left(\sum_{i=1}^N\alpha_i x_i\right)\le\sum_{i=1}^N\alpha_i\gamma_i f\left(\frac{x_i}{\gamma_i}\right). \tag{9}$$

Proof. For any $x_1,\dots,x_N\in\mathbb{R}^{n_x}$ and $\alpha\in\mathbb{R}^N_+$, let $\gamma\in\mathbb{R}^N_+$ satisfy $\sum_{i=1}^N\alpha_i\gamma_i=1$. Then, we have

$$f\left(\sum_{i=1}^N\alpha_i x_i\right)=f\left(\sum_{i=1}^N\frac{\alpha_i\gamma_i}{\gamma_i}x_i\right)\le\sum_{i=1}^N\alpha_i\gamma_i f\left(\frac{x_i}{\gamma_i}\right),$$

where the inequality follows from Jensen's inequality, which applies because f is convex, and $\sum_{i=1}^N\alpha_i\gamma_i=1$, where $\alpha_i\gamma_i\in[0,1]$, for all $i=1,\dots,N$. □

We now show that a feasible primal linear decision rule is directly obtained from variables that constitute a solution to (8).

Theorem 2 (Primal Linear Decision Rule). *If x, y_j, $j=0,\dots,n_\zeta$ are feasible for (8), then x, $y(\zeta)=y_0+\sum_{j=1}^{n_\zeta}y_j\zeta_j$ is feasible for the primal linear decision rule Model (1), and its objective value is at most the objective value of (8).*

Proof. Suppose x, y_j, $j=0,\dots,n_\zeta$ are feasible for (8). We first show that x, together with linear decision rule $y(\zeta)=y_0+\sum_{j=1}^{n_\zeta}y_j\zeta_j$, results in an objective value that is at most as high as the solution for (8). Let $\zeta\in\mathcal{U}$, $\Psi_0\in\mathbb{R}^{n_y\times p}$, and let $\gamma_{0j}\ge 0$, $j=0,\dots,n_\zeta$ such that $\gamma_{00}+\sum_{j=1}^{n_\zeta}\gamma_{0j}\zeta_j=1$. Then, we have

$$\begin{aligned}f_0(x)+g_0(y(\zeta))&=f_0(x)+g_0\left(y_0+\sum_{j=1}^{n_\zeta}y_j\zeta_j\right)\\&=f_0(x)+g_0\left(y_0+\Psi_0 d+\sum_{j=1}^{n_\zeta}(y_j-\Psi_0D_{\cdot j})\zeta_j\right)\\&\le f_0(x)+\gamma_{00}g_0\left(\frac{y_0+\Psi_0d}{\gamma_{00}}\right)\\&\quad+\sum_{j=1}^{n_\zeta}\gamma_{0j}\zeta_{0j}g_0\left(\frac{y_j-\Psi_0D_{\cdot j}}{\gamma_{0j}}\right).\end{aligned}$$

For the second equality, we used the fact that for any $\Psi_0\in\mathbb{R}^{n_y\times p}$, we have $\Psi_0 d-\sum_{j=1}^{n_\zeta}\Psi_0D_{\cdot j}\zeta_j=0$ because for any $\zeta\in\mathcal{U}$, we have $D\zeta=d$. The last inequality follows from Lemma 1. Using this relation, we can further derive that

$$\begin{aligned}&\sup_{\zeta\in\mathcal{U}}\{f_0(x)+g_0(y(\zeta))\}\\&\le\sup_{\zeta\in\mathcal{U}}\inf_{\gamma_0\ge0}\left\{f_0(x)+\gamma_{00}g_0\left(\frac{y_0+\Psi_0d}{\gamma_{00}}\right)\right.\\&\quad\left.+\sum_{j=1}^{n_\zeta}\gamma_{0j}\zeta_jg_0\left(\frac{y_j-\Psi_0D_{\cdot j}}{\gamma_{0j}}\right)\middle|\,\gamma_{00}+\sum_{j=1}^{n_\zeta}\gamma_{0j}\zeta_j=1\right\}\\&\le\inf_{\gamma_0\ge0}\sup_{\zeta\in\mathcal{U}}\left\{f_0(x)+\gamma_{00}g_0\left(\frac{y_0+\Psi_0d}{\gamma_{00}}\right)\right.\\&\quad\left.+\sum_{j=1}^{n_\zeta}\gamma_{0j}\zeta_jg_0\left(\frac{y_j-\Psi_0D_{\cdot j}}{\gamma_{0j}}\right)\middle|\,\gamma_{00}+\sum_{j=1}^{n_\zeta}\gamma_{0j}\zeta_j=1\right\},\end{aligned}$$

where in the second inequality, we used weak duality. The obtained minimax problem is still intractable. However, it can be conservatively approximated by the following robust optimization problem:

$$\begin{aligned}&\inf_{\gamma_0\ge0}\sup_{\zeta_0\in\mathcal{U}}\left\{f_0(x)+\gamma_{00}g_0\left(\frac{y_0+\Psi_0d}{\gamma_{00}}\right)\right.\\&\quad\left.+\sum_{j=1}^{n_\zeta}\gamma_{0j}\zeta_{0j}g_0\left(\frac{y_j-\Psi_0D_{\cdot j}}{\gamma_{0j}}\right)\middle|\,\forall\zeta\in\mathcal{U}:\gamma_{00}+\sum_{j=1}^{n_\zeta}\gamma_{0j}\zeta_j=1\right\}\\=&\inf_{\gamma_0\ge0,t_0,\eta_0}\left\{f_0(x)+\gamma_{00}g_0\left(\frac{y_0+\Psi_0d}{\gamma_{00}}\right)\right.\\&\quad+d^\top t_0\,\middle|\,\gamma_{0j}g_0\left(\frac{y_j-\Psi_0D_{\cdot j}}{\gamma_{0j}}\right)\le D_{\cdot j}^\top t_0,\ j=1,\dots,n_\zeta\\&\quad\left.\gamma_{00}+\sum_{k=1}^p\eta_{0k}d_k=1,\ \gamma_{0j}=D_{\cdot j}^\top\eta_0,\ j=1,\dots,n_\zeta\right\}.\end{aligned}$$

The resulting objective function and constraint are contained in (8). One can apply the same approximation steps to show feasibility of the constraints. That is, analogously, it can be derived that for $i=1,\dots,m_1$, the ith constraint

$$\zeta^\top F_{i\cdot}(x)+f_i(x)+g_i(y(\zeta))\le 0$$

is satisfied because the following set of constraints is satisfied in (8) for some $\Psi_i\in\mathbb{R}^{n_y\times p}$ and $\eta_i\in\mathbb{R}^{n_\zeta}$:

$$\begin{aligned}&f_i(x)+\gamma_{i0}g_i\left(\frac{y_0+\Psi_id}{\gamma_{i0}}\right)+d^\top t_i\le0\\&F_{ij}(x)+\gamma_{ij}g_i\left(\frac{y_j-\Psi_iD_{\cdot j}}{\gamma_{ij}}\right)\le D_{\cdot j}^\top t_i\qquad j=1,\dots,n_\zeta\\&\gamma_{i0}+d^\top\eta_i=1\\&\gamma_{ij}=D_{\cdot j}^\top\eta_i\qquad j=1,\dots,n_\zeta.\end{aligned}$$

Finally, using standard techniques in robust optimization, without perspective relaxation, one can show that

$A(x)\zeta + By(\zeta) = b(x)$ is satisfied whenever there exists $\Phi \in \mathbb{R}^{m_2 \times p}$ that satisfies the remaining constraints of (8):

$$\begin{aligned} &By_0 + \Phi d = b(x) \\ &A_{\cdot j}(x) + By_j = \Phi D_{\cdot j} \qquad j = 1, \ldots, n_\zeta. \quad \square \end{aligned}$$

To relate the objective value of the primal linear decision rule Model (1) to the dual linear decision rule Model (8), we use several conservative approximations in the proof of Theorem 2. Hence, the true objective value of the primal linear decision rule model could be lower than the value obtained of (8); see Remark 1 in Section 3 of the e-companion for a numerical demonstration.

4. Lower Bounds on the Optimal Value

A model with a finite sample of scenarios can provide a lower bound on the optimal value of (1). The sampled version of the dualized model is

$$\begin{aligned} \inf_{\substack{\tau, x \in \mathcal{X} \\ \lambda^1, \ldots, \lambda^S \geq 0}} \quad & \tau \\ \text{s.t.} \quad & f_0(x) + \sum_{i=1}^{m_1} v_i^s f_i(x) + d^\top \lambda^s - (w^s)^\top b(x) \\ & - \sum_{i=0}^{m_1} z_i^s \leq \tau \quad \forall s = 1, \ldots, S \\ & \sum_{s=1}^{p} D_{kj} \lambda_k^s \geq (w^s)^\top A_{\cdot j}(x) \\ & + \sum_{i=1}^{m_1} (v_i^s) F_{i,l}(x) \quad \forall j = 1, \ldots, n_\zeta, \; s = 1, \ldots, S, \end{aligned} \tag{10}$$

where $\{(u^1, w^1, v^1, z^1), \ldots, (u^S, w^S, v^S, z^S)\}$ is a finite subset $\mathcal{V}$ with a single optimization variable λ^s for each scenario $s = 1, \ldots, S$. Note that this is a standard convex optimization model but only guarantees feasibility of the here and now decisions for a small set of scenarios. The question is of course how to choose scenarios to get strong lower bounds. One way to obtain an effective finite set is described by Hadjiyiannis et al. (2011).

If we have a set of scenarios $\{(u^1, w^1, v^1, z^1), \ldots, (u^S, w^S, v^S, z^S)\}$ for the sampled version of the dualized model, we can link and recover primal scenarios $\{\zeta^1, \ldots, \zeta^S\}$ to obtain stronger lower bounds. To establish the link, we first dualize over $\lambda_1, \ldots, \lambda_K$ in (10), which yields

$$\begin{aligned} \inf_{x \in \mathcal{X}} \sup_{\zeta \in \mathcal{U}} \sup_{1 \leq s \leq S} \Bigg\{ & f_0(x) + \sum_{i=1}^{m_1} v_i^s (\zeta^\top F_{i\cdot}(x) + f_i(x)) \\ & + (A(x)\zeta - b(x))^\top w^s - \sum_{i=0}^{m_1} z_i^s \Bigg\}. \end{aligned} \tag{11}$$

For a fixed x, we can now obtain primal scenarios ζ^s for each s as the maximizers of model (11):

$$\zeta^s \in \arg\max_{\zeta \in \mathcal{U}} \left\{ \sum_{i=1}^{m_1} v_i^s (\zeta^\top F_{i\cdot}(x)) + (w^s)^\top (A(x)\zeta - b(x)) \right\}. \tag{12}$$

The resulting set of scenarios $\{\zeta^1, \ldots, \zeta^s\}$ can then be used in a sampled model of (1).

A special case arises for *right-hand-side* uncertainty, where primal scenarios obtained by (12) provide stronger bounds than the dual scenarios. We say that there is only right-hand-side uncertainty if there is no direct interaction between the here and now decisions x and ζ. The more formal definition is given.

Definition 1 (Right-Hand-Side Uncertainty). Model (1) has right-hand-side uncertainty if there exist $\bar{F}_{i\cdot} \in \mathbb{R}^{n_\zeta}$ and $\bar{A} \in \mathbb{R}^{m_2 \times n_\zeta}$ such that $A(x) = \bar{A}$ and $F_{i\cdot}(x) = \bar{F}_{i\cdot}$ for all $x \in \mathcal{X}$, $i = 1, \ldots, m_1$.

Using this definition, we can now formally prove that primal scenarios obtained from dual scenarios yield stronger lower bounds for right-hand-side uncertainty.

Theorem 3 (Primal-Dual Scenarios). *Let $\{(u^1, w^1, v^1, z^1), \ldots, (u^S, w^S, v^S, z^S)\}$ be a finite set of dual scenarios and $\{\zeta^1, \ldots, \zeta^S\}$ be a set of primal scenarios obtained from (12). If there is only right-hand-side uncertainty in Model (1), then the objective value of*

$$\begin{aligned} \inf_{\substack{\tau, x \in \mathcal{X} \\ y^1, \ldots, y^S}} \quad & \tau \\ \text{s.t.} \quad & f_0(x) + g_0(y^s) \leq \tau \qquad \forall s = 1, \ldots, S \\ & (\zeta^s)^\top \bar{F}_{i\cdot} + f_i(x) + g_i(y^s) \leq 0 \\ & \qquad \forall i = 1, \ldots, m_1, \; s = 1, \ldots, S \\ & \bar{A}\zeta^s + By^s = b(x) \qquad \forall s = 1, \ldots, S \end{aligned} \tag{13}$$

is at least as high as the objective value of (10).

Proof. By duality for linear programming, (10) is equivalent to (11). The latter formulation can be written as

$$\begin{aligned} \inf_{x \in \mathcal{X}} \sup_{s \in \{1, \ldots, S\}} \Bigg\{ & f_0(x) + \sum_{i=1}^{m_1} v_i^s ((\zeta^s)^\top \bar{F}_{i\cdot} + f_i(x)) \\ & + (w^s)^\top (\bar{A}\zeta^s - b(x)) - \sum_{i=0}^{m_1} z_i^s \Bigg\}, \end{aligned} \tag{14}$$

where ζ^s are the primal scenarios obtained by (12). Because (u^s, w^s, v^s, z^s) are in $\mathcal{V}$ for all $s = 1, \ldots, S$, the value of (14) must be smaller than or equal to

$$\begin{aligned} \inf_{x \in \mathcal{X}} \sup_{s} \sup_{(u^s, w^s, v^s, z^s) \in \mathcal{V}} \Bigg\{ & f_0(x) + \sum_{i=1}^{m_1} v_i^s ((\zeta^s)^\top \bar{F}_{i\cdot} + f_i(x)) \\ & + (w^s)^\top (\bar{A}\zeta^s - b(x)) - \sum_{i=0}^{m_1} z_i^s \Bigg\}, \end{aligned}$$

because we maximize over (u^s, w^s, v^s, z^s) in the full $\mathcal{V}$ instead of a subset. The value of this optimization problem is, by dualizing over (u^s, w^s, v^s, z^s), equivalent to (13). Hence, the optimal objective value is at least as high as the optimal objective value of (10). □

The intuition behind the strength of the primal scenarios for right-hand-side uncertainty can be found in the fact that primal scenarios have no direct interaction with here and now decisions. That is, for right-hand-side uncertainty only, there are no terms in which both x and ζ appear. The dual model always includes the interaction terms with here and now decisions via the terms $\sum_{i=1}^{m_1} v_i^s f_i(x)$ and $(w^s)^\top b(x)$, even with right-hand-side uncertainty in the primal sampled model. Therefore, dual scenarios could be strong for some here and now decision x but very weak for other here and now decisions. In that case, the feasible region of the dual sampled model is larger and therefore, results in a lower objective value and thus, a weaker lower bound.

For linear adjustable robust optimization models, Theorem 3 can also significantly improve lower bounds. In Section 5 of the e-companion, we evaluate the performance of the lower-bounding scheme proposed in this subsection using the same numerical experiment considered in Bertsimas and de Ruiter (2016). Original optimality gaps reported for the larger instances were more than halved when primal scenarios were obtained using (12). For the largest instance the linked primal scenarios reduced the gap from 10.7% to 5.2%. We do note that the numerical examples all satisfied the assumption of right-hand-side uncertainty. If there is no right-hand-side uncertainty, then a dual sampled model can yield tighter lower bounds than its primal sampled counterpart. A very small example showing this is given in Section 2 of the e-companion. Therefore, the assumption of right-hand-side uncertainty is crucial in Theorem 3.

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