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INSTANTIAL NEIGHBOURHOOD LOGIC

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Abstract. This paper explores a new language of neighborhood structures where existential information can be given about what kind of worlds occur in a neighborhood of a current world. The resulting system of 'instantial neighborhood logic' INL has a non-trivial mix of features from relational semantics and from neighborhood semantics. We explore some basic model-theoretic behavior, provide a matching notion of bisimulation, and give a complete axiom system for which we prove completeness by a new normal form technique. In addition, we relate INL to other modal logics by means of translations, determine its precise SAT complexity, and formulate an adequate tableau calculus that can be used for automated deduction supporting INL inference. As for a broader setting, we discuss some motivations for INL in dynamic logics of evidence, consider some special cases in the realm of topology and of powers for players in games, and we point at general model-theoretic and especially, coalgebraic backgrounds for what is achieved in this paper. Many of these final themes suggest follow-up work of independent interest.

1. Introduction

Neighborhood semantics for modal logic [22, 27, 11, 19] is a generalization of both standard relational semantics and topological semantics, in which worlds are related to sets of worlds called 'neighborhoods', and a universal modality $\Box\varphi$ holds if at least one neighborhood has only points satisfying φ. There are many motivations for this generalization, from a desire to model weaker modal logics than the usual minimal system K to representing and reasoning about significant semantic structures that are not simply graph-like. However, when we extend a realm of semantic structures for a language, there is always an inevitable issue of whether that language is still adequate for capturing all structural features of the richer models. In particular, neighborhood models suggest that the composition of neighborhoods may be important, and for that, we need not just universal, but also existential information: what different kinds of worlds occur in given neighborhoods? There are several motivations for making such a move to an appropriate richer language, as we shall see in this paper: ranging from topology to games, and from modeling notions of evidence to belief revision. Their common core is a new 'instantial neighborhood modality' to be defined in the following section.

We will study the resulting extension INL of basic modal logic in quite some detail, and show that it has significant properties, many of them due to the fact that INL mixes features of neighborhood modal logic with features from relational semantics. To set the scene, in Section 2, we discuss basics of the system: language, truth definition, bisimulation, and some typical model constructions such as tree unraveling. In Section 3 we prove a Hennessy-Milner result for bisimulation, and also give some applications to undefinability. Section 4 presents a sound minimal logic for reasoning with the neighborhood modality. Section 5 follows up with a completeness proof based on normal forms and matching canonical models. Section 6 introduces semantic tableaux for INL which adds further information on validity reductions, and potential Gentzen-style calculi as well as automated deduction with the system. More details on this tableau analysis are are presented in the appendix following our bibliography. Section 7 positions INL in between the basic neighborhood modal logic

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and normal modal logics on relational models via translations that also allow us to determine the precise complexity of validity. Several further directions are then considered in Section 8, including links with topology, game logics, dynamic logic of evidence change, and especially, promising extensions toward coalgebra. Section 9 gives our general conclusion.

2. A new neighbourhood language

2.1. **Basic definitions.** In this section we introduce a new system of 'instantial neighbourhood logic' INL that generalizes basic modal logic over neighborhood models, using largely standard modal techniques, though with a number of non-trivial adaptations.

Let Prop be the set of propositional letters. We assume that the propositional language has primitive connectives $\neg, \wedge$, while $\bot, \vee, \to, \leftrightarrow$ are defined in the standard way. The language of INL extends the propositional language by a modal operator $\Box$ creating new formulas

$$\Box(\varphi_1, \ldots, \varphi_k; \psi)$$

for any INL-formula ψ and any (possibly empty) finite sequence $\varphi_1, \ldots, \varphi_k$ of INL-formulas.

Definition 2.1 (Modal depth)**.** *The* modal depth *of a formula is defined by induction as follows:* $md(p) = md(\bot) = 0$, $md(\varphi \wedge \psi) = max\{md(\varphi), md(\psi)\}$, $md(\neg\varphi) = md(\varphi)$, $md(\Box(\varphi_1, \ldots, \varphi_k; \psi)) = 1 + max\{md(\varphi_1), \ldots, md(\varphi_n), md(\psi)\}$.

Definition 2.2 (Neighbourhood frames and models)**.** *A* neighbourhood frame *is a pair* $\mathfrak{F} = (W, N)$ *where* W *is a non-empty set called a* domain*, and* $N : W \to \mathcal{P}\mathcal{P}(W)$ *is a map called a* neighborhood function*, where* $\mathcal{P}(W)$ *is the powerset of* W*. A* valuation *on* W *is a map* $V : \mathsf{Prop} \to \mathcal{P}W$*.*

A triple $\mathfrak{M} = (W, N, V)$ *is called a* neighbourhood model *or a* neighbourhood model based on (W, N) *if* (W, N) *is a neighbourhood frame and* V *is a valuation on it.*

Definition 2.3 (Truth of INL-formulas)**.** *Let* $\mathfrak{M} = (W, N, V)$ *be a neighbourhood model,* $w \in W$*, and* α *a formula. We inductively define when* $\mathfrak{M}$ satisfies α in a state w, *written* $\mathfrak{M}, w \models \alpha$:

$\mathfrak{M}, w \models p$	*iff*	$w \in V(p)$,
$\mathfrak{M}, w \models \neg\alpha$	*iff*	$\mathfrak{M}, w \not\models \alpha$,
$\mathfrak{M}, w \models \alpha \wedge \beta$	*iff*	$\mathfrak{M}, w \models \alpha$ *and* $\mathfrak{M}, w \models \beta$,
$\mathfrak{M}, w \models \Box(\varphi_1, ..., \varphi_k; \psi)$	*iff*	*there is* $S \in N(w)$ *such that for all* $s \in S$ *we have* $\mathfrak{M}, s \models \psi$ *and for all* $i \in \{1, ..., k\}$ *there is* $s_i \in S$ *such that* $\mathfrak{M}, s_i \models \varphi_i$.

We say that a formula α is *satisfiable* in $\mathfrak{M} = (W, N, V)$ if there is $w \in W$ such that $\mathfrak{M}, w \models \alpha$. We say that α is *satisfiable* if there is a neighbourhood model $\mathfrak{M} = (W, N, V)$ satisfying α. We say that α is valid in a neighbourhood frame (W, N) if for each model $\mathfrak{M}$ based on (W, N) and each $w \in W$ we have $\mathfrak{M}, w \models \alpha$, and α is valid if it is valid in every neighbourhood frame.

Remark 2.4. We also recall the truth of the standard modal operators in neighbourhood models.

$\mathfrak{M}, w \models \Box\varphi$ iff there is $S \in N(w)$ such that for all $s \in S$ we have $\mathfrak{M}, s \models \varphi$,
$\mathfrak{M}, w \models \Diamond\varphi$ iff for any $S \in N(w)$ there is $s \in S$ such that $\mathfrak{M}, s \models \varphi$.

Bisimulations for neighborhood models that fit our language are defined as follows:

Definition 2.5 (INL-Bisimulation)**.** *Let* $\mathfrak{M} = (W, N, V)$ *and* $\mathfrak{M}' = (W', N', V')$ *be neighbourhood models. A binary relation* $Z \subseteq W \times W'$ *is called a* INL-bisimulation *if for each* $(w, w') \in Z$ *(alternative notation* $w \rightleftharpoons_Z w'$*) we have:*

1. $\mathfrak{M}, w \models p$ *iff* $\mathfrak{M}', w' \models p$ *for each proposition letter* p*,*
2. $\forall S \in N(w)\ \exists S' \in N'(w') \begin{cases} \forall s' \in S' \exists s \in S (s \rightleftharpoons_Z s') \text{ and} \\ \forall s \in S \exists s' \in S' (s \rightleftharpoons_Z s') \end{cases}$
3. $\forall S' \in N'(w')\ \exists S \in N(w) \begin{cases} \forall s \in S \exists s' \in S' (s \rightleftharpoons_Z s') \text{ and} \\ \forall s' \in S' \exists s \in S (s \rightleftharpoons_Z s') \end{cases}$

Following standard terminology, states (or worlds – we will use both of these common terms interchangeably in this paper) w and w' are called INL-bisimilar (*written,* $w \rightleftharpoons w'$) *if there exists an* INL*-bisimulation* Z *such that* $w \rightleftharpoons_Z w'$.

2.2. **Warm-up results.** We begin by proving some basic semantic results about INL. Our first result is completely as expected: formulas of INL are invariant under INL-bisimulations.

Definition 2.6 (Modal Equivalence)**.** *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models and let $w \in W$ and $w' \in W'$. We say that w and w' are* modally equivalent (*written,* $w \leftrightsquigarrow w'$) *if for any* INL*-formula α we have $\mathfrak{M}, w \models \alpha$ iff $\mathfrak{M}', w' \models \alpha$.*

Theorem 2.7. *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models with $w \in W$ and $w' \in W'$. Then $w \rightleftharpoons w'$ implies $w \leftrightsquigarrow w'$.*

Proof. Suppose there is a bisimulation Z such that $w \rightleftharpoons_Z w'$. We show, by induction on the complexity of INL-formulas as defined earlier, that w and w' agree on all INL-formulas. The cases for proposition letters, $\neg$ and $\wedge$ are trivial. Now suppose $\mathfrak{M}, w \models \Box(\varphi_1, ..., \varphi_k; \psi)$. Then there is $U \in N(w)$ such that for all $u \in U$ we have $\mathfrak{M}, u \models \psi$ and for all $i \in \{1, \dots, k\}$ there is $u_i \in U$ such that $\mathfrak{M}, u_i \models \varphi_i$. Since $w \rightleftharpoons_Z w'$ and $U \in N(w)$, there is $U' \in N'(w')$ such that for all $u' \in U'$ there is $u \in U$ with $u \rightleftharpoons_Z u'$ and for all $u \in U$ there is $u' \in U'$ with $u \rightleftharpoons_Z u'$. Let v be an arbitrary state in U'. Because Z is a bisimulation, there exists $u \in U$ with $u \rightleftharpoons_Z v$. Since for all $u \in U$ we have $\mathfrak{M}, u \models \psi$, and the complexity of ψ is smaller than the complexity of $\Box(\varphi_1, ..., \varphi_k; \psi)$, by the induction hypothesis, we obtain that $\mathfrak{M}', v \models \psi$. Furthermore, for any $i \in \{1, \dots, k\}$, we have that $\mathfrak{M}, u_i \models \varphi_i$ for some $u_i \in U$, and as Z is a bisimulation, for each u_i there is $u'_i \in U'$ such that $u_i \rightleftharpoons_Z u'_i$. The same argument as above entails that for each $i \in \{1, \dots, k\}$ we have $\mathfrak{M}', u'_i \models \varphi_i$. In conclusion, we showed that $\mathfrak{M}', w' \models \Box(\varphi_1, ..., \varphi_k; \psi)$. The other direction is similar. Therefore, we obtain that $w \leftrightsquigarrow w'$. □

Similarly to bounded morphisms (p-morphisms) in relational semantics, we may also consider special functions that preserve structure of neighborhood models:

Definition 2.8. *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighborhood models. Then a map $f : W \to W'$ is called a* bounded morphism *if, for all $u \in W$:*

- *$u \in V(p)$ iff $f(u) \in V'(p)$, for every proposition letter p,*
- *$N'(f(u)) = \{f[Z] \mid Z \in N(u)\}$, where $f[Z]$ is the image of Z for the map f.*

This definition gives the expected result:

Proposition 2.9. *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighborhood models. Then a map $f : W \to W'$ is a bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}'$ if and only if its graph is an* INL*-bisimulation.*

We have the following immediate corollary of Proposition 2.9 and Theorem 2.7.

Corollary 2.10. *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighborhood models, and a map $f : W \to W'$ a bounded morphism from $\mathfrak{M}$ to $\mathfrak{M}'$. Then for each $w \in W$ and each* INL*-formula φ,*

$$\mathfrak{M}, w \models \varphi \text{ if and only if } \mathfrak{M}', f(w) \models \varphi.$$

Next, we show that some of the basic model theoretic constructions familiar from standard relational semantics have simple counterparts in neighborhood semantics. For this purpose we shall require the following notion:

Definition 2.11. *Given a neighborhood model $\mathfrak{M} = (W, N, V)$, we define the* support *relation $S_{\mathfrak{M}} \subseteq W \times W$ by setting, for $u, v \in W$:*

$$(u, v) \in S_{\mathfrak{M}} \text{ if and only if } v \in \bigcup N(u)$$

We let $S^\star_{\mathfrak{M}}$ denote the reflexive-transitive closure of $S_{\mathfrak{M}}$. Moreover, if there exists some $u \in W$ such that, for all $v \in W$ there is a unique $S_{\mathfrak{M}}$-path from u to v, then we say that the model $\mathfrak{M}$ is tree-like *and* rooted at u.

Using this definition we can introduce an obvious analogue of the standard notion of generated submodels known from basic modal logic:

Definition 2.12. *Let $\mathfrak{M} = (W, N, V)$ be any neighborhood model, and let $u \in W$. Then the* point-generated submodel *of $\mathfrak{M}$ at u, denoted $\mathfrak{M}[u]$, is defined to be the structure (W', N', V') where:*

- $W' = \{v \in W \mid (u, v) \in S^\star_{\mathfrak{M}}\}$,
- $N' = N \restriction_{W'}$,
- $V' = V \restriction_{W'}$.

Proposition 2.13. *For any $u \in W$, the obvious inclusion map $\iota : W' \to W$ is a bounded morphism from $\mathfrak{M}[u]$ to $\mathfrak{M}$. Hence, for all formulas φ, we have:*

$$\mathfrak{M}, u \models \varphi \text{ iff } \mathfrak{M}[u], u \models \varphi$$

Next, we turn to tree unraveling. Here our goal is to show that INL has the "tree model property," i.e. every satisfiable formula is satisfiable in some tree-like model. The question is exactly what "tree-like" means in this context. Since trees are characterized by the property that there is some distinguished node, the root, from which there is a unique path to any other given node, the question we really need to answer here is what is a "path" in a neighborhood model. A first answer is to say that a path in a model $\mathfrak{M}$ is any tuple $(w_0, ..., w_n)$ where, for each $i < n$, we have $(w_i, w_{i+1}) \in S_{\mathfrak{M}}$. This suggests the following definition:

Definition 2.14. *A model $\mathfrak{M}$ is said to be* tree-like *if the support relation $S_{\mathfrak{M}}$ is a tree.*

Definition 2.15. *Let $\mathfrak{M} = (W, N, V)$ be any neighborhood model and let $u \in W$. We define the* tree-unraveling *of $\mathfrak{M}$ at u, denoted $\mathfrak{M}^U_u$, to be the structure (W', N', V') where:*

- *W' is the set of all finite sequences $(v_0, ..., v_n)$ over W with $v_0 = u$ and $(v_i, v_{i+1}) \in S_{\mathfrak{M}}$ for $0 \leq i < n$.*
- *Given a sequence $(v_0, ..., v_n)$ and $Z' \subseteq W'$ we set $Z' \in N'(v_0, ..., v_n)$ if and only if there is some $Z \in N(v_n)$ such that*
$$Z' = \{(v_0, ..., v_n, w) \mid w \in Z\}.$$
- *The valuation V' is defined by $(v_0, ..., v_n) \in V'(p)$ iff $v_n \in V(p)$.*

Clearly $\mathfrak{M}^U_u$ is a tree-like model, with root u. Furthermore, we have:

Proposition 2.16. *There is a bounded morphism $f : \mathfrak{M}^U_u \to \mathfrak{M}$ given by the assignment:*

$$(v_0, ..., v_n) \mapsto v_n$$

From this the desired tree model property follows immediately:

Proposition 2.17. *Any satisfiable formula is satisfiable at the root of some tree-like model.*

A second possibility is to say that a path in $\mathfrak{M}$ is any tuple

$$(w_0, Z_0, w_1, Z_1, ..., Z_{n-1}, w_n)$$

where, for each $i < n$, Z_i is a neighborhood of w_i and $w_{i+1} \in Z_i$. Let us call such a tuple a *strong* path from w_0 to w_n in $\mathfrak{M}$.

Definition 2.18. *A model $\mathfrak{M}$ is said to be* strongly tree-like *if there is some $w \in W$ such that for every $w' \in W$, there is a unique strong path from w to w' in $\mathfrak{M}$.*

It is not hard to see that every strongly tree-like model is tree-like, but the converse implication does not hold. Now let us consider tree unraveling again.

Definition 2.19. *Let $\mathfrak{M} = (W, N, V)$ be any neighborhood model and let $u \in W$. We define the* strong tree-unraveling *of $\mathfrak{M}$ at u, denoted $\mathfrak{M}^S_u$, to be the structure (W', N', V') where:*

- *W' is the set of all strong paths in $\mathfrak{M}$ beginning with u.*
- *Given a strong path $(v_0, Z_0, ..., Z_{n-1}, v_n)$ and $Z' \subseteq W'$ we set $Z' \in N'(v_0, Z_0, ..., Z_{n-1}, v_n)$ if and only if there is some $Z \in N(v_n)$ such that*

$$Z' = \{(v_0, Z_0, ..., Z_{n-1}, v_n, Z, w) \mid w \in Z\}.$$

- *The valuation V' is defined by $(v_0, Z_0, ..., Z_{n-1}, v_n) \in V'(p)$ iff $v_n \in V(p)$.*

The strong tree-unraveling of any model is strongly tree-like.

Proposition 2.20. *There is a bounded morphism $f : \mathfrak{M}^S_u \to \mathfrak{M}$ given by the assignment:*

$$(v_0, Z_0, ..., Z_{n-1}, v_n) \mapsto v_n$$

We now get the following strengthened version of the tree model property:

Proposition 2.21 (Strong tree model property)**.** *Any satisfiable formula is satisfiable at the root of some strongly tree-like model.*

Finally, we establish a finite-depth property for instantial neighborhood logic:

Definition 2.22. *Let $\mathfrak{M} = (W, N, V)$ be any neighborhood model, let $u \in W$, and let k be any integer. Then the* depth k point-generated submodel *of $\mathfrak{M}$ at u, denoted by $\mathfrak{M}[u, k]$, is defined to be the structure (W', N', V') where:*

- *W' is the set of all $v \in W$ such that there is a $S_{\mathfrak{M}}$-path from u to v of length $\leq k$.*
- *For $v \in W'$, we set $N'(v) = N(v)$ if there is a $S_{\mathfrak{M}}$-path from u to v of length $< k$, and $N'(v) = \emptyset$ otherwise. That is, $N'(v) = \emptyset$ if the shortest $S_{\mathfrak{M}}$-path from u to v has length k.*
- *$V' = V \restriction_{W'}$.*

A fairly simple argument will establish the following:

Fact 2.23. *Let φ be any formula of modal depth $\leq k$. Then:*

$$\mathfrak{M}, u \models \varphi \text{ iff } \mathfrak{M}[u, k], u \models \varphi$$

Corollary 2.24 (Finite-height model property)**.** *Any satisfiable formula of modal depth k is satisfiable at the root of some tree-like model with an underlying tree of height at most k.*

3. Bisimulation characterization

In this section we show that our notion of bisimulation makes sense, by proving a Hennessy-Milner theorem for INL. More precisely, we show that over finite models, the relation of modal equivalence is an INL-bisimulation.

Theorem 3.1. *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be finite neighbourhood models with $w \in W$ and $w' \in W'$. Then $w \leftrightsquigarrow w'$ iff $w \rightleftharpoons w'$.*

Proof. Our aim is to show that the relation $\leftrightsquigarrow$ is an INL-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. We show only one direction of the back-and-forth conditions, since the other direction can be proved using a symmetric argument.

First, pick any $w \in W$ and $w' \in W'$. We define a formula $\delta(w, w')$ as follows:

- If $w \leftrightsquigarrow w'$, set $\delta(w, w') = \top$.
- Otherwise, pick some formula φ such that $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}', w' \models \neg\varphi$ and set $\delta(w, w') = \varphi$.

So $\delta(w, w')$ either provides some witness that w and w' are *not* modally equivalent, or is set to $\top$ if they are modally equivalent. Now, for $w \in W$, set

$$\chi(w) := \bigwedge\{\delta(w, w') \mid w' \in W'\}$$

This conjunction is well defined because W' is a finite set. Note that $\mathfrak{M}, w \vDash \chi(w)$ for all $w \in W$. It is easy to prove the following claim:

Claim: For all $w \in W$ and all $w' \in W'$, we have:

$$w \leftrightsquigarrow w' \text{ iff } \mathfrak{M}', w' \models \chi(w).$$

Now suppose that $w \leftrightsquigarrow w'$, for $w \in W$ and $w' \in W'$. Let U be a neighborhood of w, say $U = \{u_0, ..., u_k\}$. It is easy to see that we have:

$$\mathfrak{M}, w \models \Box(\chi(u_0), ..., \chi(u_k); \chi(u_0) \vee ... \vee \chi(u_k)).$$

Hence, we must also have

$$\mathfrak{M}', w' \models \Box(\chi(u_0), ..., \chi(u_k); \chi(u_0) \vee ... \vee \chi(u_k)).$$

This means there must be some neighborhood U' of w' such that every formula $\chi(u_i)$ is true somewhere in U', and conversely every member of U' satisfies $\chi(u_i)$ for some $i \leq k$. By the Claim, it follows that the relation $\leftrightsquigarrow$ satisfies the appropriate back-and-forth conditions with respect to U and U'. $\Box$

Much more can be said about INL-bisimulations. For further notions and results, we refer to Section 8. The formulas that appear in our proof of the Hennessy-Milner theorem bear a strong resemblance to the normal forms for INL-formular that we etablish later in Section 5, which play a crucial role in the completeness proof for our axiomatization of INL. The proof also has a coalgebraic flavor, and this theme will be taken up in Section 8.4.

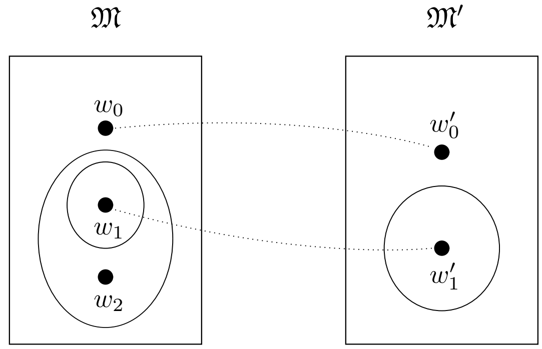
For now, however, we pursue another question. How does INL relate to more standard modal languages on neighborhood models? To see this, we recall the notion of bisimulation for the standard basic modal language on neighbourhood models.

Definition 3.2 (Standard Bisimulation). *Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models. A binary relation $Z \subseteq W \times W'$ is called a* standard bisimulation *if for each $(w, w') \in Z$ (alternative notation $w \leftrightarrows_Z w'$) we have:*

1. *$\mathfrak{M}, w \models p$ iff $\mathfrak{M}', w' \models p$ for each proposition letter p.*
2. *$\forall S \in N(w)\ \exists S' \in N'(w') \forall s' \in S'\ \exists s \in S$ such that $(s \leftrightarrows_Z s')$.*
3. *$\forall S' \in N'(w')\ \exists S \in N(w) \forall s \in S\ \exists s' \in S'$ such that $(s \leftrightarrows_Z s')$.*

It is easy to check that each INL-bisimulation is also a standard bisimulation. Below we will give an example that the converse, in general, is not true. Since standard bisimilarity entails modal equivalence for the basic modal language, this implies that over the neighbourhood models INL is more expressive than the standard modal language.

Example 3.3. We will now give an example of two neighbourhood models and states in these models which are bisimilar in the standard sense, but which are not INL-bisimilar. Start by choosing $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ where $W = \{w_0, w_1, w_2\}$, $N(w_0) = \{\{w_1\}, \{w_1, w_2\}\}$, $V(p) = \{w_1, w_2\}$, $V(q) = \{w_2\}$ and $W' = \{w'_0, w'_1\}$, $N'(w'_0) = \{w'_1\}$, $V'(p) = \{w'_1\}$, $V'(q) = \emptyset$. Then it is easy to see that w_0 and $w_{0'}$ are standard bisimilar (consider a relation linking w_0 with w'_0 and w_1 with w'_1), but not INL-bisimilar. To see this, by Theorem 3.1, it is sufficient to note that $\mathfrak{M}, w_0 \models \Box(q; p)$, but $\mathfrak{M}', w'_0 \models \Box(q; p)$.



From Example 3.3 we immediately derive that

Proposition 3.4. *Over neighbourhood models* INL *is not definable in the basic modal language.*

Next we will show that the universal modality is not definable in the INL-language. Recall that the truth condition for the *universal modalities E and A* is as follows:

$$\begin{array}{lll} \mathfrak{M}, w \models E\varphi & \text{iff} & \text{there is } s \text{ in } \mathfrak{M} \text{ such that } \mathfrak{M}, s \models \varphi. \\ \mathfrak{M}, w \models A\varphi & \text{iff} & \text{for any } s \text{ in } \mathfrak{M} \text{ we have } \mathfrak{M}, s \models \varphi. \end{array}$$

Example 3.5. Let $\mathfrak{M} = (W, N, V)$ and $\mathfrak{M}' = (W', N', V')$ be neighbourhood models such that $W = \{w, v\}$, $N(w) = \emptyset$, $V(p) = \{v\}$, and $W' = \{w'\}$, $N(w') = \emptyset$, $V'(p) = \emptyset$. Then it is easy to see that $Z = \{(w, w')\}$ is an INL-bisimulation. On the other hand, $\mathfrak{M}, w \models Ep$, while $\mathfrak{M}, w' \not\models Ep$.

Thus, we arrive at the following proposition.

Proposition 3.6. *Over neighbourhood models the universal modality is not definable in* INL.

4. Axiomatization and soundness

In this section we provide a Hilbert-style proof system for INL, prove soundness and derive some basic properties. Completeness of the axioms will be shown later in Section 5.

Our basic proof calculus has as axioms schemes all propositional tautologies, together with the following axiom schemes:

(NW) $\Box(\gamma_1, ..., \gamma_j; \psi) \to \Box(\gamma_1, ..., \gamma_j; \psi \vee \chi)$,

(SW) $\Box(\gamma_1, ..., \gamma_j, \alpha; \psi) \to \Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta; \psi)$,

(SR) $\Box(\gamma_1, ..., \gamma_j, \varphi; \psi) \to \Box(\gamma_1, ..., \gamma_j, \varphi \wedge \psi; \psi)$,

(SC) $\neg\Box(\bot; \psi)$,

(NT) $\Box(\gamma_1, ..., \gamma_j; \psi) \to \Box(\gamma_1, ..., \gamma_j, \delta; \psi) \vee \Box(\gamma_1, ..., \gamma_j; \psi \wedge \neg\delta)$,

(AD) $\Box(\gamma_1, ..., \gamma_j, \varphi, \delta_1, ..., \delta_n; \psi) \to \Box(\gamma_1, ..., \gamma_j, \delta_1, ..., \delta_n; \psi)$,

(AI) $\Box(\gamma_1, ..., \gamma_j, \delta_1, ..., \delta_n; \psi) \to \Box(\gamma_1, ..., \gamma_j, \varphi, \delta_1, ..., \delta_n; \psi)$, provided $\varphi \in \{\gamma_1, ..., \gamma_j, \delta_1, ..., \delta_n\}$.

The system has only two rules, modus ponens and substitution of equivalents:

$$\text{(MP)} \quad \frac{\alpha \to \beta \quad \alpha}{\beta},$$

$$\text{(RE)} \quad \frac{\alpha \leftrightarrow \beta \quad \varphi}{\varphi[\alpha/\beta]}.$$

where $\varphi[\alpha/\beta]$ is the result of possibly replacing some occurrences of α in φ by β.

Explanation. To understand these principles intuitively, note that some just express the set character of the finite sequence of instances at the start of our modalities, others state obvious upward monotonicity properties in all arguments, while we also have distribution over disjunction for instanced formulas. In addition, (SR) expresses the compatibility of the universal condition with all instances, while (NT) essentially shows how we can either add instances to a given modality witnessing a formula δ or strengthen the universal condition of the modality with the negation of δ.

The following standard lemma will simplify some of our proofs.

Lemma 4.1. *The following rules are admissible in the proof system:*

$$(TR) \quad \frac{\alpha \to \beta \quad \beta \to \gamma}{\alpha \to \gamma}$$

$$(MT) \quad \frac{\alpha \to \beta \quad \neg\beta}{\neg\alpha}$$

We write $\vdash \varphi$ to say that the formula φ is provable in this system, and $\Gamma \vdash \varphi$ as a shorthand for $\vdash \bigwedge \Gamma \to \varphi$ if Γ is a finite set of formulas. We are now ready to derive some basic schemes and admissible rule schemes of the logic INL.

Lemma 4.2. *The following theorem and rule schemes are provable ('admissible') in* INL*.*

$$(EX) \quad \Box(\gamma_1, ..., \gamma_j, \theta, \varphi, \delta_1, ..., \delta_m; \psi) \to \Box(\gamma_1, ..., \gamma_j, \varphi, \theta, \delta_1, ..., \delta_m; \psi),$$

$$(SC') \quad \neg\Box(\gamma_1, ..., \gamma_j, \bot, \delta_1, ..., \delta_m; \psi),$$

$$(NC) \quad \neg\Box(\gamma; \bot),$$

$$(NW') \quad \frac{\psi \to \chi}{\Box(\gamma_1, ..., \gamma_j; \psi) \to \Box(\gamma_1, ..., \gamma_j; \chi)},$$

$$(SW') \quad \frac{\alpha \to \beta}{\Box(\gamma_1, ..., \gamma_j, \alpha; \psi) \to \Box(\gamma_1, ..., \gamma_j, \beta; \psi)},$$

$$(SS) \quad \Box(\gamma_1, ..., \gamma_j, \alpha \vee \beta; \psi) \to \Box(\gamma_1, ..., \gamma_j, \alpha; \psi) \vee \Box(\gamma_1, ..., \gamma_j, \beta; \psi).$$

Proof. For (EX), use (TR) to chain together the following implications:

$$\begin{array}{lcll} \Box(\gamma_1, ..., \gamma_j, \theta, \varphi, \delta_1, ..., \delta_m; \psi) & \vdash & \Box(\gamma_1, ..., \gamma_j, \varphi, \theta, \varphi, \delta_1, ..., \delta_m; \psi) & \text{(AI)} \\ \Box(\gamma_1, ..., \gamma_j, \varphi, \theta, \varphi, \delta_1, ..., \delta_m; \psi) & \vdash & \Box(\gamma_1, ..., \gamma_j, \varphi, \theta, \delta_1, ..., \delta_m; \psi) & \text{(AD)} \end{array}$$

For (SC'), just apply (MT) to the axiom $\neg\Box(\bot, \psi)$ and the following instance of (AD):

$$\Box(\gamma_1, ..., \gamma_j, \bot, \delta_1, ..., \delta_m; \psi) \to \Box(\bot, \psi)$$

We derive (NC) as follows:

$$\begin{array}{rlll} \Box(\gamma;\bot) & \vdash & \Box(\gamma\wedge\bot;\bot) & \text{(SR)} \\ & \vdash & \gamma\wedge\bot\leftrightarrow\bot & \text{Classical logic} \\ \Box(\gamma;\bot) & \vdash & \Box(\bot,\bot) & \text{(RE)} \\ & \vdash & \neg\Box(\bot;\bot) & \text{(SC)} \\ & \vdash & \neg\Box(\gamma;\bot) & \text{(MT)} \end{array}$$

For (NW'), suppose that $\vdash \varphi\rightarrow\chi$. We derive the conclusion of the rule as a theorem as follows:

$$\begin{array}{rlll} \Box(\gamma_1,...,\gamma_j;\varphi) & \vdash & \Box(\gamma_1,...,\gamma_j;\varphi\vee\chi) & \text{(NW)} \\ & \vdash & \chi\leftrightarrow\varphi\vee\chi & \text{Classical logic} \\ \Box(\gamma_1,...,\gamma_j;\varphi) & \vdash & \Box(\gamma_1,...,\gamma_j;\chi) & \text{(RE)} \end{array}$$

(SW') can be derived similarly. Finally, for (SS):

$$\begin{array}{ll} \vdash \Box(\gamma_1,...,\gamma_j,\alpha\vee\beta;\psi)\rightarrow\Box(\gamma_1,...,\gamma_j,\alpha\vee\beta,\alpha;\psi)\vee\Box(\gamma_1,...,\gamma_j,\alpha\vee\beta;\psi\wedge\neg\alpha) & \text{(NT), (1)} \\ \vdash \Box(\gamma_1,...,\gamma_j,\alpha\vee\beta,\alpha;\psi)\rightarrow\Box(\gamma_1,...,\gamma_j,\alpha;\psi) & \text{(AD)} \\ \vdash \Box(\gamma_1,...,\gamma_j,\alpha\vee\beta;\psi\wedge\neg\alpha)\rightarrow\Box(\gamma_1,...,\gamma_j,(\alpha\vee\beta)\wedge(\psi\wedge\neg\alpha);\psi\wedge\neg\alpha) & \text{(SR), (2)} \\ \vdash (\alpha\vee\beta)\wedge(\psi\wedge\neg\alpha)\rightarrow\beta & \text{Classical logic} \\ \vdash (\psi\wedge\neg\alpha)\rightarrow\psi & \text{Classical logic} \\ \vdash \Box(\gamma_1,...,\gamma_j,(\alpha\vee\beta)\wedge(\psi\wedge\neg\alpha);\psi\wedge\neg\alpha)\rightarrow\Box(\gamma_1,...,\gamma_j,\beta;\psi) & \text{(NW'),(SW')} \\ \vdash \Box(\gamma_1,...,\gamma_j,\alpha\vee\beta;\psi\wedge\neg\alpha)\rightarrow\Box(\gamma_1,...,\gamma_j,\beta;\psi) & \text{(TR), (3)} \\ \vdash \Box(\gamma_1,...,\gamma_j,\alpha\vee\beta;\psi)\rightarrow\Box(\gamma_1,...,\gamma_j,\alpha;\psi)\vee\Box(\gamma_1,...,\gamma_j,\beta;\psi) & \text{(1) - (3)} \end{array}$$

□

The proof system for INL presented above is sound and complete. Completeness will be proved in the next section. Here, we prove that the system is sound:

Theorem 4.3 (Soundness of INL)**.** *If* $\vdash \varphi$*, then* φ *is valid in the neighborhood semantics.*

Proof. Since our two inference rules clearly preserve validity, we only have to check that all our axioms are valid. Most of the axioms in the above list are straightforward to check, so we give only the proofs for (SR) and (NT).

For (SR), we want to show that $\Box(\gamma_1,...,\gamma_j,\varphi;\psi)\rightarrow\Box(\gamma_1,...,\gamma_j,\varphi\wedge\psi;\psi)$ is a valid formula. Take an arbitrary model $\mathfrak{M}=(W,N,V)$ and some state $w\in W$. Assume $w\models\Box(\gamma_1,...,\gamma_j,\varphi;\psi)$, then there is $S\in N(w)$ such that (1) ψ is satisfied at each state in S, (2) γ_i is satisfied at some state in S for each $i\in\{1,...,j\}$, and (3) ψ is satisfied at some state, say u in S. Since $u\in S$, by (1) we also have $u\models\psi$, and hence $u\models\varphi\wedge\psi$. Therefore, $w\models\Box(\gamma_1,...,\gamma_j,\varphi\wedge\psi;\psi)$.

For (NT), we want to show validity of the formula

$$\Box(\gamma_1,...,\gamma_j;\psi)\rightarrow\Box(\gamma_1,...,\gamma_j,\delta;\psi)\vee\Box(\gamma_1,...,\gamma_j;\psi\wedge\neg\delta)$$

Again take an arbitrary model $\mathfrak{M}=(W,N,V)$ and state $w\in W$. Assume that $w\models\Box(\gamma_1,...,\gamma_j;\psi)$. Then there is $S\in N(w)$ such that (1) ψ is satisfied at each state in S, (2) γ_i is satisfied at some state in S for each $i\in\{1,...,j\}$. Now, assume furthermore that the formula $\Box(\gamma_1,...,\gamma_j,\delta;\psi)$ is *not* satisfied at w. Then it must be the case that δ is satisfied nowhere in S, so $\neg\delta$ is true everywhere in S, giving $w\models\Box(\gamma_1,...,\gamma_j,\psi\wedge\neg\delta)$. From the assumption that $w\models\Box(\gamma_1,...,\gamma_j;\psi)$ we have now derived that

$$w\models\Box(\gamma_1,...,\gamma_j,\delta;\psi)\vee\Box(\gamma_1,...,\gamma_j;\psi\wedge\neg\delta),$$

which concludes the argument. □

5. Normal forms and completeness

5.1. **A normal form theorem.** Our goal in this section is to prove a completeness theorem for the axiom system presented above. The proof will proceed via a normal form theorem.

Throughout the section, we assume a fixed well-ordering over the set of all formulas of INL. With this in mind, the following notation will be useful: given a finite set of formulas Γ and a formula φ, we denote by $\Box(\Gamma;\varphi)$ the formula $\Box(\psi_1,...,\psi_n;\varphi)$ where $\psi_1,...,\psi_n$ is a list of all the formulas in Γ, without repetitions, in the order dictated by our fixed well-ordering over the language.

Lemma 5.1. *For any formula of the form $\Box(\psi_1,...,\psi_n;\varphi)$, we have:*

$$\vdash \Box(\psi_1,...,\psi_n;\varphi) \leftrightarrow \Box(\{\psi_1,...,\psi_n\};\varphi)$$

Proof. Use (AD) and (AI) to remove repetitions, and use (EX) to rearrange the formulas $\psi_1,...,\psi_n$ in the order consistent with the fixed well-order over the language. □

As a first application this leads to the following lemma:

Lemma 5.2. *Let $i \in \omega$ and let P be a finite set of propositional variables. Then there are, up to provable equivalence, only finitely many formulas of modal depth $\leq i$ and variables in P.*

Proof. A straightforward induction on the modal depth of a formula. The crucial step is to show that there are only finitely many formulas up to provable equivalence of the form $\Box(\psi_1,...,\psi_n;\varphi)$ where $\psi_1,...,\psi_n,\varphi$ are all of depth $\leq k$. But by the previous lemma, each such formula is equivalent to a formula of the form $\Box(\Gamma;\varphi)$ where φ is of depth $\leq k$ and Γ is a finite *set* of formulas of depth $\leq k$. It is now easy see that the number of formulas up to provable equivalence of the form $\Box(\psi_1,...,\psi_n;\varphi)$ where $\psi_1,...,\psi_n,\varphi$ are all of modal depth $\leq k$ is bounded by $2^m \times m$, where m is the number of formulas of modal depth $\leq k$ up to provable equivalence. □

Definition 5.3. *Let P be a finite set of propositional variables, and let i be any natural number. An i-type over P is a consistent set of formulas Γ of modal depth $\leq i$ and variables in P such that, for every formula ψ of depth $\leq i$ and variables in P, we have $\psi \in \Gamma$ or $\neg\psi \in \Gamma$. In other words: an i-type over P is a maximal consistent set of formulas of modal depth $\leq i$ and variables in P.*

We denote the set of i-types over P by $T(i,P)$. We can associate a formula with each i-type as follows: for each formula in τ, pick the least formula that is provably equivalent to it. The collection of all these formulas is then finite by Lemma 5.2, so we can take the conjunction of all of them and denote it by $\hat{\tau}$. If Γ is a set of i-types then we write $\hat{\Gamma} = \{\hat{\tau} \mid \tau \in \Gamma\}$.

Lemma 5.4. *Let φ be any formula with variables from P and modal depth at most i. Then:*

$$\vdash \varphi \leftrightarrow \bigvee\{\hat{\tau} \mid \tau \in T(i,P) \;\&\; \varphi \in \tau\}$$

We are now ready for our normal form theorem:

Theorem 5.5. *Let $\varphi,\psi_1,...,\psi_k$ be formulas with variables in P and all of modal depth $\leq i$. Then the formula $\Box(\{\psi_1,...,\psi_k\},\varphi)$ is provably equivalent to a formula of the form:*

$$\bigvee\{\Box(\hat{\Gamma};\bigvee\hat{\Gamma}) \mid \Gamma \in F\}$$

where F is a family of sets of i-types over P such that, for each $\Gamma \in F$, the following facts hold:

(1) *For each $j \in \{1,...,k\}$ there is some $\tau \in \Gamma$ with $\psi_j \in \tau$.*
(2) *For each $\tau \in \Gamma$ we have $\varphi \in \tau$.*

Proof. We know that $\Box(\psi_1,...,\psi_k;\varphi)$ is provably equivalent to $\Box(\{\psi_1,...,\psi_k\};\varphi)$. Note that we can assume that $\Box(\{\psi_1,...,\psi_k\};\varphi)$ is consistent, since otherwise it is provably equivalent to $\bigvee\emptyset$, which is a disjunction of the right shape. We perform the reduction in two steps.

Claim 1: The formula $\Box(\{\psi_1, ..., \psi_k\}; \varphi)$ is provably equivalent to a formula of the form:

$$\bigvee\{\Box(\hat{\Gamma}; \varphi) \mid \Gamma \in F\}$$

where F is a family of sets of i-types over P such that conditions (1) and (2) hold.

Claim 2: Any formula of the form $\Box(\hat{\Gamma}; \varphi)$, where Γ is a set of i-types over P each containing φ, is provably equivalent to a disjunction of formulas of the form $\Box(\hat{\Phi}; \bigvee \hat{\Phi})$, where each Φ is a set of i-types over P for which conditions (1) and (2) hold.

The theorem obviously follows from these two claims, so we prove them one by one.

For Claim 1: First, we use Lemma 5.4 together with the uniform substitution rule to rewrite the formula $\Box(\{\psi_1, ..., \psi_k\}; \varphi)$ as:

$$\Box(\{\bigvee \hat{\Psi}_1, ..., \bigvee \hat{\Psi}_k\}; \varphi)$$

where, for each $j \in \{1, ..., k\}$, we denote by Ψ_j the set $\{\tau \in T(i, P) \mid \psi_j \in \tau\}$. At this point, we can repeatedly apply the theorem scheme (SS) to obtain the equivalent disjunction

$$\bigvee\{\Box(\{\hat{\tau}_1, ..., \hat{\tau}_k\}; \varphi) \mid \tau_1 \in \Psi_1 \;\&\; ... \;\&\; \tau_k \in \Psi_k\}$$

Clearly condition (1) holds for each set $\{\tau_1, ..., \tau_k\}$. All that is left to do to finish Claim 1 is to remove those disjuncts of the form $\Box(\{\hat{\tau}_1, ..., \hat{\tau}_k\}; \varphi)$ where for some $j \in \{1, ..., k\}$, $\varphi \notin \tau_j$. But for each such disjunct, we have $\neg\varphi \in \tau_j$, so

$$\vdash \Box(\{\hat{\tau}_1, ..., \hat{\tau}_k\}; \varphi) \to \Box(\neg\varphi; \varphi)$$

But then we can apply (SR) together with (SC) to show that the disjunct provably entails $\bot$. Hence, the whole disjunction is provably equivalent to the smaller disjunction where this disjunct is removed, by basic propositional logic. With this observation, we are finished with Claim 1.

For Claim 2: Let $\Box(\hat{\Gamma}; \varphi)$ be a formula of the shape described in the premise of the claim. Again, we apply Lemma 5.4 to rewrite the formula as $\Box(\hat{\Gamma}; \bigvee \hat{\Phi})$ where $\Phi = \{\tau \in T(i, P) \mid \varphi \in \tau\}$. Generally, call a formula of the shape $\Box(\hat{\Gamma}; \bigvee \hat{\Phi})$ where Γ and Φ are sets of i-types over P such that $\Gamma \subseteq \Phi$ an *almost normal box formula*. For any almost normal box formula $\Box(\hat{\Gamma}; \bigvee \hat{\Phi})$, let the *error degree* of $\Box(\hat{\Gamma}; \bigvee \hat{\Phi})$ be defined as the size of the set $\Phi \setminus \Gamma$. Since there are only finitely many i-types over P, the error degree of an almost normal box formula is always a finite ordinal, which equals 0 if and only if $\Gamma = \Phi$. So, to prove Claim 2, it now suffices to show:

Claim 3: Any almost normal box formula $\Box(\hat{\Gamma}; \bigvee \hat{\Phi})$ of error degree m, where $m > 0$, is provably equivalent to a disjunction of almost normal box formulas of error degree $m - 1$. Also, for each disjunct $\Box(\hat{\Gamma}'; \bigvee \hat{\Phi}')$ of this disjunction, we have $\Gamma \subseteq \Gamma'$ and $\Phi' \subseteq \Phi$.

To prove Claim 3, let $\Box(\hat{\Gamma}; \bigvee \hat{\Phi})$ be an almost normal box formula of error degree m. Since $m > 0$ there is some $\tau \in \Phi \setminus \Gamma$. Using the axiom (NT) we find:

$$\vdash \Box(\hat{\Gamma}; \bigvee \hat{\Phi}) \leftrightarrow (\Box(\hat{\Gamma} \cup \{\hat{\tau}\}; \bigvee \hat{\Phi}) \vee \Box(\hat{\Gamma}; \bigvee \hat{\Phi} \wedge \neg\hat{\tau}))$$

But clearly, the formula $\bigvee \hat{\Phi} \wedge \neg\hat{\tau}$ is provably equivalent to $\bigvee(\hat{\Phi} \setminus \{\hat{\tau}\})$ just by propositional logic. So the whole disjunction on the right side takes the shape:

$$\Box(\hat{\Gamma} \cup \{\hat{\tau}\}; \bigvee \hat{\Phi}) \vee \Box(\hat{\Gamma}; \bigvee(\hat{\Phi} \setminus \{\hat{\tau}\})$$

Each of these two disjuncts is an almost normal box formula of error degree $m - 1$, and so we are done with the proof of Claim 3, and the proof of the theorem is complete. $\Box$

5.2. **The model construction.** Fix a finite set of propositional variables P. We construct a canonical model $\mathfrak{M} = (W, N, V)$ as follows. Let W be the (disjoint) union of all the type sets $T(i, P)$ for $i \in \omega$. The neighborhood structure N is now defined by setting $N(\tau) = \emptyset$ for $\tau \in T(0, P)$ and, for $\tau \in T(i, P)$ with $i > 0$:

$$N(\tau) = \{\Gamma \subseteq T(i-1, P) \mid \Box(\hat{\Gamma}; \bigvee \hat{\Gamma}) \in \tau\}$$

Finally, for $p \in P$ we set $V(p) = \{\tau \in W \mid p \in \tau\}$.

Lemma 5.6 (Truth lemma)**.** *Let φ be a formula with propositional variables only in the set P, and of modal depth at most i. Then for any $\tau \in T(i, P)$, we have:*

$$\mathfrak{M}, \tau \models \varphi \textit{ iff } \varphi \in \tau$$

Proof. The proof proceeds by induction on i. The case for $i = 0$ is simple and left to the reader.

Suppose the lemma holds for i, and consider an $i+1$-type τ. It suffices to show that the lemma holds for all formulas of the form $\Box(\psi_1, ..., \psi_k; \varphi)$, where $\psi_1, ..., \psi_k$ and φ are all of depth i. A simple argument will then extend the claim to arbitrary formulas of depth $i + 1$.

First, suppose that $\Box(\psi_1, ..., \psi_k; \varphi) \in \tau$. By Theorem 5.5, this formula is then equivalent to a disjunction of the form

$$\bigvee\{\Box(\hat{\Gamma}; \bigvee \hat{\Gamma}) \mid \Gamma \in F\}$$

where F is a family of sets of i-types over P satisfying conditions (1) and (2). So at least one of these disjuncts $\Box(\hat{\Gamma}; \bigvee \hat{\Gamma})$ must be in τ since it is an $i+1$-type, which means that $\Gamma \in N(\tau)$. Using conditions (1) and (2) for Γ, together with the induction hypothesis applied to all i-types, we now easily show that $\mathfrak{M}, \tau \models \Box(\psi_1, ..., \psi_k; \varphi)$.

Conversely, suppose that $\mathfrak{M}, \tau \models \Box(\psi_1, ..., \psi_k; \varphi)$. Using the induction hypothesis for i-types, we see that there is a set of i-types $\Gamma \in N(\tau)$ such that $\varphi \in \bigcap \Gamma$ and, for every $j \in \{1, ..., k\}$, we there is some $\tau \in \Gamma$ with $\psi_i \in \tau$. But then $\Box(\hat{\Gamma}; \bigvee \hat{\Gamma}) \in \tau$. To show that $\Box(\psi_1, ..., \psi_k; \varphi) \in \tau$ it now suffices to show the following:

$$\vdash \Box(\hat{\Gamma}; \bigvee \hat{\Gamma}) \rightarrow \Box(\psi_1, ..., \psi_k; \varphi)$$

But since $\varphi \in \bigcap \Gamma$ we clearly have $\vdash \bigvee \hat{\Gamma} \rightarrow \varphi$, and if $\psi_j \in \tau$ then $\models \tau \rightarrow \psi_j$. We now only have to apply (NW), (SW) and (AD) to obtain the required implication, and so we are done. $\Box$

Theorem 5.7 (Completeness)**.** *For any formula φ, if $\models \varphi$ then $\vdash \varphi$.*

Proof. Suppose $\nvdash \varphi$, so that $\neg\varphi$ is consistent. Let i be the modal depth of φ and let P be the propositional variables that occur in φ. Construct the model $\mathfrak{M}$ as above. By Lemma 5.4, there is an i-type τ with $\neg\varphi \in \tau$, hence $\varphi \notin \tau$, and by the Truth Lemma, we have $(\mathfrak{M}, \tau) \nvDash \varphi$. Therefore $\nvDash \varphi$, which completes the proof. $\Box$

This completeness proof was inspired by normal form techniques from [14, 23], suitably adapted to modal neighborhood semantics. We have been unable to find a more standard Henkin construction to establish our main result, and leave this as an open problem.

Finally, our proof method clearly produces finite counter-examples to non-derivable formulas φ, whose size can be effectively computed from that of φ. Therefore we have established an effective finite model property, and in particular:

Corollary 5.8 (Decidability)**.** *Derivability and validity for the logic* INL *are decidable.*

6. Semantic tableau system

We have now given a semantic and a proof-theoretic analysis of INL, even including its decidability. But the true combinatoric behavior of a logical system often comes out more concretely with a *semantic tableau system* that allows for controlled search of counter-examples and proofs, and even for automated deduction analysis. Virtues of tableaux also include providing cut-free Gentzen calculi for logics, and their attendant combinatorial methods.

However, since tableau systems are notorious for requiring many syntactic book-keeping details, we have divided this material into two. In this section, we sketch a tableau system for INL, and state its basic properties. For further details of the actual mechanism, based on the format of Fitting [15], we refer to the Appendix at the end, which contains more information on the definition of tableaux, including various kinds of nodes and subtrees, plus the complete set of rules producing these. The Appendix also presents some concrete examples. To understand what follows, it is enough if the reader has some understanding of classical and modal tableau systems (cf. [6, 15, 18]).

6.1. **Modal tableau rules and validity reductions.** Tableaux for modal logics search for counterexamples using standard decomposition rules for the Boolean operators, but the heart of what they do is the tableau rule that describes decomposition of modalities. Typically, at a tableau node where all Boolean rules have been applied, one has some proposition letters to made true and others to be made false, while some modalized formulas are to be made true and some others false. In this case, the tableau rule creates a subtree for a counterexample, since more than one accessible world or neighborhood may need to be created for the world corresponding to the current node. In the description of the starting nodes for the successors in the subtree, one layer of modalities has disappeared, so syntactic complexity goes down, and termination of tableaux is assured.

Before we state the key modal tableau rule for our INL system, we sketch some background putting it in perspective. It is known that the related simpler systems of modal logic over neighborhood models or over relational models satisfy some very strong reductions of validity that match their tableau rules. As a warm-up then, consider sequent validity, with antecedents read conjunctively, and consequents disjunctively as usual. The following results are in [6].

Proposition 6.1. *Let $p_1, \dots p_n, q_1, \dots, q_m$ be propositional variables and let $\varphi_1, \dots, \varphi_n$ $\psi_1, \dots, \psi_m$ be modal formulas. Then*

(1) *Over relational models, $p_1, \dots p_n, \Box\varphi_1, \dots, \Box\varphi_n \models q_1, \dots, q_m, \Box\psi_1, \dots, \Box\psi_m$ iff $\vec{p} \cap \vec{q} \neq \emptyset$, or there is some $i \leq m$ such that $\vec{\varphi} \models \psi_i$.*

(2) *Over neighbourhood models, $p_1, \dots p_n, \Box\varphi_1, \dots, \Box\varphi_n \models q_1, \dots, q_m, \Box\psi_1, \dots, \Box\psi_m$ iff $\vec{p} \cap \vec{q} \neq \emptyset$, or there are $i \leq m$ and $j \leq n$ such that $\varphi_j \models \psi_i$.*

Note what this says. For the basic normal modal logic, modal consequences only hold when, right underneath the top boxes, the antecedents together imply at least one consequent. For the basic modal logic on neighborhood models, something even more startling holds: a modalized consequence holds only if, right underneath the top boxes, some antecedent implies some consequent. [1]

While we have stated these principles as validity reductions, they do have an immediate connection to tableau rules. Read in one direction, they drive a tableau rule for finding counter-examples. If the modalized sequent consequence is invalid, then the negation of its reduction statement provides a tableau decomposition rule. But also, if a tableau closes, and we can assume inductively that the sequents in the reduction statement are already derivable, then the original modalized sequent is derivable in a cut-free Gentzen system read off from the tableau system.

There is a similar reduction principle for our system INL, but it is much more complex syntactically, owing to the combination of distributive and merely monotonic behavior of arguments in

[1] That is, modalized antecedents cannot even work together to produce significant conclusions, an extreme form on 'non-distribution' of modalities over conjunctions.

our modalities. In this case, we merely display the essential modal tableau rule, where we use some technical terminology about book-keeping nodes whose details need not concern us here:

Modal tableau rule. If a path has an unmarked node r labeled by $\Box(\alpha_1, ..., \alpha_j; \alpha_0)$ and nodes $r^1, ..., r^k$ such that, for each $m \in \{1, ..., k\}$, the node r^m is labeled by the formula $\neg\Box(\beta_1^m, ..., \beta_{j_m}^m; \beta_0^m)$, then do the following: Put a mark at each unmarked regular node on the path for that path, and for each $I \in \prod_{i=1}^{k}\{0, 1, ..., j_i\}$, add a hyper node h_I together with a hyper edge from the end node of the path to h_I, where h_I is

$$\left\{\alpha_0 \wedge \sigma \wedge \bigwedge_{I(i)\neq 0} (\beta_0^i \wedge \neg\beta_{I(i)}^i) \,:\, \sigma \in \{\alpha_1, ..., \alpha_j\} \cup \{\neg\beta_0^i \,|\, I(i) = 0\}\right\}.$$

A tableau can be seen as an attempt to refute the root formula, and is considered succesful if each branch closes. Thus we should understand the modal rule scheme as follows: if we have some unmarked node labelled $\Box(\alpha_1, ..., \alpha_j, \alpha_0)$, and a set of unmarked nodes $r_1, ..., r_k$ with each r_m labelled $\neg\Box(\beta_1^m, ..., \beta_{j_m}^m), \beta_0^m)$, then intuitively we want to show that *every* attempt to create a neighborhood witnessing truth of the formula $\Box(\alpha_1, ..., \alpha_j, \alpha_0)$, in a way which is consistent with the information $\neg\Box(\beta_1^m, ..., \beta_{j_m}^m), \beta_0^m)$ for $m \in \{1, ..., m\}$, must fail. So we consider all the possibilities in which such a neighborhood could be constructed, and create a hyper-node corresponding to each of the available options with the aim of constructing a closed sub-tableau from each of these nodes. This is where the combinatorial heart of the tableau system lies: the modal rule tells us that we have one possibility to consider for each $I \in \prod_{i=1}^{k}\{0, ..., j_i\}$, in the sense that each such I corresponds to one way in which a neighborhood U of the required sort might be created, and it tells us precisely what formulas should be true at each of the worlds that U is composed of.

6.2. **Properties of the tableau system.** As we shall show in our Appendix, the tableau system for INL based on the preceding rule satisfies the usual meta-theoretical adequacy properties, where we use the obvious notions of tableaux being closed or staying open. [2]

Theorem 6.2. *Open* INL *tableaux contain a subtree structure that induces a neighborhood model which satisfies the formula placed at the top node.* [3]

Next, there is also a well-known connection with proofs for valid laws.

Theorem 6.3. *Closed* INL *tableaux can be transformed effectively into Gentzen-style sequent proofs for the negations of the formulas placed at their top node.*

For proofs of these results, which involve more combinatorial details than those for standard modal tableaux, we refer to the Appendix. In combination with the fact that tableaux either close or stay open, this results in adequacy of the algorithm that constructs tableaux.

Theorem 6.4. *An* INL *formula is satisfiable if and only if none of its tableaux are closed. A formula is valid if and only if its negation has a closed tableau.*

The usual surplus of the tableau-based Gentzen proof system is its cut-free nature, which also leads to proofs of interpolation theorems, that we do not pursue here.[4]

However, there is also an obvious issue of how tableau-based proofs relate to the complete axiomatic proof system that we have defined and studied in the preceding sections. The two

[2]We already stated that tableaux terminate, unlike the case for first-order logic that admits infinite branches.

[3]Beth's original tableau format has a list of formulas to be made true, and a list for formulas to be made false.

[4]We even suspect that there are concrete links between our tableau calculus and the proof systems for coalgebraic logic studied in [26]. For more connections with coalgebraic approaches see Section 8.4.

formats can be transformed effectively into each other, but for details, again, we refer to the Appendix which proves all of the results mentioned here in a parsimonious way.[5]

7. Translations and complexity

Our new logic extends earlier systems of modal logic, but what is the precise formal relation? The answer lies in a number of faithful translations. In this section we show how to embed the basic modal logic K and the basic neighbourhood logic E into INL. We also embed INL into the bimodal logic $\mathsf{K} \oplus \mathsf{K}$ and eventually even into K itself.

All the proofs to follow are based on semantic arguments, and could be used as semantic validity reductions. However, as all the aforementioned logics are complete with respect to the corresponding (relational or neighbourhood) semantics, we formulate our results in proof-theoretic terms.

Recall that K is the basic modal logic. For the next translation it is easier to use the $\Diamond$-presentation of the basic modal language. Later in this section we will also use $\Box$-presentations. We define a translation δ from the basic modal language into the language of INL as follows:

- $\delta(p) = p$ for each propositional variable p.
- δ preserves Boolean connectives.
- $\delta(\Diamond\varphi) = \Box(\delta(\varphi); \top)$.

Theorem 7.1. *For each formula α of basic modal language we have:*

$$\mathsf{K} \vdash \alpha \text{ iff } \mathsf{INL} \vdash \delta(\alpha).$$

Proof. Suppose $\mathsf{K} \nvdash \alpha$. Then by the completeness of K with respect to relational models, there is a relational model $\mathfrak{M} = (W, R, V)$ and $w \in W$ such that $\mathfrak{M}, w \not\models \alpha$. For each $x \in W$ we let $N(x) = \{R[x]\}$. Then $\mathfrak{M}' = (W, N, V)$ is a neighbourhood model. It is easy to show by induction on the complexity of formulas that for each β in the basic modal language we have

$$\mathfrak{M}, w \models \beta \text{ iff } \mathfrak{M}', w \models \delta(\beta).$$

So $\mathfrak{M}', w \not\models \delta(\alpha)$, and $\mathsf{INL} \nvdash \delta(\alpha)$.

Conversely, suppose $\mathsf{INL} \nvdash \delta(\alpha)$. Then by the completeness of INT with respect to neighbourhood models, there is $\mathfrak{M} = (W, N, V)$ and $w \in W$ such that $\mathfrak{M}, w \not\models \delta(\alpha)$. We define R on W by setting for each $x, y \in W$, xRy if $y \in U$ for some $U \in N(x)$. Then $\mathfrak{M}'' = (W, R, V)$ is relational model. It is easy to show by induction on the complexity of formulas that, for each β in the basic modal language, we have [6]

$$\mathfrak{M}'', w \models \beta \text{ iff } \mathfrak{M}, w \models \delta(\beta).$$

Thus, $\mathfrak{M}'', w \not\models \alpha$ and $\mathsf{K} \nvdash \alpha$. $\Box$

Recall that E is the basic neighbourhood logic. We consider the following translation τ from the basic modal language into the language of INL:

- $\tau(p) = p$ for each propositional variable p,
- τ preserves Boolean connectives,
- $\tau(\Box\varphi) = \Box(\emptyset; \tau(\varphi))$.

Lemma 7.2. *For any neighbourhood model $\mathfrak{M} = (W, N, V)$ and $w \in W$ we have*

$$\mathfrak{M}, w \models \alpha \text{ iff } \mathfrak{M}, w \models \tau(\alpha).$$

Proof. An easy induction on the complexity of α. $\Box$

Theorem 7.3. *For each formula α of basic modal logic we have:*

[5] Behind this result lies a Cut Elimination Theorem for INL that we do not spell out in this paper.

[6] We emphasize that the crux of this construction is as follows: the relation R defined above works thanks to the very special nature of the translated formulas that the model has to preserve.

$$\mathsf{E} \vdash \alpha \textit{ iff } \mathsf{INL} \vdash \tau(\alpha).$$

Proof. Suppose $\mathsf{E} \nvdash \alpha$. Then by the completeness of E with respect to neighbourhood models, there is a neighbourhood model $\mathfrak{M} = (W, N, V)$ and $w \in W$ such that $\mathfrak{M}, w \not\models \alpha$. By Lemma 7.2 $\mathfrak{M}, w \not\models \tau(\alpha)$. So $\mathsf{INL} \nvdash \tau(\alpha)$. The converse direction is similar and follows from the completeness of INL with respect to neighbourhood models. □

It is important to see that the above result is not trivial. E is weaker than INL. So it is not surprising that there is an embedding of E in INL. But the crucial further point is that our embedding is also faithful.

Let $\mathsf{K} \oplus \mathsf{K}$ be the fusion of K with itself. We define the following translation of the language of INL into the bimodal language with two unary modalities.[7] Let

- $\sigma(p) = p$ for each propositional variable p.
- σ commutes with the Boolean connectives.
- $\sigma(\Box(\psi_1, \dots, \psi_n; \varphi)) = \Diamond_1((\Diamond_2\sigma(\psi_1) \wedge \cdots \wedge \Diamond_2\sigma(\psi_n)) \wedge \Box_2\sigma(\varphi))$.

Theorem 7.4. *For each formula α in the language of INL we have:*

$$\mathsf{INL} \vdash \alpha \textit{ iff } \mathsf{K} \oplus \mathsf{K} \vdash \sigma(\alpha).$$

Proof. Suppose $\mathsf{INL} \nvdash \alpha$. Then by the completeness of INL with respect to neighbourhood models, there is a neighbourhood model $\mathfrak{M} = (W, N, V)$ and $w \in W$ such that $\mathfrak{M}, w \not\models \alpha$. We let $X = W \cup \mathcal{P}(W)$. For each $x, y \in X$ we set xR_1y if $x \in W$, $y \in \mathcal{P}(W)$ and $y \in N(x)$ and xR_2y if $x \in \mathcal{P}(W)$, $y \in X$ and $y \in x$. We also let $V'(p) = V(p)$. Then $\mathfrak{M}' = (X, R_1, R_2, V')$ is a birelational model of bimodal logic. Moreover, by an easy induction on the complexity of formulas one can show that for each formula β in the language of INL we have

$$\mathfrak{M}, w \models \beta \text{ iff } \mathfrak{M}', w \models \sigma(\beta).$$

Therefore, $\mathfrak{M}', w \not\models \sigma(\alpha)$ and $\mathsf{K} \oplus \mathsf{K} \nvdash \sigma(\alpha)$.

Conversely, suppose $\mathsf{K} \oplus \mathsf{K} \nvdash \sigma(\alpha)$. Then by the completeness of $\mathsf{K} \oplus \mathsf{K}$ with respect to birelational models, there is a birelational model $\mathfrak{M} = (W, R_1, R_2, V)$ and $w \in W$ such that $\mathfrak{M}, w \not\models \sigma(\alpha)$. For each $x \in W$ we let $N(x) = \{U : U = R_2[y] \text{ for some } y \in W \text{ with } xR_1y\}$. Then $\mathfrak{M}' = (W, N, V)$ is a neighbourhood model. Moreover, by an easy induction on the complexity of formulas one can show that for each formula β in the language of INL we have

$$\mathfrak{M}', w \models \beta \text{ iff } \mathfrak{M}, w \models \sigma(\beta).$$

Thus, $\mathfrak{M}', w \not\models \alpha$ and so $\mathsf{INL} \nvdash \alpha$.

□

The preceding translation is in line with the natural two-sorted nature of neighborhood models when viewed as generalized Henkin models for second-order logic. However, we can do better in a way that may be surprising to the reader.

Consider the special case of σ when $\Diamond_1 = \Diamond_2$, and write this modality simply as $\Diamond$, while still using σ for the translation. Then σ embeds the language of INL into the uni-modal language.

Theorem 7.5. *For each formula α in the language of INL we have:*

$$\mathsf{INL} \vdash \alpha \textit{ iff } \mathsf{K} \vdash \sigma(\alpha).$$

Proof. The proof is similar to the proof of Theorem 7.4. For the right to left direction, with any neighbourhood model $\mathfrak{M} = (W, N, V)$, as above, we associate a relational model (X, R, V'), where $R = R_1 \cup R_2$.[8] Conversely, given a relational model (W, R, V) we construct a neighbourhood model (W, N, V), as above, by assuming $R_1 = R_2 = R$. The rest is a matter of a routine verification. □

[7] A similar translation for the basic neighborhood language can be found in Parikh [24], and van Benthem [5].

[8] Here, technically, we consider worlds and sets of worlds as disjoint sets of objects, to avoid confusion.

Translations go only so far. The preceding result may look like equating INL with the language of normal modal logic, but this is not quite right. The special semantic and proof-theoretic behavior of instantial neighborhood logic only comes out when we study its concrete presentation.

Still, syntactic translations often do give useful information. For instance, the above observations allow us to determine the precise computational complexity of our new neighborhood logic.

Corollary 7.6. *The complexity of the satisfiability problem for* INL *is* PSpace*-complete.*

Proof. We "sandwich" INL between two PSpace-complete logics. That the satisfiability problem for INL is in PSpace follows from Theorems 7.4 or 7.5, since satisfiability for K and $\mathsf{K} \oplus \mathsf{K}$ is in PSpace [10, 16]. That it is PSpace-hard follows from Theorem 7.1, since K is PSpace-hard [10]. □

However, not every special feature of our logic follows automatically by translations. For instance, our results on bisimulation and on axiomatic completeness required additional serious labor.

8. Further directions

This paper has presented the modal basics of the system INL of instantial neighborhood logic. Even so, we have not exhausted all standard topics in modal logic.

One such topic is *correspondence theory* for special INL axioms on neighborhood frames, which would require an extension of the correspondence analysis in [19].[9] Another standard topic would be algebraic analysis and extensions of modal algebras with operations that are monotonic in some arguments and distributive in others. See [12] and the references therein for generalizations of modal algebra in this direction.

However, in this final section, we list some further directions that may be slightly less obvious, with illustrations of what lies ahead, although we cannot pursue these directions in depth in the short compass of this paper. We start with a few concretizations of INL, and after that, we consider language redesign, as well as other more general issues.

8.1. **Specializing to concerete settings, reductions and expansions.**

The system INL presented in this paper is a general modal logic over arbitrary neighborhood structures. Yet, it is also of interest to see what becomes of it in more concrete settings. We consider a few such cases.

Ordinary relational semantics. As we have seen already in our Section 7 on translations, every standard relational model $\mathfrak{M} = (W, R, V)$ induces a neighborhood structure as follows. For each world w, we set NwX if $R[w] \subseteq X$. In these special neighborhood models, our INL modality $\Box(\varphi; \psi)$ becomes definable as follows, where on the right-hand side, the box modality is standard relational, and E is the existential modality over any worlds in the model, accessible or not.

Proposition 8.1. *Let* $\mathfrak{M}$ *be as above. For each* $w \in W$ *and any* INL*-formula* $\varphi_1, \ldots, \varphi_n, \psi$ *we have*

$$\mathfrak{M}, w \models \Box(\varphi_1, \ldots, \varphi_n; \psi) \leftrightarrow \Box\psi \wedge \bigwedge_{i=1}^{n} E(\varphi_i \wedge \psi)$$

Proof. The direction from left to right is obvious by the truth definition for the INL modality. From right to left, we have that is true in all R-successors of the current world, and if we add just k worlds satisfying the $\varphi_i \wedge \psi$ $(1 \leq i \leq k)$ to this set then we obtain a neighborhood of the current world that witnesses $\Box(\varphi_1, \ldots, \varphi_n; \psi)$ as required. □

What we see in this special case is that INL is definable in the basic logic of $\Box$ and E. And something even stronger holds.

Proposition 8.2. *Over relational models,* INL *is equivalent with the basic logic of* $\Box$ *and* E*.*

[9]For instance, a Sahlqvist theorem now gets harder because of the underlying $\Diamond\Box$ logical form of antecedents with INL modalities: even one iteration will produce modality sequences going beyond the usual Sahlqvist syntax.

Proof. The existential modality $E\varphi$ can be defined as $\Box(\varphi;\top)$. □

The reason why the preceding definition works is that the whole set of worlds W is a neighborhood of every world, while neighborhoods are also closed under supersets. We will look at further reasons for reducibility after looking at one more example in the next section.[10]

Topological semantics. Next consider another well-known source for neighborhood models: topological spaces.

Proposition 8.3. *Over topological spaces,* INL *is definitionally equivalent with* $\mathrm{ML}(\Box, E)$.

Proof. The existential modality E in INL is defined as in Proposition 8.2, while $\Box\varphi = \Box(\emptyset;\varphi)$. Conversely, in $\mathrm{ML}(\Box, E)$ over topological spaces, the INL modality $\Box(\varphi_1,\ldots,\varphi_n;\psi)$ is definable as

$$E(\varphi_1 \wedge \Box\psi) \wedge \cdots \wedge E(\varphi_n \wedge \Box\psi) \wedge \Box\psi.$$

It is easy to show that this reduction holds, and this time, the main reason behind it is the fact that topological opens are closed under unions – while also the whole space is an open set. □

We note that $\mathrm{ML}(\Box, E)$ is a natural logic for describing topological structure [1], and INL may be viewed as an interesting notational variant. Another way of seeing the close connection is this observation that we state without proof:

Proposition 8.4. *On topological models, the* INL*-bisimulations of this paper are precisely the total topo-bisimulations in the sense of* [1].

Yet other specializations. Neighborhood structures abound in many places. INL can also be studied in specific areas such as agency, including logics for belief revision based on partially ordered similarity spheres (Girard [17]), or logics for players' powers in games (Parikh [24]). In the latter case, neighborhoods of the root in a game are those sets of endpoints that a player can force the game to end in by playing one of her strategies against any play by the opponent. On this interpretation, one thinks of minimal such sets: each point in it represents an actual outcome of the game that one's opponent could achieve by following a suitable strategy of his own. In that case, the above structural conditions on neighborhoods for reducing the INL modality fail.[11]

In such settings of non-reducibility, we can also think differently about the utility of INL. Consider any logic that has been proposed over neighborhood models, but using only the basic modal language. Have we perhaps missed generality, and could there be extended completeness theorems or other relevant results if we introduce the full language of INL?

One such case is the game logic of players' powers we mentioned earlier (see the extended exposition in van Benthem [7]). Its standard "power modalities" $\{G, i\}\varphi$ say that in game G, player i has a strategy forcing the game to end in a set of outcomes all of which satisfy φ, whatever the other players do. This emphasizes a power of player i. However, it is natural here to also introduce an extended INL notation $\{G, i\}(\varphi;\psi)$ that gives information about limits to the player's power: namely, what might still happen at the end of the game, which is a mixture of the player i exercising her strategic powers while other players also influence outcomes by choosing moves.

Proposition 8.5. *Dynamic game logic with* INL *modalities is completely axiomtizable.*

Proof. It is easy to adapt the usual reduction axioms of dynamic game logic, that work for the basic game constructions $G \cup G'$ (choice), $G; G'$ (sequence) and G^d (dual), to full INL versions. □

[10] We could also reformulate the definability results in this subsection in terms of syntactic translations between logics, whose definition will be clear from our text.

[11] There might still be reductions for the INL modality here if we add further reachability modalities on game trees.

Remark 8.6. We do not want to suggest that logics of agency are our only or even main paradigm for INL. For instance, neighborhood structures also arise naturally in the form of *hypergraphs*, that is, families of subsets of a domain, where each set is a "generalized arrow" [9, 29]. [12] In this setting, once more, no obvious reducibilities arise to other modal logics, and the INL fragment of hypergraph theory may be well-worth exploring.

8.2. **Redesigning the basic language.** The language of INL is not sacrosanct, and there are options.

Fragments. A more linguistic way of specializing the logic is by looking at fragments. While the full language of INL allows arbitrary finite sets of instances, a natural restriction would be to having just one *single instance*, in a format $\Box(\varphi;\psi)$. It is easy to see that the main proofs in this paper specialize to this fragment, so we have completeness, and even Pspace-complexity, since the lower bound reduction that we gave only uses the single-instance fragment.

Of slightly greater interest is the matching notion of bisimulation Z. Here we modify Definition 2.5 as follows.

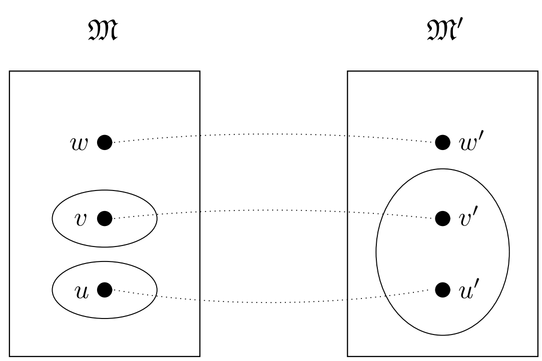
Definition 8.7. *Given two models $\mathfrak{M}$ and $\mathfrak{N}$, a relation $Z \subseteq W \times W'$ is said to be a* single-instance bisimulation *if, whenever wZw':*

1. $\mathfrak{M}, w \models p$ *iff* $\mathfrak{M}', w' \models p$ *for each proposition letter p,*
2. $\forall S \in N(w) \forall t \in S\ \exists S' \in N'(w') \begin{cases} \forall s' \in S' \exists s \in S(sZs') \text{ and} \\ \exists t' \in S'(tZt') \end{cases}$
3. $\forall S' \in N'(w') \forall t' \in S'\ \exists S \in N(w) \begin{cases} \forall s \in S \exists s' \in S'(s \rightleftharpoons_Z s') \text{ and} \\ \exists t \in S(tZt') \end{cases}$

We can easily prove an invariance result for the single-instance fragment of INL with respect to single-instance bisimulations, and we leave the details to the reader. With this invariance result in place, we can prove an undefinability result showing that INL has strictly greater expressive power than the single-instance fragment:

Proposition 8.8. *The single-instance fragment cannot define the two-instance formula $\Box(p,q;r)$.*

Proof. Consider the models $\mathfrak{M} = (W,N,V)$ and $\mathfrak{M}' = (W',N',V')$ such that $W = \{w,v,u\}$, $W' = \{w',v',u'\}$, $N(w) = \{\{v\},\{u\}\}$, $N(w') = \{\{v',u'\}\}$, $V(p) = \{v\}$, $V(q) = \{u\}$, $V'(p) = \{v'\}$, and $V'(q) = \{u'\}$ (see figure below). Then it is easy to see that w and w' are single-fragment INL-bisimilar (consider a relation linking w with w', v with v' and u with u'). Consequently they satisfy the same formulas in the single-fragment of INL. However, they are not INL-bisimilar. In particular, it is easy to see that $\mathfrak{M}', w' \models \Box(p,q;\top)$, but $\mathfrak{M}, w \not\models \Box(p,q;\top)$.



[12] Hypergraphs as generalized graphs have their origins in mathematics, but they have also been proposed for modeling 'hyperintensionality' in philosophy: Leitgeb [21].

□

Similarly, we could define the "k-instance fragment" of INL for each positive integer k, and the proof we gave above could be modified without trouble to prove that INL does not collapse into any of its k-instance fragments. In other words, the k-instance fragments of INL form a strict hierarchy.

Language extensions. Going in the opposite direction, as in the case of normal modal logics over relational models, various language extensions make sense. In particular, it is easy to see that our bisimulations preserve much more than just our base language. We can also add Kleene iteration, or even a full propositional dynamic logic version of our language with a family of atomic neighborhood relations and complex operations over these. Van Benthem [3] has several results in this setting, including a discussion of operations forming new neighborhood relations that are "safe" for basic neighborhood simulations. It would be of interest to extend this analysis to our extended INL bisimulations. Another line of extension would be a mu-calculus over INL, which would be a natural extension of current work on mu-calculus over basic neighborhood models [24, 13].

There are also less obvious extensions. For instance, we can translate INL into a *two-sorted first-order logic*, with one sort for points and one sort for neighborhoods, plus two distinguished relations: one for the neighborhood relation, and one for the (abstract) element relation. This framework would allow us to formulate a preservation theorem complementing our bisimulation analysis, showing how INL is the bisimulation-invariant fragment of a suitable first-order logic for neighborhood models. (But cf. Pauly [25] for possible difficulties along this line.) [13]

Dynamics of model change A further important language extension goes right back to the original motivation for introducing INL. Van Benthem and Pacuit [8] introduces "evidence models" where the neighborhood relation links worlds to sets of worlds that are the output of some device generating evidence. This evidence can be of any sort, no structural closure conditions are imposed, and it is even allowed, for instance, that two evidence sets are disjoint: the sources then contradict each other. The original static evidence logic had a standard neighborhood modality describing what pieces of evidence are available, as well as a universal modality standing in, to a first approximation, for knowledge that the agent has.

But in inquiry, evidence is in constant flux: it can be added, deleted, or modified. Van Benthem and Pacuit [8] therefore introduce 'dynamic modalities' that refer to what is true in a model after evidence has changed. In particular, there are three basic operations for such change.

First of all, an event $!\varphi$ of *hard information* that φ is the case intersects all available evidence sets with $[[\varphi]]$ and keeps all non-empty ones. A dynamic modality $[!\varphi]\psi$ then says in the current model $(\mathfrak{M}, s)$ that φ is true at the current world s after this change has taken place, producing an updated model $\mathfrak{M}!\varphi$. Now the heart of an axiomatization for such a dynamic modality is a "recursion law" that states what new static neighborhood modalities will hold after the update. And significantly, this recursion law leads to an extension of the basic neighborhood language with a conditionalized neighborhood modality:

$$[!\varphi]\Box\psi \leftrightarrow \Box^{\varphi}[!\varphi]\psi.$$

The latter says that there was evidence compatible with φ such that ψ holds after the $!\varphi$ update. But $\Box^{\varphi}\alpha$ is of course a formula from the single-instance fragment of INL: $\Box(\varphi;\alpha)$. Indeed, the following can be shown:

Proposition 8.9. *The closure of the basic neighborhood language under* $[!\varphi]$ *is contained in the single-instance fragment of* INL*, and this fragment is closed under* $[!\varphi]$*.*

We have not been able to determine yet whether the single-instance fragment of INL is in fact the closure of the basic neighborhood language under the modality $[!\varphi]$.

[13]There might even be natural *guarded fragments* on neighborhood models in this first-order setting.

A similar analysis can be given for a second basic operation $+\varphi$ of *soft information* that φ is the case which adds the set of φ-worlds as an evidence set to the current model.

However, the origins of full INL in this setting arise with the third basic dynamic operation $-\varphi$, that of *retracting* a proposition φ, which removes all evidence for φ, branch and root. What this does is remove all evidence sets from the model that are contained in $[[\varphi]]$. Now, when we write a recursion axiom for the basic neighborhood modality $[-\varphi]\Box\psi$, we need to state the existence of evidence sets in the original model that are compatible with $-\varphi$, and that validate $[-\varphi]\psi$. This is in the single-instance fragment of INL. But this time, when we apply the dynamic retraction modality once more to close the system, we need two instances, as is easily seen when analyzing a formula $[-\varphi]\Box(\alpha;\psi)$. In general, we then get:

Proposition 8.10. *The closure of the basic neighborhood language under $[-\varphi]$ is contained in INL, and INL is closed under applying the dynamic modality $[-\varphi]$.*

Again, we have an open problem:

Question 8.11. *Is INL equal to the closure of the basic neighborhood language under the dynamic retraction modality?*

This would be of great interest, since it would motivate a static modal language like INL in terms of dynamic considerations of model change. However, we have no proof for this assertion, and our current conjecture is that the answer to the question is negative. In that case, dynamic extensions of INL might well be *sui generis*.[14]

8.3. **Further model theory and infinitary INL languages.** Our analysis of INL-bisimulation can be extended in a standard style, involving yet one more language extension. This time consider an infinitary INL, allowing arbitrary set conjunctions and disjunctions, and arbitrary sets of instances in the instantial part of the INL-modality. The following results can be proved by straightforward adaptations of results for modal logic over relational models, keeping in mind our analysis of INL-bisimulations in Section 3.

Proposition 8.12. *INL-bisimulation preserves infinitary INL-formulas on all pointed models.*

Proposition 8.13. *For each model $\mathfrak{M}, s$, there exists an infinitary INL-formula $\delta(\mathfrak{M}, s)$ that defines $\mathfrak{M}, s$ up to INL-bisimulation.*

The proof of this result is a standard construction of so-called Scott formulas [20, 2] in standard semantics for first-order logic or modal logic. In particular, going along the ordinals, in the successor step $\alpha+1$, we enumerate INL descriptions up to depth α for all neighborhoods of the current world s, and add a statement that no other neighborhoods exist.[15]

Discussion. There are some interesting connections here with our completeness proof in Section 5. The normal form of disjunctions of complete types that we used there may be compared to the following statement in the present setting. Every infinitary INL-formula is equivalent to a disjunction of Scott formulas describing available types in the model. But note that the inductive step used in this model description is not of the form used earlier, but 'one level up'. Scott sentences describe what sort of neighborhoods are available, they do not confine themselves to describing what is exactly the case in one neighborhood. The proper setting for understanding these analogies and differences may be the coalgebraic perspective of Section 8.4 below. [16]

[14] As above, it is also natural to have several evidence relations, for instance, when modeling what evidence a group of agents has available, and what notions of collective evidence make sense.

[15] The size of the set of instances can be bounded by the cardinality of the model. The limit fixed-point argument producing the total description is as usual.

[16] For finite models, we can do better. The Scott construction stops at some finite stage. But there are alternative descriptions. One describes the given model in terms of unique proposition letters for each world, encoding the

8.4. **Co-algebraic perspectives.** Instantial neighborhood semantics has strong connections with coalgebra. First of all, readers familiar with universal coalgebra may have noticed that neighborhood frames are coalgebras for the double covariant powerset functor, $\mathcal{P} \circ \mathcal{P} : \mathbf{Set} \to \mathbf{Set}$, and more importantly our definition of a bounded morphism $f : \mathfrak{M} \to \mathfrak{M}'$ of neighborhood models requires precisely that the map f is a coalgebra morphism for the underlying neighborhood frames. The usual equation for coalgebra morphisms takes the form:

$$N' \circ f = \mathcal{P}\mathcal{P}f \circ N$$

where N, N' are the neighborhood maps of the two neighborhood frames, now viewed as coalgebras of the form $N : W \to \mathcal{P}\mathcal{P}W$ and $N' : W' \to \mathcal{P}\mathcal{P}W'$. The usual commutative square depicting this condition is displayed below:

$$\begin{tikzcd}
\mathcal{P}\mathcal{P}W \arrow[r, "\mathcal{P}\mathcal{P}f"] & \mathcal{P}\mathcal{P}W' \\
W \arrow[u, "N"] \arrow[r, "f"'] & W' \arrow[u, "N'"']
\end{tikzcd}$$

Our definition of INL-bisimulations now simply follows as a consequence of this coalgebraic analysis: the functor $\mathcal{P} \circ \mathcal{P}$ preserves weak pullback squares, and so comes equipped with a relation lifting known as the Barr extension. Generally, for a set functor T, the Barr extension $\overline{\mathsf{T}}$ of T assigns to every binary relation $R \subseteq X \times Y$ the "lifted" relation $\overline{\mathsf{T}}R \subseteq \mathsf{T}X \times \mathsf{T}Y$ given by:

$$(\alpha, \beta) \in \overline{T}R \text{ iff, for some } \gamma \in \mathsf{T}R : \mathsf{T}\pi_X(\gamma) = \alpha \text{ and } \mathsf{T}\pi_Y(\gamma) = \beta$$

where $\pi_X : R \to X$ and $\pi_Y : R \to Y$ are the projection maps. A T-bisimulation between T-coalgebras (X, f) and (Y, g) is then defined to be a relation $R \subseteq X \times Y$ such that:

$$(u, v) \in R \;\Rightarrow\; (f(u), g(v)) \in \overline{\mathsf{T}}R$$

For the special case of $\mathsf{T} = \mathcal{P} \circ \mathcal{P}$, this condition amounts to precisely the back-and-forth conditions used in our definition of bisimulations. We refer to [28] for more details.

Turning to the language, it is easy to see that the box modalities we use to construct formulas $\Box(\gamma_1, ..., \gamma_k; \psi)$ correspond to $k+1$-ary predicate liftings for $\mathcal{P} \circ \mathcal{P}$ in the usual sense. More interestingly, since $\mathcal{P} \circ \mathcal{P}$ preserves weak pullback squares we may also introduce a Moss-style nabla modality whose semantics is based on the Barr extension for $\mathcal{P} \circ \mathcal{P}$. This language is defined so that, for every finite collection $\Gamma_1, ..., \Gamma_k$ of finite sets of formulas, we may construct the formula

$$\nabla\{\Gamma_1, ..., \Gamma_k\}$$

which then gets the following semantics: given a neighborhood model $\mathfrak{M} = (W, N, V)$ and $w \in W$, we have $\mathfrak{M}, w \models \nabla\{\Gamma_1, ..., \Gamma_k\}$ if and only if:

Forth: For every $Z \in N(w)$ there is some Γ_i such that:
- For every $u \in Z$ there is some $\psi \in \Gamma_i$ with $u \models \psi$
- For every $\psi \in \Gamma_i$ there is some $u \in Z$ with $u \models \psi$

Back: For every Γ_i there is some $Z \in N(w)$ such that:
- For every $u \in Z$ there is some $\psi \in \Gamma_i$ with $u \models \psi$
- For every $\psi \in \Gamma_i$ there is some $u \in Z$ with $u \models \psi$

We plan to further explore this language, and its relation to the "box-modality" based language of neighborhood models presented here, in a future paper.

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accessibilities between them, and prefixing it all by one PDL iteration modality, or alternatively, a greatest fixed-point operator, [4]. We can also do this with existential second-order quantifiers over the world-defining p-predicates. In the modal case, this reduces to mu-calculus form by the Janin-Walukiewicz theorem. Is there an analogue for this result in neighborhood semantics? And how would such an analysis extend to infinite models?

9. Conclusions

In this paper we introduced *instantial neighbourhood logic* INL. This logic provides a new modal language and new semantics to reason about neighbourhood frames and models. We defined INL-bisimulations and showed that these bisimulations preserve the truth of INL-formulas and that for finite models the converse is also true. We also provided a sound and complete axiomatization for INL on neighbourhood frames. By embedding INL into well-known modal logics we showed that it is PSpace-complete. We also reviewed a number of other connections of INL with existing logical formalisms and outlined a number of future research directions in this area.

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10. Appendix: Tableau system

While we have given a semantic and a proof-theoretic analysis of INL, the true combinatorics of a logical system often come out more concretely with a tableau system that allows for automated analysis. Virtues of tableaux also include providing cut-free Gentzen calculi, and attendant combinatorial methods. In this Appendix, we sketch how a tableau system for INL works, and state its basic properties.

10.1. **Defining the tableaux.**

Tableau systems are notorious for requiring many syntactic bookkeeping details. The system to be presented here is in the line of the well-known textbook [15], but we will rigorously suppress details wherever possible.

Our tableau system $\mathcal{T}_{\mathsf{INL}}$ involves a collection of rule schemes working on tableau trees. Each 'regular node' (or node for short) of the tree is labeled by a formula. In addition, there are 'hyper nodes', finite (possibly empty) sets of regular nodes. Likewise, there are 'regular edges': directed links from a regular node to another, and 'hyper edges': directed links from a regular node to a hyper node. When extending a tableau, regular nodes/edges are created by applications of Boolean rules, and hyper nodes/edges are created by applications of modal rules. A series of regular nodes linearly linked by regular edges is called a path. Each regular node can be marked for each path it resides in, indicating that the node has been treated on that path. Defined as usual, paths and nodes can be open or closed. Specifically, a hyper node is closed if one of its elements is so. To find a counter-example for given formula φ, we start with a unmarked root regular node labeled by $\neg\varphi$, and then start applying rules creating more nodes and edges, of either sort, putting marks at nodes that have been treated.

There are four rule schemes in $\mathcal{T}_{\mathsf{INL}}$, three Boolean and one modal. The Boolean schemes are standard, namely, (a) suppression of double negations, (b) following up on a conjunction with both its conjuncts, and (c) splitting a tableau path for a true disjunction. The heart of the tableau system is the scheme for the instantial modality.

Modal rule scheme. If a path has an unmarked node r labeled by $\Box(\alpha_1, ..., \alpha_j; \alpha_0)$ and nodes $r^1, ..., r^k$ such that, for each $m \in \{1, ..., k\}$, the node r^m is labeled by the formula $\neg\Box(\beta_1^m, ..., \beta_{j_m}^m; \beta_0^m)$, then do the following: Put a mark at each unmarked regular node on the path for that path, and for each $I \in \prod_{i=1}^{k}\{0, 1, ..., j_i\}$, add a hyper node h_I together with a hyper edge from the end node of the path to h_I, where h_I is

$$\left\{\alpha_0 \wedge \sigma \wedge \bigwedge_{I(i)\neq 0} (\beta_0^i \wedge \neg\beta_{I(i)}^i) \,:\, \sigma \in \{\alpha_1, ..., \alpha_j\} \cup \{\neg\beta_0^i \mid I(i) = 0\}\right\}.$$

This scheme reflects the complexity of reasoning with instantial neighborhood modalities. [17]

Definition 10.1. *A tableau $\mathcal{T}$ is said to be* complete *if none of the tableau rules apply to any of its unmarked nodes. A branch of a tableau $\mathcal{T}$ is said to be* closed *if it contains some unmarked regular node labeled with p and some unmarked regular node labeled $\neg p$, for some propositional letter p.*

A formula φ is said to be provable *in the tableau system $\mathcal{T}_{\mathsf{INL}}$ if it is possible to construct a complete tableau with root labeled $\neg\varphi$, in which every branch is closed.*

A tableau can be seen as an attempt to refute the root formula, and is considered succesful if each branch closes. Thus we should understand the modal rule scheme as follows: if we have some unmarked node labelled $\Box(\alpha_1, ..., \alpha_j, \alpha_0)$, and a set of unmarked nodes $r_1, ..., r_k$ with each r_m labelled $\neg\Box(\beta_1^m, ..., \beta_{j_m}^m), \beta_0^m)$, then intuitively we want to show that *every* attempt to create a

[17] Analyzing the tableaux that arise in this way yields some facts that can help work with the system. For instance, no regular node can have both regular successors and hyper successors.

neighborhood witnessing truth of the formula $\Box(\alpha_1, ..., \alpha_j, \alpha_0)$, in a way which is consistent with the information $\neg\Box(\beta_1^m, ..., \beta_{j_m}^m), \beta_0^m)$ for $m \in \{1, ..., m\}$, must fail. So we consider all the possibilities in which such a neighborhood could be constructed, and create a hyper-node corresponding to each of the available options with the aim of constructing a closed sub-tableau from each of these nodes. This is where the combinatorial heart of the tableau system lies: the modal rule tells us that we have one possibility to consider for each $I \in \prod_{i=1}^k \{0, ..., j_i\}$, in the sense that each such I corresponds to one way in which a neighborhood U of the required sort might be created, and it tells us precisely what formulas should be true at each of the worlds that U is composed of.

No simple figure can display the tableau system in a perspicuous manner, but the reader can check some simple examples for herself. We recommend, for instance, searching for a proof (i.e., a failed refutation attempt) for the formula $\Box(;\alpha) \to \Box(\beta \vee \neg\beta; \alpha)$. This involves a special case where $j = 0$, where the tableau starts with its root label $\neg\neg(\Box(;\alpha) \wedge \neg\Box(\neg(\neg\beta \wedge \neg\neg\beta); \alpha))$. For an example of non-validity, one can write a tableau for refuting the formula $\neg\Box(\Box(\alpha;\beta); \neg\beta)$.

As in general tableau systems, sometimes the choice of the rule to be applied to a path is forced upon us, but sometimes also there are choices available. We do only one example in greater detail to show what can happen.

Example 10.2. We prove $\Box(\alpha \vee \theta; \theta) \wedge \Box(\alpha \vee \beta; \theta) \to \Box(\alpha; \theta) \vee \Box(\beta; \theta)$ in $\mathcal{T}_{\mathsf{INL}}$.

Writing the formula with $\neg, \wedge, \Box$, we have

$$\neg(\Box(\neg(\neg\alpha \wedge \neg\theta); \theta) \wedge \Box(\neg(\neg\alpha \wedge \neg\beta); \theta) \wedge \neg\Box(\alpha; \theta) \wedge \neg\Box(\beta; \theta)).$$

Starting a tableau with the negation of this formula and applying Boolean rule schemes, we get 4 unmarked regular nodes on the only path: n_1 with label $\Box(\neg(\neg\alpha \wedge \neg\theta); \theta)$, n_2 with label $\Box(\neg(\neg\alpha \wedge \neg\beta); \theta)$, n_3 with label $\neg\Box(\alpha; \theta)$, and n_4 with label $\neg\Box(\beta; \theta)$.

Since no Boolean rule schemes can be applied at this stage, we are forced to try the modal one. According to the modal rule scheme, exactly one of n_1 and n_2, as well as a subset of $\{n_3, n_4\}$ are active, and hence there are 8 options. We take n_2 with $\{n_3, n_4\}$ to proceed, and readers may try other options to see where they lead.

Taking n_2 with $\{n_3, n_4\}$, the modal rule scheme is applied with parameters $k = 2$ and $j_1 = j_2 = 1$, and hence we get 4 hyper successors.

hyper successors	number of regular nodes in	labels of regular nodes
$h_{(0,0)}$	3	$\left\{ \begin{array}{c} \theta \wedge \neg(\neg\alpha \wedge \neg\beta), \\ \theta \wedge \neg\theta, \\ \theta \wedge \neg\theta \end{array} \right\}$
$h_{(0,1)}$	2	$\left\{ \begin{array}{c} \theta \wedge \neg(\neg\alpha \wedge \neg\beta) \wedge \theta \wedge \neg\beta, \\ \theta \wedge \neg\theta \wedge \theta \wedge \neg\beta \end{array} \right\}$
$h_{(1,0)}$	2	$\left\{ \begin{array}{c} \theta \wedge \neg(\neg\alpha \wedge \neg\beta) \wedge \theta \wedge \neg\alpha, \\ \theta \wedge \neg\theta \wedge \theta \wedge \neg\alpha \end{array} \right\}$
$h_{(1,1)}$	1	$\{\theta \wedge \neg(\neg\alpha \wedge \neg\beta) \wedge \theta \wedge \neg\alpha \wedge \theta \wedge \neg\beta\}$

In each of these hyper nodes, there is obviously a regular node that can be closed by applying Boolean rule schemes on ($h_{(1,1)}$ requires a disjunctive branching). This closes the main path as well as the whole tableau.

10.2. **Termination and adequacy.** Our tableaux for INL must always terminate. This can be shown using the following auxiliary notion measuring syntactic complexity.

Definition 10.3 (Degree)**.** *Inductively define the* degree of a formula *as follows:*

- $dg(p) := 1.$
- $dg(\neg\varphi) := dg(\varphi) + 1.$

- $dg(\varphi \wedge \psi) := dg(\varphi) + dg(\psi) + 1$.
- $dg(\Box(\alpha_1, ..., \alpha_j; \alpha_0)) := (\sum_{i=0}^{j} dg(\alpha_i)) \cdot \omega$, *where ω is the minimal infinite ordinal.*

The degree of a regular node on a path *is* 0 *if that node is marked for that path, and the degree of its label otherwise. The* degree of a path *is the sum of the degrees of all the regular nodes on it.*

Theorem 10.4 (Termination Theorem)**.** *Starting with a node labeled by a formula, no matter witch strategy is employed, the procedure of constructing a tableau terminates.*

Proof. For each of Boolean rule schemes, an application either reduces the degree of a path, or breaks a path into two with lower degrees.

For the modal rule schemes, recall that in its definition, we have a bunch of α's and β's. Let e be the maximum of degrees enjoyed by α's and β's. An application of a modal rule puts marks on each unmarked node in a path for that path. Hence, the modal rule scheme can be applied only once on each path. Since Boolean rules do not put marks on nodes with $\Box$-prefixed or $\neg\Box$-prefixed labels, all active formulas for that application (where α's and β's occur) is unmarked when the rule is to be applied. So in one hand, that application defuses a path with degree at least $e \cdot \omega$. On the other hand, that application creates finitely many hyper nodes, each consists of finitely many regular nodes. That is to say, that application creates finitely many new paths each has exactly one nodes. Each of these regular nodes is labeled by a formula, which in turn is a (finite) Boolean combination of α's and β's. Hence degrees of these formulas, as well as these new paths, are strictly smaller than $e \cdot \omega$. In summary, an application of the modal rule scheme defuses a path, and feeds back with finitely many paths with lower degrees. $\Box$

Our next major result says that tableaux indeed test for the right semantic notion. There are two halves to this: (1) From the overall but failed attempt to construct a closed tableau from a root, we can collect information from open tableaux and construct a counter-model of the formula that labels the root; (2) Closed tableaux correspond to validity of the negation of the formula labeled at their root. It is possible to work these closed tableaux into a Gentzen-style sequent calculus for our logic INL, though we leave this task for another occasion.

For the first half, we have:

Theorem 10.5. *If φ is valid, then φ is provable in $\mathcal{T}_{\mathsf{INL}}$.*

This theorem is proved constructively by offering a counter-model of φ. Due to space limits, we give here only main ideas and refer to extended version of this paper for a complete proof.

The tableau system itself is semantically orientated. For a general idea, one images a regular node as a state and a hyper node as a neighborhood. Regular edges are totally Boolean, and hence a path displays Boolean facts of a state. As usual, branching of regular edges (by disjunction) displays multiple possibilities of Boolean facts of a state. A hyper edge from a regular node (state) to a hyper node (neighborhood) displays that the neighborhood is enjoyed by the state.

Following the above idea, one can extract a model from the assumption that all tableaux for a formula are open. Similar to tableau systems for modal logics, our modal rule scheme is destructive, as in each application, only one $\Box$-prefixed formula is active, whereas all $\Box$-prefixed formulas get inactivated on that branch. In general, applying this scheme loses information, and even if a tableau can be closed, one may miss that chance by choosing the wrong active formulas. [18]

Readers can try other options when applying the modal tableau scheme in Example 10.2 to see what may happen.

In order to overcome this difficulty, one should extend tableaux in such a way that best postpones applications of modal scheme, and once that scheme is to be applied on a path, one should make

[18]This observation is connected with the higher computational complexity of modal logic over propositional logic.

copies of the current tableau and apply parallel on each $\Box$-prefixed formula. If a tableau cannot be closed, then each of this parallel application leads to an open end, and offers counter-submodels via IH. In general, for each $\Box$-prefixed formula we need a counter-submodel typically for it.

For the second half, we show that refutability of negation of a formula in $\mathcal{T}_{\mathsf{INL}}$ implies axiomatical provability of the formula. Combined with soundness of INL, we get validity of the formula.

Theorem 10.6. *If φ is provable in $\mathcal{T}_{\mathsf{INL}}$, then $\vdash \varphi$ (in INL).*

Again, we briefly sketch the idea here, and leave the full proof for later.

Along with the development of a closed tableau, associate a (finite) set of formulas to each regular node. Then, by an induction, show that if a regular node is closed, then the associated set is inconsistent. The only interesting step is for the modal rule scheme, in which we need to show from inconsistencies of sub-goals to that of the main goal. In our formulation of the modal rule scheme, it is sufficient to show

$$\vdash \Box(\alpha_1, ..., \alpha_j; \alpha_0) \to \bigvee_{z=1}^{k} \Box(\beta_1^z, ..., \beta_{j_z}^z; \beta_0^z).$$

from

$$\forall I \in \prod_{i=1}^{k}\{0, ..., j_i\} \begin{cases} \exists l \in \{1, ..., j\}, \vdash \alpha_0 \wedge \alpha_l \wedge \bigwedge_{i\in\{1,...,k\}}^{I(i)\neq 0} (\beta_0^i \wedge \neg\beta_{I(i)}^i) \to \bot \quad \text{or} \\ \exists x \in \{1, ..., k\}, I(x) = 0 \text{ and } \vdash \alpha_0 \wedge \neg\beta_0^x \wedge \bigwedge_{i\in\{1,...,k\}}^{I(i)\neq 0} (\beta_0^i \wedge \neg\beta_{I(i)}^i) \to \bot \ . \end{cases}$$

This is done by another induction which is not presented here due to space limits.

Combining the above two theorems with INL soundness, we have the equivalence of (a) φ is INL provable; (b) $\neg\varphi$ has a closed $\mathcal{T}_{\mathsf{INL}}$ tableau; and (c) φ is valid. This gives tableau adequacy as well as an alternative proof of completeness.

We conclude with some statements about what this machinery achieves over and above what we had in our main text. Tableaux test for validity in a terminating precise mechanical manner, that is provably correct for the intended semantic interpretation of our logic INL. Tableaux also suggest an alternative proof theory based on Gentzen sequence rather than in Hilbert style. Indeed, it would be a very interesting proof-theoretic exercise to provide direct back and forth translations between axiomatic proofs in our earlier sense and closed tableaux.

Moreover, there is also a computational benefit to the present approach. While at work, tableaux break down validity problems into Boolean combinations of simpler validity problems of lower degree (in some suitable sense of the term 'degree'). Indeed, as we remarked in Section 6, it would be possible to work this into a reduction of valid sequences like the ones that we displayed in Section 2 for the basic modal neighborhood logic, and the basic normal modal logic on relational models. This provides another proof of the computational complexity for our system that we found indirectly via translation in Section 7, but it may also be the key to computational implementations.