Generating Picard modular forms by means of invariant theory

Cléry, F.; van der Geer, G.

DOI
10.48550/arXiv.2110.00849
10.4310/PAMQ.2023.V19.N1.A6

Publication date
2023

Document Version
Submitted manuscript

Published in
Pure and Applied Mathematics Quarterly

License
CC BY

GENERATING PICARD MODULAR FORMS BY MEANS OF INVARIANT THEORY

FABIEN CLÉRY AND GERARD VAN DER GEER

Dedicated to Don Zagier on the occasion of his 70th birthday

ABSTRACT. We use the description of the Picard modular surface for discriminant -3 as a moduli space of curves of genus 3 to generate all vector-valued Picard modular forms from bi-covariants for the action of GL_2 on the space of pairs of binary forms of bidegree $(4,1)$. The universal binary forms of degree 4 and 1 correspond to a meromorphic modular form of weight $(4,-2)$ and a holomorphic Eisenstein series of weight $(1,1)$.

1. INTRODUCTION

Some Shimura varieties can be interpreted as moduli spaces of curves and such an interpretation offers extra ways to study these Shimura varieties. More precisely, in a number of cases a dense open part of the Shimura variety is the image of a moduli space of curves under a morphism of finite degree. Examples are the moduli space of principally polarized abelian varieties of dimension 2 (resp. 3) where we have the Torelli map $\mathcal{M}_2 \to \mathcal{A}_2$ (resp. $\mathcal{M}_3 \to \mathcal{A}_3$) from the moduli space of curves of genus 2 (resp. 3). Igusa [15] used this to describe the generators for the rings of scalar-valued Siegel modular forms of degree 2 and later Tsuyumine [31] extended this to the case of degree 3. In joint work with Carel Faber [2, 3] we used the description of $\mathcal{M}_2$ as a stack quotient of GL_2 to extend the work of Igusa by describing how invariant theory makes it possible to efficiently generate all vector-valued Siegel modular forms (of level 1) of degree 2 from one universal vector-valued meromorphic Siegel modular form $\chi_{6,-2}$ and one scalar-valued holomorphic form χ_{10}. Similarly in [4] we used the description of an open part of $\mathcal{M}_3$ as a stack quotient of GL_3 to generate all Siegel and Teichmüller modular forms from a universal meromorphic Teichmüller modular form $\chi_{4,0,-1}$ of genus 3 and the form χ_9, a square root of a Siegel modular form χ_{18}. These universal vector-valued modular forms $\chi_{6,-2}$ for genus 2 and $\chi_{4,0,-1}$ for genus 3 can be seen as giving the equation of the universal curve over the moduli space while the scalar-valued ones χ_{10} and χ_{18} are related to the discriminants of these equations.

It is natural to try to extend this to other Shimura varieties. In [28] Shimura gave a list of arithmetic ball quotients that are moduli spaces of curves. This list was extended to a complete list by Rohde, see [25, 18].

Here we treat one case of Shimura's list, a quotient of the 2-ball that gives the moduli of genus 3 curves that are triple cyclic covers of the projective line. The period domain of such curves is a Picard modular surface associated to the group of unitary similitudes

2020 *Mathematics Subject Classification.* 11F46,11F70,14J15.

$\mathrm{GU}(2,1,\mathbb{Q}(\sqrt{-3}))$. These periods were first studied by Picard in the late 19th century in a series of papers [22, 23, 24].

We show how all vector-valued modular forms on the moduli space in question can be generated by invariant theory from two universal modular forms, one meromorphic form $\chi_{4,-2}$ of weight $(4,-2)$, and a holomorphic Eisenstein series $E_{1,1}$ of weight $(1,1)$. Multiplication of $\chi_{4,-2}$ by the scalar-valued modular form ζ, related to the discriminant, makes $\chi_{4,-2}$ holomorphic. These three forms are Teichmüller modular forms, but can be viewed as Picard modular forms on an appropriate congruence subgroup. The two vector-valued forms $\chi_{4,-2}$ and $E_{1,1}$ can be interpreted as the quartic and the linear term f_4 and f_1 in the equation of the universal canonical curve over the moduli space

$$y^3 f_1 = f_4 \,.$$

Like in the cases of Siegel modular forms of degree 2 and 3, the interpretation of our moduli space as a stack quotient enables the use of invariant theory. This moduli space is a stack quotient of a twisted version of the action of GL_2 on $V_4 \times V_1$, where V_1 is the standard representation of GL_2 and $V_4 = \mathrm{Sym}^4(V_1)$. The invariant theory used is that of covariants (or more precisely, bi-covariants) for this action. The generators of the ring of bi-covariants are known classically. The construction of modular forms is realized by substituting the coordinates of the basic forms $\chi_{4,-2}$ and $E_{1,1}$ in the covariants. In general a covariant yields a meromorphic modular form with possible poles only along the curve T_1 where the scalar-valued form ζ vanishes. This curve T_1 is the locus where the Jacobian of our genus 3 curve is a product of an abelian surface and a fixed elliptic curve with multiplication by third roots of unity.

In order to apply this effectively we need to construct explicitly Fourier-Jacobi expansions of the generating modular forms ζ, $\chi_{4,-2}$ and $E_{1,1}$. We use gradients of theta functions to construct these basic forms.

To check holomorphicity of the modular forms obtained from covariants we need also the Taylor expansions of these generating forms along the modular curve T_1 on our Picard modular surface.

As an application we show how to construct the generators of rings of scalar-valued modular forms and of modules of vector-valued modular forms from invariants and covariants. In particular, we determine generators of modules of vector-valued Picard modular forms of weight $(4,k)$.

As a possible further application we mention that the description of modular forms by covariants should allow a description and construction of these Picard modular forms in positive characteristic.

It is a great pleasure to dedicate this paper to Don Zagier who through his work and in his contacts with us has been a source of inspiration for both of us.

Acknowledgements

The first author was supported by Simons Foundation Award 546235 at the Institute for Computational and Experimental Research in Mathematics at Brown University. He thanks Loughborough University for hospitality during the preparation of this paper.

The second author thanks the Mittag-Leffler Institute for excellent working conditions during the final preparations of this paper. He also thanks YMSC of Tsinghua University and the University of Luxembourg for support.

Contents

2. Picard modular forms

We briefly recall the notion of Picard modular forms on the 2-ball. We refer to [9, 5] for more details. Let $F = \mathbb{Q}(\sqrt{-3})$ with ring of integers $O_F = \mathbb{Z}[\rho]$ for a primitive third root of unity ρ and units $O_F^\times = \mu_6$. We consider the non-degenerate Hermitian form h of signature $(2, 1)$ on the F-vector space $Z = F^3$ given by

$$z_1 z_2' + z_1' z_2 + z_3 z_3' ,$$

where the prime indicates the Galois conjugate. It defines an algebraic group G over $\mathbb{Q}$ consisting of the similitudes of h

$$G(\mathbb{Q}) = \{g \in \mathrm{GL}(3, F) : h(gz) = \eta(g)h(z)\}$$

with multiplyer homomorphism $\eta : G \to \mathbb{G}_m$. This is a group of type $\mathrm{GU}(2, 1, F)$. We let $G^0 = \ker(\eta)$. The two arithmetic groups of interest are

$$\Gamma = G^0(\mathbb{Z}), \quad \Gamma_1 = G^0(\mathbb{Z}) \cap \ker \det \ .$$

After choosing an embedding $F \hookrightarrow \mathbb{C}$ we can identify $F \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}$ and $G(\mathbb{R})$ acts on the complex vector space $Z_{\mathbb{R}} = Z \otimes_{\mathbb{Q}} \mathbb{R}$ via the standard representation. An element g of $G^+(\mathbb{R}) = \{g \in G(\mathbb{R}) : \eta(g) > 0\}$ preserves the set of negative complex lines

$$\mathfrak{B} = \{L : L \subset Z \otimes_{\mathbb{Q}} \mathbb{R}, \dim_{\mathbb{C}} L = 1, h_{|L} < 0\}\,.$$

The action can be given explicitly by first identifying $\mathfrak{B}$ via $u = z_3/z_2$ and $v = z_1/z_2$ with a complex 2-ball

$$\mathfrak{B} = \{(u, v) \in \mathbb{C}^2 : v + \bar{v} + u\bar{u} < 0\}\,.$$

Then an element $g = (g_{ij})$ acts by

$$g \cdot (u, v) = \left(\frac{g_{31}v + g_{32} + g_{33}u}{g_{21}v + g_{22} + g_{23}u}, \frac{g_{11}v + g_{12} + g_{13}u}{g_{21}v + g_{22} + g_{23}u} \right)\,.$$

The quotient $X_\Gamma = \Gamma\backslash\mathfrak{B}$ is called a Picard modular surface. It is not compact, but can be compactified by adding one cusp. It was studied in detail by Holzapfel and Feustel, see [12, 8]. The two congruence subgroups

$$\Gamma[\sqrt{-3}] = \{\gamma \in \Gamma : \gamma \equiv 1_3 \,(\mathrm{mod}\sqrt{-3})\} \quad \text{and} \quad \Gamma_1[\sqrt{-3}] = \Gamma[\sqrt{-3}] \cap \Gamma_1$$

will also play a role here. For later use we record the following lemma, see [27, p. 329].

Lemma 2.1. *The following six elements generate the group* $\Gamma[\sqrt{-3}]$*:* $g_0 = \rho\, 1_3$ *and*

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 0 & 0 \\ \sqrt{-3} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, g_3 = \begin{pmatrix} 1 & 0 & 0 \\ \rho-1 & 1 & \rho-1 \\ 1-\rho^2 & 0 & 1 \end{pmatrix}, g_4 = \begin{pmatrix} 1 & \sqrt{-3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, g_5 = \begin{pmatrix} 1 & \rho-1 & \rho-1 \\ 0 & 1 & 0 \\ 0 & 1-\rho^2 & 1 \end{pmatrix}.$$

The quotient $X_{\Gamma_1[\sqrt{-3}]} = \Gamma_1[\sqrt{-3}]\backslash\mathfrak{B}$ can be compactified by adding four cusps represented by $[1:0:0]$, $[0:1:0]$ $[\rho:1:1]$ and $[\rho:1:-1]$. We have an isomorphism

$$\Gamma/\Gamma_1[\sqrt{-3}] \cong \mathfrak{S}_4 \times \mu_6, \quad g \mapsto (\sigma(g), \det(g))\,,$$

where $\mathfrak{S}_4$ is the symmetric group and $\sigma(g)$ is the permutation of the four cusps. The μ_6–part is generated by -1_3 and $g_1 = \mathrm{diag}(1, 1, \rho)$, while the $\mathfrak{S}_4$–part is generated by:

$$r_1 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad r_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad r_3 = \begin{pmatrix} 1 & \rho^2 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Note that $\Gamma_1/\Gamma_1[\sqrt{-3}] \simeq \mathfrak{S}_4$ and the three elements r_1, r_2 and r_3 correspond to the permutations $(12), (34)$ and (234) in $\mathfrak{S}_4$.

The action of $G^+(\mathbb{R})$ on $\mathfrak{B}$ defines two factors of automorphy:

$$j_1(g, u, v) = g_{21}v + g_{22} + g_{23}u\,, \quad j_2(g, u, v) = \det(g)^{-1} \begin{pmatrix} G_{32}u + G_{33} & G_{32}v + G_{31} \\ G_{12}u + G_{13} & G_{12}v + G_{11} \end{pmatrix}\,,$$

where G_{ij} denotes the minor of g_{ij}. We have

$$\det(j_2(g, u, v)) = j_1(g, u, v)/\det(g)\,.$$

The factor of automorphy j_2 agrees with the canonical factor of automorphy as defined by Satake, see [26, Chapter II.5]; see also [29].

For a pair (j,k) of integers and $g \in G^+(\mathbb{R})$, we define a slash operator on functions $f : \mathfrak{B} \to \mathrm{Sym}^j(\mathbb{C}^2)$,

$$(f|_{j,k}g)(u,v) = j_1(g,u,v)^{-k}\mathrm{Sym}^j(j_2(g,u,v)^{-1})f(g\cdot(u,v)).$$

For a discrete subgroup Γ' of $G^+(\mathbb{R}) \cap \ker\eta$ and a character χ of Γ' of finite order, we define the space of modular forms of weight (j,k) and character χ on Γ' as

$$M_{j,k}(\Gamma',\chi) = \left\{ f : B \to \mathrm{Sym}^j(\mathbb{C}^2) \mid f \text{ holomorphic}, f|_{j,k}g = \chi(g)\, f \text{ for any } g \in \Gamma' \right\}.$$

We denote by $S_{j,k}(\Gamma',\chi)$ the subspace of cusp forms of $M_{j,k}(\Gamma',\chi)$. For $j=0$, that is, for scalar-valued forms, we shorten these notations by just writing $M_k(\Gamma',\chi)$ and $S_k(\Gamma',\chi)$, and $M_k(\Gamma')$ and $S_k(\Gamma')$ if χ is trivial. We have the graded ring of modular forms on Γ' with

$$M(\Gamma') = \bigoplus_{k\geqslant 0} M_k(\Gamma').$$

We apply this to the case where Γ' equals to one of the groups $\Gamma, \Gamma_1, \Gamma[\sqrt{-3}]$ and $\Gamma_1[\sqrt{-3}]$.

Remark 2.2. The isomorphism $\Gamma[\sqrt{-3}]/\Gamma_1[\sqrt{-3}] \cong \mu_3$ via $g \mapsto \det(g)$ gives a decomposition

$$M_{j,k}(\Gamma_1[\sqrt{-3}]) = \oplus_{l=0}^2 M_{j,k}(\Gamma[\sqrt{-3}], \det^l),$$

and similarly, $\Gamma/\Gamma_1 \cong \mu_6$ via $g \mapsto \det(g)$ gives

$$M_{j,k}(\Gamma_1) = \oplus_{l=0}^5 M_{j,k}(\Gamma, \det^l).$$

However, since $-1_3 \in \Gamma$ acts by $(-1)^{j+k}$ on $M_{j,k}(\Gamma_1)$ here we may restrict l by $j+k \equiv l (\mathrm{mod} 2)$, that is, $l \in \{0,2,4\}$ or $l \in \{1,3,5\}$. But note that if we view a modular form on Γ_1 as a modular form on $\Gamma_1[\sqrt{-3}]$ the notation of the character may change since $\Gamma/\Gamma_1 \cong \mu_6$, but $\Gamma[\sqrt{-3}]/\Gamma_1[\sqrt{-3}] \cong \mu_3$.

We have two order 2 characters on Γ. The first one is $\det^3$, while the second one, denoted ϵ, comes from the isomorphism $\Gamma/\Gamma_1[\sqrt{-3}] \cong \mathfrak{S}_4 \times \mu_6$ and the map $\mathfrak{S}_4 \times \mu_6 \to \{\pm 1\}$ given by $(\sigma, z) \mapsto \mathrm{sgn}(\sigma)$ with sgn the sign character on $\mathfrak{S}_4$.

The isomorphism $\Gamma_1/\Gamma_1[\sqrt{-3}] \cong \mathfrak{S}_4$ makes $M_{j,k}(\Gamma_1[\sqrt{-3}])$ into a representation of $\mathfrak{S}_4$ and we have

$$M_{j,k}(\Gamma_1, \epsilon) = M_{j,k}(\Gamma_1[\sqrt{-3}])^{s[1^4]},$$

with $s[1^4]$ the alternating $\mathfrak{S}_4$-representation and with $M_{j,k}(\Gamma_1[\sqrt{-3}])^{s[1^4]}$ denoting the subspace of $M_{j,k}(\Gamma_1[\sqrt{-3}])$ where $\mathfrak{S}_4$ acts via the alternating character.

By [12] the Baily-Borel compactification $X^*_{\Gamma[\sqrt{-3}]}$ of $X_{\Gamma[\sqrt{-3}]} = \Gamma[\sqrt{-3}]\backslash\mathfrak{B}$ can be identified with $\mathbb{P}^2 \subset \mathbb{P}^3$ given by the hyperplane $x_1+x_2+x_3+x_4=0$ with the action of $\Gamma/\Gamma[\sqrt{-3}] \cong \mathfrak{S}_4 \times \mu_2$ given by $x_i \mapsto \mathrm{sgn}(\sigma)\, x_{\sigma(i)}$ and μ_2 acting trivially. Moreover, $X^*_{\Gamma_1[\sqrt{-3}]}$ can be identified with the 3-fold cover given by $\zeta^3 = \prod_{1\leq i<j\leq 4}(x_i - x_j)$.

The factor of automorphy j_1 corresponds to an orbifold line bundle L on $\Gamma\backslash\mathfrak{B}$ and the factor j_2 to a rank 2 orbifold vector bundle U. If we define $j_3 = \det(g)$ to be the third factor of automorphy we have $\det(j_2) = j_1/j_3$. This factor j_3 corresponds to $R = \det(U)^{-1} \otimes L$, see [1]. Note that R is a torsion line bundle.

When we speak of weight (j,k,l) we refer to the factor of automorphy $j_1^k\,\mathrm{Sym}^j(j_2)\,j_3^l$. The $M(\Gamma)$-module

$$\mathbb{M}(\Gamma) = \bigoplus_{j,k\in\mathbb{Z}_{\geq 0}} M_{j,k}(\Gamma)$$

can be made into a ring; indeed a modular form on Γ of weight (j,k,l) is a section of $\mathrm{Sym}^j(U)\otimes L^k\otimes R^l$ on $\Gamma\backslash\mathfrak{B}$ and the canonical projection $\mathrm{Sym}^a(U)\otimes\mathrm{Sym}^b(U)\to\mathrm{Sym}^{a+b}(U)$ and the usual multiplication of line bundles determines the ring structure. Similarly, we have a ring structure on the $M(\Gamma_1)$-module $\mathbb{M}(\Gamma_1)=\oplus M_{j,k}(\Gamma_1)$.

We now briefly summarize what is known about Picard modular forms on the groups in question. Shiga studied in the sixties Picard modular forms using theta functions in [27]. In the eighties Feustel and Holzapfel determined a few rings of scalar-valued modular forms, see below. In the nineties Finis constructed a number of scalar-valued Hecke eigenforms of small weight and determined Hecke eigenvalues in [9]. Shintani discussed the notion of vector-valued modular forms in [30].

Bergström and one of us studied in [1] the cohomology of local systems on the arithmetic quotient $\Gamma_1[\sqrt{-3}]\backslash\mathfrak{B}$ and gave dimension formulas for the spaces $S_{j,k,l}(\Gamma[\sqrt{-3}])$. The interpretation of the Picard modular surface as a moduli of curves was used there to determine experimentally by counting points over finite fields Hecke eigenvalues of Picard modular forms on $\Gamma_1[\sqrt{-3}]$. Motivated by the early experimental results of [1] we constructed in [5] a number of vector-valued modular forms and determined the structure of a few modules of vector-valued modular forms.

We finish this section by recalling some results of Feustel and Holzapfel, [8, 12] on the structure of some graded rings of scalar-valued modular forms. There exist modular forms $\varphi_i\in M_3(\Gamma[\sqrt{-3}])$ for $i=0,1,2$, and $\zeta\in S_6(\Gamma[\sqrt{-3}],\det)$ such that

$$M(\Gamma[\sqrt{-3}])=\mathbb{C}[\varphi_0,\varphi_1,\varphi_2], \quad \text{and} \quad M(\Gamma_1[\sqrt{-3}])=\mathbb{C}[\varphi_0,\varphi_1,\varphi_2,\zeta]/(R),$$

where (R) is the ideal generated by the relation

$$\zeta^3 = -\frac{\rho}{3^7\sqrt{-3}}\varphi_0\varphi_1\varphi_2(\varphi_1-\varphi_0)(\varphi_2-\varphi_0)(\varphi_2-\varphi_1)\,. \tag{1}$$

The constant $-\rho/3^7\sqrt{-3}$ is due to our normalizations, see later. The φ_i are related with the coordinates x_i of the Baily-Borel compactification $X^*_{\Gamma[\sqrt{-3}]}$ via

$$x_1=\varphi_0+\varphi_1+\varphi_2,\quad x_2=-3\,\varphi_0+\varphi_1+\varphi_2,\quad x_3=\varphi_0-3\,\varphi_1+\varphi_2,\quad x_4=\varphi_0+\varphi_1-3\,\varphi_2,$$

and the action of $\mathfrak{S}_4$ by $x_i\mapsto \mathrm{sgn}(\sigma)x_{\sigma(i)}$ makes $M_3(\Gamma[\sqrt{-3}])$ into the $\mathfrak{S}_4$-representation $s[2,1^2]$ corresponding to the partition $(2,1,1)$ of 4.

The form $\zeta\in S_6(\Gamma[\sqrt{-3}],\det)$ is $\mathfrak{S}_4$-anti-invariant. We thus can view ζ as an element of $S_6(\Gamma_1,\epsilon)$.

One defines Eisenstein series E_i of weight i on the group Γ or a smaller group by

$$\begin{aligned} E_6 &= \varphi_0^2+\varphi_1^2+\varphi_2^2-\frac{2}{3}(\varphi_0\varphi_1+\varphi_0\varphi_2+\varphi_1\varphi_2) \in M_6(\Gamma), \\ E_9 &= (-\varphi_0+\varphi_1+\varphi_2)(\varphi_0-\varphi_1+\varphi_2)(\varphi_0+\varphi_1-\varphi_2) \in M_9(\Gamma[\sqrt{-3}]) \cap M_9(\Gamma_1,\epsilon), \\ E_{12} &= -\frac{1}{3}(\varphi_0+\varphi_1+\varphi_2)(-3\varphi_0+\varphi_1+\varphi_2)(\varphi_0-3\varphi_1+\varphi_2)(\varphi_0+\varphi_1-3\varphi_2) \in M_{12}(\Gamma). \end{aligned}$$

Then we can describe the rings of modular forms on Γ and Γ_1:

$$M(\Gamma) = \mathbb{C}[E_6, E_{12}, E_9^2]\,, \quad \text{and} \quad M(\Gamma_1) = \mathbb{C}[E_6, E_{12}, E_9^2, \zeta E_9, \zeta^2]/(R_1),$$

with the ideal (R_1) generated by the relation $(E_9\,\zeta)^2 = E_9^2\zeta^2$ and

$$\rho\, 2^{16} 3^{12} \zeta^6 = 9\, E_6^4 E_{12} - 8\, E_6^3 E_{12}^2 + 6\, E_6^2 E_{12}^2 - 24\, E_6 E_9^2 E_{12} + 16\, E_9^4 + E_{12}^3\,. \tag{2}$$

We also have

$$M(\Gamma_1,\epsilon) = \mathbb{C}[E_6, E_9, E_{12}, \zeta^2]/(R')$$

with (R') generated by the relation (2).

3. A modular embedding

The arithmetic quotient $X_\Gamma = \Gamma\backslash\mathfrak{B}$ parametrizes principally polarized abelian threefolds with multiplication by O_F. Therefore there is a morphism $X_\Gamma \to \mathcal{A}_3(\mathbb{C})$. We now describe the corresponding modular embedding $\Gamma\backslash\mathfrak{B} \to \mathrm{Sp}(6,\mathbb{Z})\backslash\mathfrak{H}_3$ with $\mathfrak{H}_3$ the Siegel upper half space of degree 3

$$\mathfrak{H}_3 = \{\tau \in \mathrm{Mat}(3,\mathbb{C}) : \tau^t = \tau, \mathrm{Im}(\tau) > 0\}\,.$$

Such modular embeddings were considered by Picard, Shiga and Holzapfel, see [22, 27, 13].

The lattice O_F^3 with Hermitian form $h = z_1z_2' + z_2z_1' + z_3z_3'$ determines an alternating form $(2/\sqrt{3})\mathrm{Im}(h)$ and by taking as $\mathbb{Z}$-basis of this lattice

$$e_1 = (\rho^2,0,0),\ e_2 = (0,\rho^2,0),\ e_3 = (0,0,\rho^2), \quad f_1 = (0,\rho,0),\ f_2 = (\rho,0,0),\ f_3 = (0,0,\rho)$$

we can identify it with the standard symplectic lattice generated by $e_1, e_2, e_3, f_1, f_2, f_3$ with $\langle e_i, e_j\rangle = 0$, $\langle f_i, f_j\rangle = 0$, $\langle e_i, f_j\rangle = \delta_{ij}$. Here $\langle\,,\,\rangle$ denotes the alternating form that is the imaginary part of the Hermitian form. This defines an embedding $\Gamma \to \mathrm{Sp}(6,\mathbb{Z})$.

If we take instead the symplectic basis $(e_1, e_3, -f_2, f_1, f_3, e_2)$ we get the following modular embedding

$$\iota : \mathfrak{B} \to \mathfrak{H}_3, \quad \sigma : \Gamma \to \mathrm{Sp}(6,\mathbb{Z})$$

given by

$$\iota(u,v) = \begin{pmatrix} \frac{u^2+2\rho^2 v}{1-\rho} & \rho^2 u & \frac{\rho u^2-\rho^2 v}{1-\rho} \\ \rho^2 u & -\rho^2 & u \\ \frac{\rho u^2-\rho^2 v}{1-\rho} & u & \frac{\rho^2 u^2+2\rho^2 v}{1-\rho} \end{pmatrix}$$

and for $g = (a_{ij} + \rho\, b_{ij})$

$$\sigma(g) = \begin{pmatrix} a_{11}-b_{11} & a_{13}-b_{13} & -b_{11} & b_{12} & b_{13} & a_{12}-b_{12} \\ a_{31}-b_{31} & a_{33}-b_{33} & -b_{31} & b_{32} & b_{33} & a_{32}-b_{32} \\ b_{11} & b_{13} & a_{11} & -a_{12} & -a_{13} & b_{12} \\ -b_{21} & -b_{23} & -a_{21} & a_{22} & a_{23} & -b_{22} \\ -b_{31} & -b_{33} & -a_{31} & a_{32} & a_{33} & -b_{32} \\ a_{21}-b_{21} & a_{23}-b_{23} & -b_{21} & b_{22} & b_{23} & a_{22}-b_{22} \end{pmatrix}$$

The pullback of the stabilizer in $\mathrm{Sp}(6,\mathbb{Z})$ of $\iota(\mathfrak{B})$ is the group Γ. This can be derived from the Torelli theorem applied to curves of genus 3 that are triple cyclic covers of $\mathbb{P}^1$.

Let $\mathbb{E}$ be the Hodge bundle on $\mathcal{A}_3(\mathbb{C})$, that is, the cotangent bundle of the universal abelian threefold along the zero section. Via the map ι we can pull back $\mathbb{E}$ to $\Gamma\backslash\mathfrak{B}$.

We wish to express the pull back of the Hodge bundle in terms of the automorphic bundles L, U and R associated to the factors of automorphy j_1, j_2 and j_3.

Lemma 3.1. *The pullback of the Hodge bundle* $\mathbb{E}$ *over* $\mathcal{A}_3(\mathbb{C})$ *to* $\Gamma_1\backslash\mathfrak{B}$ *is isomorphic to* $U \oplus L$*. The pullback of* $\det(\mathbb{E})$ *is* $L^2 \otimes R^{-1}$*.*

Proof. The second statement follows from the first when one uses $\det(U) = L \otimes R^{-1}$. In order to prove the first one, we observe that the Hodge bundle corresponds to the factor of automorphy $(c\tau + d)$ for $\mathrm{Sp}(6,\mathbb{R})$ acting on $\mathfrak{H}_3$. With $\tau = \iota(u,v)$ and $\sigma(g) = (a,b;c,d)$ we find for diagonal matrices $g = \mathrm{diag}(g_1,g_2,g_3)$ with $g_i = a_i + \rho b_i$ that $c\,\iota(u,v) + d$ equals

$$\begin{pmatrix} a_2 & 0 & -b_2 \\ -b_3\rho^2 u & \bar{g}_3 & -b_3 u \\ b_2 & 0 & -b_2 \end{pmatrix}$$

with characteristic polynomial $(X - g_2)(X - \bar{g}_3)(X - \bar{g}_2)$. Since g respects the Hermitian form, we have $g_1\bar{g}_2 = g_3\bar{g}_3 = 1$, and therefore $j_2(g,(u,v)) = \mathrm{diag}(\bar{g}_3, \bar{g}_2)$ and $j_1(g,(u,v)) = g_2$. Hence, up to a base change we have $c\tau + d = j_2(g,(u,v)) \oplus j_1(g,(u,v))$. Since arbitrary Hermitian matrices can be diagonalized the lemma follows. $\square$

4. Modular curves

Picard modular surfaces contain modular curves defined by positive vectors in the lattice O_F^3. Though these curves were considered by Feustel, Kudla, Cogdell and others, their geometry on these surfaces did not yet get the attention that their counterparts on Hilbert modular surfaces got. Here we need just two curves that play a role.

A vector $w = (a,b,c) \in O_F^3$ with positive norm $ab' + a'b + cc'$ defines a 1-ball $\mathfrak{B}_w$ inside $\mathfrak{B} = \{L : L \subset Z_{\mathbb{R}} : \dim_{\mathbb{C}} L = 1, h_{|L} < 0\}$ by the condition $L \perp w$, or equivalently by

$$a' + b'v + c'u = 0\,. \tag{3}$$

This defines a curve in $\Gamma\backslash\mathfrak{B}$ and also in the Baily-Borel compactification X_Γ^*. We can define a modular curve T_N in $\Gamma\backslash\mathfrak{B}$ as the union of all curves defined by equations (3) with $ab' + a'b + cc' = N$. Its closure in X_Γ^* is also denoted T_N.

In the following we need the two curves T_1 and T_2. The curve T_2 was studied in [21].

The curve T_1 has one irreducible component on X_Γ^* as one sees by verifying that the action of Γ on positive vectors (a, b, c) with $ab' + a'b + cc' = 1$ is transitive using the generators of Γ, see Section 2. It can be defined by $u = 0$ and viewed as a quotient of the upper half plane $\mathfrak{H}$ embedded in $\mathfrak{B}$ by $\tau \mapsto (0, \sqrt{-3}\tau)$. The image in $\Gamma_1[\sqrt{-3}]\backslash\mathfrak{B}$ is isomorphic to $\Gamma_0(3)\backslash\mathfrak{H}$ with $\Gamma_0(3)$ the usual congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$. The modular form ζ vanishes on T_1 since $\zeta(-u, v) = -\zeta(u, v)$.

The curve T_2 is a Shimura curve associated to the unit group of a maximal order in the quaternion algebra $\left(\frac{-3,2}{\mathbb{Q}}\right)$ of discriminant 6. The curve T_2 has one irreducible component on X_Γ^* and it can be defined by $v = -1$ and is the fixed point locus of the involution

$$\xi : (u, v) \mapsto (-u/v, 1/v)$$

that is induced by the symmetry $(z_1, z_2, z_3) \mapsto (z_2, z_1, -z_3)$ of our Hermitian form $z_1 z_2' + z_1' z_2 + z_3 z_3'$. The involution ξ induces an action on spaces of modular forms. The action on a modular form $f \in M_k(\Gamma[\sqrt{-3}])$ restricted to $v = -1$ is by multiplication by $(-1)^k$. In particular, the Eisenstein series E_9 vanishes on the fixed point locus of ξ.

More precisely, on X_Γ^* the modular forms ζ^6 and E_9^2 give rise to the cycle relations:

$$6\,\lambda_1 = [T_1], \qquad 9\,\lambda_1 = [T_2]\,,$$

where λ_1 represents the first Chern class of L and the classes $[T_1]$ and $[T_2]$ are Q-classes on the orbifold X_Γ^* in the sense of Mumford [20]. Indeed, the modular forms ζ^6 and E_9^2 that live on X_Γ have divisors $6\,T_1$ and $2\,T_2$, where the multiplicities come from the fact that a generic point of T_1 (resp. T_2) has a stabilizer of order 6 (resp. of order 2). Equivalently, one can also work on $X^*_{\Gamma_1[\sqrt{-3}]}$ where one has the modular forms ζ and E_9 with divisors T_1 and T_2; see for example the Taylor expansion of ζ along $u = 0$ in Section 15. The volume form on the orbifold defines a class T_0, see [6].

Corollary 4.1. *If $[T_N]$ denotes for $N \in \mathbb{Z}_{\geq 0}$ the Q-class of the curve T_N on X_Γ^*, then the series $\sum_{N=0}^{\infty} [T_N]\, q^N$ equals $F \otimes \lambda_1$ with*

$$F = -1/6 + 6\,q + 9\,q^2 + 42\,q^3 + 78\,q^4 + O(q^5)$$

a modular form in $M_3(\Gamma_0(3), \left(\frac{\cdot}{3}\right))$.

Proof. We can work on the minimal resolution of singularities $\tilde{X}_{\Gamma_1[\sqrt{-3}]}$ of $X^*_{\Gamma_1[\sqrt{-3}]}$ and consider the classes $[T_N^c]$ there that are defined by a linear combination (with $\mathbb{Q}$-coefficients) of T_N plus a sum of resolution curves such that T_N^c is orthogonal to the cusp resolutions. Then we can use the result of Cogdell [6, Thm. on page 126], the analogue for Picard modular surfaces of the Hirzebruch-Zagier theorem on curves on Hilbert modular surfaces. It says that $\sum_N [T_N^c] q^N$ is a modular form of weight 3 on $\Gamma_0(3)$ with Dirichlet character. Since $\dim M_3(\Gamma_0(3), \left(\frac{\cdot}{3}\right)) = 2$ and we know the coefficients of q and q^2, this identifies the modular form. $\square$

In the Baily-Borel compactification $X^*_{\Gamma[\sqrt{-3}]}$, identified with $\mathbb{P}^2$ viewed as the hyperplane $x_1 + x_2 + x_3 + x_4 = 0$ in $\mathbb{P}^3$ and with the action of $\mathfrak{S}_4$ given by $x_i \mapsto \mathrm{sgn}(\sigma) x_{\sigma(i)}$, the lines $x_i = x_j$ describe the six components of T_1. Similarly, the curve T_2 has three components and is given by $x_i + x_j = 0$ for $1 \leq i < j \leq 4$.

We now describe the image of T_1 under the modular embedding ι constructed in the preceding section. The image in $\mathfrak{H}_3$ under ι of the curve given by $u = 0$ is

$$\left\{ \begin{pmatrix} \tau_{11} & 0 & \tau_{12} \\ 0 & 1+\rho & 0 \\ \tau_{12} & 0 & \tau_{22} \end{pmatrix} \in \mathfrak{H}_3 : \tau_{11} = \tau_{22} = -2\tau_{12} \right\} .$$

In particular, $\iota(T_1) \subset \mathcal{A}_{2,1} \subset \mathcal{A}_3$, with $\mathcal{A}_{2,1}$ the moduli of abelian varieties that are products. The equations $\tau_{11} = \tau_{22}$ and $\tau_{11} + 2\tau_{12} = 0$ define two Humbert surfaces of discriminant 4 in $\mathcal{A}_2$, cf. [10, p. 210].

The fact that $\iota(T_1)$ is contained in $\mathcal{A}_{2,1}$ means that an abelian threefold X representing a point of T_1 splits as a product $X = X_2 \times X_1$ with X_2 a principally polarized abelian surface and X_1 an elliptic curve. Since X_1 has multiplication by ρ the curve X_1 is rigid. This means that the Hodge bundle $\mathbb{E}$ restricted to an irreducible component of T_1 on $X_{\Gamma_1[\sqrt{-3}]}$ has a trivial factor. By Lemma 3.1 the Hodge bundle $\mathbb{E}$ splits as $U \oplus L$ and since the action of ρ on the 2-dimensional factor X_2 has eigenvalues (ρ, ρ^2) (see [1, Section 5.5]), we see that this constant factor is contained in U.

This means that the bundle U restricted to an irreducible component T of our modular curve T_1 on $X_{\Gamma_1[\sqrt{-3}]}$ is of the form $\mathcal{O}_T \oplus N$ with N the line bundle obtained by the restriction of $\det(U)$; its sections correspond to modular forms of weight 1.

However, the curve T_1 on $\Gamma_1[\sqrt{-3}]\backslash\mathfrak{B}$ is reducible with six smooth irreducible components meeting in ordinary double points.

Lemma 4.2. *Let f be a meromorphic modular form of weight (j,k) on $\Gamma_1[\sqrt{-3}]$ that is holomorphic outside the curve T_1. If f has order r along T_1, then the first non-zero Taylor term of f along T_1 is an element of*

$$\oplus_{i=0}^{j} M_{i+k+r}^{(r)}(\Gamma_1(3)) ,$$

with $M_k^{(s)}(\Gamma_1(3))$ the space of meromorphic modular forms of weight k on $\Gamma_1(3)$ that are holomorphic outside the orbit of $\tau_0 = (1-\rho^2)/3 \in \mathfrak{H}$ and have order at least s at τ_0.

Proof. Restricting the vector bundle $\mathrm{Sym}^j(U) \otimes L^k$ to an irreducible component T of the modular curve T_1 gives the vector bundle $B = \oplus_{i=0}^{j} N^{\otimes(i+k)}$. Moreover, the conormal space of the component T of T_1 in $\Gamma_1[\sqrt{-3}]\backslash\mathfrak{B}$ when pulled back to $\mathfrak{H}$ can be identified with a fibre of the line bundle N, as one sees by looking at the action of ρ on the deformation space of an abelian threefold $X = X_2 \times X_1$ representing a point of T. Thus the conormal bundle of T can be identified with N. The rth term in the Taylor expansion along T of f, viewed as a section of $\mathrm{Sym}^j(U) \otimes L^k$, is a section of $B \otimes N^{\otimes r}$. Correcting for the double point of T_1 lying on a component T, that is represented by τ_0, implies the result. $\square$

For later use we discuss the Taylor development of modular forms along the curve T_1. Recall that this curve is represented by $u = 0$ in $\mathfrak{B}$. We can apply Proposition 8.4 of [5] that we recall for convenience: Let $f \in M_{j,k}(\Gamma[\sqrt{-3}], \det^l)$ and write

$$f(u, \sqrt{-3}\tau) = \sum_{n \geqslant 0} \begin{bmatrix} f_n^{(0)}(\tau) \\ \vdots \\ f_n^{(j)}(\tau) \end{bmatrix} u^n .$$

We write $\Gamma(3)$ for the principal congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$, and $\Gamma_1(3)$, $\Gamma_0(3)$ for the usual congruence subgroups.

Proposition 4.3. *The first component $f_n^{(0)}$ is a modular form of weight $k+n$ on $\Gamma_1(3)$ and a cusp form if $n > 0$. Moreover $f_n^{(m)}$ vanishes unless $n + j - m \equiv l \bmod 3$. The function $f_0^{(m)}$ is a modular form of weight $k+m$ on $\Gamma_1(3)$, while for $n > 0$ the function $f_n^{(m)}$ is a quasi-modular form of weight $k+m+n$ on $\Gamma_1(3)$.*

The proof was not given in [5]. Since we use this proposition and a variant later, we give some details. The modular embedding of T_1 on $X_{\Gamma[\sqrt{-3}]}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \sqrt{-3}b & 0 \\ c/\sqrt{-3} & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tau \mapsto (0, \sqrt{-3}\,\tau) .$$

We write $F \in M_{j,k}(\Gamma[\sqrt{-3}], \det^l)$ as

$$F(u, v) = \begin{pmatrix} F^{(0)} \\ \vdots \\ F^{(j)} \end{pmatrix} \qquad \text{with} \quad F^{(m)} = \sum_{n=0}^{\infty} F_n^{(m)}(v) u^n .$$

Changing coordinates by setting $f_n^{(m)}(\tau) = F_n^{(m)}(\sqrt{-3}\tau)$, the modularity of F implies the following equation.

Equation 4.4.

$$\sum_{n=0}^{\infty} \begin{pmatrix} f_n^{(0)}(\tau) \\ \vdots \\ f_n^{(j)}(\tau) \end{pmatrix} u^n = (c\tau+d)^{-k-j} \mathrm{Sym}^j \begin{pmatrix} c\tau + d & 0 \\ -cu/\sqrt{-3} & 1 \end{pmatrix} \sum_{n=0}^{\infty} (c\tau+d)^{-n} \begin{pmatrix} f^{(0)}(\frac{a\tau+b}{c\tau+d}) \\ \vdots \\ f^{(j)}(\frac{a\tau+b}{c\tau+d}) \end{pmatrix} u^n$$

Here the matrix $\mathrm{Sym}^j \begin{pmatrix} c\tau + d & 0 \\ -cu/\sqrt{-3} & 1 \end{pmatrix}$ is a lower diagonal matrix with entry on place (r, s) for $r \geq s$ equal to

$$\binom{j}{j+1-r} \left(\frac{-c}{\sqrt{-3}} \right)^{r-s} (c\tau+d)^{j+1-r} .$$

From this it follows that $F_n^{(m)}$ is a modular form (and not only quasi-modular) if $F_\nu^{(\mu)} = 0$ for all $\mu < m$ and $\nu < n$.

Modular forms on $\Gamma_1(3)$. For later use we recall some facts about elliptic modular forms of level 3. Recall that the ring of modular forms on $\Gamma(3)$ equals

$$M(\Gamma(3)) = \mathbb{C}[\vartheta, \psi]\,, \quad \text{where} \quad \vartheta(\tau) = \sum_{\alpha \in O_F} q^{N(\alpha)} \quad \text{and} \quad \psi(\tau) = \frac{\eta(3\tau)^3}{\eta(\tau)}$$

with ϑ and ψ of weight 1 and $\eta(\tau) = q^{1/24} \prod_{n \geqslant 1} (1 - q^n)$ the Dedekind eta-function, and for $\tau \in \mathfrak{H}$ we set as usual $q = e^{2\pi i \tau}$. Since $\Gamma_1(3)/\Gamma(3)$ is cyclic of order 3 generated by $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\psi|_1 T = \rho\, \psi$, we get the structure of the ring of modular forms on $\Gamma_1(3)$

$$M(\Gamma_1(3)) = \mathbb{C}[\vartheta, \psi^3].$$

For example, the form F of Corollary 4.1 is $(54\, \psi^3 - \vartheta^3)/6$. To lighten notation we will sometimes use the relation

$$\psi(\vartheta^3 - \psi^3) = \eta^8\,.$$

Note that

$$M_{2k}(\Gamma_1(3)) = M_{2k}(\Gamma_0(3)) \quad \text{and} \quad M_{2k+1}(\Gamma_1(3)) = M_{2k+1}(\Gamma_0(3), \left(\frac{\cdot}{3}\right))\,.$$

By a result of Kaneko-Zagier (see [17, Proposition 1, part b]) we know that the graded ring $\widetilde{M}(\Gamma_1(3))$ of quasi-modular forms on $\Gamma_1(3)$ is given by

$$\widetilde{M}(\Gamma_1(3)) = M(\Gamma_1(3)) \otimes \mathbb{C}[e_2] \simeq \mathbb{C}[\vartheta, \psi^3, e_2]\,,$$

where e_2 is the Eisenstein series of weight 2 on $\mathrm{SL}(2, \mathbb{Z})$. We normalise e_2 such that its Fourier expansion is given by

$$e_2(\tau) = 1 - 24 \sum_{n \geqslant 1} \sigma_1(n) q^n.$$

Examples of modular forms of level 3 are given by $\Theta_j(\tau) = \sum_{\alpha \in O_F} \alpha^j q^{N(\alpha)} \in M_{j+1}(\Gamma_1(3))$. Observe that Θ_j is a cusp form as soon as $j > 1$ and identically zero if $j \not\equiv 0 \bmod 6$. For example, we have $\Theta_0 = \vartheta$ and

$$\Theta_6 = 6\vartheta\psi^3(\vartheta^3 - 27\psi^3) = 6\,\vartheta\psi^2\eta^8, \quad \Theta_{12} = \vartheta\eta^8 \left(\eta^{16} + 18\psi^4\eta^8 + 729\psi^8\right)\,.$$

Recall that $M_k^{(r)}(\Gamma_1(3))$ is the space of meromorphic modular forms of weight k that are holomorphic outside the orbit of $\tau_0 = (1 - \rho^2)/3$ and have order at least r at τ_0.

The form ϑ is non-zero outside the orbit of τ_0. Multiplication by ϑ^{-r} provides an isomorphism for $r \in \mathbb{Z}$

$$M_r^{(r)}(\Gamma_1(3)) \xrightarrow{\sim} M_0(\Gamma_1(3)) = \mathbb{C}\,. \tag{4}$$

5. A stack quotient

In this section we discuss the moduli stack of curves of genus 3 that are a cyclic cover of degree 3 of the projective line. We consider smooth projective curves C over $\mathbb{C}$ of genus 3 together an automorphism α of order 3 such that the eigenvalues on $H^0(C,\Omega^1_C)$ are ρ,ρ,ρ^2. An isomorphism $(C,\alpha)\longrightarrow(C',\alpha')$ is an isomorphism $\nu: C\to C'$ such that $\nu\alpha=\alpha'\nu$. We let $\mathcal{N}$ denote the moduli stack over $\mathbb{C}$ of such curves.

A choice ω_1,ω_2 of a basis of the ρ-eigenspace $H^0(C,\Omega^1_C)^\rho$ defines a morphism of degree 3 to $C/\alpha=\mathbb{P}^1$. By the holomorphic Lefschetz fixed point formula the automorphism α has five fixed points on C, four of which have action by ρ on the tangent space and one with action by ρ^2.

The ρ-eigenspace of $\mathrm{Sym}^4(H^0(C,\Omega^1_C))$ has dimension 7, whereas the ρ-eigenspace of $H^0(C,(\Omega^1_C)^{\otimes 4})$ has dimension 6. (This follows from the holomorphic Lefschetz formula; or by the simple argument that the ternary quartic defining the canonical image of C in $\mathbb{P}^2$ must lie in an eigenspace and all elements with eigenvalue 1 or ρ^2 are divisible by η, a generator of $H^0(C,\Omega^1_C)^{\rho^2}$, and this would give a reducible equation.) After choosing a generator η of $H^0(C,\Omega^1_C)^{\rho^2}$ we thus find a non-trivial relation

$$b_1\eta^3\omega_1+b_2\eta^3\omega_2=\sum_{i=0}^4 a_i\,\omega_1^{4-i}\omega_2$$

with $b_i,a_j\in\mathbb{C}$. By setting $f_1=b_1x_1+b_2x_2$ and $f_4=\sum_{i=0}^4 a_ix_1^{4-i}x_2^i$ and observing that f_1 is not identically zero, we obtain an equation

$$y^3f_1=f_4\,. \tag{5}$$

This represents the canonical image of C. A different normalization is obtained by putting $y=\tilde{y}/f_1$ which gives $\tilde{y}^3=f_4f_1^2$. By putting the zero of f_1 at infinity we find yet another normalization: an affine equation $u^3=f$ with f of degree 4 in v. The map α corresponds to the field exension $\mathbb{C}(u,v)/\mathbb{C}(v)$.

Changing the choice of basis of $H^0(C,\Omega^1_C)^\rho$ corresponds to an action of GL_2. Changing the basis η corresponds to an action of $\mathbb{G}_m$. Together this defines an action of the subgroup $\mathcal{G}=\mathrm{GL}_1\times\mathrm{GL}_2\subset\mathrm{GL}_3$ on $H^0(C,\Omega^1_C)$ that preserves the decomposition in eigenspaces for α.

Let V be the 2-dimensional $\mathbb{C}$-vector space generated by elements x_1,x_2. We view V as the standard representation of GL_2. We consider elements $f_1\in V$ and $f_4\in\mathrm{Sym}^4(V)$. If the discriminant of f_4f_1 does not vanish, the equation $y^3f_1=f_4$ defines an equation of a smooth projective curve C of genus 3 with an automorphism α given by $y\mapsto\rho y$. The space $H^0(C,\Omega^1_C)$ comes with a basis consisting of the forms (in affine coordinates) $\eta=dx/f_1y$, $\omega_1=dx/y^2$, $\omega_2=dx/f_1y^2$.

An element $(a,b;c,d)\in\mathrm{GL}_2$ acts on f_4 and f_1 by

$$f_4(x_1,x_2)\mapsto f_4(ax_1+bx_2,cx_1+dx_2),\quad f_1(x_1,x_2)\mapsto f_1(ax_1+bx_2,cx_1+dx_2),$$

and we can define an action

$$y\mapsto y/(cx_1+dx_2)\,.$$

However, in order to get the right stack quotient we need to consider a twisted action. We define $V_{m,n}$ for $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$ as the GL_2-representation

$$\mathrm{Sym}^m(V) \otimes \det(V)^{\otimes n}\,.$$

The underlying space of $V_{m,n}$ can and will be identified with $\mathrm{Sym}^m(V)$, but the action of GL_2 is different.

We define an action of $\mathbb{G}_m$ on $V_{4,-2} \oplus V_{1,1}$ by letting $t \in \mathbb{G}_m$ act via $(x_1, x_2) \mapsto (tx_1, tx_2)$. Via $y \mapsto t\,y$ this leaves the equation (5) unchanged. It corresponds to the action of the diagonal $\mathbb{G}_m$ in $\mathcal{G}$. Then the action of the diagonal $\mathbb{G}_m$ in GL_2 on $V_{4,-2} \oplus V_{1,1}$ is given by $(f_4, f_1) \mapsto (f_4, t^3 f_1)$; hence the central $\mu_3 \subset \mathrm{GL}_2$ acts trivially.

We let $\mathcal{Y}$ be the subset of $V_{4,-2} \times V_{1,1}$ of pairs (f_4, f_1) such that the discriminant of $f_4 f_1$ is not zero. Moreover, we let $\mathbb{P}(\mathcal{Y})$ be the image in $\mathbb{P}(V_{4,-2} \oplus V_{1,1})$. The stack quotient that we need is obtained by first dividing by the diagonal $\mathbb{G}_m$ in $\mathcal{G}$ to get $\mathbb{P}(\mathcal{Y})$ and then dividing by the action of $P\mathcal{G} \subset \mathrm{PGL}(3)$ on $\mathbb{P}(\mathcal{Y})$. Equivalently, we directly take the stack quotient $[\mathcal{Y}/\mathcal{G}]$. Summarizing we get the following result.

Proposition 5.1. *The stack quotient $[\mathcal{Y}/\mathcal{G}]$ represents the moduli stack of curves $\mathcal{N}$.*

Note that the central $\mu_6 \,\mathrm{Id}_V$ acts trivially on the equation (5) but $-1\,\mathrm{Id}_V$ acts by $f_1 \mapsto -f_1$. Hence the stabilizer of a generic element is μ_3 as it should.

We can extend $\mathcal{Y}$ to the open subset $\mathcal{Y}'$ of $V_{4,-2} \oplus V_{1,1}$ consisting of pairs (f_4, f_1) such that either

(1) f_4 has one double zero: $f_4 = h_1^2 h_2$ with $\deg(h_i) = i$ and $\mathrm{disc}(h_2 h_1 f_1) \neq 0$, or
(2) $f_4 = f_1 h_3$ with h_3 of degree 3 and $\mathrm{disc}(f_4) \neq 0$.

The locus $\mathcal{Y}'$ has a complement of codimension 2 in $V_{4,-2} \oplus V_{1,1}$.

In Case (1) the equation $y^3 f_1 = h_1^2 h_2$ (or equivalently $y^3 = h_1^2 h_2 f_1^2$) defines a curve of genus 2 which is a triple cyclic cover of $\mathbb{P}^1$. In Case (2) the equation $y^3 = h_3$ defines a 3-pointed genus 1 curve C_1 with as marked points the three points of the fibre of $C_1 \to \mathbb{P}^1$ defined by $f_1 = 0$.

The space $\mathcal{N}$ can be viewed as a Hurwitz space and can be compactified as such. We will deal with this in the next section.

Remark 5.2. **Relation with ternary quartics**. We conclude this section by giving the relationship with a stack quotient description of the moduli of non-hyperelliptic curves of genus 3. It is well-known that the moduli space $\mathcal{M}_3^{\mathrm{nh}}$ of non-hyperelliptic curves of genus 3 can be described as a stack quotient associated to the action of GL_3 on ternary quartics. Since we are using canonical curves as in (5) we get an embedding of stacks $\mathcal{N} \to \mathcal{M}_3^{\mathrm{nh}}$.

Let W be a 3-dimensional vector space and let

$$W_{4,0,-1} = \mathrm{Sym}^4(W) \otimes \det(W)^{-1}\,.$$

This space can be regarded as the space of ternary quartics with a twisted $\mathrm{GL}(W)$ action. The element $t\,\mathrm{id}_W$ in the diagonal $\mathbb{G}_m$ in $\mathrm{GL}(W)$ acts via $f \mapsto t\,f$ for a quartic

$f \in W_{4,0,-1}$. We let $\mathcal{Z} \subset W_{4,0,-1}$ be the subset of quartics with non-zero discriminants. The stack quotient $[\mathcal{Z}/\mathrm{GL}(W)]$ can be identified with $\mathcal{M}_3^{\mathrm{nh}}$, see [4].

To connect it to our case we write $W = W' \oplus W''$ with $W' = \langle x_1, x_2 \rangle$ and $W'' = \langle y \rangle$. The element $\mathrm{diag}(1,1,\rho)$ acts $W_{4,0,-1}$ and the ρ^2-eigenspace is

$$\mathrm{Sym}^4(W') \otimes \det(W')^{-1} \otimes (W'')^{-1} \bigoplus W' \otimes \det(W')^{-1} \otimes (W'')^{\otimes 2}$$

By putting $\det(W') = W''$ we find the $\mathrm{GL}(W')$ representation $(W')_{4,-2} \oplus (W')_{1,1}$. Note that $\det(W')$ and W'' differ by the action of the diagonal μ_3 in GL_2 when viewed as subgroup of $\mathcal{G}$.

6. The Hurwitz space

We briefly discuss a compactification of $\mathcal{N}$ as a Hurwitz space. We consider admissible triple cyclic covers $f : C \to P$ where C is a nodal curve of genus 3 and P a stable curve of genus 0 with marked points $p_0, \{p_1, \ldots, p_4\}$ together with an order 3 automorphism α such that C/α is isomorphic to P, f equals the map $C \to C/\alpha$ and α fixes the p_i while acting by ρ^2 (resp. by ρ) on the tangent space of p_0 (resp. of p_i with $i \neq 0$). Here admissible is taken in the sense of Harris-Mumford, see for example [11, p. 175]. The marking of $p_1, \ldots, p_4$ is taken unordered, that is, modulo the action of the symmetric group $\mathfrak{S}_4$. We denote this space by $\overline{\mathcal{N}}$.

It allows a morphism $\overline{\mathcal{N}} \to \overline{\mathcal{M}}_{0,1+4} = \overline{\mathcal{M}}_{0,5}/\mathfrak{S}_4$ with $\overline{\mathcal{M}}_{g,n}$ the usual Deligne-Mumford moduli stack of stable n-pointed curves of genus g.

Note that the moduli space $\overline{\mathcal{M}}_{0,1+4}$ of marked stable curves of genus 0 has a stratification with five strata according to the topological type of the genus 0 curve.

There is a corresponding stratification of $\overline{\mathcal{N}}$. We now describe the five types of curves (C, P, α) corresponding to the strata of $\overline{\mathcal{M}}_{0,1+4}$. These are:

1. C is smooth.
2. C is a union $C_1 \cup C_2$ of curves C_i of genus i with automorphisms α_1 and α_2 of order 3. The unique node is a fixed point of α_1 and α_2. Moreover, this point is of type ρ^2 for α_2 and of type ρ for α_1.
3. C is a linear chain of three curves C_i of genus 1 with automorphisms α_i $(i = 1, 2, 3)$ and the two nodes are fixed points. Moreover the action of α_1 and α_3 is by ρ, while for the middle one the action by α_2 is by ρ^2.
4. C is a join of a genus 1 curve C_1 with an automorphism α_1 that acts by ρ and a rational curve $C_0 = \mathbb{P}^1$ with an automorphism α_0 that acts by $x \mapsto \rho x$. The curve C is obtained by identifying the three points of an α_1-orbit of length 3 with $1, \rho, \rho^2$ on $\mathbb{P}^1$.
5. C consists of the union of a genus 1 curve C_1 with an order 3 automorphism α_1 and two $\mathbb{P}^1$'s with automorphism $x \mapsto 1/(1-x)$, say C_0 and C_0', that intersect each other in 0, 1 and ∞ such that C_1 and C_0' are disjoint, while C_1 and C_0 intersect in a fixed point of α_1. Moreover, the action of α_1 is by ρ.

The corresponding strata are denoted by $\mathcal{N}_i$ for $i = 1, \ldots, 5$ with $\mathcal{N}_1 = \mathcal{N}$.

The first three cases represent curves whose generalized Jacobian is an abelian variety. The dimensions of the strata are 2, 1, 0, 1, 0 respectively.

The strata $\mathcal{N}_2$ and $\mathcal{N}_4$ correspond to the two cases (1) and (2) of the preceding section. To connect it with the quartics discussed there, one considers for the case $\mathcal{N}_2$ the space $H^0(C_2, \Omega^1_{C_2}(2P))$ with P the point of C_2 shared with C_1. This space has dimension 3 and the action of α_2 on it has eigenvalues ρ, ρ, ρ^2. A choice of basis can be used to generate an equation of type $y^3 f_1 = h_1^2 h_2$ as in the smooth case. For $\mathcal{N}_4$ one considers $H^0(C_1, O(Q))$ with Q the degree 3 divisor of intersection points.

On the Hurwitz space $\overline{\mathcal{N}}$ we have the Hodge bundle $\mathbb{E}$. It allows a decomposition in ρ and ρ^2-eigenspaces of dimension 2 and 1.

7. The Torelli morphism

The homology $H_1(C, \mathbb{Z})$ of a smooth curve C given by an equation (4), or in other words of type (1) of the preceding section, is a projective $\mathbb{Z}[\rho]$-module of rank 3, hence isomorphic to a direct sum of ideals of $\mathbb{Z}[\rho]$, and since $F = \mathbb{Q}(\rho)$ has class number 1, it can be identified with $\Lambda = O_F^3$ with $O_F = \mathbb{Z}[\rho]$. Moreover, if have chosen an embedding $F \hookrightarrow \mathbb{C}$ we obtain a 3-dimensional complex vector space $W = H_1(C, \mathbb{R}) = H_1(C, \mathbb{Z}) \otimes_{\mathbb{Q}} \mathbb{R}$. The Jacobian variety of such a curve C is a principally polarized abelian threefold W/Λ with complex multiplication by the ring of integers O_F. The polarization defines an alternating integral form on the lattice Λ. The corresponding Hermitian form on W may be normalized to the form

$$z_1 z_2' + z_1' z_2 + z_3 z_3',$$

where $x \mapsto x'$ corresponds to the Galois automorphism of $F/\mathbb{Q}$. This form has signature $(2, 1)$.

The Torelli map that associates to a curve C its Jacobian defines a map

$$\tau : \mathcal{N} \to \Gamma \backslash \mathfrak{B}\,.$$

Note that our generic curve, given by an equation $y^3 f_1 = f_4$, has an automorphism group of order 3, while the generic Jacobian of such a curve has an automorphism group of order 6.

Recall that on $\Gamma \backslash \mathfrak{B}$ we have the basic orbifold vector bundle U corresponding to the factor of automorphy j_2. We may identify $\mathcal{N}$ with the stack quotient $[\mathcal{Y}/\mathcal{G}]$ by Proposition 5.1 and consider the pullback of U on $\mathcal{Y} \subset V_{4,-2} \times V_{1,1}$.

Proposition 7.1. *The Torelli map τ induces an orbifold morphism of degree 2 with the property that the pullback of the orbifold bundle U is the equivariant bundle V. The image of τ is the open part where the cusp form ζ does not vanish.*

Proof. The first statement follows from the construction in Section 5. The second statement follows from [12, Thm. 6.1.3] or even already from Picard's papers [22, 23, 24]. □

8. Teichmüller modular forms

We begin by noting that we have two notions of modular forms here, Picard modular forms and Teichmüller modular forms. On the Hurwitz space $\overline{\mathcal{N}}$ we have the Hodge bundle that agrees on

$$\mathcal{N}^{\mathrm{ct}} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$$

(where 'ct' refers to compact type) with the pullback $\tau^*(\mathbb{E})$ under the Torelli morphism and thus admits a decomposition $\tau^*(\mathbb{E}) = \tau^*(U) \oplus \tau^*(L)$. Furthermore, $\det(\tau^*(U))$ and $\tau^*(L)$ differ by a torsion line bundle $\tau^*(R)^{-1}$.

We will denote $\tau^*(\mathbb{E})$ again by $\mathbb{E}$. We thus can speak of Teichmüller modular forms with a character: a Teichmüller modular form of weight (j, k, l) is a section on $\mathcal{N}^{\mathrm{ct}}$ of

$$\mathbb{E}_{j,k,l} = \mathrm{Sym}^j(\tau^*(U)) \otimes \tau^*(L)^k \otimes \tau^*(R)^l\,.$$

Proposition 8.1. *A section of $\mathbb{E}_{j,k,l}$ over $\mathcal{N}^{\mathrm{ct}}$ extends to a section of $\mathbb{E}_{j,k,l}$ over $\overline{\mathcal{N}}$.*

The proof is a slight adaptation of the proof of Proposition 14.1 in [4] and is omitted.

We can pull back Picard modular forms via the Torelli map. Since the Torelli map is of degree 2, there can be more Teichmüller modular forms than Picard modular forms, that is, Teichmüller modular forms that are not pullbacks of Picard modular forms.

An example of a Teichmüller modular form that is not a Picard modular form on Γ is the form $\zeta^3 \in S_{18}(\Gamma[\sqrt{-3}])$. Since -1_3 acts trivially on $\mathfrak{B}$ and ζ changes sign under -1_3, the form ζ^3 does not live on Γ. But it lives on $\mathcal{N}$ as we now show.

Lemma 8.2. *The form ζ^3 is a Teichmüller modular form of weight* $(0, 18, 3)$

Proof. On $\mathcal{M}_3$ we have a Teichmüller form of weight 9, see [4]. The pullback under the morphism $\mathcal{N}^{\mathrm{ct}} \to \mathcal{M}_3$ of χ_9 under the morphism $\mathcal{N} \to \mathcal{M}_3$ gives a Teichmüller form ζ' of weight $(0, 18, 3)$ on $\mathcal{N}^{\mathrm{ct}}$ that does not vanish outside the divisor of ζ. As a calculation shows, the pullback of χ_9 via $\iota : \mathfrak{B} \to \mathfrak{H}_3$ to $\Gamma[\sqrt{-3}]\backslash\mathfrak{B}$ gives a non-zero multiple of the modular form $\zeta^3 \in S_{18}(\Gamma[\sqrt{-3}])$. Hence ζ' coincides with a non-zero multiple of ζ^3. □

Remark 8.3. The form ζ^3 can be constructed algebraically as Ichikawa does in [14, p. 1059] for χ_9. One observes that the natural map of rank 6 sheaves $\mathrm{Sym}^2(\mathbb{E}) \to \pi_*(\omega_{\mathcal{C}/\mathcal{N}^{\mathrm{ct}}}^{\otimes 2})$ with $\pi : \mathcal{C} \to \mathcal{N}^{\mathrm{ct}}$ the universal curve, is an isomorphism on $\mathcal{N}$, and by [19, Thm 5.10] taking the determinant thus gives a morphism $L^8 \otimes R^{-4} \to L^{26} \otimes R^{-13}$. This morphism extends over $\mathcal{N}^{\mathrm{ct}}$, but vanishes on $\mathcal{N}_2$. This gives a section of $L^{18} \otimes R^3$.

Remark 8.4. If we consider the moduli stack $\mathcal{N}[\Gamma_1]$ of curves of genus 3 that are a triple cyclic cover of $\mathbb{P}^1$ with a Jacobian with a Γ_1-level structure, then ζ is a Teichmüller modular form on $\mathcal{N}[\Gamma_1]$ of weight $(0, 6, 1)$, but it is not a Picard modular form on Γ_1. Its square ζ^2 is a Picard modular form on Γ_1.

The involution -1 (fibrewise) on $\mathbb{E}$ induces an involution θ on the space $H^0(\mathcal{N}^{\mathrm{ct}}, \mathbb{E}_{j,k,l})$ and on $H^0(\overline{\mathcal{N}}, \mathbb{E}_{j,k,l})$. The pullback of $M_{j,k,l}(\Gamma)$ to $\mathcal{N}^{\mathrm{ct}}$ lands in the $(+1)$-eigenspace $H^0(\overline{\mathcal{N}}, \mathbb{E}_{j,k,l})^+$ of θ.

Lemma 8.5. *We have $H^0(\overline{\mathcal{N}}, \mathbb{E}_{j,k,l})^+ = \tau^*(M_{j,k,l}(\Gamma))$.*

Proof. We have $H^0(\mathcal{N}^{\mathrm{ct}}, \tau^*(\mathbb{E}_{j,k,l}))^+ = \tau^*(H^0(\tau(\mathcal{N}^{\mathrm{ct}}), \mathbb{E}_{j,k,l})) = \tau^*(M_{j,k,l}(\Gamma))$, where the last equality follows from the Koecher principle for Picard modular forms. Proposition 8.1 concludes the proof. □

We now show that Teichmüller modular forms can be viewed as Picard modular forms on a congruence subgroup.

Corollary 8.6. *For the (-1)-eigenspace of θ we have*

$$H^0(\overline{\mathcal{N}}, \mathbb{E}_{j,k,l})^- \cong M_{j,k,l}(\Gamma[\sqrt{-3}])^{s[1^4]}\,.$$

Proof. Multiplication by $\zeta \in S_{0,6,1}(\Gamma[\sqrt{-3}])$ maps θ-anti-invariant forms to θ-invariant ones, that is, Picard modular forms. The fact that ζ is $\mathfrak{S}_4$ anti-invariant completes the proof. □

Recall the character ϵ obtained from the sign character on $\mathfrak{S}_4$ as defined in Section 2. We have

$$M_{j,k}(\Gamma_1, \epsilon) = M_{j,k}(\Gamma_1[\sqrt{-3}])^{s[1^4]}\,.$$

We define an index 2 subgroup $\tilde{\Gamma}_1$ of Γ_1 as the kernel of ϵ. Thus a Teichmüller form can be viewed as a Picard modular form on $\tilde{\Gamma}_1$.

9. Covariants of pairs of binary forms of degree 4 and 1

We recall some classical invariant theory. As before we have the $\mathbb{C}$-vector space $V = \langle x_1, x_2\rangle$ and we write V_n for $\mathrm{Sym}^n(V)$, the space of binary forms of degree n. Consider for a given tuple $(n_1, \ldots, n_r)$ the GL_2-representation

$$\mathcal{V} = V_{n_1} \oplus \cdots \oplus V_{n_r}\,.$$

A covariant of $\mathcal{V}$ of order m and degree d is an equivariant polynomial map $\phi : \mathcal{V} \to V_m$ that is homogeneous of degree d:

$$\phi(g \cdot v) = g \cdot \phi(v), \quad \phi(tv) = t^d\, \phi(v) \quad \text{for all } v \in \mathcal{V},\, t \in \mathbb{G}_m.$$

The covariants form a doubly graded ring

$$\mathcal{C}(\mathcal{V}) = \oplus_{d,m} \mathcal{C}(\mathcal{V})_{d,m}\,.$$

A covariant of order $m = 0$ is called an invariant. One can view it as a polynomial in the coefficients of the r-tuple $(f_1, \ldots, f_r) \in \mathcal{V}$ of binary forms that is invariant under the action of SL_2.

Classical invariant theory provides a SL_2-equivariant linear map from $V_m \otimes V_n \to V_{m+n-2k}$ via $f \otimes g \mapsto (f, g)_k$, where the expression $(f, g)_k$ is called the kth-transvectant and it is given by

$$(f,g)_k = \frac{(m-k)!\,(n-k)!}{m!\,n!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\partial^k f}{\partial x_1^{k-j} \partial x_2^j} \frac{\partial^k g}{\partial x_1^j \partial x_2^{k-j}}\,.$$

Covariants of $\mathcal{V}$ can be identified with the invariants of $\mathcal{V} \oplus V_1 \cong \mathcal{V} \oplus V_1^\vee$ via the map that associates to a covariant of order m the transvectant $(\phi(v), l^m)_m$ with $l \in V_1$. For a good reference we refer to Draisma [7].

We are interested in the action of GL_2 on $V_4 \oplus V_1$ and the corresponding covariants. Equivalently, we can look at the invariants of the action on $V_4 \oplus 2\,V_1$. We write f for the covariant which is the universal binary quartic (corresponding to the identity map on V_4) and h and l for the universal linear terms:

$$f = a_0x_1^4 + a_1x_1^3x_2 + a_2x_1^2x_2^2 + a_3x_1x_2^3 + a_4x_2^4, \quad h = b_0x_1 + b_1x_2, \quad \text{and} \quad l = l_0x_1 + l_1x_2,$$

The following result is classical; we refer to [7].

Proposition 9.1. *The* 20 *generating invariants of* $V_4 \oplus 2\,V_1$ *are given by:*

$I_{2,1} = (f,f)_4$, $I_{2,2} = (h,l)_1$, $I_{3,1} = (f,(f,f)_2)_4$, $I_{5,1} = (f,h^4)_4$, $I_{5,2} = (f,h^3l)_4$, $I_{5,3} = (f,h^2l^2)_4$, $I_{5,4} = (f,hl^3)_4$, $I_{5,5} = (f,l^4)_4$, $I_{6,1} = ((f,f)_2,h^4)_4$, $I_{6,2} = ((f,f)_2,h^3l)_4$, $I_{6,3} = ((f,f)_2,h^2l^2)_4$, $I_{6,4} = ((f,f)_2,hl^3)_4$, $I_{6,5} = ((f,f)_2,l^4)_4$, $I_{9,1} = ((f,(f,f)_2)_1,h^6)_6$, $I_{9,2} = ((f,(f,f)_2)_1,h^5l)_6$, $I_{9,3} = ((f,(f,f)_2)_1,h^4l^2)_6$, $I_{9,4} = ((f,(f,f)_2)_1,h^3l^3)_6$, $I_{9,5} = ((f,(f,f)_2)_1,h^2l^4)_6$, $I_{9,6} = ((f,(f,f)_2)_1,hl^5)_6$, $I_{9,7} = ((f,(f,f)_2)_1,l^6)_6$.

We get the generating covariants of $V_4 \oplus V_1$ by substituting $l_0 = -x_2$ and $l_1 = x_1$ in the generating invariants of $V_4 \oplus 2\,V_1$; we denote these covariants by $J_{a,b,c}$, where a is degree in the coefficients of f, b is the degree in the coefficients of h and c is the degree in x_1 and x_2. If we write

$$s : \mathcal{C}(V_4 \oplus 2\,V_1) \to \mathcal{C}(V_4 \oplus V_1)$$

for this substitution, we find the following table for the images under s of the twenty generators:

$I_{2,1}$	$I_{2,2}$	$I_{3,1}$	$I_{5,1}$	$I_{5,2}$	$I_{5,3}$	$I_{5,4}$	$I_{5,5}$	$I_{6,1}$	$I_{6,2}$
$J_{2,0,0}$	$J_{0,1,1}$	$J_{3,0,0}$	$J_{1,4,0}$	$J_{1,3,1}$	$J_{1,2,2}$	$J_{1,1,3}$	$J_{1,0,4}$	$J_{2,4,0}$	$J_{2,3,1}$
$I_{6,3}$	$I_{6,4}$	$I_{6,5}$	$I_{9,1}$	$I_{9,2}$	$I_{9,3}$	$I_{9,4}$	$I_{9,5}$	$I_{9,6}$	$I_{9,7}$
$J_{2,2,2}$	$J_{2,1,3}$	$J_{2,0,4}$	$J_{3,6,0}$	$J_{3,5,1}$	$J_{3,4,2}$	$J_{3,3,3}$	$J_{3,2,4}$	$J_{3,1,5}$	$J_{3,0,6}$

Some simple examples are

$$J_{2,0,0} = (12\,a_0a_4 - 3\,a_1a_3 + a_2^2)/6\,,$$

and $J_{0,1,1} = h$, $J_{1,0,4} = f/70$. The discriminants of f and fh are given by

$$32\,(J_{2,0,0}^3 - 6J_{3,0,0}^2) \quad \text{and} \quad 32\,(J_{2,0,0}^3 - 6J_{3,0,0}^2)J_{1,4,0}^2\,.$$

These invariants satisfy many relations, for example we have

$$5250\,J_{2,4,0}^3 + 26136\,J_{3,6,0}^2 + 1750\,J_{1,4,0}^3J_{3,0,0} - 2625\,J_{1,4,0}^2J_{2,0,0}J_{2,4,0} = 0\,. \tag{6}$$

Remark 9.2. We may view V as the dual of $V^\vee$; then we can view $\mathrm{Sym}^m(V)$ as the set of homogeneous polynomial maps $V^\vee \to \mathbb{C}$ of degree m, and as such it carries a natural left action of $\mathrm{GL}(V)$ by composition. Instead of the representation $\mathcal{V} = \oplus V_{n_i}$ we can also consider twisted cases $\mathcal{V} = \oplus_{i=1}^r V_{n_i,m_i}$ with, as before, $V_{n,m} = \mathrm{Sym}^n(V) \otimes \det(V)^{\otimes m}$ and consider covariants for this GL_2-representation. That is, taking $\mathcal{V}^\vee = \oplus V_{n_i,m_i}^\vee$ we look at GL_2-equivariant embeddings

$$V_{j,k} \hookrightarrow \mathcal{O}(\mathcal{V}^\vee) = \oplus_m \mathrm{Sym}^m(\mathcal{V})\,.$$

10. From covariants to modular forms

A Teichmüller modular form of weight (j,k,l) is a section of $\mathrm{Sym}^j(U)\otimes\det(U)^k\otimes R^{l+k}$ on $\mathcal{N}^{\mathrm{ct}}$. Here we write again U for $\tau^*(U)$. We identify the Hurwitz space $\mathcal{N}$ with the quotient stack $[\mathcal{Y}/\mathcal{G}]$. Under the Torelli morphism $\tau:[\mathcal{Y}/\mathcal{G}]\cong\mathcal{N}\to\Gamma\backslash\mathfrak{B}$ the pullback of the bundle U is the equivariant bundle V. Therefore modular forms pull back to bi-covariants for the action of $\mathcal{G}$ on $\mathcal{Y}$. If the modular form is of weight (j,k,l), that is, a section of $\mathrm{Sym}^j(U)\otimes\det(U)^k\otimes R^{l+k}$, the corresponding bi-covariant lies by Remark 9.2 in a space of bi-covariants that is given as the image of GL_2-equivariant map

$$V_{j,k}\to\mathcal{O}(V_{4,-2}\oplus V_{1,1})\,,$$

where $\mathcal{O}(V_{4,-2}\oplus V_{1,1})$ is the ring of polynomial functions on $V_{4,-2}\oplus V_{1,1}$. The character of the modular form can be read off from the action of the diagonal $\mathbb{G}_m\subset\mathrm{GL}_2$.

Thus we see that a section of $\mathrm{Sym}^j(U)\otimes L^k\otimes R^l$ on $\Gamma\backslash\mathfrak{B}$ pulls back to a covariant for the action of GL_2 on $V_{4,-2}\oplus V_{1,1}$, and this covariant can be identified with a covariant for the (untwisted) action on $V_4\oplus V_1$. Moreover, we are identifying covariants of the action of GL_2 on $V_4\oplus V_1$ with invariants of the action on $V_4\oplus V_1\oplus V_1$ as explained in Section 9. Thus our section provides a covariant $J_{a,b,c}$, where the index (a,b,c) indicates that it has degree a in the a_i, degree b in the b_i and degree c in x_1,x_2. Clearly, we have $j=c$. Moreover, we find $k=(3b-c)/2$, since the action of the diagonal $\mathbb{G}_m\subset\mathrm{GL}_2$ is by t^0 on $V_{4,-2}$, by t^3 on $V_{1,1}$, by t^{-1} on the component V_1, the dual of $V_{1,1}$ but twisted back by $\det^{-1}$, and by t^2 on $\det(V)$. If we start with a Picard modular form, then $a+b+c$ is even.

Let

$$\mathbb{M}=\oplus_{j,k,l}H^0(\overline{\mathcal{N}},\mathbb{E}_{j,k,l})$$

be the ring of modular forms, where the ring structure is obtained in a similar way as for Picard modular forms, see Section 2. Restricting to $\mathcal{N}$ we get a map

$$\mathbb{M}\xrightarrow{\mu}\mathcal{C}(V_4\oplus V_1)\,.$$

Since the image of the Torelli map on $\mathcal{N}$ is the complement of T_1, the locus where the cusp form ζ vanishes, as discussed in Section 4, a covariant defines a meromorphic modular form that is holomorphic outside this divisor. Thus we can complement the map μ by a ring homomorphism

$$\mathbb{M}\xrightarrow{\mu}\mathcal{C}(V_4\oplus V_1)\xrightarrow{\nu}\mathbb{M}[1/\zeta] \tag{7}$$

with the property that $\nu\circ\mu=\mathrm{id}_{\mathbb{M}}$. Note that ζ is a Teichmüller modular form of weight 6 with character det as explained in Remark 8.4.

On our quotient stack $\mathcal{N}$ we have two diagonal sections corresponding to the universal quartic f_4 and universal linear form f_1. We put

$$\chi_{4,-2}=\nu(f_4),\quad \chi_{1,1}=\nu(f_1)\,.$$

Here $\chi_{4,-2}$ (resp. $\chi_{1,1}$) is a meromorphic Teichmüller modular form of weight $(j,k,l)=(4,-2,1)$ (resp. $(1,1,1)$) and we wish to identify these meromorphic modular forms. For this we need an estimate on the pole orders along the curve T_1. By Corollary 8.6 we may view these Teichmüller forms as Picard modular forms on $\Gamma_1[\sqrt{-3}]$.

Lemma 10.1. *The meromorphic modular form $\chi_{4,-2}$ has order -1 along the curve T_1. The meromorphic modular form $\chi_{1,1}$ is holomorphic.*

Proof. In order to prove that the order of both $\chi_{1,1}$ and $\chi_{4,2}$ is at least -1 we may use the restriction of the Teichmüller modular form $\chi_{4,0,-1}$ constructed in [4]. It is known that it has a pole of order 1 along the hyperelliptic locus in $\mathcal{M}_3$. The relation between $\chi_{4,0,-1}$ and the pair $(\chi_{4,-2}, \chi_{1,1})$ is provided by Remark 5.2. From this we can conclude the result since the order of $\chi_{1,1}$ satisfies a congruence condition, see (8) below. But we shall give a direct argument that gives more information.

We may view $\chi_{1,1}$ and $\chi_{4,-2}$ as meromorphic Picard modular forms on $\Gamma_1[\sqrt{-3}]$. We start with the Taylor expansion along T_1 given by $u = 0$

$$\chi_{1,1} = \begin{pmatrix} \chi_{1,1}^{(0)} \\ \chi_{1,1}^{(1)} \end{pmatrix} \qquad \text{with} \quad \chi_{1,1}^{(m)} = \sum_{n \geq r} f_n^{(m)} u^n \,.$$

We assume that $\chi_{1,1}$ has order r along T_1. Using the action of $(1,1,\rho)$ and -1_3 we see that a non-zero term $f_n^{(m)}$ satisfies the congruence condition

$$n \equiv m \,(\mathrm{mod}\, 6)\,. \tag{8}$$

In particular, by a slight variant of Proposition 4.3, a non-zero term $f_r^{(m)}$ is a meromorphic modular form of weight $1+m+r$ on $\Gamma_1(3)$, regular outside (the orbit of) $\tau_0 = (1-\rho^2)/3$ and with order at least r at τ_0. The space $M_1^{(1)}(\Gamma_1(3))$ is generated by $\vartheta = \sum_{\alpha \in O_F} q^{N(\alpha)}$ and we know that $M_r^{(r)}(\Gamma_1(3))$ is generated by ϑ^r, as explained in (4) at the end of Section 3. This implies that $f_r^{(m)}$ is divisible by ϑ^r, so $f_r^{(m)} = \vartheta^r \varphi$ with $\varphi \in M_{m+1}(\Gamma_1(3))$ for $m = 0$ or 1. But then φ is a non-zero multiple of ϑ^{m+1}, implying that $f_r^{(m)}$ is a non-zero multiple of ϑ^{r+m+1}. The anti-invariance of $\chi_{1,1}$ implies that $r \equiv 0 \,(\mathrm{mod}\, 6)$. Thus the order of $\chi_{1,1}$ equals the order of $\chi_{1,1}^{(0)}$.

Since holomorphic Picard modular forms have weight $(j \geq 0, k \geq 0)$, the order s of $\chi_{4,-2}$ along T_1 is negative. If we write $\chi_{4,-2}^{(m)} = \sum_{n \geq s} g_n^{(m)} u^n$ for $m = 0, \ldots, 4$, then for non-zero $g_n^{(m)}$ we have

$$n \equiv m + 3 \,(\mathrm{mod}\, 6)\,. \tag{9}$$

Moreover, $g_s^{(m)} \in M_{-2+s+m}^{(s)}(\Gamma_1(3))$. For non-zero $M_k^{(s)}(\Gamma_1(3))$ we need $-2+s+m \geq s$, hence using the congruence restriction (9), we see $m = 2$ for non-zero $g_s^{(m)}$ and it is a non-zero multiple of ϑ^s. We then have that $g_{s+1}^{(m)} = 0$ unless $m = 3$, and in fact it is quasi-modular and one observes that it is not zero by applying the Equation 4.4 to ϑ^s. Similarly $g_{s+2}^{(m)} = 0$ unless $m = 4$. Using again Equation 4.4 we see $\mathrm{ord}(\chi_{4,-2}^{(4)}) = \mathrm{ord}(\chi_{4,-2}^{(2)}) + 2$.

The discriminant of f_4 and of $f_4 f_1$ are invariants that define scalar-valued modular forms. These invariants are given by $\Delta_4 = J_{2,0,0}^3 - 6J_{3,0,0}^2$ and $\Delta_4\, J_{1,4,0}^2$ up to non-zero multiplicative scalars. The weight of $\nu(\Delta_4)$ is 0 and that of $\nu(J_{1,4,0})$ is 6, and these are units outside T_1. Therefore $\nu(\Delta_4)$ is constant and $\nu(J_{1,4,0})$ must be a multiple of ζ. Now we have $J_{1,4,0} = a_0 b_1^4 - a_1 b_0 b_1^3 + a_2 b_0^2 b_1^2 - a_3 b_0^3 b_1 + a_4 b_0^4$ and we can vary a_4 and b_0, while keeping b_1 and $a_0, \ldots, a_3$ fixed. Then the term $a_4 b_0^4$ must yield under ν a regular expression and we infer that

$$\mathrm{ord}(g^{(4)}) + 4\,\mathrm{ord}(f^{(0)}) = s + 2 + 4\,r \geq 0\,,$$

where we write $g^{(4)} = \chi_{4,-2}^{(4)}$ and $f^{(0)} = \chi_{1,1}^{(0)}$, hence $r \geq 0$ and $\chi_{1,1}$ is holomorphic. Since $\dim M_{1,1}(\Gamma[\sqrt{-3}], \det) = 1$ we can identify $\chi_{1,1}$ with a non-zero multiple of a generator $E_{1,1}$ of $M_{1,1}(\Gamma[\sqrt{-3}], \det)$ constructed in [5]. We conclude from the development given there that it has order 0 with $f_0^{(0)}$ being a non-zero multiple of ϑ and that $\chi_{1,1}^{(1)}$ has order 1.

Keeping now b_0, b_1 fixed and varying one of $a_0, \ldots, a_4$, we see that all terms $a_0 b_1^4$, $a_1 b_0 b_1^3$, $a_2 b_0^2 b_1^2$, $a_3 b_0^3 b_1$ in $J_{1,4,0}$ must give regular forms. Using this regularity we find that

$$\mathrm{ord}(g^{(0)}, \ldots, g^{(4)}) = (\geq 1, \geq 1, \geq -1, \geq 0, \geq 1)\,.$$

But using the congruence condition (9) we see

$$\mathrm{ord}(g^{(0)}, \ldots, g^{(4)}) = (\geq 3, \geq 4, = -1, \geq 0, \geq 1)\,,$$

which proves that the order of $\chi_{4,-2}$ along T_1 equals -1. □

Corollary 10.2. *The modular form $\chi_{1,1}$ generates $M_{1,1}(\Gamma[\sqrt{-3}], \det)$. The modular form $\zeta\chi_{4,-2}$ generates $M_{4,4}(\Gamma[\sqrt{-3}], \det^2)$.*

Proof. By [5] we know that $\dim M_{1,1}(\Gamma[\sqrt{-3}], \det) = 1$. In [1] it was shown that $\dim S_{4,4}(\Gamma[\sqrt{-3}], \det^2) = 1$. As ζ vanishes along the curve T_1 it follows that $\zeta\chi_{4,-2}$ is regular and generates $S_{4,4}(\Gamma[\sqrt{-3}], \det^2)$. □

A generator $\chi_{4,4}$ of $S_{4,4}(\Gamma[\sqrt{-3}], \det^2)$ will be constructed explicitly in Section 12. In the paper [5] we constructed explicitly an Eisenstein series $E_{1,1} \in M_{1,1}(\Gamma[\sqrt{-3}], \det)$. Hence up to a non-zero multiplicative constant $\chi_{1,1}$ agrees with $E_{1,1}$.

We can write the meromorphic modular form $\chi_{4,-2}$, when viewed as a meromorphic Picard modular form on $\Gamma_1[\sqrt{-3}]$, as

$$\chi_{4,-2} = \sum_{i=0}^{4} \alpha_i X_1^{4-i} X_2^i\,, \tag{10}$$

where the X_1, X_2 are dummy variables to indicate the coordinates of V and the α_i are meromorphic functions on the the 2-ball $\mathfrak{B}$. Similarly, we can write $E_{1,1}$ as

$$E_{1,1} = \beta_1 X_1 + \beta_2 X_2 \tag{11}$$

with β_i holomorphic on $\mathfrak{B}$.

In Section 15 we shall derive the beginning of the Taylor expansion along $u = 0$ of the generators $\chi_{4,4}$ of $S_{4,4}(\Gamma, \det^2)$ and $E_{1,1}$ of $M_{1,1}(\Gamma, \det)$. This gives the orders, see Corollary 15.3. As a corollary we find the orders of the α_i and β_i along the curve given by $u = 0$.

Corollary 10.3. *The orders of $(\alpha_0, \ldots, \alpha_4)$ and (β_1, β_2) along T_1 are*

$$\mathrm{ord}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3, 4, -1, 0, 1)\,, \quad \mathrm{ord}(\beta_1, \beta_2) = (0, 1)\,.$$

Proof. As observed in Section 4 the modular form ζ vanishes simply along the components of T_1 on $\Gamma_1[\sqrt{-3}]\backslash\mathfrak{B}$. Together with the orders of $\chi_{4,4}$ this proves the result. □

Now we can describe the map ν. Recall that we write a covariant as a polynomial of degree a in the coefficients a_i of f_4, of degree b in the coefficients of f_1 and degree c in x_1, x_2. The map ν amounts to substituting the coordinates α_i ($i = 0, \ldots, 4$) and β_i ($i = 0, 1$) in a covariant. For simplicity we will view the elements of $\mathbb{M}[1/\zeta]$ as Picard modular forms on $\Gamma_1[\sqrt{-3}]$, see the description in Section 8.

Theorem 10.4. *The map $\nu : \mathcal{C}(V_4 \oplus V_1) \to \mathbb{M}[1/\zeta]$ is given by substituting α_i for a_i, β_i for b_i and X_i for x_i in a covariant. The map ν sends an invariant $J_{a,b,c}$ of degree a in the a_i, degree b in the b_i and degree c in x_1, x_2 to a meromorphic modular form of weight $(j, k, l) = (c, (3b - c)/2, 2(a + b + c))$. The form $\nu(f)$ is $\mathfrak{S}_4$-invariant if $a + b$ even, and $\mathfrak{S}_4$-anti-invariant if $a + b$ odd.*

Proof. This follows from the identities (10) and (11). □

Corollary 10.5. *All modular forms on Γ can be constructed by substituting the coordinates of $\chi_{4,-2}$ and $E_{1,1}$ in covariants.*

Proof. The composition $\mathbb{M} \to \mathcal{C}(V_4 \oplus V_1) \to \mathbb{M}[1/\zeta]$ is the identity on $\mathbb{M}$. Indeed, the map μ interprets modular forms in terms of bi-covariants and the map ν re-interprets a bi-covariant as a (a priori meromorphic) Teichmüller modular form. But ν is given by substituting the coordinates of $\chi_{4,-2}$ and $\chi_{1,1}$. The form $\chi_{1,1}$ is a non-zero multiple of $E_{1,1}$. □

Remark 10.6. Given generators $\chi_{4,4}$ of $S_{4,4}(\Gamma[\sqrt{-3}], \det^2)$ and $E_{1,1}$ of $M_{1,1}(\Gamma[\sqrt{-3}], \det)$ one can show directly using the modular behavior of $\chi_{4,4}, E_{1,1}$ and ζ, that if we write

$$\chi_{4,4}/\zeta = \sum_{i=0}^{4} \alpha'_i X_1^{4-i} X_2^i, \quad \chi_{1,1} = \beta'_1 X_1 + \beta'_2 X_2\,,$$

the substitution of α'_i for a_i, β'_i for b_i and X_i for x_i in a covariant of multi-degree (a, b, c) gives a modular form of weight $(c, (3b - c)/2, 2(a + b + c))$.

The orders of the modular forms $\nu(J)$ along the curve T_1 for the generating invariants given in Section 9 can be deduced from Corollary 10.3. We give a table.

$J_{2,0,0}$	$J_{0,1,1}$	$J_{3,0,0}$	$J_{1,4,0}$	$J_{1,3,1}$	$J_{1,2,2}$	$J_{1,1,3}$	$J_{1,0,4}$	$J_{2,4,0}$	$J_{2,3,1}$
-2	0	-3	1	0	-1	-1	-1	0	-1
$J_{2,2,2}$	$J_{2,1,3}$	$J_{2,0,4}$	$J_{3,6,0}$	$J_{3,5,1}$	$J_{3,4,2}$	$J_{3,3,3}$	$J_{3,2,4}$	$J_{3,1,5}$	$J_{3,0,6}$
-2	-2	-2	0	-1	-1	-1	-1	-1	-1

11. Gradients of theta functions

In order to use Theorem 10.4 effectively it is important to know the Fourier-Jacobi expansions of ζ, $\chi_{4,-2}$ (or $\chi_{4,4}$) and $E_{1,1}$ quite well. In this section and the next one we construct these modular forms and give part of their Fourier-Jacobi expansion. We will use gradients of theta series to construct modular forms.

Recall the definition of theta series with characteristics (see [16], p.49): let $g \in \mathbb{Z}_{\geqslant 1}$, $(\mu_1, \ldots, \mu_g) \in \mathbb{R}^g$, $(\nu_1, \ldots, \nu_g) \in \mathbb{R}^g$ and set for $\tau \in \mathfrak{H}_g$, the Siegel upper half space of degree g, and $z = (z_1, \ldots, z_g) \in \mathbb{C}^g$

$$\vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau, z) = \vartheta_{\left[\begin{smallmatrix}\mu_1,\ldots,\mu_g\\ \nu_1,\ldots,\nu_g\end{smallmatrix}\right]}(\tau, z) = \sum_{n=(n_1,\ldots,n_g)\in\mathbb{Z}^g} e^{\pi i(n+\mu)\left(\tau(n+\mu)^t+2(z+\nu)^t\right)} .$$

We simply call this series a theta series with characteristics. Its restriction to $z = 0$ is called a theta constant, and we omit the variable z in this case: $\vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau, 0) = \vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau)$. We have the formulas (see loc. cit.)

$$\vartheta_{-\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau, z) = \vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau, -z) \qquad \text{and} \qquad \vartheta_{\left[\begin{smallmatrix}\mu+m\\ \nu+n\end{smallmatrix}\right]}(\tau, z) = e^{2\pi i\mu n^t}\vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau, z)$$

for any $(m, n) \in \mathbb{Z}^g \times \mathbb{Z}^g$.

From [16, p. 85, p. 176] we deduce the following transformation formula for the gradient $\nabla\vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}$ (written as a column vector) of the theta function $\vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}$ as a function of z.

Proposition 11.1. *Let $\gamma = (a, b; c, d) \in \mathrm{Sp}(2g, \mathbb{Z})$ then we have*

$$\nabla\vartheta_{\gamma\cdot\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\gamma\tau, z(c\tau+d)^{-1}) = j(\gamma, \tau, z)\,\left((c\tau + d)\nabla\vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau, z) + \vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\tau, z)\nabla(e^{\pi i z(c\tau+d)^{-1}c\,z^t})\right) ,$$

where

$$j(\gamma, \tau, z) = \kappa(\gamma)e^{\pi i\phi(\gamma,\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right])}\det(c\tau + d)^{\frac{1}{2}}e^{\pi i z(c\tau+d)^{-1}c\,z^t}$$

with

$$\phi(\gamma, \left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]) = -\mu d^t b\mu^t + 2\mu b^t c\nu^t - \nu c^t a\nu^t + (\mu d^t - \nu c^t)(ab^t)_0$$

$$\gamma\cdot\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right] = \left[\left(\begin{smallmatrix}d & -c\\ -b & a\end{smallmatrix}\right)\left(\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right) + \frac{1}{2}\left(\begin{smallmatrix}(cd^t)_0\\ (ab^t)_0\end{smallmatrix}\right)\right] = \left[\begin{smallmatrix}d\mu-c\nu+(cd^t/2)_0\\ -b\mu+a\nu+(ab^t/2)_0\end{smallmatrix}\right].$$

Moreover $\kappa(\gamma)$ is an eighth root of unity (depending only on γ) and the symbol X_0 denotes the diagonal (column) vector (in its natural order) of a matrix X.

As a direct corollary of Proposition 11.1, we have for theta functions vanishing on $\mathfrak{B} \times (0) \subset \mathfrak{B} \times \mathbb{C}^3$ the following.

Corollary 11.2. *Assume that $\vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\iota(u, v), 0) = 0$ for any $(u, v) \in \mathfrak{B}$. Then for any $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{Sp}(6, \mathbb{Z})$ we have*

$$\nabla\vartheta_{\gamma\cdot\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\gamma\,\iota(u, v), 0) = \kappa(\gamma)e^{\pi i\phi(\gamma,\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right])}\det(c\,\iota(u, v) + d)^{\frac{1}{2}}(c\,\iota(u, v) + d)\nabla\vartheta_{\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right]}(\iota(u, v), 0).$$

If C is a smooth projective curve of genus 3 that is a triple cyclic cover of $C/\alpha = \mathbb{P}^1$, the kernel of multiplication by $1 - \alpha$ on its Jacobian $\mathrm{Jac}(C)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$. This is a totally isotropic subspace for the Weil pairing on $\mathrm{Jac}(C)[3]$. If the ramification points of $C \to \mathbb{P}^1$ are $p_0, p_1, \ldots, p_4$, with p_0 the unique point with ρ^2 action, we have a surjective homomorphism

$$\phi : (\mathbb{Z}/3\mathbb{Z})^4 \to \mathrm{Jac}(C)[1 - \alpha], \quad (c_1, \ldots, c_4) \to \sum c_i(p_i - p_0)$$

with kernel given by $\sum_{i=1}^{4} c_i = 0$. The group $\mathfrak{S}_4 \times \mu_2$ acts and can be seen as the orthogonal group for $(O_F/\sqrt{-3}\,O_F)^3$ with the form $ab' + ab' + cc'$.

We can identify the canonical theta divisor $\Theta \subset \mathrm{Pic}^{(2)}(C)$ with a theta divisor in $\mathrm{Jac}(C)$ by translation over $\kappa = 2p_0$, a half-canonical divisor. There are fifteen $(1-\alpha)$-torsion points lying on a theta divisor $\Theta - 2p_0 \subset \mathrm{Jac}(C)$, namely: $p_i + p_j - 2p_0$ for $0 \leq i \leq j \leq 4$ and $(i,j) \neq (0,0)$. Note the linear equivalence $p_1 + p_2 + p_3 + p_4 \sim 4p_0$.

The set of 15 torsion points on the theta divisor can be divided into three sets:

$$C_1 = \{0, \pm(p_i - p_0), i = 1, \ldots 4\}, \quad C_2 = \{p_i + p_j - 2p_0 : 1 \leq i < j \leq 4\}$$

and C_0 the complement of $C_1 \sqcup C_2$ with cardinalities $\#C_0 = 12$, $\#C_1 = 9$ and $\#C_2 = 6$.

If we let the theta characteristic $2p_0$ correspond to $\left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right] = \left[\begin{smallmatrix}0 & 1/2 & 0\\ 0 & 1/2 & 0\end{smallmatrix}\right]$, then using the embedding σ of $\Gamma[\sqrt{-3}]$ in $\mathrm{Sp}(6,\mathbb{Z})$ and the property that

$$\sigma(g) \cdot \left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right] \equiv \left[\begin{smallmatrix}\mu\\ \nu\end{smallmatrix}\right] \bmod \mathbb{Z}$$

for any of $g \in \Gamma[\sqrt{-3}]$, we find that the set C_i of $(1-\alpha)$-torsion points corresponds to the set of theta characteristics of degree 3

$$\left\{ \left[\begin{smallmatrix} k/3 & (2l+1)/6 & -k/3 \\ m/3 & (2l+1)/6 & m/3 \end{smallmatrix}\right] : 0 \leq k, l, m \leq 2 : km - (l-1)^2 \equiv i - 1 (\mathrm{mod} 3) \right\}.$$

We will abbreviate such a characteristic by $[klm]$.

Lemma 11.3. *Writing* $\sigma(g) = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ *for* $g \in \Gamma[\sqrt{-3}]$ *and* $\iota(u,v) = \tau$ *for* $(u,v) \in \mathfrak{B}$ *we have*

$$(c\tau + d) \begin{bmatrix} v_1 \\ \left(\begin{smallmatrix}1 & 0\\ 0 & -\rho^2\end{smallmatrix}\right)\left[\begin{smallmatrix}v_2\\ v_1\end{smallmatrix}\right] \end{bmatrix} = \begin{bmatrix} \mathrm{pr}_2(j_2(g,(u,v))\left[\begin{smallmatrix}v_2\\ v_1\end{smallmatrix}\right]) \\ \left(\begin{smallmatrix}1 & 0\\ 0 & -\rho^2\end{smallmatrix}\right) j_2(g,(u,v))\left[\begin{smallmatrix}v_2\\ v_1\end{smallmatrix}\right] \end{bmatrix},$$

where pr_2 *denotes the projection* $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right] \mapsto y$.

The proof can be carried out by checking it on the generators of $\Gamma[\sqrt{-3}]$. A set of generators was given in Lemma 2.1.

From the set C_1 we take representatives modulo changing sign

$$C_1' = \{[011], [110], [101], [202], [010]\}$$

and for $\lambda \in C_1'$ with $g_0 = \rho 1_3$ and $\sigma(g_0) = (a_0, b_0; c_0, d_0) \in \mathrm{Sp}(6,\mathbb{Z})$ we have

$$\nabla\vartheta_\lambda(\iota(u,v),0) = \rho(c_0\iota(u,v) + d_0)\nabla\vartheta_\lambda(\iota(u,v),0) = \left[\begin{smallmatrix} 0 & 0 & -\rho \\ -u & 1 & -\rho u \\ \rho & 0 & -\rho \end{smallmatrix}\right] \nabla\vartheta_\lambda(\iota(u,v),0)$$

and this implies that

$$\frac{\partial\vartheta_\lambda}{\partial z_3}(\iota(u,v),0) = -\rho^2\,\frac{\partial\vartheta_\lambda}{\partial z_1}(\iota(u,v),0).$$

So for the characteristics in the set C_1, only the last component of the gradient of ϑ_λ is dependent of the previous one. This gives a $(2+1)$–decomposition and therefore good hopes to get vector-valued modular forms.

For the five elements $\lambda_i \in C_1'$ $(i = 0, \ldots, 4)$ listed in the order above we put

$$F_i(u,v) = c^{-1} \begin{bmatrix} \frac{\partial\vartheta_{\lambda_i}}{\partial z_2}(\iota(u,v),0) \\ \frac{\partial\vartheta_{\lambda_i}}{\partial z_1}(\iota(u,v),0) \end{bmatrix} \tag{12}$$

with the constant c given by

$$c = \vartheta_{\left[\begin{smallmatrix}1/6\\1/6\end{smallmatrix}\right]}(-\rho^2, 0) = \frac{3^{3/8}}{2\pi}\Gamma(1/3)^{3/2} e^{\frac{5\pi i}{72}}\,. \tag{13}$$

Using Corollary 11.2 we then obtain:

Lemma 11.4. *Let $\lambda = [klm] \in C_1'$ and $g \in \Gamma[\sqrt{-3}]$ and write $\sigma(g)\cdot\lambda = \lambda + \left[\begin{smallmatrix}m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{smallmatrix}\right]$. Then $F_i(g\cdot(u,v)) = A_i(g,u,v)\,F_i(u,v)$ with $A_i(g,u,v)$ given by*

$$\kappa(\sigma(g))\, e^{\pi\sqrt{-1}\phi(\sigma(g),c_{1i})}\, e^{-\frac{2\pi\sqrt{-1}}{6}(2(n_1-n_3)k+n_2(2l+1))}\, j_1(g,(u,v))j_2(g,(u,v))\,.$$

This gives us the transformation behavior of the F_i under the generators g_i of the group $\Gamma[\sqrt{-3}]$ given in Lemma 2.1.

Corollary 11.5. *We have $F_{i|1,1}g_j = c(i,j)F_i$ with $c(i,j)$ given in the following diagram.*

$i\backslash j$	0	1	2	3	4	5
0	1	ρ	ρ	ρ^2	1	1
1	1	ρ	1	1	ρ	ρ^2
2	1	1	ρ	ρ	ρ	ρ
3	1	1	ρ	1	ρ	1
4	1	ρ	1	1	1	1

12. Construction of $\chi_{4,4}$ and $E_{1,1}$.

Finis constructed in [9] elliptic modular functions X, Y, Z for the elliptic curve $\mathbb{C}/\Lambda$ with $\Lambda = \sqrt{-3}\,O_F$, that satisfy the relation

$$X^3 = \rho\,(Y^3 - Z^3)\,.$$

Here X, Y and Z are normalized theta functions satisfying

$$f(z+\alpha) = \exp\left(\frac{2\pi}{\sqrt{3}}(\bar{\alpha}z - \rho N(\alpha))\right) f(z) \qquad (\alpha \in \Lambda)$$

and $Z(z) = Y(-z)$. The zero divisor of X is the degree 3 divisor O_F mod Λ and that of Y is $1/\sqrt{-3} + O_F$ mod Λ. These functions can be defined by

$$X(z) = \frac{1}{c} e^{\pi z^2/\sqrt{3}} \vartheta_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(-\rho^2, z), \quad Y(z) = \frac{1}{c} e^{\pi z^2/\sqrt{3}} \vartheta_{\left[\begin{smallmatrix}1/6\\1/6\end{smallmatrix}\right]}(-\rho^2, z),$$

with $c = \vartheta_{\left[\begin{smallmatrix}1/6\\1/6\end{smallmatrix}\right]}(-\rho^2, 0)$ as given in (13). The functions X and Y satisfy for $\alpha \in O_F$

$$X(z+\alpha) = e^{\frac{2\pi}{\sqrt{3}}(\bar{\alpha}z-\rho N(\alpha))} X(z), \qquad Y(z+\alpha) = e^{\frac{2\pi}{\sqrt{3}}(\bar{\alpha}z-\rho N(\alpha))} \rho^{\mathrm{Tr}(\alpha)} Y(z)\,.$$

and $X(\epsilon z) = \epsilon X(z)$ for $\epsilon \in O_F^\times$ and $Y(\rho z) = Y(z)$. Moreover, we have the identity

$$X(\sqrt{-3}\,z) = \sqrt{-3}\,X(z)Y(z)Z(z)\,.$$

We can develop the theta functions ϑ_λ for $\lambda \in C_1'$ in Fourier-Jacobi series. Working this out and substituting this in the F_i as defined in equation (12), one finds after some amount of calculation the following Fourier-Jacobi series of the F_i.

We set

$$\xi = (\rho^2 - 1)/3 \qquad \text{and} \qquad q_v = e^{2\pi v/\sqrt{3}}$$

and obtain

$$\begin{aligned}
F_0(u,v) &= \sum_{\alpha \in O_F} \rho^{-Tr(\alpha)} \begin{bmatrix} X'(\alpha u) \\ \frac{2\pi}{\sqrt{3}}\bar{\alpha} X(\alpha u) \end{bmatrix} q_v^{N(\alpha)}, \\
F_1(u,v) &= \sum_{\alpha \in O_F + \xi} \begin{bmatrix} X'(\alpha u) \\ \frac{2\pi}{\sqrt{3}}\bar{\alpha} X(\alpha u) \end{bmatrix} q_v^{N(\alpha)}, \\
F_2(u,v) &= \sum_{\alpha \in O_F + \xi} e^{\frac{2\pi}{\sqrt{3}}(\alpha\bar{\xi} - \bar{\alpha}\xi)} \begin{bmatrix} Y'(\alpha u) \\ \frac{2\pi}{\sqrt{3}}\bar{\alpha} Y(\alpha u) \end{bmatrix} q_v^{N(\alpha)}, \\
F_3(u,v) &= \sum_{\alpha \in O_F + \xi} e^{\frac{2\pi}{\sqrt{3}}(\alpha\bar{\xi} - \bar{\alpha}\xi)} \begin{bmatrix} Z'(\alpha u) \\ \frac{2\pi}{\sqrt{3}}\bar{\alpha} Z(\alpha u) \end{bmatrix} q_v^{N(\alpha)}, \\
F_4(u,v) &= \sum_{\alpha \in O_F} \begin{bmatrix} X'(\alpha u) \\ \frac{2\pi}{\sqrt{3}}\bar{\alpha} X(\alpha u) \end{bmatrix} q_v^{N(\alpha)}.
\end{aligned}$$

Using the transformation behavior of X, Y, Z one can calculate the transformation of the F_i under the generators r_k for $k = 1, 2, 3$ of the $\mathfrak{S}_4$-part, see Section 2. Putting all of that together, we get

i	$F_0\|_{1,1}r_i^{-1}$	$F_1\|_{1,1}r_i^{-1}$	$F_2\|_{1,1}r_i^{-1}$	$F_3\|_{1,1}r_i^{-1}$	$F_4\|_{1,1}r_i^{-1}$
1	$-F_1$	$-F_0$	$-F_2$	$-F_3$	$-F_4$
2	$-F_0$	$-F_1$	$-F_3$	$-F_2$	$-F_4$
3	F_0	$e^{5\pi i/9}F_2$	$e^{-2\pi i/3}F_3$	$e^{\pi i/9}F_1$	F_4

After these preparations we can construct the two basic modular forms. We put

$$\chi_{4,4} = \mathrm{Sym}^4(F_0, F_1, F_2, F_3) \quad \text{and} \quad E_{1,1} = F_4\,.$$

Corollary 12.1. *We have $\chi_{4,4} \in S_{4,4}(\Gamma[\sqrt{-3}], \det^2)$ and $E_{1,1} \in M_{1,1}(\Gamma[\sqrt{-3}], \det)$.*

Remark 12.2. Both $\chi_{4,4}$ and $E_{1,1}$ are Hecke eigenforms. But the eigenvalues of $E_{1,1}$ are not always integral, see [5, Remark 12.3].

In [1, Section 11.3,Case 1] it is conjectured that there is a lift from $S_8^-(\Gamma_0(9))$ (see loc. cit., Section 11.1 for the notation) to the $\mathfrak{S}_4$-invariant part of the space $S_{4,4}(\Gamma[\sqrt{-3}], \det^2)$. The space $S_8^-(\Gamma_0(9))$ is one-dimensional and generated by one form, say f, whose Fourier expansion $f(\tau) = \sum_{n\geqslant 1} a_n(f)q^n$ can be normalised as

$$f(\tau) = q + 232\,q^4 + 260\,q^7 - 5760\,q^{10} + 6890\,q^{13} + 7744\,q^{16} + 33176\,q^{19} + \dots\,.$$

The claim that $\chi_{4,4}$ is a lift of f is supported by the fact the Hecke eigenvalue of $\chi_{4,4}$ at $1 + 3\rho$ (of norm 7) is given by

$$\lambda_{1+3\rho}(\chi_{4,4}) = 309 - 882\rho.$$

This Hecke eigenvalue is computed directly by using the Fourier-Jacobi expansion of the last component of $\chi_{4,4}$ (recall that the last component of a vector-valued Picard modular Hecke eigenform is sufficient to compute its Hecke eigenvalues). Since the Fourier-Jacobi expansion of $\chi_{4,4}^{(4)}$ (after suitable normalisation) starts with $X^2 q_v^2$, for computing its Hecke eigenvalue at $1+3\rho$, we need its Fourier-Jacobi coefficient at q_v^{14}. This Fourier-Jacobi coefficient will be given in the next section and this gives

$$\lambda_{1+3\rho}(\chi_{4,4}) = 309 - 882\rho = 260 + (1+3\rho^5)(1+3\rho^2)^2 = a_7(f) + (1+3\rho^5)(1+3\rho^2)^2.$$

13. The Fourier-Jacobi expansion of $\chi_{1,1}$

Here we develop the form $E_{1,1}$ in a Fourier-Jacobi series. Recall the definition of $E_{1,1}$ with $q_v = e^{2\pi v/\sqrt{3}}$

$$E_{1,1}(u,v) = \sum_{\alpha \in O_F} \begin{bmatrix} X'(\alpha u) \\ \frac{2\pi}{\sqrt{3}} \bar{\alpha} X(\alpha u) \end{bmatrix} q_v^{N(\alpha)}.$$

Using the relation $X(\epsilon u) = \epsilon X(u)$ for $\epsilon \in O_F^\times$, we get

$$E_{1,1}(u,v) = \begin{bmatrix} X'(0) \\ 0 \end{bmatrix} + 6 \begin{bmatrix} X'(u) \\ \frac{2\pi}{\sqrt{3}} X(u) \end{bmatrix} q_v + 6 \begin{bmatrix} X'(u\sqrt{-3}) \\ -\frac{2\pi}{\sqrt{3}} \sqrt{-3} X(u\sqrt{-3}) \end{bmatrix} q_v^3 + \dots$$

and using the Shintani operators (see [5, §4]), we can rewrite this as

$$E_{1,1}(u,v) = \begin{bmatrix} X'(0) \\ 0 \end{bmatrix} + 6 \begin{bmatrix} X'(u) \\ \frac{2\pi}{\sqrt{3}} X(u) \end{bmatrix} q_v + 6 \begin{bmatrix} (XYZ)'(u) \\ \frac{2\pi}{\sqrt{3}} 3XYZ(u) \end{bmatrix} q_v^3 + \dots$$

We set

$$P_n(X,Y,Z) = \sum_{\alpha \in N_n} \bar{\alpha} X(\alpha u),$$

where we use the notation

$$N_n = \{\alpha = a + b\rho \in O_F \mid N(\alpha) = a^2 - ab + b^2 = n\}.$$

If $P_n'(X,Y,Z)$ denotes the derivative of $P_n(X,Y,Z)$ with respect to the variable u

$$P_n'(X,Y,Z) = \Big(\sum_{\alpha \in N_n} \bar{\alpha} X(\alpha u)\Big)' = \sum_{\alpha \in N_n} \alpha \bar{\alpha} X'(\alpha u) = n \sum_{\alpha \in N_n} X'(\alpha u),$$

we have

$$E_{1,1}(u,v) = \begin{bmatrix} X'(0) \\ 0 \end{bmatrix} + \sum_{n \geqslant 1} \begin{bmatrix} P_n'(X,Y,Z)/n \\ \frac{2\pi}{\sqrt{3}} P_n(X,Y,Z) \end{bmatrix} q_v^n.$$

In order to get more terms in the Fourier-Jacobi expansion of $E_{1,1}$ we write the set N_n as $N_n = \alpha_1 O_F^\times \sqcup \alpha_2 O_F^\times \sqcup \ldots \sqcup \alpha_j O_F^\times$ and split P_n according to this decomposition:

$$P_n(X,Y,Z) = \sum_{\alpha \in N_n} \bar{\alpha} X(\alpha u) = \sum_{i=1}^{j} \sum_{\epsilon \in O_F^\times} \bar{\epsilon} \bar{\alpha}_i X(\epsilon \alpha_i u) = \sum_{i=1}^{j} \sum_{\epsilon \in O_F^\times} \bar{\alpha}_i X(\alpha_i u) = 6 \sum_{i=1}^{j} \bar{\alpha}_i X(\alpha_i u)$$

The polynomials P_n are homogeneous of degree n in X, Y and Z and the first few of them are given by

$$\begin{aligned} P_1 &= 6\,X; \quad P_3 = 18\,XYZ; \quad P_4 = 12\,X(Y^3+Z^3); \\ P_7 &= -6\,X(Y^6-16Y^3Z^3+Z^6); \quad P_9 = 54\,XYZ(Y^6-Y^3Z^3+Z^6); \\ P_{12} &= -36\,XYZ(2Y^9-3Y^6Z^3-3Y^3Z^6+2YZ^9); \\ P_{13} &= 6\,X(5Y^{12}-7Y^9Z^3+30Y^6Z^6-7Y^3Z^9+5Z^{12})\,. \end{aligned}$$

14. The Fourier-Jacobi expansion of $\chi_{4,4}$

The Fourier-Jacobi expansion of $\chi_{4,4}$ is determined by those of $F_0, \ldots, F_3$. We determine these and start with F_0. We set

$$Q_n(X,Y,Z) = \sum_{\alpha \in N_n} \rho^{-Tr(\alpha)} \bar{\alpha} X(\alpha u)\,,$$

which gives for the derivative of $Q_n(X,Y,Z)$ with respect to the variable u

$$Q_n'(X,Y,Z) = n \sum_{\alpha \in N_n} \rho^{-Tr(\alpha)} X'(\alpha u)$$

and this leads to the expansion

$$F_0(u,v) = \sum_{\alpha \in O_F} \rho^{-Tr(\alpha)} \begin{bmatrix} X'(\alpha u) \\ \frac{2\pi}{\sqrt{3}} \bar{\alpha} X(\alpha u) \end{bmatrix} q_v^{N(\alpha)} = \begin{bmatrix} X'(0) \\ 0 \end{bmatrix} + \sum_{n \geqslant 1} \begin{bmatrix} Q_n'(X,Y,Z)/n \\ \frac{2\pi}{\sqrt{3}} Q_n(X,Y,Z) \end{bmatrix} q_v^n\,.$$

Lemma 14.1. *We have $Q_n = P_n$ if $n \equiv 0 \pmod 3$, else $Q_n = -P_n/2$.*

Proof. Using $X(\epsilon u) = \epsilon X(u)$ and the decomposition $N_n = \sqcup_{i=1}^j \alpha_i\, O_F^\times$ as above we get

$$Q_n(X,Y,Z) = \sum_{i=1}^{j} \Big(\sum_{\epsilon \in O_F^\times} \rho^{-Tr(\epsilon\alpha_i)} \Big) \bar{\alpha}_i X(\alpha_i u)$$

and writing $\alpha_i = a_i + \rho b_i$ with $a_i, b_i \in \mathbb{Z}$, we have

$$\sum_{\epsilon \in O_F^\times} \rho^{-Tr(\epsilon\alpha_i)} = 3(\rho^{a_i+b_i} + \rho^{2(a_i+b_i)}) = \begin{cases} 6 & \text{if } a_i + b_i \equiv 0 \pmod 3 \\ -3 & \text{if } a_i + b_i \not\equiv 0 \pmod 3\,. \end{cases}$$

Noticing that $N(\alpha_i) \equiv (a_i + b_i)^2 \pmod 3$, we get the desired result. □

For F_1 we set $N_n(\xi) = \xi \cdot \{\alpha \in O_F \,|\, N(\alpha) = n, \alpha \equiv 1 \pmod{\sqrt{-3}}\}$ and note that the map $\alpha + \xi \mapsto \xi(\alpha(\rho-1)+1)$ is a bijection from $\{\alpha + \xi \,|\, \alpha \in O_F, N(\alpha+\xi) = n/3\}$ to $N_n(\xi)$. We define

$$R_n(X,Y,Z) = \sum_{\alpha \in N_n(\xi)} \bar{\alpha} X(\alpha u)\,,$$

so we have

$$R_n'(X,Y,Z) = \frac{n}{3} \sum_{\alpha \in N_n(\xi)} X'(\alpha u)\,,$$

and we can write

$$F_1(u,v) = \sum_{\alpha \in O_F+\xi} \left[\begin{smallmatrix} X'(\alpha u) \\ \frac{2\pi}{\sqrt{3}} \bar{\alpha} X(\alpha u) \end{smallmatrix} \right] q_v^{N(\alpha)} = \sum_{n \geqslant 1} \left[\begin{smallmatrix} 3\, R_n'(X,Y,Z)/n \\ \frac{2\pi}{\sqrt{3}} R_n(X,Y,Z) \end{smallmatrix} \right] q_v^{n/3}\,.$$

We relate the polynomials R_n and P_n. For $n \equiv 0 \,(\mathrm{mod}\, 3)$ we have $R_n = 0$; if $n \equiv 1 \,(\mathrm{mod}\, 3)$ we write

$$N_n(\xi) = \sqcup_{i=1}^{j} \xi \alpha_i \{1, \rho, \rho^2\}$$

with $\alpha_i \equiv 1 (\mathrm{mod} \sqrt{-3})$. We thus get

$$R_n(X,Y,Z) = \bar{\xi} P_n(X_0,Y_0,Z_0)/2 \qquad \text{with } X_0 = X(\xi u), Y_0 = Y(\xi u) \text{ and } Z_0 = Z(\xi u).$$

For F_2 we use

$$S_n(X,Y,Z) = \sum_{\alpha \in N_n(\xi)} e^{\frac{2\pi}{\sqrt{3}}(\alpha\bar{\xi} - \bar{\alpha}\xi)} \bar{\alpha} Y(\alpha u)\,,$$

so we obtain

$$S_n'(X,Y,Z) = \frac{n}{3} \sum_{\alpha \in N_n(\xi)} e^{\frac{2\pi}{\sqrt{3}}(\alpha\bar{\xi} - \bar{\alpha}\xi)} Y'(\alpha u)\,.$$

We then have

$$F_2(u,v) = \sum_{\alpha \in O_F+\xi} e^{\frac{2\pi}{\sqrt{3}}(\alpha\bar{\xi} - \bar{\alpha}\xi)} \left[\begin{smallmatrix} Y'(\alpha u) \\ \frac{2\pi}{\sqrt{3}} \bar{\alpha} Y(\alpha u) \end{smallmatrix} \right] q_v^{N(\alpha)} = \sum_{n \geqslant 1} \left[\begin{smallmatrix} 3\, S_n'(X,Y,Z)/n \\ \frac{2\pi}{\sqrt{3}} S_n(X,Y,Z) \end{smallmatrix} \right] q_v^{n/3}.$$

Writing again $N_n(\xi) = \sqcup_{i=1}^{j} \xi \alpha_i \,\{1, \rho, \rho^2\}$ with $\alpha_i - 1 \in (3)$ we find

$$S_n(X,Y,Z) = 3\,\xi \sum_{i=1}^{j} \bar{\alpha}_i \, Y(\alpha_i \xi u)\,.$$

Using the Shintani operators we find the first few S_n, where again we use the notation $X_0 = X(\xi u)$, $Y_0 = Y(\xi u)$ and $Z_0 = Z(\xi u)$:

$$S_1 = 3\,\bar{\xi} Y_0, \qquad S_4 = -6\,\bar{\xi} Y_0(-Y_0^3 + 2\,Z_0^3), \qquad S_7 = -3\,\bar{\xi} Y_0(Y_0^6 + 14\,Y_0^3 Z_0^3 - 14\,Z_0^6),$$
$$S_{13} = 3\,\bar{\xi} Y_0(5\,Y_0^{12} - 13\,Y_0^9 Z_0^3 + 39\,Y_0^6 Z_0^6 - 52\,Y_0^3 Z_0^9 + 26\,Z_0^{12})\,.$$

Finally for F_3 we use $F_3 = -F_2|_{1,1} r_2^{-1}$ and thus put $T_n(X,Y,Z) = S_n(X,Z,Y)$ and then can write

$$F_3(u,v) = -\sum_{\alpha \in O_F+\xi} e^{\frac{2\pi}{\sqrt{3}}(\alpha\bar{\xi} - \bar{\alpha}\xi)} \left[\begin{smallmatrix} Z'(\alpha u) \\ \frac{2\pi}{\sqrt{3}} \bar{\alpha} Z(\alpha u) \end{smallmatrix} \right] q_v^{N(\alpha)} = -\sum_{n \geqslant 1} \left[\begin{smallmatrix} 3\, T_n'(X,Y,Z)/n \\ \frac{2\pi}{\sqrt{3}} T_n(X,Y,Z) \end{smallmatrix} \right] q_v^{n/3}.$$

After these preparations we can calculate the beginning of the Fourier-Jacobi expansion of $\chi_{4,4}$. The result is, after suitable normalisation,

$$\chi_{4,4}=\begin{bmatrix} (\frac{\sqrt{3}}{2\pi})^4(3\,c_1\rho^2(X_0'Y_0'Z_0')\,q_v+O(q_v^2)) \\ (\frac{\sqrt{3}}{2\pi})^3(c_1(2+\rho)(X_0Y_0'Z_0'+X_0'Y_0'Z_0+X_0Y_0'Z_0')\,q_v+O(q_v^2)) \\ (\frac{\sqrt{3}}{2\pi})^2(-c_1X'q_v+O(q_v^2)) \\ \frac{\sqrt{3}}{2\pi}(-\frac{c_1}{3}Xq_v+4XX'q_v^2+O(q_v^4)) \\ X^2q_v^2-6\,X^2YZq_v^4+O(q_v^5) \end{bmatrix},$$

where $c_1=X'(0)$. For the last coordinate one can calculate more terms. Indeed, this involves only the second coordinates of the F_i $(i=0,\ldots,3)$ and one finds:

$$\begin{aligned}\chi_{4,4}^{(4)} =X^2\Big(& q_v^2-6\,YZq_v^4-20\,(Y^3+Z^3)q_v^5+81\,Y^2Z^2q_v^6+132\,(Y^4Z+YZ^4)q_v^7\\ &+(122\,Y^6-800\,Y^3Z^3+122\,Z^6)q_v^8+(-1020\,Y^7Z+1470\,Y^4Z^4-1020\,YZ^7)q_v^{10}\\ &+(-76\,Y^9+1140\,Y^6Z^3+1140\,Y^3Z^6-76\,Z^9)q_v^{11}\\ &+(-486\,Y^8Z^2+486\,Y^5Z^5-486\,Y^2Z^8)q_v^{12}\\ &+(3012\,Y^{10}Z-3924\,Y^7Z^4-3924\,Y^4Z^7+3012\,YZ^{10})q_v^{13}\\ &+(-1261\,Y^{12}+266\,Y^9Z^3+4782\,Y^6Z^6+266\,Y^3Z^9-1261\,Z^{12})q_v^{14}+\ldots\Big).\end{aligned}$$

Remark 14.2. To identify the terms in the Fourier-Jacobi expansions we use the fact that we know a basis for the space of theta functions of degree $3n$ on the elliptic curve $\mathbb{C}/\Lambda$:

$$\{X^aY^bZ^c : 0\le a\le 2,\, 0\le b\le n-a,\, a+b+c=n\}\,.$$

We can use the Taylor expansion of an element in this space around the origin to express it in terms of such a basis.

15. Restriction to the curve T_1

In order to know which covariants yield holomorphic modular forms we need the expansion of the modular forms ζ, $\chi_{4,4}$ and $E_{1,1}$ along T_1, given by $u=0$, the zero locus of ζ.

We start with the expansions of the elliptic functions X, Y and Z near the origin of the elliptic curve $\mathbb{C}/\Lambda$. These have the form

$$X(z)=\sum_{j\geqslant 0}c_{6j+1}z^{6j+1},\quad Y(z)=\sum_{j\geqslant 0}d_{3j}z^{3j},\quad Z(z)=\sum_{j\geqslant 0}(-1)^j\,d_{3j}z^{3j}. \tag{14}$$

Note that the functions Y and Z are normalised such that $d_0=1$. Let $\xi=(\rho^2-1)/3=\rho\sqrt{-3}/3$ as before. By [9, Lemma 9, formulas (68–69)] these functions satisfy

$$Y^3(\xi z)=\frac{1}{\rho-1}(\rho Y(z)-Z(z))\quad\text{and}\quad Z^3(\xi u)=\frac{1}{\rho-1}(-Y(z)+\rho Z(z))\,,$$

and with $X^3 = \rho(Y^3 - Z^3)$, we have $X^3(\xi z) = -\xi\,(Y(z) - Z(z))$. This relation provides links between the numbers c_{6j+1} and d_{6j+3}, while the relation $Y^3(\xi z) + Z^3(\xi z) = Y(z) + Z(z)$ provides the relation between the c_{6j+1} and d_{6j}. If follows that the numbers c_{6j+1} and d_{3j} can be expressed in terms of powers of c_1. For example, we have:

$$
\begin{aligned}
&c_7 = 6\,\rho\, c_1^7/7!, \quad c_{13} = -6^3\rho^2\, c_1^{13}/13!, \quad c_{19} = -2^7 3^4 5\cdot 23\, c_1^{19}/19!,\\
&d_3 = \rho^2 c_1^3/3!, \quad d_6 = -2\,\rho c_1^6/6!, \quad d_9 = -8\, c_1^9/9!, \quad d_{12} = -2^3\, 19\,\rho^2 c_1^{12}/12!,\\
&d_{15} = 2^3 5\cdot 31\,\rho\, c_1^{15}/15!\,.
\end{aligned}
$$

Here the constant c_1 is given by

$$
c_1 = \Gamma(1/3)^3\, e^{-17i\pi/18}/(2\,\pi)\,. \tag{15}
$$

Remark 15.1. These numbers are related to the development of the modular form ϑ around its zero $\tau_0 = (1-\rho^2)/3$, see [32, Prop. 17].

For the restriction to the curve T_1 of the modular forms ζ, $E_{1,1}$ and $\chi_{4,4}$ we can apply Proposition 4.3.

The definition of the cusp form $\zeta \in S_6(\Gamma[\sqrt{-3}, \det])$ yields by (14) with $q_v = e^{2\pi\, v/\sqrt{3}}$

$$
\zeta(u, \sqrt{-3}\tau) = 1/6 \sum_{\alpha\in O_F} \alpha^5 X(\alpha u) q^{N(\alpha)} = 1/6 \sum_{j\geqslant 0} a_{6j+1}\Theta_{6(j+1)}(\tau) u^{6j+1}\,,
$$

where $\Theta_j(\tau) = \sum_{\alpha\in O_F} \alpha^j q^{N(\alpha)} \in M_{j+1}(\Gamma_1(3))$, as introduced in Section 4. Therefore with $w = c_1 u$ the Taylor expansion of ζ about $u = 0$ starts with

$$
\begin{aligned}
\zeta(u, \sqrt{-3}\tau) &= c_1\, \vartheta\eta^8\psi^2(\tau) u + c_7\, \vartheta\eta^8\left(\eta^{16} + 18\psi^4\eta^8 + 729\psi^8\right) u^7 + \ldots\\
&= \vartheta\eta^8(\tau)\Big(\psi^2(\tau) w + (\rho/840)\,(\eta^{16} + 18\psi^4\eta^8 + 729\psi^8)(\tau) w^7 + \ldots\Big)\,.
\end{aligned}
$$

In a similar way, we obtain the development along the curve T_1 of the Eisenstein series $E_{1,1}$:

$$
\begin{aligned}
E_{1,1}(u, \sqrt{-3}\tau) &= \begin{bmatrix} E_{1,1}^{(0)}(u,\sqrt{-3}\tau) \\ E_{1,1}^{(1)}(u,\sqrt{-3}\tau)\end{bmatrix} = \sum_{\alpha\in\mathcal{O}} \begin{bmatrix} X'(\alpha u) \\ \frac{2\pi}{\sqrt{3}}\bar{\alpha} X(\alpha u)\end{bmatrix} q^{N(\alpha)}\\
&= \sum_{j\geqslant 0} c_{6j+1}\Big(\begin{bmatrix} (6j+1)\Theta_{6j}(\tau) \\ 0\end{bmatrix} u^{6j} + \begin{bmatrix} 0 \\ \frac{\Theta'_{6j}(\tau)}{\sqrt{-3}}\end{bmatrix} u^{6j+1}\Big)\,,
\end{aligned}
$$

where the components $E_{1,1}^{(i)}$ have the following expansions in which the variable τ is omitted:

$$
\begin{aligned}
E_{1,1}^{(0)}(u, \sqrt{-3}\tau) &= c_1\big(\vartheta + \rho\, \vartheta\psi^2\eta^8/20\, w^6 + \ldots\big)\\
E_{1,1}^{(1)}(u, \sqrt{-3}\tau) &= \frac{\pi}{6\sqrt{3}}\big((108\psi^3 + \vartheta(e_2 - \vartheta^2))w + (\frac{\rho}{140})\,\psi\,\eta^8(5\eta^8 + 243\psi^4 + 7e_2\psi\vartheta)\, w^7 + \ldots\big)
\end{aligned}
$$

For the cusp form $\chi_{4,4}$, we get

$$\chi_{4,4}(u,\sqrt{-3}\tau)=\begin{bmatrix}0\\0\\h_0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\h_1\\0\end{bmatrix}u+\begin{bmatrix}0\\0\\0\\0\\h_2\end{bmatrix}u^2+\begin{bmatrix}h_4\\0\\0\\0\\0\end{bmatrix}u^4+\begin{bmatrix}0\\h_5\\0\\0\\0\end{bmatrix}u^5+\dots,$$

where h_0 and h_4 are cusp forms on $\Gamma_1(3)$, while h_1, h_2 and h_5 are quasi-modular forms on $\Gamma_1(3)$. We set

$$\gamma = 2\pi/\sqrt{3}\,. \tag{16}$$

Then the h_i are given by

$$h_0=-\frac{c_1^2}{\gamma^2}\,\eta^8\psi^2;\quad h_1=-\frac{c_1^2}{6\,\gamma}\,\eta^8\psi^2(\vartheta^2+e_2);\quad h_2=\frac{c_1^2}{144}\,\eta^8\psi^2(3\,\vartheta^2+e_2)(\vartheta^2-e_2)$$
$$h_4=-\rho\,\frac{c_1^8}{12\,\gamma^4}\eta^8\psi^2\vartheta^2;\quad h_5=\rho\,\frac{c_1^8}{180\,\gamma^3}\eta^8\psi^2\vartheta(4\,\vartheta^3+5\,\vartheta e_2+54\,\psi^3).$$

Remark 15.2. The cusp form h_0 is proportional to the cusp form $f_6(\tau)=\eta^6(\tau)\eta^6(3\tau)$ of weight 6 on $\Gamma_0(3)$ with Fourier expansion

$$f_6(\tau)=q-6\,q^2+9\,q^3+4\,q^4+6\,q^5-54\,q^6-40\,q^7+168\,q^8+\dots.$$

This cusp form is one of the famous eta products and plays a similar role for $\Gamma_0(3)$ as the discriminant form Δ for $\mathrm{SL}(2,\mathbb{Z})$.

From the above expansions we derive the order of vanishing along T_1 of $\chi_{4,4}$ and $\chi_{1,1}$. This is used in Corollary 10.3.

Corollary 15.3. *The order of the five coordinates of $\chi_{4,4}$ along $u=0$ is $(4,5,0,1,2)$. The order of the two coordinates of $E_{1,1}$ along $u=0$ is $(0,1)$.*

16. Construction of Modular Forms from Invariants

In this section we shall use the map $\nu:\mathcal{C}(V_4\oplus V_1)\to\mathbb{M}[1/\zeta]$ to construct modular forms. Under ν a covariant $J_{a,b,c}$ of degree (a,b,c) in the variables (a_i,b_i,x_i) maps to a meromorphic modular form of weight $(c,(3b-c)/2)$ on $\Gamma[\sqrt{-3}]$ with character $\epsilon^{a+b}\circ\det^{2a+2b+2c}$, that is,

$$\nu(J_{a,b,c})\in\widetilde{M}_{c,3b/2,-a-b}(\Gamma[\sqrt{-3}])$$

with the property that

$$\nu(J_{a,b,c})\quad\text{is}\quad\begin{cases}\mathfrak{S}_4\text{-invariant if } a+b\equiv 0 \bmod 2\\ \mathfrak{S}_4\text{-anti-invariant if } a+b\equiv 1 \bmod 2\,.\end{cases}$$

Here the tilda on M refers to the meromorphicity along T_1. In the following table we give for the twenty generating covariants $J_{a,b,c}$ the weight (j,k,l), the index $e=a+b(\bmod 2)$ and the order of the coordinates of the meromorphic modular form $\nu(J_{a,b,c})$ along T_1.

(a,b,c)	j	k	l	e	order along T_1
$(2,0,0)$	0	0	1	0	-2
$(0,1,1)$	1	1	1	1	$[0,1]$
$(3,0,0)$	0	0	0	1	-3
$(1,4,0)$	0	6	1	1	1
$(1,3,1)$	1	4	1	0	$[0,1]$
$(1,2,2)$	2	2	1	1	$[-1,0,1]$
$(1,1,3)$	3	0	1	0	$[4,-1,0,1]$
$(1,0,4)$	4	-2	1	1	$[3,4,-1,0,1]$
$(2,4,0)$	0	6	0	0	0
$(2,3,1)$	1	4	0	1	$[-1,0]$
$(2,2,2)$	2	2	0	0	$[-2,-1,0]$
$(2,1,3)$	3	0	0	1	$[3,-2,-1,0]$
$(2,0,4)$	4	-2	0	0	$[2,3,-2,-1,0]$
$(3,6,0)$	0	9	0	1	0
$(3,5,1)$	1	7	0	0	$[-1,0]$
$(3,4,2)$	2	5	0	1	$[4,-1,0]$
$(3,3,3)$	3	3	0	0	$[3,4,-1,0]$
$(3,2,4)$	4	1	0	1	$[2,3,4,-1,0]$
$(3,1,5)$	5	-1	0	0	$[1,2,3,4,-1,0]$
$(3,0,6)$	6	-3	0	1	$[6,1,2,3,4,-1,0]$

As we saw in Section 2 we have $M(\Gamma)=\mathbb{C}[E_6,E_{12},E_9^2]$ and first cusp form appears in weight 12 and is given by

$$\chi_{12}=(E_6^2-E_{12})/5184\,.$$

The cusp form χ_{12} is a Kudla lift of an element in $S_{11}^{-}(\Gamma_1(3))$ (see [9, Prop. 10], or [1, Section 11.3, Case 2b]). Moreover, there is a cusp form

$$\chi_{18}=(E_6^3-E_9^2)/3888\,.$$

By calculating the expansions for some of these $\nu(J_{a,b,c})$ one can identify sometimes the resulting modular forms. We use here what was mentioned in Remark 14.2

Doing this for the generators $J_{a,b,0}$ one obtains the following proposition. Recall $\gamma=2\pi/\sqrt{3}$ and $c_1=X'(0)$ as given in equations (15) and (16).

Proposition 16.1. *The images under ν of the generators $J_{a,b,0}$ are;*

$$\nu(J_{1,4,0})=\frac{3\,c_1^4}{70}\zeta\,,\qquad \nu(J_{2,0,0})=\frac{c_1^4}{6\gamma^4}\frac{\chi_{12}}{\zeta^2}\,,\qquad \nu(J_{3,0,0})=\frac{c_1^6}{864\,\gamma^6}\frac{\chi_{18}-E_6\chi_{12}}{\zeta^3}\,,$$

$$\nu(J_{2,4,0})=-\frac{c_1^6}{5040\,\gamma^2}\,E_6\,,\qquad \nu(J_{3,6,0})=\frac{c_1^9}{798336\,\gamma^3}\,E_9\,.$$

Remark 16.2. As a check, one may apply ν to the relation

$$3J_{2,4,0}^3+\frac{13068}{875}\,J_{3,6,0}^2+J_{1,4,0}^3J_{3,0,0}-\frac{3}{2}\,J_{2,4,0}J_{1,4,0}^2J_{2,0,0}=0. \tag{17}$$

(see (6) in Section 9) and obtain

$$\frac{c_1^{18}}{42674688000\,\gamma^6}(-E_6^3+E_9^2+3888\chi_{18})=0\,, \tag{18}$$

in agreement with the definition of χ_{18}.

Remark 16.3. The image of the discriminant $32(J_{2,0,0}^3-6\,J_{3,0,0}^2)$ of the quartic polynomial f_4 under ν is constant:

$$\nu(32(J_{2,0,0}^3-6\,J_{3,0,0}^2))=-\frac{\rho\, c_1^{12}}{3^3\,\gamma^{12}}\,.$$

This comes about by the fact that the moduli space is obtained by blowing up of the discriminant locus, cf. the diagram in [3, p. 6] and the discussion there.

We finish this section with a result on the module of scalar-valued cusp forms on Γ. Since the group Γ has a unique cusp, we have $\dim S_k(\Gamma)=\dim M_k(\Gamma)-1$ if $\dim M_k(\Gamma)>0$ and the generating series for the dimension of the spaces $S_k(\Gamma)$ is given by

$$\sum_{k\geqslant 0}\dim S_k(\Gamma)t^k=\frac{t^{12}+t^{18}-t^{30}}{(1-t^6)(1-t^{12})(1-t^{18})}=t^{12}+2\,t^{18}+3\,t^{24}+4\,t^{30}+6\,t^{36}+\dots$$

The first cusp form appears in weight 12, namely χ_{12}. Then we have

$$S_{18}(\Gamma)=\mathrm{Span}_{\mathbb{C}}(E_6\chi_{12},\chi_{18}),\quad S_{24}(\Gamma)=\mathrm{Span}_{\mathbb{C}}(E_6^2\chi_{12},E_6\chi_{18},\chi_{12}^2),$$
$$S_{30}(\Gamma)=\mathrm{Span}_{\mathbb{C}}(E_6^3\chi_{12},E_9^2\chi_{12},\chi_{12}\chi_{18},\chi_{12}^2,E_6^2\chi_{18},E_6^2\chi_{12}^2)\,,$$

but in the last case, we have the relation $\chi_{12}(3888\,\chi_{18}-(E_6^3-E_9^2))=0$ which comes from the relation (18) multiplied by $J_{1,4,0}^2J_{2,0,0}$ which corresponds to χ_{12}. Here this relation actually counts as a relation between cusp forms.

Corollary 16.4. *The $M(\Gamma)$-module $\Sigma(\Gamma)=\oplus_k S_k(\Gamma)$ of cusp forms on Γ is generated by the forms χ_{12} and χ_{18} with the relation $3888\,\chi_{18}\chi_{12}-(E_6^3-E_9^2)\chi_{12}=0$ in weight 30.*

17. The structure of a module for $j=4$

To show the feasibility of constructing modular forms by covariants, as an application we determine the structure of the $M(\Gamma)$-module

$$\mathcal{M}_4^2(\Gamma)=\oplus_{k\geqslant 0}M_{4,k}(\Gamma,\det{}^2)\,.$$

With the same method one can also treat the modules

$$\mathcal{M}_4^l(\Gamma)=\oplus_{k\geqslant 0}M_{4,k}(\Gamma,\det{}^l)$$

for $l=0$ and $l=1$, but we refrain from giving the details.

The structure of the modules $\mathcal{M}_j^l(\Gamma)=\oplus_k M_{j,k}(\Gamma,\det{}^l)$ for $j<4$ was determined in [5] in a different manner, but it would be very difficult to go beyond these cases that way. Invariant theory provides a good way to build generators.

For the cusp forms, we use the notation $\Sigma_4^l(\Gamma)$ or $\Sigma_4^l(\Gamma[\sqrt{-3}])=\oplus_{k\geqslant 0}S_{4,k}(\Gamma[\sqrt{-3}],\det{}^l)$. Note (see [5, Proposition 5.1]) that

$$\mathcal{M}_4^l=\Sigma_4^l\quad\text{if } l\not\equiv 1 \bmod 3.$$

Recall that if $k \not\equiv 1 \bmod 3$ then $M_{4,k}(\Gamma[\sqrt{-3}], \det^l) = (0)$.

Theorem 17.1. *The $M(\Gamma)$-module Σ_4^2 is freely generated by cusp forms of weight $(4,4)$, $(4,10)$, $(4,16)$, $(4,22)$ and $(4,28)$.*

Proof. We begin the proof by deducing the Hilbert–Poincaré series for the module Σ_4^2. For this we start with $\Gamma[\sqrt{-3}]$. The dimension of the space $S_{4,1+3k}(\Gamma[\sqrt{-3}], \det^2)$ is given by (see [1, Thm. 4.7])

$$\dim S_{4,1+3k}(\Gamma[\sqrt{-3}], \det^2) = k(5k+1)/2 - 2$$

for $k \geqslant 1$. One can show that $\dim S_{4,1,2}(\Gamma[\sqrt{-3}]) = 0$, for example by the following argument. By multiplication with E_6 and the knowledge of $S_{4,7,2}(\Gamma[\sqrt{-3}])$ as a $\mathfrak{S}_4$-module, we see that only the $s[3,1]$ and $s[2,1,1]$ components can be non-zero. Restricting to a component of T_1 is injective since such a component is the zero locus of a form of weight 1, and dividing would give a non-zero form of weight $(4,0)$ on some congruence subgroup. The fact that $\dim S_3(\Gamma_0(3), \left(\frac{\cdot}{3}\right))) = 2$ and $\dim s[3,1] = \dim s[2,1,1] = 3$, now shows that $\dim S_{4,1,2}(\Gamma[\sqrt{-3}]) = 0$.

The Hilbert–Poincaré series for the dimensions is therefore given by

$$\sum_{k\geqslant 0} \dim S_{4,1+3k}(\Gamma[\sqrt{-3}], \det^2) t^{1+3k} = \frac{t^4 + 6\,t^7 - 2\,t^{10}}{(1-t^3)^3}\,.$$

Lemma 17.2. *The $M(\Gamma[\sqrt{-3}])$-module $\Sigma_4^2(\Gamma[\sqrt{-3}])$ is generated by a generator of type $s[4]$ in weight $(4,4)$, generators of type $s[3,1]$ and $s[2,1,1]$ in weight $(4,7)$ and a relation of type $s[2,2]$ in weight $(4,10)$.*

Proof. This can be proved using results of [1] as in [5]. □

Writing the isotypical decomposition of $M_{3k}(\Gamma[\sqrt{-3}]) = \mathrm{Sym}^k(s[2,1,1])$ as

$$\mathrm{Sym}^k(s[2,1,1]) = a_k\, s[4] + b_k\, s[3,1] + c_k\, s[2,2] + d_k\, s[2,1,1] + e_k\, s[1,1,1,1]$$

we get by Lemma 17.2 for $k \geqslant 1$

$$S_{4,6k-2}(\Gamma[\sqrt{-3}], \det^2) = (a_{2k-2} + b_{2k-3} + d_{2k-3} - c_{2k-4})\, s[4] + \ldots$$

The generating series of the numbers a_k, b_k, c_k, d_k and e_k are given by the generating series $N/(1-t)(1-t^2)(1-t^3)(1-t^4)$ with N as in the next table.

	a_k	b_k	c_k	d_k	e_k
N	$(1-t)(1-t^3+t^6)$	$t^2(1-t^3)$	$(1-t)t^2(1+t^2)$	$(t-t^2+t^3)(1-t^3)$	$t^3(1-t)$

This leads to the following generating series for the dimension of the spaces $S_{4,6k-2}(\Gamma, \det^2)$:

$$\begin{aligned}\sum_{k\geqslant 1} \dim S_{4,6k-2}(\Gamma, \det^2) t^{6k-2} &= \frac{t^4 + t^{10} + t^{16} + t^{22} + t^{28}}{(1-t^6)(1-t^{12})(1-t^{18})} \\ &= t^4 + 2\,t^{10} + 4\,t^{16} + 7\,t^{22} + 11\,t^{28} + O(t^{34})\,.\end{aligned}$$

Now we turn to the construction by covariants of the generators of weights $(4,4)$, $(4,10)$, $(4,16)$, $(4,22)$ and $(4,28)$.

The form of weight $(4,4)$ is already available:

$$\chi_{4,4} = \frac{4900}{3\,c_1^4}\,\nu(J_{1,0,4}J_{1,4,0})$$

and for later use we observe that the Fourier-Jacobi of its last component starts with

$$\begin{aligned}\chi_{4,4}^{(4)}(u,v) = X^2\big(&q_v^2 - 6\,YZ\,q_v^4 - 20\,(Y^3+Z^3)\,q_v^5 + 81\,Y^2Z^2\,q_v^6 + 132(Y^4Z+YZ^4)\,q_v^7\\ &+ (122\,Y^6 - 800\,Y^3Z^3 + 122\,Z^6)\,q_v^8 - (1020\,Y^7Z - 1470\,Y^4Z^4 + 1020\,YZ^7)\,q_v^{10}\\ &+ (-76\,Y^9 + 1140\,Y^6Z^3 + 1140\,Y^3Z^6 - 76\,Z^9)\,q_v^{11} + \dots\big)\,.\end{aligned}$$

Next we construct a generator $\chi_{4,10}$ of weight 10. Note that $\dim S_{4,10}(\Gamma,\det^2)=2$ and we know already a form of weight $(4,10)$, namely $E_6\,\chi_{4,4}$.

The three covariants $J_{2,4,0}J_{1,4,0}J_{1,0,4}$, $J_{1,4,0}^2J_{2,0,4}$ and $J_{1,4,0}J_{0,1,1}J_{3,3,3}$ produce modular forms in $S_{4,10}(\Gamma,\det^2)$. But we have the following relation

$$132\,J_{0,1,1}J_{3,3,3} + 175(J_{2,4,0}J_{1,0,4} - J_{1,4,0}J_{2,0,4}) = 0\,.$$

We know $\nu(J_{2,4,0}) = -(c_1^6/5040\gamma^2)E_6$. We set

$$\chi_{4,10} = -\frac{2744000\,\gamma^2}{c_1^{10}}\nu(J_{1,4,0}^2J_{2,0,4}).$$

One checks holomorphicity using the table in Section 12. The Fourier-Jacobi expansion of its last component starts with

$$\begin{aligned}\chi_{4,10}^{(4)} = X^2\big(&q_v^2 + 54\,YZ\,q_v^4 - 272(Y^3+Z^3)\,q_v^5 + 405\,Y^2Z^2\,q_v^6 + 3024(Y^4Z+YZ^4)\,q_v^7\\ &+ (4406\,Y^6 - 15560\,Y^3Z^3 + 4406\,Z^6)\,q_v^8 - 23328(Y^5Z^2+Y^2Z^5)\,q_v^9\\ &- (62748\,Y^7Z - 221022\,Y^4Z^4 + 62748\,YZ^7)\,q_v^{10}\\ &- (22000\,Y^9 - 16368\,Y^6Z^3 - 16368\,Y^3Z^6 + 22000\,Z^9)\,q_v^{11} + \dots\big)\,.\end{aligned}$$

The Fourier-Jacobi expansions of the last components of $E_6\chi_{4,4}$ and $\chi_{4,10}$ start with

$$E_6\chi_{4,4}^{(4)} = X^2q_v^2 + 750\,X^2YZq_v^4 + \dots, \qquad \chi_{4,10}^{(4)} = X^2q_v^2 + 54\,X^2YZq_v^4 + \dots\,.$$

and this shows that they generate the space $S_{4,10}(\Gamma,\det^2)$.

For the generator of weight $(4,16)$ we note that $\dim S_{4,16}(\Gamma,\det^2)=4$ and we have already three linearly independent elements $E_6^2\chi_{4,4}$, $E_{12}\chi_{4,4}$ and $E_6\chi_{4,10}$. We now put

$$\chi_{4,16} = \frac{2304960000\,\gamma^4}{c_1^{16}}\nu(J_{1,4,0}^2(6\,J_{2,1,3}J_{2,3,1} - J_{2,0,0}J_{1,1,3}J_{1,3,1}))\,.$$

We observe that the Taylor expansion of $\nu(6\,J_{2,1,3}J_{2,3,1} - J_{2,0,0}J_{1,1,3}J_{1,3,1})$ along T_1 starts with

$$\nu(6\,J_{2,1,3}J_{2,3,1} - J_{2,0,0}J_{1,1,3}J_{1,3,1})(u,\sqrt{-3}\tau) =$$
$$\frac{c_1^8}{705600\gamma^6}\left(\begin{bmatrix}0\\0\\\vartheta^4\\0\\0\end{bmatrix}u^{-2} + \gamma/6\begin{bmatrix}0\\0\\0\\\vartheta^3(3\vartheta^3+\vartheta e_2-108\psi^3)\\0\end{bmatrix}u^{-1} + \dots\right),$$

so the multiplication by $\nu(J_{1,4,0})^2$, proportional to ζ^2, makes it holomorphic along T_1.

The Fourier-Jacobi expansion of the last component of $\chi_{4,16}$ starts with

$$\begin{aligned}\chi_{4,16}^{(4)} =& X^2\big(q_v^2 + 162\,YZq_v^4 + 3040(Y^3+Z^3)q_v^5 + 43497\,Y^2Z^2q_v^6 - 2592(Y^4Z+YZ^4)q_v^7 \\ & - (298462\,Y^6 + 263600\,Y^3Z^3 + 298462\,Z^6)q_v^8 - 839808(Y^5Z^2+Y^2Z^5)q_v^9 \\ & + (2185380\,Y^7Z - 127170\,Y^4Z^4 + 2185380\,YZ^7)q_v^{10} \\ & + (4366688\,Y^9 + 8413152\,Y^6Z^3 + 8413152\,Y^3Z^6 + 4366688\,Z^9)q_v^{11} + \dots\big)\,.\end{aligned}$$

The Fourier-Jacobi expansions of the last component of $E_6^2\chi_{4,4}$, $E_{12}\chi_{4,4}$, $E_6\chi_{4,10}$ and $\chi_{4,16}$ start with

$$\begin{aligned} E_6^2\chi_{4,4}^{(4)} &= X^2q_v^2 + 1506\,X^2YZq_v^4 + 4012\,X^2(Y^3+Z^3)q_v^5 + 603369\,X^2Y^2Z^2q_v^6 + \dots \\ E_{12}\chi_{4,4}^{(4)} &= X^2q_v^2 - 3678\,X^2YZq_v^4 + 35116\,X^2(Y^3+Z^3)q_v^5 + 354537\,X^2Y^2Z^2q_v^6 + \dots \\ E_6\chi_{4,10}^{(4)} &= X^2q_v^2 + 810\,X^2YZq_v^4 + 1744\,X^2(Y^3+Z^3)q_v^5 + 61641\,X^2Y^2Z^2q_v^6 + \dots \\ \chi_{4,16}^{(4)} &= X^2q_v^2 + 162\,X^2YZq_v^4 + 3040\,X^2(Y^3+Z^3)q_v^5 + 43497\,X^2Y^2Z^2q_v^6 + \dots \end{aligned}$$

showing that these generate $S_{4,16}(\Gamma,\det^2)$.

Before we construct the last two generators we need a lemma.

Lemma 17.3. *We have*

$$\begin{aligned} \nu(J_{1,4,0}^2(J_{3,0,0}J_{1,3,1} - J_{2,0,0}J_{2,3,1}) &\in M_{1,16}(\Gamma)\,, \\ \nu(J_{1,4,0}J_{1,1,3}) &\in M_{3,6}(\Gamma,\det^2)\,, \\ \nu(J_{1,4,0}J_{0,1,1}J_{3,4,2}) &\in M_{3,12}(\Gamma,\det^2)\,, \end{aligned}$$

and these three forms are $\mathfrak{S}_4$-invariant.

For the proof one calculates the Taylor expansion along T_1 as done for the examples above.

In order to get a form of weight $(4,22)$ we set

$$\chi_{4,22} = \frac{53782400000\,\gamma^6}{3\,c_1^{22}}\nu(J_{1,4,0}^3J_{1,1,3}(J_{3,0,0}J_{1,3,1} - J_{2,0,0}J_{2,3,1}))$$

and by applying Lemma 17.3, we see that $\chi_{4,22} \in S_{4,22}(\Gamma,\det^2)$.

The Fourier-Jacobi expansion of its last component starts with

$$\begin{aligned}\chi_{4,22}^{(4)} =& X^2\big(YZq_v^4 + 9(Y^3+Z^3)\,q_v^5 + 60\,Y^2Z^2q_v^6 - 277(Y^4Z+YZ^4)\,q_v^7 \\ & - (6363\,Y^6 - 9468\,Y^3Z^3 + 6363\,Z^6)\,q_v^8 + 2106(Y^5Z^2+Y^2Z^5)\,q_v^9 \\ & + (15128\,Y^7Z + 27844\,Y^4Z^4 + 15128\,YZ^7)\,q_v^{10} + \\ & (276471\,Y^9 - 212895\,Y^6Z^3 - 212895\,Y^3Z^6 + 276471\,Z^9)\,q_v^{11} + \dots\big)\,.\end{aligned}$$

By using the Fourier-Jacobi of the last component of $E_6^3\chi_{4,4}$, $E_6E_{12}\chi_{4,4}$, $E_9^2\chi_{4,4}$, $E_6^2\chi_{4,10}$, $E_{12}\chi_{4,10}$, $E_6\chi_{4,16}$ and $\chi_{4,22}$, we check that they are linearly independent so they span the space $S_{4,22}(\Gamma,\det^2)$ that is of dimension 7.

For the generator of weight $(4, 28)$ we put

$$\chi_{4,28} = -\frac{51114792960000\,\gamma^8}{c_1^{28}}\nu(J_{1,4,0}^3 J_{0,1,1} J_{3,4,2}\,(J_{3,0,0}J_{1,3,1} - J_{2,0,0}J_{2,3,1}))$$

and by applying Lemma 17.3 we see that $\chi_{4,28} \in S_{4,28}(\Gamma, \det^2)$. The Fourier-Jacobi expansion of its last component starts with

$$\begin{aligned}\chi_{4,28}^{(4)} =& X^2\big(Y Z\, q_v^4 + 9(Y^3 + Z^3)\, q_v^5 - 384\, Y^2 Z^2 q_v^6 - 7117(Y^4 Z + Y Z^4)\, q_v^7 \\ &+ (-31959\, Y^6 - 92592\, Y^3 Z^3 - 31959\, Z^6) q_v^8 - 274698(Y^5 Z^2 + Y^2 Z^5)\, q_v^9 \\ &+ (3511880\, Y^7 Z - 4338416\, Y^4 Z^4 + 3511880\, Y Z^7)\, q_v^{10} \\ &+ (18226071\, Y^9 - 5450355\, Y^6 Z^3 - 5450355\, Y^3 Z^6 + 18226071\, Z^9)\, q_v^{11} + \dots\big)\,.\end{aligned}$$

By using the Fourier-Jacobi of the last components of $E_6^4\chi_{4,4}$, $E_6^2 E_{12}\chi_{4,4}$, $E_6 E_9^2\chi_{4,4}$, $E_{12}^2\chi_{4,4}$, $E_6^3\chi_{4,10}$, $E_6 E_{12}\chi_{4,10}$, $E_9^2\chi_{4,10}$, $E_6^2\chi_{4,16}$, $E_{12}\chi_{4,16}$, $E_6\chi_{4,22}$ and $\chi_{4,28}$, we check that they are linearly independent, so they span the 11-dimensional space $S_{4,28}(\Gamma, \det^2)$.

Lemma 17.4. *The exterior product of our generators satisfies*

$$\bigwedge_{k=1}^{5} \chi_{4,6k-2} = -\frac{c_1^{10}}{2\,\gamma^{10}}\,\rho^2\,\zeta^{12}\,E_9^2\,.$$

Proof. We first note that the exterior product of the five forms $\chi_{4,6k-2}$ for $k \in \{1, 2, 3, 4, 5\}$, which take values in $\mathrm{Sym}^4(\mathbb{C}^2) \simeq \mathbb{C}^5$, can be viewed as the determinant of the five components of these five forms, and viewing a covariant of degree 4 in x_1, x_2 as a vector of size 5 whose $(i + 1)$th component is the coefficient of $x_1^{4-i}x_2^i$, $0 \leqslant i \leqslant 4$, we have

$$\begin{aligned}\bigwedge_{k=1}^{5} \chi_{4,6k-2} &= 2^{41}\cdot 3^2\cdot 5^{18}\cdot 7^{19}\cdot 11\frac{\gamma^{20}}{c_1^{80}}\,\nu(J_{1,4,0})^{11}\times \\ &\quad \nu(\det(J_{1,0,4}J_{2,0,4}J_{1,1,3}J_{0,1,1}J_{3,4,2}(6\,J_{2,1,3}J_{2,3,1} - J_{2,0,0}J_{1,1,3}J_{1,3,1})(J_{3,0,0}J_{1,3,1} - J_{2,0,0}J_{2,3,1})^2)) \\ &= -2^{35}\cdot 3^2\cdot 5^{12}\cdot 7^{14}\cdot 11^2\frac{\gamma^{20}}{c_1^{80}}\,\nu(J_{1,4,0}^{12}J_{3,6,0}^2(J_{2,0,0}^3 - 6J_{3,0,0}^2)^2)\,.\end{aligned}$$

We have seen in Proposition 16.1 that

$$\nu(J_{1,4,0)} = \frac{3c_1^4}{70}\,\zeta, \quad \nu(J_{3,6,0}) = \frac{c_1^9}{798336\,\gamma^3}E_9, \quad \nu(32(J_{2,0,0}^3 - 6\,J_{3,0,0}^2)) = -\rho\frac{c_1^{12}}{3^3\,\gamma^{12}}$$

and this implies

$$\bigwedge_{k=1}^{5} \chi_{4,6k-2} = -\frac{c_1^{10}}{2\,\gamma^{10}}\,\rho^2\,\zeta^{12}\,E_9^2\,,$$

thus proving the lemma. ∎

We can now conclude the proof of Theorem 17.1. The modular forms $\chi_{4,k}$ with $k \in \{4, 10, 16, 22, 28\}$ are algebraically independent over $M(\Gamma)$ because of Lemma 17.4. Since they generate a submodule with Hilbert–Poincaré series equal to that of Σ_4^2 the result follows. $\square$

In a similar way one can treat the cases $l = 1$ and $l = 0$. We intend to come back to these cases in another paper.

References

[1] J. Bergström, G. van der Geer: *Picard modular forms and the cohomology of local systems on a Picard modular surface.* arXiv: To appear in Commentarii Math. Helvetici. 5, 6, 10, 22, 27, 34, 36

[2] F. Cléry, C. Faber, and G. van der Geer: *Covariants of binary sextics and vector-valued Siegel modular forms of genus* 2. Math. Ann. **369**(3—4), 1649—1669 (2017). 1

[3] F. Cléry, C. Faber, and G. van der Geer: *Covariants of binary sextics and modularforms of degree* 2 *with character.* Math. Comput. **88**(319), 2423-–2441 (2019). 1, 35

[4] F. Cléry, C. Faber, and G. van der Geer: *Concomitants of ternary quartics and vector-valued Siegel and Teichmüller modular forms of genus three.* Selecta Math. (2020) 1, 15, 17, 21

[5] F. Cléry, G. van der Geer: *Generators for modules of vector-valued Picard modular forms.* Nagoya Math. J. Volume **212**, (2013), 19–57. 3, 6, 10, 11, 22, 27, 28, 35, 36

[6] J. Cogdell: *Arithmetic cycles on Picard modular surfaces and modular forms of Nebentypus.* J. Reine Angew. Math. **357**, 115–137 (1985). 9

[7] M. Draisma: *Invariants of binary forms.* Inauguraldissertation, University of Basel, 2014. 18, 19

[8] J.M. Feustel: *Ringe automorpher Formen auf der komplexen Einheitskugel und ihre Erzeugung durch Theta-konstanten.* Preprint Ser. Akad. Wiss. DDR, P-Math-13, (1986). 4, 6

[9] T. Finis: *Some computational results on Hecke eigenvalues of modular forms on a unitary group.* Manuscripta Math. **96** (1998), 149–180. 3, 6, 26, 31, 34

[10] G. van der Geer: *Hilbert modular surfaces.* Ergebnisse der Mathematik 16. Springer Verlag 1988. 10

[11] J. Harris, I. Morrison: *Moduli of Curves.* Graduate Texts in Math. 187. Springer Verlag 1998. 15

[12] R.P. Holzapfel: *Around Euler Partial Differential Equations. VEB Deutscher Verlag der Wissenschaften, Berlin, 1986.* 4, 5, 6, 16

[13] R.P. Holzapfel: *The ball and some Hilbert problems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, (1995). 7

[14] T. Ichikawa: *Teichmüller modular forms of degree* 3. American Journal of Mathematics **117** (1995), 1057–1061. 17

[15] J.-I. Igusa: *Modular forms and projective invariants.* American Journal of Mathematics **89**, 817-–855(1967). 1

[16] J.-I. Igusa: *Theta Functions.* Berlin-Heidelberg-New York, Springer-Verlag, 1972. 24

[17] M. Kaneko, D. Zagier: *A generalized Jacobi theta function and quasimodular forms.* In: The moduli space of curves. Proceedings of the conference held on Texel Island, Netherlands, April 1994. Basel: Birkhäuser. Prog. Math. 129, 165–172 (1995). 12

[18] B. Moonen, F. Oort: *The Torelli locus and special subvarieties.* In: Handbook of Moduli, (Eds. G. Farkas and I. Morrison), pp. 549–594. Adv. Lect. Math. (ALM), 25, Int. Press, Somerville, MA, 2013. Higher Education Press, Beijing, 2013. 1

[19] D. Mumford: *Stability of projective varieties.* L'Enseignement Mathématique **23** (1977), 39–110. 17

[20] D. Mumford: *Towards an enumerative geometry of the moduli space of curves.* In: Arithmetic and Geometry, edited by M. Artin, J. Tate. Birkhauser-Boston, 1983, 271–326. 9

[21] M. Petkova, H. Shiga: *A new interpretation of the Shimura curve with discriminant* 6 *in terms of Picard modular forms.* Arch. Math. **96** (2011), 335—348. 8

[22] É. Picard: *Sur une classe de groupes discontinus de substitutions linéaires et sur les fonctions de deux variables indépendantes restant invariables par ces substitutions.* Acta Math. **1** (1882), 297–320. 2, 7, 16

[23] É. Picard: *Sur des fonctions de deux variables indépendantes analogues aux fonctions modulaires.* Acta Math. **2** (1883), 114–135. 2, 16

[24] É. Picard: *Sur les formes quadratiques ternaires indéfinies à indéterminées conjuguées et sur les fonctions hyperfuchsiennes correspondantes.* Acta Math. **5** (1884), 121–182. 2, 16

[25] J. Rohde: *Cyclic coverings, Calabi-Yau manifolds and complex multiplication.* Lecture Notes in Math. **1975**, (2000). Springer Verlag. 1

[26] I. Satake: *Algebraic Structures of Symmetric Domains.* Kano Memorial Lectures, 4. Iwanami Shoten, Tokyo; Princeton University Press, Princeton, N.J., 1980. 4

[27] H. Shiga: *On the representation of Picard modular function by θ constants I-II.* Publ. RIMS, Kyoto University **24** (1963), 311–360. 4, 6, 7

[28] G. Shimura: *On purely transcendental fields of automorphic functions of several variables.* Osaka Math. Journal **1** (1964), 1–14. 1

[29] G. Shimura: *The arithmetic of automorphic forms with respect to a unitary group. Ann. of Math.* **(2) 107** (1978), 569–605. 4

[30] T. Shintani: *On automorphic forms on unitary groups of order* 3. Unpublished manuscript, 1979. 6

[31] S. Tsuyumine: *On Siegel modular forms of degree three.* Am. J. Math. **108**,755—862(1986). *Addendum to "On Siegel modular forms of degree three."* Am. J. Math. **108**, 1001-–1003 (1986). 1

[32] D. Zagier: *Elliptic modular forms and their applications.* The 1-2-3 of modular forms, 1—103, Universitext, Springer, Berlin, 2008. 32

INSTITUTE OF COMPUTATIONAL AND EXPERIMENTAL RESEARCH IN MATHEMATICS, 121 SOUTH MAIN STREET, PROVIDENCE, RI 02903, USA
Email address: `cleryfabien@gmail.com`

KORTEWEG-DE VRIES INSTITUUT, UNIVERSITEIT VAN AMSTERDAM, POSTBUS 94248, THE NETHERLANDS
Email address: `g.b.m.vandergeer@uva.nl`