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The Cohomology of the Moduli Space of Abelian Varieties

Gerard van der Geer

Introduction

That the moduli spaces of abelian varieties are a rich source of arithmetic and geometric information slowly emerged from the work of Kronecker, Klein, Fricke and many others at the end of the 19th century. Since the 20th century we know that the first place to dig for these hidden treasures is the cohomology of these moduli spaces.

In this survey we are concerned with the cohomology of the moduli space of abelian varieties. Since this is an extensive and widely ramified topic that connects to many branches of algebraic geometry and number theory we will have to limit ourselves. I have chosen to stick to the moduli spaces of principally polarized abelian varieties, leaving aside the moduli spaces of abelian varieties with non-principal polarizations and the other variations like the moduli spaces with extra structure (like conditions on their endomorphism rings) since often the principles are the same, but the variations are clad in heavy notation.

The emphasis in this survey is on the tautological ring of the moduli space of principally polarized abelian varieties. We discuss the cycle classes of the Ekedahl-Oort stratification, that can be expressed in tautological classes, and discuss differential forms on the moduli space. We also discuss complete subvarieties of \mathcal{A}_g . Finally, we discuss Siegel modular forms and its relations to the cohomology of these moduli spaces. We sketch the approach developed jointly with Faber and Bergström to calculate the traces of the Hecke operators by counting curves of genus ≤ 3 over finite fields, an approach that opens a new window on Siegel modular forms.

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1. The Moduli Space of Principally Polarized Abelian Varieties

We shall assume the existence of the moduli space of principally polarized abelian varieties as given. So throughout this survey \mathcal{A}_g will denote the Deligne-Mumford stack of principally polarized abelian varieties of dimension g . It is a smooth Deligne-Mumford stack over $\text{Spec}(\mathbb{Z})$ of relative dimension $g(g+1)/2$, see [25].

Over the complex numbers this moduli space can be described as the arithmetic quotient (orbifold) $\text{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$ of the Siegel upper half space by the symplectic group. This generalizes the well-known description of the moduli of complex elliptic curves as $\text{SL}(2, \mathbb{Z}) \backslash \mathfrak{H}$ with $\mathfrak{H} = \mathfrak{H}_1$ the usual upper half plane of the complex plane. We refer to Milne's account in this Handbook for the general theory of Shimura varieties.

The stack \mathcal{A}_g comes with a universal family of principally polarized abelian varieties $\pi : \mathcal{X}_g \rightarrow \mathcal{A}_g$. Since abelian varieties can degenerate the stack \mathcal{A}_g is not proper or complete.

The moduli space \mathcal{A}_g admits several compactifications. The first one is the Satake compactification or Baily-Borel compactification. It is defined by considering the vector space of Siegel modular forms of sufficiently high weight, by using these to map \mathcal{A}_g to projective space and then by taking the closure of the image of \mathcal{A}_g in the receiving projective space. This construction was first done by Satake and by Baily-Borel over the field of complex numbers, cf. [5]. The Satake compactification \mathcal{A}_g^* is very singular for $g \geq 2$. It has a stratification

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1}^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0.$$

In an attempt to construct non-singular compactifications of arithmetic quotients, such as $\text{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$, Mumford with a team of co-workers created a theory of

so-called toroidal compactifications of \mathcal{A}_g in [3], cf. also the new edition [4]. These compactifications are not unique but depend on a choice of combinatorial data, an admissible cone decomposition of the cone of positive (semi)-definite symmetric bilinear forms in g variables. Each such toroidal compactification admits a morphism $q : \tilde{\mathcal{A}}_g \rightarrow \mathcal{A}_g^*$. Faltings and Chai showed how Mumford's theory of compactifying quotients of bounded symmetric domains could be used to extend this to a compactification of \mathcal{A}_g over the integers, see [24, 25, 3]. This also led to the Satake compactification over the integers.

There are a number of special choices for the cone decompositions, such as the second Voronoi decomposition, the central cone decomposition or the perfect cone decomposition. Each of these choices has its advantages and disadvantages. Moreover, Alexeev has constructed a functorial compactification of \mathcal{A}_g , see [1, 2]. It has as a disadvantage that for $g \geq 4$ it is not irreducible but possesses extra components. The main component of this corresponds to $\mathcal{A}_g^{\text{Vor}}$, the toroidal compactification defined by the second Voronoi compactification. Olsson has adapted Alexeev's construction using log structures and obtained a functorial compactification. The normalization of this compactification is the compactification $\mathcal{A}_g^{\text{Vor}}$, cf [61].

The toroidal compactification $\mathcal{A}_g^{\text{perf}}$ defined by the perfect cone decomposition is a canonical model of \mathcal{A}_g for $g \geq 12$ as was shown by Shepherd-Barron, see [69] and his contribution to this Handbook.

The partial desingularization of the Satake compactification obtained by Igusa in [45] coincides with toroidal compactification corresponding to the central cone decomposition. We refer to a survey by Grushevsky ([38]) on the geometry of the moduli space of abelian varieties.

These compactifications (2nd Voronoi, central and perfect cone) agree for $g \leq 3$, but are different for higher g . For $g = 1$ one has $\tilde{\mathcal{A}}_1 = \mathcal{A}_1^* = \overline{\mathcal{M}}_{1,1}$, the Deligne-Mumford moduli space of 1-pointed stable curves of genus 1. For $g = 2$ the Torelli morphism gives an identification $\tilde{\mathcal{A}}_2 = \overline{\mathcal{M}}_2$, the Deligne-Mumford moduli space of stable curves of genus 2. The open part \mathcal{A}_2 corresponds to stable curves of genus 2 of compact type. But please note that the Torelli morphism of stacks $\mathcal{M}_g \rightarrow \mathcal{A}_g$ is a morphism of degree 2 for $g \geq 3$, since every principally polarized abelian variety possesses an automorphism of order 2, but the generic curve of genus $g \geq 3$ does not.

We shall use the term Faltings-Chai compactifications for the compactifications (over rings of integers) defined by admissible cone decompositions.

2. The Compact Dual

The moduli space $\mathcal{A}_g(\mathbb{C})$ has the analytic description as $\text{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$. The Siegel upper half space \mathfrak{H}_g can be realized in various ways, one of which is as an open subset of the so-called compact dual and the arithmetic quotient inherits

various properties from this compact quotient. For this reason we first treat the compact dual at some length.

To construct \mathfrak{H}_g , we start with an non-degenerate symplectic form on a complex vector space, necessarily of even dimension, say $2g$. To be explicit, consider the vector space \mathbb{Q}^{2g} with basis e_1, \dots, e_{2g} and symplectic form $\langle \cdot, \cdot \rangle$ given by $J(x, y) = x^t J y$ with

$$J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}.$$

We let $G = \mathrm{GSp}(2g, \mathbb{Q})$ be the algebraic group over \mathbb{Q} of symplectic similitudes of this symplectic vector space

$$G = \{g \in \mathrm{GL}(2g, \mathbb{Q}) : J(gx, gy) = \eta(g)J(x, y)\},$$

where $\eta : G \rightarrow \mathbb{G}_m$ is the so-called multiplier. Then G is the group of matrices $\gamma = (A B; C D)$ with A, B, C, D integral $g \times g$ -matrices with

$$A^t \cdot C = C^t \cdot A, \quad B^t \cdot D = D^t \cdot B \quad \text{and} \quad A^t \cdot D - C^t \cdot B = \eta(g)1_{2n}.$$

We denote the kernel of η by G^0 .

Let Y_g be the Lagrangian Grassmannian

$$Y_g = \{L \subset \mathbb{C}^{2g} : \dim(L) = g, J(x, y) = 0 \text{ for all } x, y \in L\}$$

that parametrizes all totally isotropic subspaces of our complex symplectic vector space. This is a homogeneous manifold of complex dimension $g(g+1)/2$ for the action of $G(\mathbb{C}) = \mathrm{GSp}(2g, \mathbb{C})$; in fact this group acts transitively and the quotient of $\mathrm{GSp}(2g, \mathbb{C})$ by the central $\mathbb{G}_m(\mathbb{C})$ acts effectively. We can write Y_g as a quotient $Y_g = G(\mathbb{C})/Q$, where Q is a parabolic subgroup of $G(\mathbb{C})$. More precisely, if we fix a point $y_0 = e_1 \wedge \dots \wedge e_g \in Y_g$ then we can write Q as the group of matrices $(A B; C D)$ in $\mathrm{GSp}(2g, \mathbb{C})$ with $C = 0$. This parabolic group Q has a Levi decomposition as $Q = M \times U$ with M the subgroup of G that respects the decomposition of the symplectic space as $\mathbb{Q}^g \oplus \mathbb{Q}^g$; the matrices of M are of the form $(A 0; 0 D)$ and is isomorphic to $\mathrm{GL}(g) \times \mathbb{G}_m$, while those of U are of the form $(1_g B; 0 1_g)$ with B symmetric.

There is an embedding of \mathfrak{H}_g into Y_g as follows. We consider the group $G^0(\mathbb{R})$ and the maximal compact subgroup K of elements that fix $\sqrt{-1}1_g$. Its elements are described as $(A - B; B A)$ and assigning to it the element $A + \sqrt{-1}B$ gives an isomorphism of K with the unitary group $U(g)$. One way to describe the Siegel upper half space is as the orbit $X_g = G^0(\mathbb{R})/K$ under the action of G^0 ; this can be embedded in $Y_g = (G/Q)(\mathbb{C})$ as the set of all maximal isotropic subspaces V such that $-\sqrt{-1}\langle v, \bar{v} \rangle$ is positive definite on V . Each such subspace has a basis consisting of the columns of the transpose of the matrix $(-1_g \tau)$ for a unique $\tau \in \mathfrak{H}_g$. The subgroup $G^+(\mathbb{R})$ leaves this subset invariant and this establishes the embedding of the domain \mathfrak{H}_g in its Y_g and this space Y_g is called the *compact dual*

of \mathfrak{H}_g . It contains $X_g \sim \mathfrak{H}_g$ as an open subset. The standard example (for $g = 1$) is that of the upper half plane contained in \mathbb{P}^1 .

For later use we extend this a bit by looking not only at maximal isotropic subspaces, but also at symplectic filtrations on our standard symplectic space.

Consider for $i = 1, \dots, g$ the (partial) flag variety $U_g^{(i)}$ of symplectic flags $E_i \subset \dots \subset E_{g-1} \subset E_g$, of linear subspaces E_j of \mathbb{C}^{2g} with $\dim(E_j) = j$ and E_g totally isotropic. We have $U_g^{(g)} = Y_g$. There are natural maps $\pi_i : U_g^{(i)} \rightarrow U_g^{(i+1)}$ and the fibre of π_i is a Grassmann variety of dimension i . We can represent $U_g^{(1)}$ as a quotient G/B , where B is a Borel subgroup of G . These spaces $U_g^{(i)}$ come equipped with universal flags $E_i \subset E_{i+1} \subset \dots \subset E_g$.

The manifold Y_g possesses, as all Grassmannians do, a cell decomposition. To define it, choose a fixed isotropic flag $\{0\} = Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \dots \subsetneq Z_g$; that is, $\dim Z_i = i$ and Z_g is an isotropic subspace of our symplectic space. We extend the filtration by setting $Z_{g+i} = (Z_{g-i})^\perp$ for $i = 1, \dots, g$. For general $V \in Y_g$ we expect that $V \cap Z_j = \{0\}$ for $j \leq g$. Therefore, for $\mu = (\mu_1, \dots, \mu_r)$ with μ_i non-negative integers satisfying

$$0 \leq \mu_i \leq g, \quad \mu_i - 1 \leq \mu_{i+1} \leq \mu_i \quad (1)$$

we put

$$W_\mu = \{V \in Y_g : \dim(V \cap Z_{g+1-\mu_i}) = i\}.$$

This gives a cell decomposition with cells only in even real dimension. The cell W_μ has (complex) codimension $\sum \mu_i$. Denote the set of n -tuples $\mu = (\mu_1, \dots, \mu_g)$ satisfying (1) by M_g . Then $\#M_g = 2^g$. Moreover, we have

$$W_\mu \subseteq \overline{W}_\nu \iff \mu_i \geq \nu_i \quad \text{for } 1 \leq i \leq g.$$

From the cell decomposition we find the homology of Y_g .

Proposition 2.1. *The integral homology of Y_g is generated by the cycle classes of $[\overline{W}_\mu]$ of the closed cells with $\mu \in M_g$. The Poincaré-polynomial of Y_g is given by $\sum b_{2i} t^i = (1+t)(1+t^2) \cdots (1+t^g)$.*

Note that Y_g is a rational variety and that the Chow ring R_g of Y_g is the isomorphic image of the cohomology ring of Y_g under the usual cycle class map. On Y_g we have a sequence of tautological vector bundles $0 \rightarrow E \rightarrow H \rightarrow Q \rightarrow 0$, where H is the trivial bundle of rank $2g$ defined by our fixed symplectic space and where the fibre of E_y of E over y is the isotropic subspace of dimension g corresponding to y . The bundle E corresponds to the standard representation of $K = U(g)$, see also Section 3. The tangent space to a point $e = [E]$ of Y_g is $\text{Hom}^{\text{sym}}(E, Q)$, the space of symmetric homomorphisms from E to Q ; indeed, usually the tangent space of a Grassmannian is described as $\text{Hom}(E, Q)$ (“move E a bit and it moves infinitesimally out of the kernel of $H \rightarrow Q$ ”), but we have to preserve the symplectic form that identifies E with the dual of Q ; therefore we have to take the ‘symmetric’ homomorphisms. We can identify this with $\text{Sym}^2(E^\vee)$.

We consider the Chern classes $u_i = c_i(E) \in R_g = \text{CH}^*(Y_g)$ for $i = 1, \dots, g$. We call them *tautological classes*. The symplectic form J on H can be used to identify E with the dual of the quotient bundle Q . The triviality of the bundle H implies the following relation for the Chern classes of E in R_g :

$$(1 + u_1 + u_2 + \dots + u_g)(1 - u_1 + u_2 + \dots + (-1)^g u_g) = 1.$$

These relations may be succinctly stated as

$$\text{ch}_{2k}(E) = 0 \quad (k \geq 1), \quad (2)$$

where ch_{2k} is the part of degree $2k$ of the Chern character.

Proposition 2.2. *The Chow ring R_g of Y_g is the ring generated by the u_i with as only relations $\text{ch}_{2k}(E) = 0$ for $k \geq 1$.*

One can check algebraically that the ring which is the quotient of $\mathbb{Z}[u_1, \dots, u_g]$ by the relation (2) has Betti numbers as given in Prop. 2.1. This description implies that this ring after tensoring with \mathbb{Q} is in fact generated by the u_j with j odd. Furthermore, by using induction one obtains the relations

$$u_g u_{g-1} \cdots u_{k+1} u_k^2 = 0 \quad \text{for } k = g, \dots, 1. \quad (3)$$

It follows that this ring has the following set of 2^g basis elements

$$\prod_{i \in I} u_i, \quad I \subseteq \{1, \dots, g\}.$$

The ring R_g is a so-called Gorenstein ring with socle $u_g u_{g-1} \cdots u_1$.

Using $R_g/(u_g) \cong R_{g-1}$ one finds the following properties.

Lemma 2.3. *In $R_g/(u_g)$ we have $u_1^{g(g-1)/2} \neq 0$ and $u_1^{g(g-1)/2+1} = 0$.*

Define now classes in the Chow ring of the flag spaces $U_g^{(i)}$ as follows:

$$v_j = c_1(E_j/E_{j-1}) \in \text{CH}^*(U_g^{(i)}) \quad j = i, \dots, g.$$

Moreover, we set

$$u_j^{(i)} = c_j(E_i) \in \text{CH}^*(U_g^{(i)}) \quad j = 1, \dots, i.$$

We can view $u_j^{(i)}$ as the j th symmetric function in v_1, \dots, v_i . Then we have the relations under the forgetful maps $\pi_i : U_g^{(i)} \rightarrow U_g^{(i+1)}$

$$\pi_i^*(u_j^{(i+1)}) = u_j^{(i)} + v_{i+1} u_{j-1}^{(i)} \quad j = 1, \dots, i+1, \quad (4)$$

and

$$v_j^j - u_1^{(j)} v_j^{j-1} + \dots + (-1)^j u_j^{(j)} = 0 \quad j = 1, \dots, g. \quad (5)$$

The Chow ring of $U_g^{(1)}$ has generators

$$v_1^{\eta_1} v_2^{\eta_2} \cdots v_g^{\eta_g} u_g^{\epsilon_g} u_{g-1}^{\epsilon_{g-1}} \cdots u_1^{\epsilon_1}$$

with $0 \leq \eta_i < i$ and $\epsilon_i = 0$ or 1 .

Lemma 2.4. *The following Gysin formulas hold:*

- (1) $(\pi_i)_*(c) = 0$ for all $c \in CH^*(U_g^{(i+1)})$;
- (2) $(\pi_i)_*(v_{i+1}^i) = 1$;
- (3) $(\pi_i)_*(u_i^{(i)}) = (-1)^i$.

These formulas together with (4) and (5) completely determine the image of the Gysin map $(\pi)_* = (\pi_{g-1} \cdots \pi_1)_*$.

For a sequence of $r \leq g$ positive integers μ_i with $\mu_i \geq \mu_{i+1}$ we define an element of R_g :

$$\Delta_\mu(u) = \det \left((u_{\mu_i - i + j})_{1 \leq i \leq r; 1 \leq j \leq r} \right).$$

We also define so-called Q -polynomials by the formula:

$$Q_{i,j} = u_i u_j - 2u_{i+1} u_{j-1} + \dots + (-1)^j 2u_{i+j} \text{ for } i < j.$$

We have $Q_{i0} = u_i$ for $i = 1, \dots, g$. Let μ be a strict partition. For r even we define an anti-symmetric matrix $[x_\mu] = [x_{ij}]$ as follows. Let

$$x_{i,j} = Q_{\mu_i \mu_j}(u) \quad \text{for } 1 \leq i < j \leq r.$$

We then set for even r

$$\Xi_\mu = \text{Pf}([x_\mu]),$$

while for r odd we define $\Xi_\mu = \Xi_{\mu_1 \dots \mu_r 0}$. These expressions may look a bit artificial, but their purpose is clearly demonstrated by the following Theorem, due to Pragacz [66, 31].

Theorem 2.5. (*Pragacz's formula*) *The class of the cycle $[\overline{W}_\mu]$ in the Chow ring is given by (a multiple of) $\Xi_\mu(u)$.*

From (2.4) we have the property: for partitions μ and ν with $\sum \mu_i + \sum \nu_i = g(g+1)/2$ we have

$$\Xi_\mu \Xi_\nu = \begin{cases} 1 & \text{if } \nu = \rho - \mu, \\ 0 & \text{if } \nu \neq \rho - \mu, \end{cases}$$

where $\rho = \{g, g-1, \dots, 1\}$. We have the following relations in $CH^*(Y_g)$:

- i) $u_g u_{g-1} \dots u_1 = 1$,
- ii) $u_1^N = N! \prod_{k=1}^g \frac{1}{(2k-1)!!}$,

where 1 represents the class of a point and $N = g(g+1)/2$.

Proof. For i) We have $T_{Y_g} \cong \text{Sym}^2(E^\vee)$, thus the Chern classes of the tangent bundle T_{Y_g} are expressed in the u_i . By [29], Ex. 14.5.1, p. 265) for the top Chern class of Sym^2 of a vector bundle we have

$$e(Y_g) = 2^g = 2^g \Delta_{(g,g-1,\dots,1)}(u) = 2^g \det \begin{pmatrix} u_g & 0 & 0 & \dots & 0 \\ u_{g-2} & u_{g-1} & u_g & 0 & \dots \\ & & \dots & & \\ & & & & u_1 \end{pmatrix}.$$

Developing the determinant gives

$$1 = u_g u_{g-1} \dots u_1 + u_g^2 A(u) + u_g u_{g-1}^2 B(u) + \dots,$$

from which the desired identity results by using (3). Alternatively, one may use Pragacz's formula above for the degeneracy locus given by $Z_g = V$ for some subspace V . \square

3. The Hodge Bundle

Since \mathcal{A}_g comes with a universal principally polarized abelian variety $\pi : \mathcal{X}_g \rightarrow \mathcal{A}_g$ we have a rank g vector bundle or locally free sheaf

$$\mathbb{E} = \mathbb{E}_g = \pi_* (\Omega_{\mathcal{X}_g/\mathcal{A}_g}^1),$$

called the *Hodge Bundle*. It has an alternative definition as

$$\mathbb{E} = s^* (\Omega_{\mathcal{X}_g/\mathcal{A}_g}^1),$$

i.e., as the cotangent bundle to the zero section $s : \mathcal{A}_g \rightarrow \mathcal{X}_g$. If \mathcal{X}_g^t is the dual abelian variety (isomorphic to \mathcal{X}_g because we stick to a principal polarization) then the Hodge bundle \mathbb{E}^t of \mathcal{X}_g^t satisfies

$$(\mathbb{E}^t)^\vee = \text{Lie}(\mathcal{X}_g^t) \cong R^1 \pi_* \mathcal{O}_{\mathcal{X}_g}.$$

The Hodge bundle \mathbb{E} can be extended to any toroidal compactification $\tilde{\mathcal{A}}_g$ of Faltings-Chai type. In fact, over $\tilde{\mathcal{A}}_g$ we have a universal family of semi-abelian varieties and one takes the dual of $\text{Lie}(\tilde{\mathcal{X}})$, cf. [25]. We shall denote it again by \mathbb{E} .

If we now go to a fine moduli space, say $\mathcal{A}_g[n]$ with $n \geq 3$, the moduli space of principally polarized abelian varieties with a level n structure, and take a smooth toroidal compactification then we can describe the sheaf of holomorphic 1-forms in terms of the Hodge bundle. With $D = \tilde{\mathcal{A}}_g[n] - \mathcal{A}_g[n]$, the divisor at infinity, we have the important result:

Proposition 3.1. *The Hodge bundle \mathbb{E} on $\tilde{\mathcal{A}}_g[n]$ for $n \geq 3$ satisfies the identity*

$$\text{Sym}^2(\mathbb{E}) \cong \Omega_{\tilde{\mathcal{A}}_g[n]}^1(\log D).$$

This result extends the description of the tangent space to \mathfrak{H}_g in the compact dual. It is proven in the general setting in [25], p. 86.

Recall the description of Y_g as the symplectic Grassmannian $G(\mathbb{C})/Q(\mathbb{C})$ with $G = \mathrm{GSp}(2g, \mathbb{Z})$ and Q the parabolic subgroup with Levi decomposition $M \ltimes U$ with $M = \mathrm{GL}(g) \times \mathbb{G}_m$. The Siegel upper half space \mathfrak{H}_g can be viewed as an open subset of Y_g . Put $G_0 = \mathrm{Sp}(2g, \mathbb{Z})$, $Q_0 = Q \cap G_0$ and $M_0 = M \cap G_0$. Then if $\rho : Q_0 \rightarrow \mathrm{GL}(V)$ is a finite-dimensional complex representation, we define an equivariant vector bundle \mathcal{V}_ρ on Y_g by

$$\mathcal{V}_\rho = G_0(\mathbb{C}) \times^{Q_0(\mathbb{C})} V,$$

where the contracted product is defined by the usual equivalence relation $(g, v) \sim (gq, \rho(q)^{-1}v)$ for all $g \in G_0(\mathbb{C})$ and $q \in Q_0(\mathbb{C})$. Then our group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on the bundle \mathcal{V}_ρ and the quotient is a vector bundle V_ρ in the orbifold sense on $\mathcal{A}_g(\mathbb{C}) = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$.

A representation of $\mathrm{GL}(g)$ can be lifted to a representation of Q_0 by letting it act trivially on the unipotent radical U of M_0 . Carrying out this construction with the standard (tautological) representation of $\mathrm{GL}(g)$ produces the Hodge bundle.

The Hodge bundle can be extended to a bundle over our toroidal compactification $\tilde{\mathcal{A}}_g$. Since any bundle V_ρ is obtained by applying Schur functors to powers of the Hodge bundle (see [30]) we can extend the bundle V_ρ by applying the Schur functor to the extended power of the Hodge bundle. In this way we obtain a canonical extension to $\tilde{\mathcal{A}}_g$ for all equivariant holomorphic bundles V_ρ .

4. The Tautological Ring of \mathcal{A}_g

The moduli space \mathcal{A}_g is a Deligne-Mumford stack or orbifold and as such it has a Chow ring. To be precise, consider the moduli space $\mathcal{A}_g \otimes k$ over an algebraically closed field k . For simplicity we shall write \mathcal{A}_g instead.

We are interested in the Chow rings $\mathrm{CH}(\mathcal{A}_g)$ and $\mathrm{CH}_{\mathbb{Q}}(\mathcal{A}_g)$ of this moduli space. In general these rings seem unattainable. However, they contain subrings that we can describe well and that play an important role.

We denote the Chern classes of the Hodge bundle by

$$\lambda_i := c_i(\mathbb{E}), \quad i = 1, \dots, g.$$

We define the *tautological subring* of the Chow ring $\mathrm{CH}_{\mathbb{Q}}(\mathcal{A}_g)$ as the subring generated by the Chern classes of the Hodge bundle \mathbb{E} .

The main result is a description of the tautological ring in terms of the Chow ring R_{g-1} of the compact dual Y_{g-1} (see [32]).

Theorem 4.1. *The tautological subring T_g of the Chow ring $\mathrm{CH}_{\mathbb{Q}}(\mathcal{A}_g)$ generated by the λ_i is isomorphic to the ring R_{g-1} .*

This implies that a basis of the codimension i part is given by the monomials

$$\lambda_1^{e_1} \lambda_2^{e_2} \cdots \lambda_{g-1}^{e_{g-1}} \quad e_j \in \{0, 1\} \text{ and } \sum_{j=1}^{g-1} j e_j = i.$$

This theorem follows from the following four results, each interesting in its own right.

Theorem 4.2. *The Chern classes λ_i of the Hodge bundle \mathbb{E} satisfy the relation*

$$(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g) = 1.$$

Theorem 4.3. *The top Chern class λ_g of the Hodge bundle \mathbb{E} vanishes in $\text{CH}_{\mathbb{Q}}(\mathcal{A}_g)$.*

Theorem 4.4. *The first Chern class λ_1 of \mathbb{E} is ample on \mathcal{A}_g .*

Theorem 4.5. *In characteristic $p > 0$ the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$ contains a complete subvariety of codimension g .*

To deduce Theorem 4.1 from Thms 4.2 - 4.5 we argue as follows. By Theorem 4.2 the tautological subring T_g is a quotient ring of R_g via $u_i \mapsto \lambda_i$, and then by Theorem 4.3 also of $R_{g-1} \cong R_g/(u_g)$. The ring R_{g-1} is a Gorenstein ring with socle $u_1 u_2 \cdots u_{g-1}$ and this is a non-zero multiple of $u_1^{g(g-1)/2}$. By Theorem 4.4 the ample class λ_1 satisfies $\lambda_1^j \cdot [V] \neq 0$ on any complete subvariety V of codimension j in \mathcal{A}_g . The existence of a complete subvariety of codimension g in positive characteristic implies that $\lambda_1^{g(g-1)/2}$ does not vanish in $\text{CH}_{\mathbb{Q}}(\mathcal{A}_g \otimes \mathbb{F}_p)$, hence the class $\lambda_1^{g(g-1)/2}$ does not vanish in $\text{CH}_{\mathbb{Q}}(\mathcal{A}_g)$. If the map $T_g \rightarrow R_{g-1}$ defined by $\lambda_i \mapsto u_i$ would have a non-trivial kernel then $\lambda_1^{g(g-1)/2}$ would have to vanish.

Theorem 4.2 was proved in [32]. The proof is obtained by applying the Grothendieck-Riemann-Roch formula to the theta divisor on the universal abelian variety \mathcal{X}_g over \mathcal{A}_g . In fact, take a line bundle L on \mathcal{X}_g that provides on each fibre X a theta divisor and normalize L such that it vanishes on the zero section of \mathcal{X}_g over \mathcal{A}_g . Then the Grothendieck-Riemann-Roch Theorem tells us that

$$\begin{aligned} \text{ch}(\pi_! L) &= \pi_*(\text{ch}(L) \cdot \text{Td}^{\vee}(\Omega_{\mathcal{X}_g/\mathcal{A}_g}^1)) \\ &= \pi_*(\text{ch}(L) \cdot \text{Td}^{\vee}(\pi^*(\mathbb{E}_g))) \\ &= \pi_*(\text{ch}(L)) \cdot \text{Td}^{\vee}(\mathbb{E}), \end{aligned} \tag{1}$$

where we used $\Omega_{\mathcal{X}_g/\mathcal{A}_g}^1 \cong \pi^*(\mathbb{E})$ and the projection formula. Here Td^{\vee} is defined for a line bundle with first Chern class γ by $\gamma/(e^{\gamma} - 1)$. But $R^i \pi_* L = 0$ for our relatively ample L for $i > 0$ and so $\pi_! L$ is represented by a vector bundle, and because L defines a principal polarization, by a line bundle. We write $\theta = c_1(\pi_! L)$. This gives the identity

$$\sum_{k=0}^{\infty} \frac{\theta^k}{k!} = \pi_* \left(\sum_{k=0}^{\infty} \frac{\Theta^{g+k}}{(g+k)!} \right) \text{Td}^{\vee}(\mathbb{E}). \tag{2}$$

Recall that $\mathrm{Td}^\vee(\mathbb{E}) = 1 - \lambda_1/2 + (\lambda_1^2 + \lambda_2)/12 + \dots$. We can compare terms of equal codimension; the codimension 1 term gives

$$\theta = -\lambda_1/2 + \pi_*(\Theta^{g+1}/(g+1)!). \quad (3)$$

If we now look how both sides of (2) behave when we replace L by L^n , we see immediately that the term Θ^{g+k} of the right hand side changes by a factor n^{g+k} . As to the left hand side, for a principally polarized abelian variety X the space of sections $H^0(X, L_X^{\otimes n})$ is a representation of the Heisenberg group; it is the irreducible representation of degree n^g . This implies that $\mathrm{ch}(\pi_!L) = n^g \mathrm{ch}(\pi_!L)$. We see

$$n^g \sum_{k=0}^{\infty} \frac{\theta^k}{k!} = \pi_* \left(\sum_{k=0}^{\infty} \frac{n^{g+k} \Theta^{g+k}}{(g+k)!} \right) \cdot \mathrm{Td}^\vee(\mathbb{E}).$$

Comparing the coefficients leads immediately to the following result.

Corollary 4.6. *In $\mathrm{CH}_{\mathbb{Q}}(\mathcal{A}_g)$ we have $\pi_*(\mathrm{ch}(L)) = 1$ and $\mathrm{ch}(\pi_*L) = \mathrm{Td}^\vee(\mathbb{E})$.*

In particular $\pi_*(\Theta^{g+1}) = 0$. Substituting this in (3) gives the following result.

Corollary 4.7. *(Key Formula) We have $2\theta = -\lambda_1$.*

Corollary 4.8. *We have $\mathrm{ch}_{2k}(\mathbb{E}) = 0$ for $k \geq 1$.*

Proof. The relation (1) reduces now to

$$e^{-\lambda_1/2} = \mathrm{Td}^\vee(\mathbb{E}).$$

If, say ρ_1, \dots, ρ_g are the Chern roots of \mathbb{E} , so that our λ_i is the i th elementary symmetric function in the ρ_j , then this relation is

$$\prod_{i=1}^g \frac{e^{\rho_i} - e^{-\rho_i}}{\rho_i} = 1.$$

This is equivalent with $\mathrm{ch}_{2k}(\mathbb{E}) = 0$ for $k > 0$ and also with $\mathrm{Td}(\mathbb{E} \oplus \mathbb{E}^\vee) = 0$. \square

The proof of Theorem 4.3 is also an application of Grothendieck-Riemann-Roch, this time applied to the structure sheaf (i.e., $n = 0$ in the preceding case). We find

$$\mathrm{ch}(\pi_!\mathcal{O}_{\mathcal{X}_g}) = \pi_*(\mathrm{ch}(\mathcal{O}_{\mathcal{X}_g}) \cdot \mathrm{Td}^\vee \pi^*(\mathbb{E}_g)) = \pi_*(1) \mathrm{Td}^\vee(\mathbb{E}_g) = 0.$$

For an abelian variety X the cohomology group $H^i(X, \mathcal{O}_X)$ is the i th exterior power of $H^1(X, \mathcal{O}_X)$ and we thus see that the left hand side equals (global Serre duality)

$$\mathrm{ch}(1 - \mathbb{E}^\vee + \wedge^2 \mathbb{E}^\vee - \dots + (-1)^g \wedge^g \mathbb{E}^\vee).$$

But by some general yoga (see [10]) we have for a vector bundle B of rank r the relation $\sum_{j=0}^r (-1)^j \mathrm{ch}(\wedge^j B^\vee) = c_r(B) \mathrm{Td}(B)^{-1}$. So the left hand side is $\lambda_g \mathrm{Td}(\mathbb{E})^{-1}$ and since $\mathrm{Td}(\mathbb{E})$ starts with $1 + \dots$ it follows that λ_g is zero.

Theorem 4.4 is a classical result in characteristic zero. If we present our moduli space \mathcal{A}_g over \mathbb{C} as the quotient space $\mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$ then the determinant of the Hodge bundle is a (n orbifold) line bundle that corresponds to the factor of automorphy

$$\det(c\tau + d) \quad \text{for } \tau \in \mathfrak{H}_g \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}).$$

In other words, a section of the k th power of $\det(\mathbb{E})$ gives by pull back to \mathfrak{H}_g a holomorphic function $f : \mathfrak{H}_g \rightarrow \mathbb{C}$ with the property that

$$f((a\tau + b)(c\tau + d)^{-1}) = \det(c\tau + d)^k f(\tau).$$

A very well-known theorem by Baily and Borel (see [5]) says that modular forms of sufficiently high weight k define an embedding of $\mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$ into projective space. The idea is that one can construct sufficiently many modular forms to separate points and tangent vectors on $\mathcal{A}_g(\mathbb{C})$. So λ_1 is an ample class. Clearly this holds then also in characteristic p if p is sufficiently large. So if we have a complete subvariety of codimension j in $\mathcal{A}_g \otimes \mathbb{F}_p$ for every p then λ_1^j cannot vanish when restricted to this subvariety.

Theorem 4.4 was extended to all characteristics by Moret-Bailly, see [55].

5. The Tautological Ring of $\tilde{\mathcal{A}}_g$

In this section, just as in the preceding one, we will write \mathcal{A}_g for $\mathcal{A}_g \otimes k$ with k an algebraically closed field.

The class λ_g vanishes in the rational Chow ring of \mathcal{A}_g . However, in a suitable compactification of \mathcal{A}_g this is no longer true and instead of the quotient $R_{g-1} = R_g/(u_g)$ we find a copy of R_g in the Chow ring of such a compactification.

We consider a smooth toroidal compactification $\tilde{\mathcal{A}}_g$ of \mathcal{A}_g of the type constructed by Faltings and Chai. Over $\tilde{\mathcal{A}}_g$ we have a ‘universal’ semi-abelian variety \mathcal{G} with a zero-section s , see [25]. Then $s^*\mathrm{Lie}(\mathcal{G})$ defines in a canonical way an extension of the Hodge bundle \mathbb{E} on \mathcal{A}_g to a vector bundle $\tilde{\mathbb{E}}$ on $\tilde{\mathcal{A}}_g$. We will denote this extension again by \mathbb{E} .

The relation (1) of the preceding section can now be extended to $\tilde{\mathcal{A}}_g$. A proof of this extension was given by Esnault and Viehweg, see [21]. They show that in characteristic 0 for the Deligne extension of the cohomology sheaf \mathcal{H}^1 with its Gauss-Manin connection the Chern character vanishes in degree $\neq 0$ by applying Grothendieck-Riemann-Roch to a log extension and using the action of -1 to separate weights. By specializing one finds the following result.

Theorem 5.1. *The Chern classes λ_i of the Hodge bundle \mathbb{E} on $\tilde{\mathcal{A}}_g$ satisfy the relation*

$$(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g) = 1.$$

This implies the following result:

Theorem 5.2. *The tautological subring \tilde{T}_g of $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$ is isomorphic to R_g*

Proof. Clearly, by the relation of 5.1 the tautological ring is a quotient of R_g ; since λ_1 is ample on an open dense part (\mathcal{A}_g) the socle

$$\lambda_g \lambda_{g-1} \cdots \lambda_1 = \frac{1}{(g(g+1)/2)!} \left(\prod_{k=1}^g (2k-1)!! \right) \lambda_1^{g(g+1)/2}$$

does not vanish, and the tautological ring must be isomorphic to R_g . \square

The Satake compactification \mathcal{A}_g^* of \mathcal{A}_g , defined in general as the proj of the ring of Siegel modular forms, possesses a stratification

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0.$$

By the natural map $q: \tilde{\mathcal{A}}_g \rightarrow \mathcal{A}_g^*$ this induces a stratification of $\tilde{\mathcal{A}}_g$: we let $\mathcal{A}_g^{(t)}$ be the inverse image $q^{-1}(\mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \mathcal{A}_{g-t})$, the moduli space of abelian varieties of torus rank $\leq t$. We have the following extension of Theorems 4.3 and 4.5.

Theorem 5.3. *For $t < g$ we have the relation $\lambda_g \lambda_{g-1} \cdots \lambda_{g-t} = 0$ in the Chow group $\mathrm{CH}_{\mathbb{Q}}^m(\mathcal{A}_g^{(t)})$ with $m = \sum_{i=0}^t g - i$.*

Theorem 5.4. *The moduli space $\mathcal{A}_g^{(t)} \otimes \mathbb{F}_p$ in characteristic $p > 0$ contains a complete subvariety of codimension $g - t$, namely the locus of abelian varieties of p -rank $\leq t$.*

Corollary 5.5. *We have $\lambda_1^{g(g-1)/2+t} \neq 0$ on $\mathcal{A}_g^{(t)}$.*

Proposition 5.6. *For $0 \leq k \leq g$ and $r > 0$ we have on the boundary $\mathcal{A}_k^* \subset \mathcal{A}_g^*$ of the Satake compactification*

$$\lambda_1^{k(k+1)/2} \neq 0 \quad \text{and} \quad \lambda_1^{k(k+1)/2+r} = 0.$$

This follows from the fact that λ_1 is an ample class on \mathcal{A}_g^* , that $\lambda_1|_{\mathcal{A}_k^*}$ is again λ_1 (now of course defined on \mathcal{A}_k^*) and the fact that $\dim \mathcal{A}_k^* = k(k+1)/2$.

6. The Proportionality Principle

The result on the tautological ring has the following immediate corollary, a special case of the so-called Proportionality Principle of Hirzebruch-Mumford:

Theorem 6.1. *The characteristic numbers of the Hodge bundle are proportional to those of the tautological bundle on the ‘compact dual’ Y_g :*

$$\lambda_1^{n_1} \cdots \lambda_g^{n_g}([\tilde{\mathcal{A}}_g]) = (-1)^{g(g+1)/2} \frac{1}{2^g} \left(\prod_{k=1}^g \zeta(1-2k) \right) \cdot u_1^{n_1} \cdots u_g^{n_g}([Y_g])$$

for all (n_1, \dots, n_g) with $\sum n_i = g(g+1)/2$.

Indeed, any top-dimensional class $\lambda_1^{n_1} \cdots \lambda_g^{n_g}$ is a multiple $m(n_1, \dots, n_g)$ times the top-dimensional class $\lambda_1 \cdots \lambda_g$ where the coefficient depends only on the structure of R_g . So the proportionality principle is clear, and to make it explicit one must find the value of one top-dimensional class, say $\lambda_1 \cdots \lambda_g$ on $\tilde{\mathcal{A}}_g$. The top Chern class of the bundle $\text{Sym}^2(\mathbb{E})$ on \mathcal{A}_g equals

$$2^g \lambda_1 \cdots \lambda_g.$$

We can interpret $2^g \deg(\lambda_1 \cdots \lambda_g)$ up to a sign $(-1)^{g(g+1)/2}$ as the log version of the Euler number of \mathcal{A}_g . It is known via the work of Siegel and Harder:

Theorem 6.2. (*Siegel-Harder*) *The Euler number of \mathcal{A}_g is equal to*

$$\zeta(-1)\zeta(-3) \cdots \zeta(1-2g).$$

Example 6.3. For $g = 1$ the Euler number of $\mathcal{A}_1(\mathbb{C})$ is $-1/12$. It is obtained by integrating

$$(-1) \frac{1}{2\pi i} \frac{dx \wedge dy}{y^2}$$

over the usual fundamental domain $(\{\tau = x + iy \in \mathfrak{H} : |x| \leq 1/2, x^2 + y^2 \geq 1\})$ for the action of $\text{SL}(2, \mathbb{Z})$ and (which gives $2\zeta(-1) = -1/6$) and then dividing by 2 since the group $\text{SL}(2, \mathbb{Z})$ does not act effectively (-1 acts trivially).

The Proportionality Principle immediately extends to our equivariant bundles V_ρ since these are obtained by applying Schur functors to powers of the Hodge bundle.

Corollary 6.4. (*The Proportionality Principle of Hirzebruch-Mumford*) *The characteristic numbers of the equivariant vector bundle V_ρ on $\tilde{\mathcal{A}}_g$ are proportional to those of the corresponding bundle on the compact dual Y_g*

The proportionality factor here is

$$p(g) = (-1)^{g(g+1)/2} \prod_{j=1}^g \frac{\zeta(1-2j)}{2}$$

We give a few values.

g	1	2	3	4	5
$1/p(g)$	24	5760	2903040	1393459200	367873228800

A remark about the history of this ‘principle’. Hirzebruch, inspired by a paper of Igusa, found in 1958 (see [43]) that for a discrete torsion-free group Γ of automorphisms of a bounded symmetric domain D such that the quotient $X_\Gamma = \Gamma \backslash D$ is compact, the Chern numbers of the quotient are proportional to the Chern numbers of the compact dual \hat{D} of D . In the case of a subgroup $\Gamma \subset \text{Sp}(2g, \mathbb{R})$

that acts freely on \mathfrak{H}_g this means that there is a proportionality factor e_Γ such that for all $n = (n_1, \dots, n_{g(g+1)/2})$ with $n_i \in \mathbb{Z}_{\geq 0}$ and $\sum n_i = g(g+1)/2$ we have

$$c^n(X_\Gamma) = e_\Gamma c^n(Y_g),$$

where c^n stands for the top Chern class $\prod_i c_i^{n_i}$ and Y_g is the compact dual of \mathfrak{H}_g . His argument was that the top Chern classes of X are represented by $G(\mathbb{R})$ -invariant differential forms of top degree and these are proportional to each other on D and on \hat{D} . The principle extends to all equivariant holomorphic vector bundles on \mathfrak{H}_g . Mumford has obtained (see [58]) an extension of Hirzebruch's Proportionality Principle that applies to his toroidal compactifications of \mathcal{A}_g .

7. The Chow Rings of $\tilde{\mathcal{A}}_g$ for $g = 1, 2, 3$

For low values of g the Chow rings of \mathcal{A}_g and their toroidal compactifications can be explicitly described.

Theorem 7.1. *The Chow ring $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{A}}_1)$ is isomorphic to $\mathbb{Q}[\lambda_1]/(\lambda_1^2)$.*

Mumford determined in [59] the Chow ring of $\tilde{\mathcal{A}}_2$.

Theorem 7.2. *The Chow ring $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{A}}_2)$ is generated by λ_1, λ_2 and σ_1 and isomorphic to $\mathbb{Q}[\lambda_1, \lambda_2, \sigma_1]/(I_2)$, with I_2 the ideal generated by*

$$(1 + \lambda_1 + \lambda_2)(1 - \lambda_1 + \lambda_2) - 1, \quad \lambda_2 \sigma_1, \quad \sigma_1^2 - 22\lambda_1 \sigma_1 + 120\lambda_1^2.$$

The ranks of the Chow groups are 1, 2, 2, 1.

The intersection numbers in this ring are given by

$2 \setminus 1$	λ_1	σ_1
λ_1^2	1/2880	0
$\lambda_1 \sigma_1$	0	-1/24

while the degrees of the top classes in the sigmas are

$$\deg(\sigma_3) = 1/4, \quad \deg(\sigma_2 \sigma_1) = -1/4, \quad \deg(\sigma_1^3) = -11/12.$$

For genus 3 the result is (cf. [33]):

Theorem 7.3. *The tautological ring of \mathcal{A}_3 is generated by the Chern classes λ_i and isomorphic to $\mathbb{Q}[\lambda_1]/(\lambda_1^4)$.*

Theorem 7.4. *The Chow ring $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{A}}_3)$ of $\tilde{\mathcal{A}}_3$ is generated by the $\lambda_i, i = 1, 2, 3$ and the $\sigma_i, i = 1, 2$ and is isomorphic to the graded ring (subscript is degree)*

$$\mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \sigma_1, \sigma_2]/I_3,$$

with I_3 the ideal generated by the relations

$$(1 + \lambda_1 + \lambda_2 + \lambda_3)(1 - \lambda_1 + \lambda_2 - \lambda_3) = 1,$$

$$\begin{aligned}
\lambda_3 \sigma_1 &= \lambda_3 \sigma_2 = \lambda_1^2 \sigma_2 = 0, \\
\sigma_1^3 &= 2016 \lambda_3 - 4 \lambda_1^2 \sigma_1 - 24 \lambda_1 \sigma_2 + \frac{11}{3} \sigma_2 \sigma_1, \\
\sigma_2^2 &= 360 \lambda_1^3 \sigma_1 - 45 \lambda_1^2 \sigma_1^2 + 15 \lambda_1 \sigma_2 \sigma_1, \\
\sigma_1^2 \sigma_2 &= 1080 \lambda_1^3 \sigma_1 - 165 \lambda_1^2 \sigma_1^2 + 47 \lambda_1 \sigma_2 \sigma_1.
\end{aligned}$$

The ranks of the Chow groups are: 1, 2, 4, 6, 4, 2, 1.

For the degrees of the top dimensional elements we refer to [33].

8. Some Intersection Numbers

As stated in section 1 the various compactifications employed for \mathcal{A}_g each have their own merits. For example, the toroidal compactification associated to the perfect cone decomposition has the advantage that its boundary is an irreducible divisor $D = D_g$.

By a result of Borel [9] it is known that in degrees $\leq g - 2$ the cohomology of $\mathrm{Sp}(2g, \mathbb{Q})$ (and in fact for any finite index subgroup Γ) is generated by elements in degree $2k + 4$ for $k = 0, 1, \dots, (g - 6)/2$:

$$H^*(\Gamma, \mathbb{Q}) = \mathbb{Q}[x_2, x_6, x_{10}, \dots].$$

In particular in degree 2 there is one generator. We deduce (at least for $g \geq 6$ and with special arguments also for $2 \leq g \leq 5$):

Proposition 8.1. *The Picard group of \mathcal{A}_g for $g \geq 2$ is generated by the determinant λ_1 of the Hodge bundle.*

By observing that on $\mathcal{A}_g^{(1)}$ and $\mathcal{A}_g^{\mathrm{Perf}}$ the boundary divisor is irreducible we get:

Corollary 8.2. *The Picard group of $\mathcal{A}_g^{(1)}$ and of $\mathcal{A}_g^{\mathrm{Perf}}$ is generated by λ_1 and the class of D .*

It would be nice to know the top intersection numbers $\lambda_1^n D^{G-n}$ with $G = g(g + 1)/2 = \dim \mathcal{A}_g$. It seems that these numbers are zero when n is not of the form $k(k + 1)/2$. In fact, Erdenberger, Grushevsky and Hulek formulated this as a conjecture, see [20].

Conjecture 8.3. *The intersection number $\deg \lambda_1^n D^{G-n}$ on $\mathcal{A}_g^{\mathrm{Perf}}$ vanishes unless n is of the form $n = k(k + 1)/2$ for $k \leq g$.*

From our results above it follows that

$$\deg \lambda_1^G = (-1)^G \frac{G!}{2^g} \prod_{k=1}^g \frac{\zeta(1 - 2k)}{(2k - 1)!}.$$

Erdenberger, Grushevsky and Hulek calculated the next two cases. We quote only the first

$$\deg \lambda_1^{g(g-1)/2} D^g = (-1)^{G-1} \frac{(g-1)!(G-g)!}{2} \prod_{k=1}^{g-1} \frac{\zeta(1-2k)}{(2k-1)!!}$$

and we refer for the intersection number $\deg \lambda^{(g-2)(g-1)/2} D^{2g-1}$ to loc. cit.

9. The Top Class of the Hodge Bundle

As before, we shall write here \mathcal{A}_g for $\mathcal{A}_g \otimes k$ with k an algebraically closed field.

The cycle class λ_g vanishes in $\mathrm{CH}_{\mathbb{Q}}^g(\mathcal{A}_g)$. However, it does not vanish on $\mathrm{CH}_{\mathbb{Q}}^g(\tilde{\mathcal{A}}_g)$, for example because $\lambda_1 \lambda_2 \cdots \lambda_g$ is a non-zero multiple of $\lambda_1^{g(g+1)/2}$ that has positive degree since λ_1 is ample on \mathcal{A}_g . This raises two questions. First, what is the order of λ_g in $\mathrm{CH}^g(\mathcal{A}_g)$ as a torsion class? Second, since up to torsion λ_g comes from the boundary $\tilde{\mathcal{A}}_g - \mathcal{A}_g$, one can ask for a natural supporting cycle for this class in the boundary. Since we work on stacks one has to use intersection theory on stacks; the theory is still in its infancy, but we use Kresch's approach [50]. Mumford answered the first question for $g = 1$ in [56].

In joint work with Ekedahl ([17, 18]) we considered these two questions. Let us begin with the torsion order of λ_g in $\mathrm{CH}^g(\mathcal{A}_g)$. A well-known formula of Borel and Serre (used above in Section 3) says that the Chern character of the alternating sum of the exterior powers of the Hodge bundle satisfies

$$\mathrm{ch}(\wedge^* \mathbb{E}) = (-1)^g \lambda_g \mathrm{Td}(\mathbb{E})^{-1}$$

and this implies that its terms of degree $1, \dots, g-1$ vanish and that it equals $-(g-1)! \lambda_g$ in degree g .

Lemma 9.1. *Let p be a prime and $\pi : A \rightarrow S$ be a family of abelian varieties of relative dimension g and L a line bundle on A that is of order p on all fibres of π . If $p > \min(2g, \dim S + g)$ then $p(g-1)! \lambda_g = 0$.*

Indeed, we may assume after twisting by a pull back from the base S that L^p is trivial. Let $[L]$ be its class in $K_0(A)$. Then

$$0 = [L]^p - 1 = ([L] - 1)^p + p([L] - 1)\left(1 + \frac{p-1}{2}([L] - 1) + \dots\right).$$

Now $[L] - 1$ is nilpotent because it has support of codimension ≥ 1 and so it follows that $p([L] - 1)$ lies in the ideal generated by $([L] - 1)^p$ which has codimension $\geq p$. Now if $p > 2g$ or $p > \dim S + g$ the codimension is $> 2g$, hence the image under π has codimension $> g$ or is zero. We thus can safely remove it and may assume that $p[L] = p$ in $K_0(A)$. Consider now the Poincaré bundle P on $A \times \hat{A}$; we know that $H^i(X \times \hat{X}, P)$ for an abelian variety X is zero for $i \neq g$ and 1-dimensional for $i = g$. So the derived pullback of $R\pi_* P$ along the zero section of $A \times \hat{A}$ over A is $R\pi_* \mathcal{O}_A$.

We know that $p[P] = p[L \otimes P]$ and $p[R\pi_*P] = p[R\pi_*(L \otimes P)]$ and $R\pi_*(L \otimes P)$ has support along the inverse of the section of \hat{A} corresponding to L . This section is everywhere disjoint from the zero section, so the pull back of $R\pi_*(L \otimes P)$ along the zero section is trivial: $p[R\pi_*\mathcal{O}_A] = 0$ and since $R\pi_*\mathcal{O}_A = \wedge^* R^1\pi_*\mathcal{O}_A = \wedge^*\mathbb{E}$ we find $-p(g-1)!\lambda_g = 0$.

We put

$$n_g := \gcd\{p^{2g} - 1 : p \text{ running through the primes } > 2g + 1\}.$$

Note that for a prime ℓ we have $\ell^k | n_g$ if and only if the exponent of $(\mathbb{Z}/\ell^k\mathbb{Z})^*$ divides $2g$. By Dirichlet's prime number theorem we have for $p > 2$

$$\text{ord}_p(n_g) = \begin{cases} 0 & (p-1) \nmid 2g \\ \max\{k : p^{k-1}(p-1) | 2g\} & (p-1) | 2g, \end{cases}$$

while

$$\text{ord}_2(n_g) = \max\{k : 2^{k-2} | 2g\}.$$

For example, we have

$$n_1 = 24, \quad n_2 = 240, \quad n_3 = 504, \quad n_4 = 480.$$

Theorem 9.2. *Let $\pi : \mathcal{X}'_g \rightarrow \mathcal{A}_g$ be the universal family. Then $(g-1)!n_g\lambda_g = 0$.*

For the proof we consider the commutative diagram

$$\begin{array}{ccc} \mathcal{X}'_g & \longrightarrow & \mathcal{X}_g \\ \downarrow & & \downarrow \\ \mathcal{A}'_g & \longrightarrow & \mathcal{A}_g \end{array}$$

where $\mathcal{A}'_g \rightarrow \mathcal{A}_g$ is the degree $p^{2g} - 1$ cover obtained by adding a line bundle of order p . It follows that $(g-1)!p(p^{2g} - 1)\lambda_g$ vanishes. Then the rest follows from the definition of n_g .

In [56] Mumford proved that the order of λ_1 is 12. So our result is off by a factor 2.

In the paper [17] we also gave the vanishing orders for the Chern classes of the de Rham bundle $\mathcal{H}_{\text{dR}}^1$ for the universal family $\mathcal{X}_g \rightarrow \mathcal{A}_g$. This bundle is provided with an integrable connection and so its Chern classes are torsion classes in integral cohomology. If l is a prime different from the characteristic these Chern classes are torsion too by the comparison theorems and using specialization. We denote these classes by $r_i \in H^{2i}(\mathcal{A}_g, \mathbb{Z}_l)$. We determined their orders up to a factor 2. Note that r_i vanishes for i odd as $\mathcal{H}_{\text{dR}}^1$ is a symplectic bundle.

Theorem 9.3. *For all i the class r_{2i+1} vanishes. For $1 \leq i \leq g$ the order of r_{2i} in integral (resp. l -adic) cohomology equals (resp. equals the l -part of) $n_i/2$ or n_i .*

We now turn to the question of a representative cycle in the boundary for λ_g in $\mathrm{CH}_{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$.

It is a very well-known fact that there is a cusp form Δ of weight 12 on $\mathrm{SL}(2, \mathbb{Z})$ and that it has a only one zero, a simple zero at the cusp. This leads to the relation $12\lambda_1 = \delta$, where δ is the cycle representing the class of the cusp. This formula has an analogue for higher g .

Theorem 9.4. *In the Chow group $\mathrm{CH}_{\mathbb{Q}}(\mathcal{A}_g^{(1)})$ of codimension g cycles on the moduli stack $\mathcal{A}_g^{(1)}$ of rank ≤ 1 degenerations the top Chern class λ_g satisfies the formula*

$$\lambda_g = (-1)^g \zeta(1 - 2g) \delta_g,$$

with δ_g the \mathbb{Q} -class of the locus Δ_g of semi-abelian varieties which are trivial extensions of an abelian variety of dimension $g - 1$ by the multiplicative group \mathbb{G}_m .

The number $\zeta(1 - 2g)$ is a rational number $-b_{2g}/2g$ with b_{2g} the $2g$ th Bernoulli number. For example, we have

$$12\lambda_1 = \delta_1, \quad 120\lambda_2 = \delta_2, \quad 252\lambda_3 = \delta_3, \quad 240\lambda_4 = \delta_4, \quad 132\lambda_5 = \delta_5$$

One might also wish to work on the Satake compactification \mathcal{A}_g^* , singular though as it is. Every toroidal compactification of Faltings-Chai type $\tilde{\mathcal{A}}_g$ has a canonical morphism $q : \tilde{\mathcal{A}}_g \rightarrow \mathcal{A}_g^*$. Then we can define a class ℓ_α where α is a subset of $\{1, 2, \dots, g\}$ by

$$\ell_\alpha = q_*(\lambda_\alpha) \quad \text{with } \lambda_\alpha \text{ equal to } \prod_{i \in \alpha} \lambda_i \in \mathrm{CH}_d^{\mathbb{Q}}(\tilde{\mathcal{A}}_g)$$

with $d = d(\alpha) = \sum_{i \in \alpha} i$ the degree of α and $\mathrm{CH}_d^{\mathbb{Q}}(\mathcal{A}_g^*)$ the Chow homology group. Note that this push forward does not depend on the choice of toroidal compactification as such toroidal compactifications always allow a common refinement and the λ_i are compatible with pull back. One can ask for example whether the classes of the boundary components \mathcal{A}_j^* lie in the \mathbb{Q} -vector space generated by the ℓ_α with $d(\alpha) = \mathrm{codim}_{\mathcal{A}_g^*}(\mathcal{A}_j^*)$. In [18] we made the following conjecture.

Conjecture 9.5. *In the group $\mathrm{CH}_d^{\mathbb{Q}}(\mathcal{A}_g^*)$ with $d = g(g+1)/2 - k(k+1)/2$ we have*

$$[\mathcal{A}_{g-k}^*] = \frac{(-1)^k}{\prod_{i=1}^k \zeta(2k-1-2g)} \ell_{g, g-1, \dots, g+1-k}$$

The evidence that Ekedahl and I gave is:

Theorem 9.6. *Conjecture 9.5 is true for $k = 1$ and $k = 2$ and if $\mathrm{char}(k) = p > 0$ then for all k .*

We shall see later that a multiple of the class λ_g has a beautiful representative cycle in $\mathcal{A}_g \otimes \mathbb{F}_p$, namely the locus of abelian varieties of p -rank 0.

10. Cohomology

By a result of Borel [9] the stable cohomology of the symplectic group is known; this implies that in degrees $\leq g - 2$ the cohomology of $\mathrm{Sp}(2g, \mathbb{Q})$ (and in fact for any finite index subgroup Γ) is generated by elements in degree $2k + 4$ for $k = 0, 1, \dots, (g - 6)/2$:

$$H^*(\Gamma, \mathbb{Q}) = \mathbb{Q}[x_2, x_6, x_{10}, \dots].$$

The Chern classes λ_{2k+1} of the Hodge bundle provide these classes.

There are also some results on the stable homology of the Satake compactification, see [12]; besides the λ_{2k+1} there are other classes α_{2k+1} for $k \geq 1$.

Van der Kallen and Looijenga proved in [47] that the rational homology of the pair $(\mathcal{A}_g, \mathcal{A}_{g,\mathrm{dec}})$ with $\mathcal{A}_{g,\mathrm{dec}}$ the locus of decomposable principally polarized abelian varieties, vanishes in degree $\leq g - 2$.

For low values of g the cohomology of \mathcal{A}_g is known. For $g = 1$ one has $H^0(\mathcal{A}_1, \mathbb{Q}) = \mathbb{Q}$ and $H^1 = H^2 = (0)$. For $g = 2$ one can show that $H^0(\mathcal{A}_2, \mathbb{Q}) = \mathbb{Q}$, $H^2(\mathcal{A}_2, \mathbb{Q}) = \mathbb{Q}(-1)$.

Hain determined the cohomology of \mathcal{A}_3 . His result ([39]) is:

Theorem 10.1. *The cohomology $H^*(\mathcal{A}_3, \mathbb{Q})$ is given by*

$$H^j(\mathcal{A}_3, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & j = 0 \\ \mathbb{Q}(-1) & j = 2 \\ \mathbb{Q}(-2) & j = 4 \\ E & j = 6 \\ 0 & \text{else} \end{cases}$$

where E is an extension

$$0 \rightarrow \mathbb{Q}(-3) \rightarrow E \rightarrow \mathbb{Q}(-6) \rightarrow 0$$

We find for the compactly supported cohomology a similar result

$$H_c^j(\mathcal{A}_3, \mathbb{Q}) \cong \begin{cases} \mathbb{Q}(-6) & j = 12 \\ \mathbb{Q}(-5) & j = 10 \\ \mathbb{Q}(-4) & j = 8 \\ E' & j = 6 \\ 0 & \text{else} \end{cases}$$

where E' is an extension $0 \rightarrow \mathbb{Q} \rightarrow E' \rightarrow \mathbb{Q}(-3) \rightarrow 0$.

The natural map $H_c^* \rightarrow H^*$ is the zero map. Indeed, the classes in H_c^j for $j = 12, 10$ and 8 are $\lambda_3 \lambda_1^3$, $\lambda_3 \lambda_1^2$ and $\lambda_3 \lambda_1$, while λ_3 gives a non-zero class in H_c^6 . On the other hand, $1, \lambda_1, \lambda_1$ and λ_1^3 give non-zero classes in H^0, H^2, H^4 and H^6 . Looking at the weight shows that the map is the zero map.

By calculating the cohomology of $\tilde{\mathcal{A}}_3$ Hulek and Tommasi proved in [44] that the cohomology of the Voronoi compactification for $g \leq 3$ coincides with the Chow ring (known by the Theorems of Section 7):

Theorem 10.2. *The cycle class map gives an isomorphism*

$$\mathrm{CH}_{\mathbb{Q}}^*(\mathcal{A}_g^{\mathrm{Vor}}) \cong H^*(\mathcal{A}_g^{\mathrm{Vor}}) \quad \text{for } g = 1, 2, 3.$$

11. Siegel Modular Forms

The cohomology of \mathcal{A}_g itself and of the universal family \mathcal{X}_g and its powers \mathcal{X}_g^n is closely linked to modular forms. We therefore pause to give a short description of these modular forms on $\mathrm{Sp}(2g, \mathbb{Z})$. For a general reference we refer to the book of Freitag [27] and to [34] and the references there.

Siegel modular forms generalize the notion of usual (elliptic) modular forms on $\mathrm{SL}(2, \mathbb{Z})$ and its (finite index) subgroups. We first need to generalize the notion of the weight of a modular form. To define it we need a finite-dimensional complex representation $\rho : \mathrm{GL}(g, \mathbb{C}) \rightarrow \mathrm{GL}(V)$ with V a finite-dimensional complex vector space.

Definition 11.1. A holomorphic map $f : \mathfrak{H}_g \rightarrow V$ is called a Siegel modular form of weight ρ if

$$f(\gamma(\tau)) = \rho(c\tau + d)f(\tau) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}) \text{ and all } \tau \in \mathfrak{H}_g,$$

plus for $g = 1$ the extra requirement that f is holomorphic at the cusp. (The latter means that f has a Fourier expansion $f = \sum_{n \geq 0} a(n)q^n$ with $q = e^{2\pi i\tau}$.)

Modular forms of weight ρ form a \mathbb{C} -vector space $M_{\rho}(\mathrm{Sp}(2g, \mathbb{Z}))$; as it turns out this space is finite-dimensional. This vector space can be identified with the space of holomorphic sections of the vector bundle V_{ρ} defined before (see Section 3). Since we are working on an orbifold, one has to be careful; we could replace $\mathrm{Sp}(2g, \mathbb{Z})$ by a normal congruence subgroup Γ of finite index that acts freely on \mathfrak{H}_g , take the space $M_{\rho}(\Gamma)$ of modular forms on Γ and consider the invariant subspace under the action of $\mathrm{Sp}(2g, \mathbb{Z})/\Gamma$.

Since we can decompose the representation ρ into irreducible representations it is no restriction of generality to limit ourselves to the case where ρ is irreducible. The irreducible representations ρ of $\mathrm{GL}(g, \mathbb{C})$ are given by g -tuples integers (a_1, a_2, \dots, a_g) with $a_i \geq a_{i+1}$, the highest weight of the representation.

A special case is where ρ is given by $a_i = k$, in other words

$$\rho(c\tau + d) = \det(c\tau + d)^k.$$

In this case the Siegel modular forms are called *classical Siegel modular forms* of weight k .

A modular form f has a Fourier expansion

$$f(\tau) = \sum_{n \text{ half-integral}} a(n) e^{2\pi i \text{Tr}(n\tau)},$$

where n runs over the symmetric $g \times g$ matrices that are half-integral (i.e. $2n$ is an integral matrix with even entries along the diagonal) and

$$\text{Tr}(n\tau) = \sum_{i=1}^g n_{ii} \tau_{ii} + 2 \sum_{1 \leq i < j \leq g} n_{ij} \tau_{ij}.$$

A classical result of Koecher (cf. [27]) asserts that $a(n) = 0$ for n that are not semi-positive. This is a sort of Hartogs extension theorem.

We shall use the suggestive notation q^n for $e^{2\pi i \text{Tr}(n\tau)}$ and then can write the Fourier expansion as

$$f(\tau) = \sum_{n \geq 0, \text{ half-integral}} a(n) q^n.$$

We observe the invariance property

$$a(u^t n u) = \rho(u^t) a(n) \quad \text{for all } u \in \text{GL}(g, \mathbb{Z}) \quad (1)$$

This follows by the short calculation

$$\begin{aligned} a(u^t n u) &= \int_{x \bmod 1} f(\tau) e^{-2\pi i \text{Tr}(u^t n u \tau)} dx \\ &= \rho(u^t) \int_{x \bmod 1} f(u \tau u^t) e^{-2\pi i \text{Tr}(n u \tau u^t)} dx = \rho(u^t) a(n), \end{aligned}$$

where we wrote $\tau = x + iy$.

There is a way to extend Siegel modular forms on \mathcal{A}_g to modular forms on $\mathcal{A}_g \sqcup \mathcal{A}_{g-1}$ and inductively to \mathcal{A}_g^* by means of the so-called Φ -operator of Siegel. For $f \in M_\rho$ one defines

$$\Phi(f)(\tau') = \lim_{t \rightarrow \infty} f\left(\begin{smallmatrix} \tau' & 0 \\ 0 & it \end{smallmatrix}\right) \quad \tau' \in \mathfrak{H}_{g-1}.$$

The limit is well-defined and gives a function that satisfies

$$(\Phi f)(\gamma'(\tau')) = \rho\left(\begin{smallmatrix} c\tau + d & 0 \\ 0 & 1 \end{smallmatrix}\right) (\Phi f)(\tau'),$$

where $\gamma' = (ab; cd) \in \text{Sp}(2g-2, \mathbb{Z})$ is embedded in $\text{Sp}(2g, \mathbb{Z})$ as the automorphism group of the symplectic subspace $\langle e_i, f_i : i = 1, \dots, g-1 \rangle$. That is, we get a linear map

$$M_\rho \longrightarrow M_{\rho'}, \quad f \mapsto \Phi(f),$$

for some representation ρ' ; in fact, with the representation $\rho' = (a_1, a_2, \dots, a_{g-1})$ for irreducible $\rho = (a_1, a_2, \dots, a_g)$, cf. the proof of 11.4.

Definition 11.2. A Siegel modular form $f \in M_\rho$ is called a *cuspidal form* if $\Phi(f) = 0$.

We can extend this definition by

Definition 11.3. A non-zero modular form $f \in M_\rho$ has co-rank k if $\Phi^k(f) \neq 0$ and $\Phi^{k+1}(f) = 0$.

That is, f has co-rank k if it survives (non-zero) till \mathcal{A}_{g-k} and no further. So cusp forms have co-rank 0.

For an irreducible representation ρ of GL_g with highest weight (a_1, \dots, a_g) we define the co-rank as

$$\mathrm{co-rank}(\rho) = \#\{i : 1 \leq i \leq g, a_i = a_g\}$$

Weissauer proved in [72] the following result.

Theorem 11.4. *Let ρ be irreducible. For a non-zero Siegel modular form $f \in M_\rho$ one has $\mathrm{corank}(f) \leq \mathrm{corank}(\rho)$.*

Proof. We give Weissauer's proof. Suppose that $f : \mathfrak{H}_g \rightarrow V$ is a form of co-rank k with Fourier series $\sum a(n)q^n$; then there exists a semi-definite half-integral matrix n such that

$$n = \begin{pmatrix} n' & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } n' \text{ a } (g-k) \times (g-k) \text{ matrix}$$

such that $a(n) \neq 0$. The identity (1) implies that $a(n) = \rho(u)a(n)$ for all u in the group

$$G_{g,k} = \left\{ \begin{pmatrix} 1_{g-k} & b \\ 0 & d \end{pmatrix} : d \in \mathrm{GL}_k \right\}$$

This group is a semi-direct product $\mathrm{GL}_k \ltimes N$ with N the unipotent radical. The important remark now is that the Zariski closure of $\mathrm{GL}(k, \mathbb{Z})$ in $G_{g,k}$ contains $\mathrm{SL}_k(\mathbb{C}) \ltimes N$ and we have $a(n) = \rho(u)a(n)$ for all $u \in \mathrm{SL}_k(\mathbb{C}) \ltimes N$. So the Fourier coefficient $a(n)$ lies in the subspace V^N of N -invariants. The parabolic group $P = \{(ab; 0d); a \in \mathrm{GL}_{n-k}, d \in \mathrm{GL}_k\}$, which is also a semi-direct product $(\mathrm{GL}_{n-k} \times \mathrm{GL}_k) \ltimes N$, acts on V^N via its quotient $\mathrm{GL}_{n-k} \times \mathrm{GL}_k$. Since this is a reductive group V^N decomposes into irreducible representations, each of which is a tensor product of an irreducible representation of GL_{n-k} times an irreducible representation of GL_k . If U_k denotes the subgroup of upper triangular unipotent matrices in $\mathrm{GL}(k, \mathbb{C})$, then the highest weight space is $(V^N)^{U_{g-k} \times U_k}$ and this equals V^{U_g} , and this is 1-dimensional. Thus V^N is an irreducible representation of $\mathrm{GL}_{g-k} \times \mathrm{GL}_k$ and one checks that it is given by $(a_1, \dots, a_{g-k}) \otimes (a_{g-k+1}, \dots, a_g)$. The space of invariants $V^{G_{g,k}} \subset V^N$ under $G_{g,k}$ can only contain $\mathrm{SL}_k(\mathbb{C})$ -invariant elements if the GL_k -representation is 1-dimensional. Therefore $V^{G_{g,k}}$ is zero unless (a_{g-k+1}, \dots, a_g) is a 1-dimensional representation, hence $a_{g-k+1} = \dots = a_g$. In that case the representation for GL_{g-k} is given by (a_1, \dots, a_{g-k}) . Hence the Fourier coefficients of f all have to vanish if the corank of f is greater than the corank of ρ . This proves the result. \square

12. Differential Forms on \mathcal{A}_g

Here we look first at the moduli space over the field of complex numbers $\mathcal{A}_g(\mathbb{C}) = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$. We are interested in differential forms living on $\tilde{\mathcal{A}}_g$. If η is a holomorphic p form on $\tilde{\mathcal{A}}_g$ then we can pull the form back to \mathfrak{H}_g . It will then be a section of some exterior power of $\Omega_{\mathfrak{H}_g}^1$, hence of some exterior power of the second symmetric power of the Hodge bundle. Such forms can be analytically described by vector-valued Siegel modular forms.

As we saw in 3.1, the bundle $\Omega^1(\log D)$ is associated to the second symmetric power $\mathrm{Sym}^2(\mathbb{E})$ of the Hodge bundle (at least in the stacky interpretation) and we are led to ask which irreducible representations occur in the exterior powers of the second symmetric power of the standard representation of $\mathrm{GL}(g)$. The answer is that these are exactly those irreducible representations ρ that are of the form $w\eta - \eta$, where $\eta = (g, g-1, \dots, 1)$ is half the sum of the positive roots and w runs through the 2^g Kostant representatives of $W(\mathrm{GL}_g) \backslash W(\mathrm{Sp}_{2g})$, where W is the Weyl group. They have the property that they send dominant weights for Sp_{2g} to dominant weights for GL_g .

To describe this explicitly, recall that the Weyl group W of $\mathrm{Sp}(2g, \mathbb{Z})$ is the semi-direct product $S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g$ of signed permutations. The Weyl group of GL_g is equal to the symmetric group S_g . Every left coset $S_g \backslash W$ contains exactly one Kostant element w . Such an element w corresponds one to one to an element of $(\mathbb{Z}/2\mathbb{Z})^g$ that we view a g -tuple $(\epsilon_1, \dots, \epsilon_g)$ with $\epsilon_i \in \{\pm 1\}$ and the action of this element w on the root lattice is then via

$$(a_1, \dots, a_g) \xrightarrow{w} (\epsilon_{\sigma(1)} a_{\sigma(1)}, \dots, \epsilon_{\sigma(g)} a_{\sigma(g)})$$

for all $a_1 \geq a_2 \geq \dots \geq a_g$ and with σ the unique permutation such that

$$\epsilon_{\sigma(1)} a_{\sigma(1)} \geq \dots \geq \epsilon_{\sigma(g)} a_{\sigma(g)}.$$

If f is a classical Siegel modular form of weight $k = g+1$ on the group Γ_g then $f(\tau) \prod_{i \leq j} d\tau_{ij}$ is a top differential on the smooth part of quotient space $\Gamma_g \backslash \mathcal{H}_g = \mathcal{A}_g(\mathbb{C})$. It can be extended over the smooth part of the rank-1 compactification $\mathcal{A}_g^{(1)}$ if and only if f is a cusp form. It is not difficult to see that this form can be extended as a holomorphic form to the whole smooth compactification $\tilde{\mathcal{A}}_g$.

Proposition 12.1. *The map that associates to a classical Siegel modular cusp form $f \in S_{g+1}(\Gamma_g)$ of weight $g+1$ the top differential $\omega = f(\tau) \prod_{i \leq j} d\tau_{ij}$ defines an isomorphism between $S_{g+1}(\Gamma_g)$ and the space of holomorphic top differentials $H^0(\tilde{\mathcal{A}}_g, \Omega^{g(g+1)/2})$ on any smooth compactification $\tilde{\mathcal{A}}_g$.*

Freitag and Pommerening proved in [28] the following extension result.

Theorem 12.2. *Let $p < g(g+1)/2$ and ω a holomorphic p -form on the smooth locus of $\mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$. Then it extends uniquely to a holomorphic p -form on any smooth toroidal compactification $\tilde{\mathcal{A}}_g$.*

But for $g > 1$ the singular locus has codimension ≥ 2 . This implies:

Corollary 12.3. *For $p < g(g+1)/2$ holomorphic p -forms on $\tilde{\mathcal{A}}_g$ correspond one-to-one to $\mathrm{Sp}(2g, \mathbb{Z})$ -invariant holomorphic p -forms on \mathfrak{H}_g :*

$$\Gamma(\tilde{\mathcal{A}}_g, \Omega^p) \cong (\Omega^p(\mathfrak{H}_g))^{\mathrm{Sp}(2g, \mathbb{Z})}.$$

When can holomorphic p -forms exist on $\tilde{\mathcal{A}}_g$? Weissauer proved in [72] a vanishing theorem of the following form.

Theorem 12.4. *Let $\tilde{\mathcal{A}}_g$ be a smooth compactification of \mathcal{A}_g . If p is an integer $0 \leq p < g(g+1)/2$ then the space of holomorphic p -forms on $\tilde{\mathcal{A}}_g$ vanishes unless p is of the form $g(g+1)/2 - r(r+1)/2$ with $1 \leq r \leq g$ and then $H^0(\tilde{\mathcal{A}}_g, \Omega_{\tilde{\mathcal{A}}_g}^p) \cong M_\rho(\Gamma_g)$ with $\rho = (g+1, \dots, g+1, g-r, \dots, g-r)$ with $g-r$ occurring r times.*

Weissauer deduced this from the following Vanishing Theorem for Siegel modular forms (cf. loc. cit.).

Theorem 12.5. *Let $\rho = (a_1, \dots, a_g)$ be irreducible with $\mathrm{co-rank}(\rho) < g - a_g$. If we have*

$$\#\{1 \leq i \leq g : a_i = a_g + 1\} < 2(g - a_g - \mathrm{co-rank}(\rho))$$

then $M_\rho = (0)$.

13. The Kodaira Dimension of \mathcal{A}_g

The Kodaira dimension of the moduli space \mathcal{A}_g over \mathbb{C} , the least integer $\kappa = \kappa(\mathcal{A}_g)$ such that the sequence P_m/m^κ with $P_m(\mathcal{A}_g) = \dim H^0(\mathcal{A}_g, K^{\otimes m})$, is bounded, is an important invariant. In terms of Siegel modular forms this comes down to the growth of the dimension of the space of classical Siegel modular forms f of weight $k(g+1)$ that extend to holomorphic tensors $f(\tau)\eta^{\otimes k}$ with $\eta = \wedge_{1 \leq i \leq j \leq g} d\tau_{ij}$ on $\tilde{\mathcal{A}}_g$. The first condition is that f vanishes with multiplicity $\geq k$ along the divisor at infinity. Then one has to deal with the extension over the quotient singularities. Reid and Tai independently found a criterion for the extension of pluri-canonical forms over quotient singularities.

Proposition 13.1. *(Reid-Tai Criterion) A pluri-canonical form η on \mathbb{C}^n that is invariant under a finite group G acting linearly on \mathbb{C}^n extends to a resolution of singularities if for every non-trivial element $\gamma \in G$ and every fixed point x of γ we have*

$$\sum_{j=1}^n \alpha_j \geq 1,$$

where the action of γ on the tangent space of x has eigenvalues $e^{2\pi i \alpha_j}$. If G does not possess pseudo-reflections then it extends if and only if $\sum_{j=1}^n \alpha_j \geq 1$.

Tai checked (cf. [70]) that if $g \geq 5$ then every fixed point of a non-trivially acting element of $\mathrm{Sp}(2g, \mathbb{Z})$ satisfies the requirement of the Reid-Tai criterion and thus these forms extend over the quotient singularities of \mathcal{A}_g . He also checked that the extension over the singularities in the boundary did not present difficulties. Thus he showed by calculating the dimension of the space of sections of $K^{\otimes k}$ on a level cover of \mathcal{A}_g and calculating the space over invariants under the action of $\mathrm{Sp}(2g, \mathbb{Z}/\ell\mathbb{Z})$ that for $g \geq 9$ the space \mathcal{A}_g was of general type. He thus improved earlier work of Freitag ([26]).

Mumford extended this result in [60] and proved:

Theorem 13.2. *The moduli space \mathcal{A}_g is of general type if $g \geq 7$.*

His approach was similar to the method used in his joint paper with Harris on the Kodaira dimension of \mathcal{M}_g . First he works on the moduli space $\mathcal{A}_g^{(1)}$ of rank 1-degenerations and observes that the Picard group $\mathrm{Pic}(\mathcal{A}_g^{(1)}) \otimes \mathbb{Q}$ is generated by two divisors: λ_1 and the class δ of the boundary. By Proposition 3.1 for the coarse moduli space we know the canonical class.

Proposition 13.3. *Let U be the open subset of $\mathcal{A}_g^{(1)}$ of semi-abelian varieties of torus rank ≤ 1 with automorphism group $\{\pm\}$. Then the canonical class of U is given by $(g+1)\lambda_1 - \delta$.*

Now there is an explicit effective divisor N_0 in $\mathcal{A}_g^{(1)}$, namely the divisor of principally polarized abelian varieties (X, Θ) that have a singular theta divisor. By a tricky calculation in a 1-dimensional family Mumford is able to calculate the divisor class of N_0 .

Theorem 13.4. *The divisor class of \bar{N}_0 is given by*

$$[\bar{N}_0] = \left(\frac{(g+1)!}{2} + g! \right) \lambda_1 - \frac{(g+1)!}{12} \delta.$$

The divisor N_0 is in general not irreducible and splits off the divisor θ_0 of principally polarized abelian varieties whose symmetric theta divisor has a singularity at a point of order 2. This divisor is given by the vanishing of an even theta constant (Nullwert) and so this divisor class can be easily computed. The divisor class of θ_0 equals

$$[\theta_0] = 2^{g-2}(2^g + 1) \lambda_1 - 2^{2g-5} \delta.$$

Proof. (of Theorem 13.2) We have $[\bar{N}_0] = \theta_0 + 2[R]$ with R an effective divisor given as

$$[R] = \left(\frac{(g+1)!}{4} + \frac{g!}{2} - 2^{g-3}(2^g + 1) \right) \lambda_1 - \left(\frac{(g+1)!}{24} - 2^{2g-6} \right) \delta.$$

For an expression $a\lambda_1 - b\delta$ we call the ratio a/b its slope. If R is effective with a slope \leq slope of the expression for the canonical class we know that the canonical

class is ample. From the formulas one sees that the slope of \bar{N}_0 is $6 + 12/(g + 1)$ and deduces that the inequality holds for $g \geq 7$. \square

From the relatively easy equation for the class $[\theta_0]$ one deduces that the even theta constants provide a divisor with slope $8 + 2^{3-g}$, and this gives that \mathcal{A}_g is of general type for $g \geq 8$; cf. [27].

For $g \leq 5$ we know that \mathcal{A}_g is rational or unirational. For $g = 1$ and 2 rationality was known in the 19th century; Katsylo proved rationality for $g = 3$, Clemens proved unirationality for $g = 4$ and unirationality for $g = 5$ was proved by several people (Mori-Mukai, Donagi, Verra). But the case $g = 6$ is still open.

For some results on the nef cone we refer to the survey [38] of Grushevsky.

14. Stratifications

In positive characteristic the moduli space $\mathcal{A}_g \otimes \mathbb{F}_p$ possesses stratifications that can tell us quite a lot about the moduli space; it is not clear what the characteristic zero analogues of these stratifications are. This makes this moduli space in some sense more accessible in characteristic p than in characteristic zero, a fact that may sound counterintuitive to many.

The stratifications we are referring to are the Ekedahl-Oort stratification and the Newton polygon stratification. Since the cycle classes are not known for the latter one we shall stick to the Ekedahl-Oort stratification, E-O for short. It was originally defined by Ekedahl and Oort in terms of the group scheme $X[p]$, the kernel of multiplication by p for an abelian variety in characteristic p . It has been studied intensively by Oort and many others, see e.g. [64] and [65].

The alternative definition using degeneracy loci of vector bundles was given in my paper [32] and worked out fully in joint work with Ekedahl, see [19]. In this section we shall write \mathcal{A}_g for $\mathcal{A}_g \otimes \mathbb{F}_p$. We start with the universal principally polarized abelian variety $\pi : \mathcal{X}_g \rightarrow \mathcal{A}_g$. For an abelian variety X over a field k the de Rham cohomology $H_{\text{dR}}^1(X)$ is a vector space of rank $2g$. We get by doing this in families a cohomology sheaf $\mathcal{H}_{\text{dR}}^1(\mathcal{X}_g/\mathcal{A}_g)$, the hyperdirect image $R^1\pi_*(\mathcal{O}_{\mathcal{X}_g} \rightarrow \Omega_{\mathcal{X}_g/\mathcal{A}_g}^1)$. Because of the polarization it comes with a symplectic pairing $\langle \cdot, \cdot \rangle : \mathcal{H}_{\text{dR}}^1 \times \mathcal{H}_{\text{dR}}^1 \rightarrow \mathcal{O}_{\mathcal{A}_g}$. We have the Hodge filtration of $\mathcal{H}_{\text{dR}}^1$:

$$0 \rightarrow \pi_*(\Omega_{\mathcal{X}_g/\mathcal{A}_g}^1) \rightarrow \mathcal{H}_{\text{dR}}^1 \rightarrow R^1\pi_*\mathcal{O}_{\mathcal{X}_g} \rightarrow 0,$$

where the first non-zero term is the Hodge bundle \mathbb{E}_g . In characteristic $p > 0$ we have additionally two maps

$$F : \mathcal{X}_g \rightarrow \mathcal{X}_g^{(p)}, \quad V : \mathcal{X}_g^{(p)} \rightarrow \mathcal{X}_g,$$

relative Frobenius and the Verschiebung satisfying $F \circ V = p \cdot \text{id}_{\mathcal{X}_g^{(p)}}$ and $V \circ F = p \cdot \text{id}_{\mathcal{X}_g}$.

Look at the simplest case $g = 1$. Since Frobenius is inseparable the kernel $X[p]$ of multiplication by p is not reduced and has either 1 or p physical points.

An elliptic curve in characteristic $p > 0$ is called *supersingular* if and only if $X[p]_{\text{red}} = (0)$. Equivalently this means that V is also inseparable, hence the kernel of F and V (for \mathcal{X}_g instead of $\mathcal{X}_g^{(p)}$) coincide. There are finitely many points on the j -line \mathcal{A}_1 that correspond to the supersingular elliptic curves.

So in general we will compare the relative position of the kernel of F and of V inside $\mathcal{H}_{\text{dR}}^1$. As it turns out, it is better to work with the flag space \mathcal{F}_g of symplectic flags on $\mathcal{H}_{\text{dR}}^1$; by this we mean that we consider the space of flags $(E_i)_{i=1}^{2g}$ on $\mathcal{H}_{\text{dR}}^1$ such that $\text{rank}(E_i) = i$, $E_g = \mathbb{E}$ and $E_{g-i} = E_j^\perp$. We can then introduce a second flag on $\mathcal{H}_{\text{dR}}^1$, say $(D_i)_{i=1}^{2g}$ defined by setting

$$D_g = \ker(V) = V^{-1}(0), \quad D_{g+i} = V^{-1}(E_i^{(p)}),$$

and complementing by

$$D_{g-i} = D_{g+i}^\perp \quad \text{for } i = 1, \dots, g.$$

This is called the conjugate flag. For an abelian variety X we define the *canonical flag* of X as the coarsest flag that is stable under F : if G is a member, then also $F(G^{(p)})$; usually this will not be a full flag. The conjugate flag D is a refinement of the canonical flag.

So starting with one filtration we end up with two filtrations. We then compare these two filtrations. In order to do this we need the Weyl group W_g of the symplectic group $\text{Sp}(2g, \mathbb{Z})$. This group is isomorphic to the semi-direct product

$$W_g \cong S_g \ltimes (\mathbb{Z}/2\mathbb{Z})^g = \{\sigma \in S_{2g} : \sigma(i) + \sigma(2g+1-i) = 2g+1, i = 1, \dots, g\}.$$

As is well-known each element $w \in W_g$ has a *length* $\ell(w)$. Let

$$W'_g = \{\sigma \in W_g : \sigma\{1, 2, \dots, g\} = \{1, 2, \dots, g\}\} \cong S_g.$$

Now for every coset aW'_g there is a unique element w of minimal length in aW'_g such that every $w' \in aW'_g$ can be written as $w' = wx$ with $\ell(w') = \ell(w) + \ell(x)$.

There is a partial order on W_g , the Bruhat-Chevalley order:

$$w_1 \geq w_2 \quad \text{if and only if} \quad r_{w_1}(i, j) \leq r_{w_2}(i, j) \quad \text{for all } i, j,$$

where the function $r_w(i, j)$ is defined as

$$r_w(i, j) = \#\{n \leq i : w(n) \leq j\}.$$

There are 2^g so-called *final elements* in W_g : these are the minimal elements of the cosets. In another context (see Section 12) these are called Kostant representatives. An element $\sigma \in W_g$ is final if and only if $\sigma(i) < \sigma(j)$ for $i < j \leq g$. These final elements correspond one-to-one to so-called final types: these are increasing surjective maps

$$\nu : \{0, 1, \dots, 2g\} \rightarrow \{0, 1, \dots, 2g\}$$

satisfying $\nu(2g-i) = \nu(i) + (g-i)$ for $0 \leq i \leq g$. The bijection is gotten by associating to $w \in W_g$ the final type ν_w with $\nu_w(i) = i - r_w(g, i)$.

Now the stratification on our flag space \mathcal{F}_g is defined by

$$(E., D.) \in U_w \iff \dim(E_i \cap D_j) \geq r_w(i, j).$$

There is also a scheme-theoretic definition: the two filtrations mean that we have two sections s, t of a G/B -bundle T (with structure group G) over the base with $G = \mathrm{Sp}_{2g}$ and B a Borel subgroup; locally (in the étale topology) we may assume that t is a trivializing section; then we can view s as a map of the base to G/B and we simply take U_w (resp. \bar{U}_w) to be the inverse image of the B -orbit BwB (resp. its closure). This definition is independent of the chosen trivialization. We thus find global subschemes U_w and \bar{U}_w for the $2^g(g!)$ elements of W_g .

It will turn out that for *final* elements the projection map $\mathcal{F}_g \rightarrow \mathcal{A}_g$ restricted to U_w defines a finite morphism to its image in \mathcal{A}_g , and these will be the E-O strata. But the strata on \mathcal{F}_g behave in a much better way. That is why we first study these on \mathcal{F}_g . We can extend the strata to strata over a compactification, see [19]: the Hodge bundle extends and so does the de Rham sheaf, namely as the logarithmic de Rham sheaf $R^1\pi_*(\Omega_{\tilde{\mathcal{X}}_g/\tilde{\mathcal{A}}_g})$ and then we play the same game as above.

There is a smallest stratum: \bar{U}_1 associated to the identity element of W_g .

Proposition 14.1. *Any irreducible component of any \bar{U}_w in \mathcal{F}_g contains a point of \bar{U}_1 .*

The advantage of working in the flag space comes out clearly if we consider the local structure of our strata. The idea is that locally our moduli space (flag space) looks like the space FL_g of complete symplectic flags in a $2g$ -dimensional symplectic space.

We need the notion of *height 1 maps* in characteristic $p > 0$. A closed immersion $S \rightarrow S'$ is called a height 1 map if $I^{(p)} = (0)$ with I the ideal sheaf defining S and $I^{(p)}$ the ideal generated by p -th powers of elements. If $S = \mathrm{Spec}(R)$ and $x : \mathrm{Spec}(k) \rightarrow S$ is a closed point then the height 1 neighborhood is $\mathrm{Spec}(R/m^{(p)})$. We can then introduce the notions of *height 1 isomorphic* and *height 1 smooth* in an obvious way.

Our basic result of [19] about our strata is now:

Theorem 14.2. *Let k be a perfect field. For every k -point x of \mathcal{F}_g there exists a k -point of FL_g such that their height 1 neighborhoods are isomorphic as stratified spaces.*

Here the stratification on the flag space FL_g is the usual one given by the Schubert cells. The idea of the proof is to trivialize the de Rham bundle over a height 1 neighborhood and then use infinitesimal Torelli for abelian varieties. Our theorem has some immediate consequences.

Corollary 14.3. (1) *Each stratum U_w is smooth of dimension $\ell(w)$.*

- (2) Each stratum \bar{U}_w is Cohen-Macaulay, reduced and normal; moreover \bar{U}_w is the closure of U_w .
- (3) For w a final element the projection $\mathcal{F}_g \rightarrow \mathcal{A}_g$ induces a finite étale map $U_w \rightarrow V_w$, with V_w the image of U_w .

We now descend from \mathcal{F}_g to \mathcal{A}_g to define the Ekedahl-Oort stratification.

Definition 14.4. For a final element $w \in W_g$ the E-O stratum V_w is defined to be the image of U_w under the projection $\mathcal{F}_g \rightarrow \mathcal{A}_g$.

Over a E-O stratum $V_w \subset \mathcal{A}_g$ the type of the canonical filtration (i.e., the dimensions of the filtration steps that occur) are constant.

Proposition 14.5. *The image of any U_w under the projection $\mathcal{F}_g \rightarrow \mathcal{A}_g$ is a union of strata V_v ; the image of any \bar{U}_w is a union of strata \bar{V}_v .*

We also have an irreducibility result (cf. [19]). Each final element $w \in W_g$ corresponds to a partition and a Young diagram.

Theorem 14.6. *If $w \in W_g$ is a final element whose Young diagram Y does not contain all rows of length i with $\lceil (g+1)/2 \rceil \leq i \leq g$ then \bar{V}_w is irreducible and $U_w \rightarrow V_w$ is a connected étale cover.*

Harashita proved in [40] that the other ones are in general reducible (with exceptions maybe for small characteristics).

A stratification on a space is not worth much unless one knows the cycle classes of the strata. In our case one can calculate these.

On the flag space the Chern classes λ_i of the Hodge bundle decompose in their roots:

$$c_1(E_i) = l_1 + \dots + l_i.$$

We then have

$$c_1(D_{g+i}) - c_1(D_{g+1-i}) = pl_i.$$

Theorem 14.7. *The cycle classes of \bar{U}_w are polynomials in the classes l_i with coefficients that are polynomials in p .*

We refer to [19] for an explicit formula. Using the analogues of the maps π_i of Section 2 and Lemma 2.4 we can calculate the cycle classes of the E-O strata on \mathcal{A}_g .

Instead of giving a general formula we restrict to giving the formulas for some important strata. For example, there are the p -rank strata

$$V_f := \{[X] \in \mathcal{A}_g : \#X[p](\bar{k}) \leq p^f\}$$

for $f = g, g-1, \dots, 0$. Besides these there are the a -number strata

$$T_a := \{[X] \in \mathcal{A}_g : \dim_k \text{Hom}(\alpha_p, X) \geq a\}.$$

Recall that the p -rank $f(X)$ of an abelian variety is f if and only if $\#X[p](\bar{k}) = p^f$ and $0 \leq f \leq g$ with $f = g$ being the generic case. Similarly, the a -number of $a(X)$ of X is $\dim_k(\alpha_p, X)$ and this equals the rank of $\ker(V) \cap \ker(F)$; so $0 \leq a(X) \leq g$ with $a(X) = 0$ being the generic case. The stratum V_f has codimension $g - f$ while the stratum T_a has codimension $a(a + 1)/2$. These codimensions were originally calculated by Oort and follow here easily from 14.3.

Theorem 14.8. *The cycle class of the p -rank stratum V_f is given by*

$$[V_f] = (p - 1)(p^2 - 1) \cdots (p^{g-f} - 1)\lambda_{g-f}.$$

For example, for $g = 1$ the stratum V_0 is the stratum of supersingular elliptic curves. We have $[V_0] = (p - 1)\lambda_1$. By the cycle relation $12\lambda_1 = \delta$ with δ the cycle of the boundary we find

$$[V_0] = \frac{p - 1}{12}\delta.$$

Since the degenerate elliptic curve (rational nodal curve) has two automorphisms we find (using the stacky interpretation of our formula) the Deuring Mass Formula

$$\sum_{E/\bar{k} \text{ supersingular}} \frac{1}{\#\text{Aut}_k(E)} = \frac{p - 1}{24}$$

for the number of supersingular elliptic curves in characteristic p . One may view all the formulas for the cycle classes as a generalization of the Deuring Mass Formula.

The formulas for the a -number strata are given in [32] and [19]. We have

Theorem 14.9. *The cycle class of the locus T_a of abelian varieties with a -number $\geq a$ is given by*

$$\sum Q_\beta(\mathbb{E}^{(p)}) \cdot Q_{\varrho(a) - \beta}(\mathbb{E}^\vee),$$

where Q_μ is defined as in Section 2 and the sum is over all partitions β contained in $\varrho(a) = \{a, a - 1, \dots, 1\}$.

For example, the formula for the stratum T_1 is $p\lambda_1 - \lambda_1$, which fits since a -number ≥ 1 means exactly that the p -rank is $\leq g - 1$. For $a = 2$ the formula is $[T_2] = (p - 1)(p^2 + 1)(\lambda_1\lambda_2) - (p^3 - 1)2\lambda_3$. For $a = g$ the formula reads

$$[T_g] = (p - 1)(p^2 + 1) \cdots (p^g + (-1)^g)\lambda_1\lambda_2 \cdots \lambda_g.$$

This stratum is of maximal codimension; this formula is visibly a generalization of the Deuring Mass Formula and counts the number of superspecial abelian varieties; this number was first calculated by Ekedahl in [16].

We formulate a corollary of our formulas for the p -rank strata.

Corollary 14.10. *The Chern classes λ_i of the Hodge bundle are represented by effective \mathbb{Q} -classes.*

Another nice aspect of our formulas is that when we specialize $p = 0$ in our formulas, that are polynomials in the l_i and λ_i with coefficients that are polynomials in p , we get back the formulas for the cycle classes of the Schubert cells both on the Grassmannian and the flag space, see [19].

15. Complete subvarieties of \mathcal{A}_g

The existence of complete subvarieties of relative small codimension can give us interesting information about a non-complete variety. In the case of the moduli space $\mathcal{A}_g \otimes k$ (for some field k) we know that λ_1 is an ample class and that $\lambda_1^{g(g-1)/2+1}$ vanishes. Since for a complete subvariety X of dimension d we must have $\lambda_1^d|X \neq 0$, this implies immediately a lower bound on the codimension.

Theorem 15.1. *The minimum possible codimension of a complete subvariety of $\mathcal{A}_g \otimes k$ is g .*

This lower bound can be realized in positive characteristic as was noted by Koblitz and Oort, see [49], [62]. The idea is simple: a semi-abelian variety with a positive torus rank has points of order p . So by requiring that our abelian varieties have p -rank 0 we stay inside \mathcal{A}_g and this defines the required complete variety.

Theorem 15.2. *The moduli stack $\mathcal{A}_g \otimes k$ with $\text{char}(k) = p > 0$ contains a complete substack of codimension g : the locus V_0 of abelian varieties with p -rank zero.*

A generalization is:

Theorem 15.3. *The partial Satake compactification $\mathcal{A}_g^* - \mathcal{A}_{g-(t+1)}^*$ in characteristic $p > 0$ of degenerations of torus rank t contains a complete subvariety of codimension $g - t$, namely the locus V_t of p -rank $\leq t$.*

In characteristic 0 there is no such obvious complete subvariety of codimension g , at least if $g \geq 3$. Oort conjectured in [63] that it should not exist for $g \geq 3$. This was proved by Keel and Sadun in [48].

Theorem 15.4. *If $X \subset \mathcal{A}_g(\mathbb{C})$ is a complete subvariety with the property that $\lambda_i|X$ is trivial in cohomology for some $1 \leq i \leq g$ then $\dim X \leq i(i-1)/2$ with strict inequality if $i \geq 3$.*

This theorem implies the following corollary.

Corollary 15.5. *For $g \geq 3$ the moduli space $\mathcal{A}_g \otimes \mathbb{C}$ does not possess a complete subvariety of codimension g .*

So the question arises what the maximum dimension of a complete subvariety of $\mathcal{A}_g(\mathbb{C})$ is.

One might conclude that the analogue of $\mathcal{A}_g(\mathbb{C})$ in positive characteristic in some sense is rather the locus of principally polarized abelian varieties with maximal p -rank ($= g$) than $\mathcal{A}_g \otimes \mathbb{F}_p$.

16. Cohomology of local systems and relations to modular forms

There is a close connection between the cohomology of moduli spaces of abelian varieties and modular forms. This connection was discovered in the 19th century and developed further in the work of Eichler, Shimura, Kuga, Matsushima and many others. It has developed into a central theme involving the theory of automorphic representations and the Langlands philosophy. We shall restrict here to just one aspect of this. This is work in progress that is being developed in joint work with Jonas Bergström and Carel Faber.

Let us start with $g = 1$. The space of cusp forms S_{2k} of weight $2k$ on $\mathrm{SL}(2, \mathbb{Z})$ has a cohomological interpretation. To describe it we consider the universal elliptic curve $\pi : \mathcal{X}_1 \rightarrow \mathcal{A}_1$ and let $V := R^1\pi_*\mathbb{Q}$ be the local system of rank 2 with as fibre over $[X]$ the cohomology $H^1(X, \mathbb{Q})$ of the elliptic curve X

From V we can construct other local systems: define for $a \geq 1$

$$V_a := \mathrm{Sym}^a(V).$$

This is a local system of rank $a+1$. There is also the l -adic analogue $V^{(l)} = R^1\pi_*\mathbb{Q}_l$ for l -adic étale cohomology and its variants $V_a^{(l)}$. The basic result of Eichler and Shimura says that for $a > 0$ and a even there is an isomorphism

$$H_c^1(\mathcal{A}_1 \otimes \mathbb{C}, V_a \otimes \mathbb{C}) = S_{a+2} \oplus \bar{S}_{a+2} \oplus \mathbb{C}.$$

So we might say that as a mixed Hodge module the compactly supported cohomology of the local system $V_a \otimes \mathbb{C}$ equals $S_{a+2} \oplus \bar{S}_{a+2} \oplus \mathbb{C}$.

But this identity can be stretched further. The left hand side may be replaced by other flavors of cohomology, for example by l -adic étale cohomology $H^1(\mathcal{A}_1 \otimes \bar{\mathbb{Q}}, V_a^{(l)})$ that comes with a natural Galois action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. In view of this, we replace the left hand side by an Euler characteristic

$$e_c(\mathcal{A}_1, V_a) := \sum_{i=0}^2 [H_c^i(\mathcal{A}_1, V_a)],$$

where now the cohomology groups of compactly supported cohomology are to be interpreted in an appropriate Grothendieck group, e.g. of mixed Hodge structures when we consider complex cohomology $H_c^*(\mathcal{A}_1 \otimes \mathbb{C}, V_a \otimes \mathbb{C})$ with its mixed Hodge structure, or in the Grothendieck group of Galois representations when we consider compactly supported étale l -adic cohomology $H_{c,et}^*(\mathcal{A}_1 \otimes \bar{\mathbb{Q}}, V_a^{(l)})$. On the other hand, for the right hand side Scholl defined in [67] a Chow motive $S[2k]$ associated to the space of cusp forms S_{2k} for $k > 1$. Then a sophisticated form of the Eichler-Shimura isomorphism asserts that we have an isomorphism

$$e_c(\mathcal{A}_1, V_a) = -S[a+2] - 1 \quad a \geq 2 \text{ even}.$$

We have a natural algebra of operators, the Hecke operators, acting on the spaces of cusp forms, but also on the cohomology since the Hecke operators are

defined by correspondences. Then the isomorphism above is compatible with the action of the Hecke operators.

But our moduli space \mathcal{A}_1 is defined over \mathbb{Z} . We thus can study the cohomology over \mathbb{Q} by looking at the fibres $\mathcal{A}_1 \otimes \mathbb{F}_p$ and the corresponding local systems $V_a^{(l)} \otimes \mathbb{F}_p$ (for $l \neq p$) by using comparison theorems. Now in characteristic p the Hecke operator is defined by the correspondence $X_0(p)$ of (cyclic) p -isogenies $\phi : X \rightarrow X'$ between elliptic curves (i.e. we require $\deg \phi = p$); it allows maps $q_i : X_0(p) \rightarrow \mathcal{A}_1$ ($i = 1, 2$) by sending ϕ to its source X and target X' . In characteristic p the correspondence decomposes into two components

$$X_0(p) \otimes \mathbb{F}_p = F_p + F_p^t, \quad (\text{the congruence relation})$$

where F_p is the correspondence $X \mapsto X^{(p)}$ and F_p^t its transpose. This follows since for such a p -isogeny we have that $X' \cong X^{(p)}$ or $X \cong (X')^{(p)}$. This implies a relation between the Hecke operator $T(p)$ and the action of Frobenius on $H^1(\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p, V_a^{(\ell)})$; The result is then a relation between the action of Frobenius on étale cohomology and the action of a Hecke operator on the space of cusp forms ([14], Prop. 4.8)

$$\text{Tr}(F_p, H_c^1(\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p, V_a)) = \text{Tr}(T(p), S_{a+2}) + 1.$$

We can calculate the traces of the Frobenii by counting points over finite fields. In fact, if we make a list of all elliptic curves over \mathbb{F}_p up to isomorphism over \mathbb{F}_p and we calculate the eigenvalues $\alpha_X, \bar{\alpha}_X$ of F_p acting on the étale cohomology $H^1(X, \mathbb{Q}_l)$ then we can calculate the trace of F_p on the cohomology $H_c^1(\mathcal{A}_1 \otimes \overline{\mathbb{F}}_p, V_a^{(\ell)})$ by summing the expression

$$\frac{\alpha_X^a + \alpha_X^{a-1} \bar{\alpha}_X + \dots + \bar{\alpha}_X^a}{\#\text{Aut}_{\mathbb{F}_p}(X)}$$

over all X in our list. So by counting elliptic curves over a finite field \mathbb{F}_p we can calculate the trace of the Hecke operator $T(p)$ on S_{a+2} ; in fact, once we have our list of elliptic curves over \mathbb{F}_p together with the eigenvalues $\alpha_X, \bar{\alpha}_X$ and the order of their automorphism group, we can calculate the trace of $T(p)$ on the space of cusp forms S_{a+2} for all a .

The term -1 in the Eichler-Shimura identity can be interpreted as coming from the kernel

$$-1 = \sum (-1)^i [\ker H_c^i(\mathcal{A}_1, V_a) \rightarrow H^i(\mathcal{A}_1, V_a)].$$

So to avoid this little nuisance we might replace the compactly supported cohomology by the image of compactly supported cohomology in the usual cohomology, i.e. define the *interior* cohomology $H_!^i$ as the image of compactly supported cohomology in the usual cohomology. Then the result reads

$$e_!(\mathcal{A}_1, V_a) = -S[a+2] \quad \text{for } a > 2 \text{ even.}$$

Some words about the history of the Eichler-Shimura result may be in order here. Around 1954 Eichler showed (see [15]) that for some congruence subgroup

Γ of $\mathrm{SL}(2, \mathbb{Z})$ the p -part of the zeta function of the corresponding modular curve X_Γ in characteristic p is given by the Hecke polynomial for the Hecke operator $T(p)$ acting on the space of cusp forms of weight 2 for Γ . This was generalized to some other groups by Shimura. M. Sato observed in 1962 that by combining the Eichler-Selberg trace formula for modular forms with the congruence relation (expressing the Hecke correspondence in terms of the Frobenius correspondence and its transpose) one could extend Eichler's results by expressing the Hecke polynomials in terms of the zeta functions of $\mathcal{M}_{1,n} \otimes \mathbb{F}_p$, except for problems due to the non-completeness of the moduli spaces. A little later Kuga and Shimura showed that Sato's idea worked for compact quotients of the upper half plane (parametrizing abelian surfaces). Ihara then made Sato's idea reality in 1967 by combining the Eichler-Selberg trace formula with results of Deuring and Hasse on zeta functions of elliptic curves, cf. [46], where one also finds references to the history of this problem. A year later Deligne solved in [14] the problems posed by the non-completeness of the moduli spaces. Finally Scholl proved the existence of the motive $S[k]$ for even k in 1990. A different construction of this motive was given by Consani and Faber in [13].

That the approach sketched above for calculating traces of Hecke operators by counting over finite fields is not used commonly, is due to the fact that we have an explicit formula for the traces of the Hecke operators, the Eichler-Selberg trace formula, cf. [74]. But for higher genus g , i.e. for modular forms on the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ with $g \geq 2$, no analogue of the trace formula for Siegel modular forms is known. Moreover, a closer inspection reveals that vector-valued Siegel modular forms are the good analogue for higher g of the modular forms on $\mathrm{SL}(2, \mathbb{Z})$ (rather than classical Siegel modular forms only). This suggests to try the analogue of the point counting method for genus 2 and higher. That is what Carel Faber and I did for $\mathrm{Sp}(4, \mathbb{Z})$ and in joint work with Bergström extended to genus 2 and level 2 and also to genus 3, see [22, 6, 7].

There are alternative methods that we should point out. In general one tries to compare a trace formula of Selberg type (Arthur trace formula) with the Grothendieck-Lefschetz fixed point formula. There is a topological trace formula for the trace of Hecke operators acting on the compactly supported cohomology, see for example [41], esp. the letter to Goresky and MacPherson there. Laumon gives a spectral decomposition of the cohomology of local systems for $g = 2$ (for the trivial one in [51] and in general in [52]); cf. also the work of Kottwitz in general. We also refer to Sophie Morel's book [54]. But though these methods in principle could lead to explicit results on Siegel modular forms, as far as I know it has not yet done that.

So start with the universal principally polarized abelian variety $\pi : \mathcal{X}_g \rightarrow \mathcal{A}_g$ and form the local system $V := R^1 \pi_* \mathbb{Q}$ and its l -adic counterpart $R^1 \pi_* \mathbb{Q}_l$ for étale cohomology. By abuse of notation we will write simply V . We consider π as a

morphism of stacks. For any irreducible representation λ of Sp_{2g} we can construct a corresponding local system V_λ ; it is obtained by applying a Schur functor, cf. e.g. [30]. So if we denote λ by its highest weight $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g$ then $V = V_{1,0,\dots,0}$ and $\mathrm{Sym}^a(V) = V_{a,0,\dots,0}$.

We now consider

$$e_c(\mathcal{A}_g, V_\lambda) := \sum_i (-1)^i [H_c^i(\mathcal{A}_g, V_\lambda)],$$

where as before the brackets indicate that we consider this in the Grothendieck group of the appropriate category (mixed Hodge modules, Galois representations). It is a basic result of Faltings [25, 23] that $H^i(\mathcal{A}_g \otimes \mathbb{C}, V_\lambda \otimes \mathbb{C})$ and $H_c^i(\mathcal{A}_g \otimes \mathbb{C}, V_\lambda \otimes \mathbb{C})$ carry a mixed Hodge structure. Moreover, the interior cohomology H_i^i carries a pure Hodge structure with the weights equal to the 2^g sums of any of the subsets of $\{\lambda_1 + g, \lambda_2 + g - 1, \dots, \lambda_g + 1\}$. He also shows that for regular λ , that is, when $\lambda_1 > \lambda_2 > \dots > \lambda_g > 0$, the cohomology H_i^i vanishes when $i \neq g(g+1)/2$.

These results of Faltings are the analogue for the non-compact case of results of Matsushima and Murakami in [53]; they could use Hodge theory to deduce the vanishing of cohomology groups and the decompositions indexed by pairs of elements in the Weyl group.

For $g = 2$ a local system V_λ is specified by giving $\lambda = (a, b)$ with $a \geq b \geq 0$. Then the Hodge filtration is

$$F^{a+b+3} \subseteq F^{a+2} \subseteq F^{b+1} \subseteq F^0 = H_1^3(\mathcal{A}_2 \otimes \mathbb{C}, V_\lambda \otimes \mathbb{C}).$$

Moreover, there is an identification of the first step in the Hodge filtration

$$F^{a+b+3} \cong S_{a-b, b+3},$$

with the space $S_{a-b, b+3}$ of Siegel modular cusp forms whose weight ρ is the representation $\mathrm{Sym}^{a-b} \mathrm{St} \otimes \det(\mathrm{St})^{b+3}$ with St the standard representation of $\mathrm{GL}(2, \mathbb{C})$.

Again, as for $g = 1$, we have an algebra of Hecke operators induced by geometric correspondences and it acts both on the cohomology and the modular forms compatible with the isomorphism.

Given our general ignorance of vector-valued Siegel modular forms for $g > 1$ the obvious question at this point is whether we can mimic the approach sketched above for calculating the traces of the Hecke operators by counting over finite fields.

Note that by Torelli we have morphism $\mathcal{M}_2 \rightarrow \mathcal{A}_2 = \mathcal{M}_2^{\mathrm{st}}$, where $\mathcal{M}_2^{\mathrm{st}}$ is the moduli space of curves of genus 2 of compact type. This means that we have to include besides the smooth curves of genus 2 the stable curves of genus 2 that are a union of two elliptic curves.

Can we use this to calculate the traces of the Hecke operator $T(p)$ on the spaces of vector-valued Siegel modular forms?

Two problems arise. The first is the so-called Eisenstein cohomology. This is

$$e_{\text{Eis}}(\mathcal{A}_2, V_{a,b}) := \sum (-1)^i (\ker H_c^i \rightarrow H^i)(\mathcal{A}_2, V_{a,b}).$$

In the case of genus 1 this expression was equal to the innocent -1 , but for higher g this is a more complicated term. The second problem is the *endoscopy*: the terms in the Hodge filtration that do not see the first and last step of the Hodge filtration. For torsion-free groups there is work by Schwermer (see [68]) on the Eisenstein cohomology; cf. also the work of Harder [41]. And there is an extensive literature on endoscopy. But explicit formulas were not available.

On the basis of numerical calculations Carel Faber and I guessed in [22] a formula for the Eisenstein cohomology. I was able to prove this formula in [35] for regular λ . We also made a guess for the endoscopic term. Putting this together we get the following conjectural formula for $(a, b) \neq (0, 0)$.

Conjecture 16.1. *The trace of the Hecke operator $T(p)$ on the space $S_{a-b, b+3}$ of cusp forms on $\text{Sp}(4, \mathbb{Z})$ equals*

$$-\text{Tr}(F_p, e_c(\mathcal{A}_2 \otimes \mathbb{F}_p, V_{a,b})) + \text{Tr}(F_p, e_{2, \text{extra}}(a, b)),$$

where the term $e_{2, \text{extra}}(a, b)$ is defined as

$$s_{a-b+2} - s_{a+b+4} (S[a-b+2] + 1) \mathbb{L}^{b+1} + \begin{cases} S[b+2] + 1 & a \text{ even,} \\ -S[a+3] & a \text{ odd,} \end{cases}$$

and $s_n = \dim S_n(\text{SL}(2, \mathbb{Z}))$ is the dimension of the space of cusp forms on $\text{SL}(2, \mathbb{Z})$ and $\mathbb{L} = h^2(\mathbb{P}^1)$ is the Lefschetz motive.

If $a > b = 0$ or $a = b > 0$ then one should put $s_2 = -1$ and $S[2] = -1 - \mathbb{L}$ in the formula.

In the case of regular local systems (i.e. $a > b > 0$) our conjecture can be deduced from results of Weissauer as he shows in his preprint [73]. In the case of local system of non-regular highest weight the conjecture is still open.

We have counted the curves of genus 2 of compact type for all primes $p \leq 37$. This implies that we can calculate the traces of the Hecke operator $T(p)$ for $p \leq 37$ on the space of cusp forms $S_{a-b, b+3}$ for all $a > b > 0$, and assuming the conjecture also for $a > b = 0$ and $a = b > 0$.

Our results are in accordance with results on the numerical Euler characteristic $\sum (-1)^i \dim H_c^i(\mathcal{A}_2, V_{a,b})$ due to Getzler ([36]) and with the dimension formula for the space of cusp form $S_{a-b, b+3}$ due to Tsushima ([71]).

We give an example. For $(a, b) = (11, 5)$ we have

$$e_c(\mathcal{A}_2, V_{11,5}) = -\mathbb{L}^6 - S[6, 8]$$

with $S[6, 8]$ the hypothetical motive associated to the (1-dimensional) space of cusp forms $S_{6,8}$. We list a few eigenvalues $\lambda(p)$ and $\lambda(p^2)$ of the Hecke operators.

This allows us to give the characteristic polynomial of Frobenius (and the Euler factor of the spinor L -function of the Siegel modular form)

$$1 - \lambda(p)X + (\lambda(p)^2 - \lambda(p^2) - p^{a+b+2})X^2 - \lambda(p)p^{a+b+3}X^3 + p^{2(a+b+3)}X^4$$

and its slopes.

p	$\lambda(p)$	$\lambda(p^2)$	slopes
2	0	-57344	13/2, 25/2
3	-27000	143765361	3, 7, 12, 16
5	2843100	-7734928874375	2, 7, 12, 17
7	-107822000	4057621173384801	0, 6, 13, 19

Another indication that the computer counts of curves of genus 2 are correct comes from a conjecture of Harder about congruences between Hecke eigenvalues of cusp forms for genus 1 and genus 2. Harder had the idea that there should be such congruences many years ago (cf. [41]), but our results motivated him to make his conjecture precise and explicit. He conjectured that if a (not too small, or better an ordinary) prime ℓ divides a critical value s of the L -function of an eigenform on $\mathrm{SL}(2, \mathbb{Z})$ of weight r then there should be a vector-valued Siegel modular form of prescribed weight depending on s and r and a congruence modulo ℓ between the eigenvalues under the Hecke operator $T(p)$ for f and F . We refer to [11], the papers by Harder [42] and van der Geer [34] there, for an account of this fascinating story. These congruences generalize the famous congruence

$$\tau(n) \equiv p^{11} + 1 \pmod{691}$$

for the Hecke eigenvalues $\tau(p)$ (p a prime) of the modular form $\Delta = \sum_{n>0} \tau(n)q^n$ of weight 12 on $\mathrm{SL}(2, \mathbb{Z})$. One example of such a congruence is the congruence

$$\lambda(p) \equiv p^8 + a(p) + p^{13} \pmod{41}$$

where $f = \sum a(n)q^n$ is the normalized ($a(1) = 1$) cusp form of weight 22 on $\mathrm{SL}(2, \mathbb{Z})$ and the $\lambda(p)$ are the Hecke eigenvalues of the genus 2 Siegel cusp form $F \in S_{4,10}$. We checked this congruence for all primes p with $p \leq 37$. For example, $a(37) = 22191429912035222$ and $\lambda(37) = 11555498201265580$.

In joint work with Bergström and Faber we extended this to level 2. One considers the moduli space $\mathcal{A}_2[2]$ of principally polarized abelian surfaces of level 2. This moduli space contains as a dense open subset the moduli space $\mathcal{M}_2[w^6]$ of curves of genus 2 together with six Weierstrass points. It comes with an action of $S_6 \cong \mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z})$. We formulated an analogue of conjecture 16.1 for level 2. Assuming this conjecture we can calculate the traces of the Hecke operators $T(p)$ for $p \leq 37$ for the spaces of cusp forms of all level 2. Using these numerical data we could observe liftings from genus 1 to genus 2 and could make precise conjectures

about such liftings; and again we could predict and verify numerically congruences between genus 1 and genus 2 eigenforms.

We give an example. For $(a, b) = (4, 2)$ we find

$$e_c(\mathcal{A}_2[2], V_{4,2}) = -45\mathbb{L}^3 + 45 - S[\Gamma_2[2], (2, 5)]$$

and assuming our conjecture we can calculate the traces of the Hecke operators on the space of cusp forms of weight $(2, 5)$ on the level 2 congruence subgroup $\Gamma_2[2]$ of $\mathrm{Sp}(4, \mathbb{Z})$; this space is a representation of type $[2^2, 1^2]$ for the symmetric group S_6 and is generated by one Siegel modular form; for this Siegel modular form we have the eigenvalue $\lambda(23) = -323440$ for $T(23)$.

It is natural to ask how the story continues for genus 3. The first remark is that the Torelli map $\mathcal{M}_3 \rightarrow \mathcal{A}_3$ is a morphism of degree 2 in the sense of stacks. This is due to the fact that every principally polarized abelian variety has a non-trivial automorphism $-\mathrm{id}$, while the general curve of genus 3 has a trivial automorphism group. This has as a consequence that for local systems $V_{a,b,c}$ with $a + b + c$ odd the cohomology on \mathcal{A}_3 vanishes, but on \mathcal{M}_3 it need not; and in fact in general it does not.

In joint work with Bergström and Faber we managed to formulate an analogue of 16.1 for genus 3, see [7]. Assuming the conjecture we are able to calculate the traces of Hecke operators $T(p)$ on the space of cusp forms $S_{a-b,b-c,c+4}$ for all primes p for which we did the counting (at least $p \leq 19$). Again the numerical data fit with calculations of dimensions of spaces of cusp forms and numerical Euler characteristics. (These numerical Euler characteristics were calculated in [8].) And again we could observe congruences between eigenvalues for $T(p)$ for $g = 3$ eigenforms and those of genus 1 and 2. We give two examples:

$$e_c(\mathcal{A}_3, V_{10,4,0}) = -\mathbb{L}^7 + \mathbb{L} + S[6, 8],$$

the same $S[6, 8]$ for genus 2 we met above. And for example

$$e_c(\mathcal{A}_3, V_{8,4,4}) = -S[12]\mathbb{L}^6 + S[12] + S[4, 0, 8],$$

where genuine Siegel modular forms (of weight $(4, 0, 8)$) of genus 3 do occur. We refer to [7] for the details and to the Chapter by Faber and Pandharipande for the cohomology of local systems and modular forms on the moduli spaces of curves.

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