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Upper Bounds on the Distillable Randomness of Bipartite Quantum States

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***Abstract*—The distillable randomness of a bipartite quantum state is an information-theoretic quantity equal to the largest net rate at which shared randomness can be distilled from the state by means of local operations and classical communication. This quantity has been widely used as a measure of classical correlations, and one version of it is equal to the regularized Holevo information of the ensemble that results from measuring one share of the state. However, due to the regularization, the distillable randomness is difficult to compute in general. To address this problem, we define measures of classical correlations and prove a number of their properties, most importantly that they serve as upper bounds on the distillable randomness of an arbitrary bipartite state. We then further bound these measures from above by some that are efficiently computable by means of semi-definite programming, we evaluate one of them for the example of an isotropic state, and we remark on the relation to quantities previously proposed in the literature.**

Full version at https://markwilde.com/RD-bnds.pdf

I. Introduction

The distillable randomness of a bipartite quantum state is equal to the largest net rate at which uniformly random, perfectly correlated bits can be distilled from the state by means of local operations and one-way classical communication (1W-LOCC) [1]. That is, in this scenario, the net rate is the rate at which shared randomness is distilled minus the rate at which classical bits are communicated to accomplish the task. It is important to subtract the classical communication rate, as failure to do so would trivialize the task, allowing for an infinite number of random bits to be shared. This task fundamentally has its roots in classical information theory [2], [3]. Here we also extend the task to allow for general local operations and classical communication (LOCC).

The distillable randomness of a bipartite state is considered a fundamental measure of classical correlations contained in a state. There has been a large literature on this topic, starting with [4], [5] and reviewed in [6], due to the wide interest in understanding correlations present in quantum states. However, [1] is the one of the few papers to consider understanding these correlations in an information-theoretic manner (see also [7]–[13] for other perspectives and related works).

Ref. [1] provided a formal solution for the aforementioned information-theoretic task. However, like many such formal solutions in quantum Shannon theory, it does not provide a computationally efficient procedure for quantifying correlations because it is expressed as a multi-letter or regularized formula. Our goal here is to fill this void by providing upper bounds on the distillable randomness that are efficiently computable by semi-definite programming. We also justify how some of the newly proposed measures are themselves measures of classical correlations contained in a bipartite state.

Although our results here can be viewed as a static counterpart to the results in the dynamical setting from [14]–[17], their derivation requires some new insights, as we explain in detail below. Ref. [17] proved that generalizations of the dynamical measures from [14]–[16] give upper bounds on the rate at which classical messages be communicated from one party to another, whenever they have access to a bipartite channel as a communication resource. A special case of this setting is when a sender is communicating classical messages to a receiver, with the assistance of a classical feedback channel. In contrast, as described earlier, our static setting here involves two parties distilling shared randomness from a bipartite state, by means of 1W-LOCC or general LOCC.

We comment here on the novel contributions of our paper when compared to earlier works such as [14]–[17]. The main measures of classical correlations proposed here, denoted by γ, C_γ, and $\mathbf{\Gamma}$ in Section III, are inspired by the measures that have appeared earlier in [14]–[17], denoted there by β, C_β, and $\mathbf{\Upsilon}$. However, the measures proposed here are symmetric with respect to a swap of the A and B systems. This property is critical for bounding the LOCC-assisted distillable randomness from above, as accomplished in Theorems 11 and 12. Furthermore, we prove here that these measures obey the classical communication bound in Proposition 7, also known as a "non-lockability" property in prior work on correlation measures [18], [19]. Finally, Lemma 9 and Proposition 10 are critical bounds that allow us to compare the actual state at the end of a randomness distillation protocol with an ideal state, and they ultimately lead to a strong converse upper bound for the distillable randomness. For the task of randomness distillation, the target (ideal) state is a mixed state, which is in distinction to most prior works on quantum resource theories, in which the target state is necessarily pure [20]–[22]. So the approach we have taken here goes beyond methods previously applied in such contexts and could potentially be useful more generally in research on quantum resource theories. We note here that similar methods were employed in [23], [24], but Lemma 9 and Proposition 10 provide a comparison between the ideal mixed state of interest and a convex set of positive semi-definite operators, rather than a comparison between an ideal mixed state and just one other mixed state.

The rest of the paper is organized as follows. In Section II we introduce some notation and key concepts. In Section III we construct and study certain classical correlation measures, while in Section IV we define the distillable randomness of a bipartite state. Continuing, in Section V we prove that the proposed measures are upper bounds on the distillable randomness, and in Section VI we consider semi-definite restrictions of the bounds. Finally, in Section VII we conclude with a brief summary and some open directions for future research.

II. Notation

Here we establish some notation and concepts that we use throughout the paper. We point the reader to the textbook [25] and the manuscript [26] for further details. We defer to Appendix A the introduction of the notation required for the proofs presented in the rest of the appendix. Let $\overline{\Phi}^d_{AB}$ denote the maximally classically correlated state of rank d, given by

$$\overline{\Phi}^d_{AB} \coloneqq \frac{1}{d}\sum_{m=0}^{d-1} |m\rangle\!\langle m|_A \otimes |m\rangle\!\langle m|_B. \tag{1}$$

We denote the transpose map acting on the system A by $T_A(\cdot) \coloneqq \sum_{i,j=0}^{d-1} |i\rangle\!\langle j|_A(\cdot)|i\rangle\!\langle j|_A$.

Let us define a generalized divergence $\boldsymbol{D}$ of a state ρ and a positive semi-definite operator σ as a function that obeys:

1) data processing: $\boldsymbol{D}(\rho\|\sigma) \geq \boldsymbol{D}(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$, where $\mathcal{N}$ is an arbitrary quantum channel;
2) the scaling relation: $\boldsymbol{D}(\rho\|c\sigma) = \boldsymbol{D}(\rho\|\sigma) - \log_2 c$, for all $c > 0$; and
3) the zero-value property: $\boldsymbol{D}(\rho\|\rho) = 0$ for every state ρ.

We note that the scaling and zero-value properties together imply non-negativity: $\forall c \in (0,1]$, $\boldsymbol{D}(1\|c) \geq 0$. Indeed, considering that the number 1 is a state of a trivial system, we have that $\boldsymbol{D}(1\|c) = \boldsymbol{D}(1\|1) - \log_2 c = -\log_2 c \geq 0$.

III. Classical Correlation Measures

We now define some classical correlation measures that are used later on, in the application of bounding the distillable randomness from above. For a positive semi-definite, bipartite operator σ_{AB}, let us define

$$\gamma(\sigma_{AB}) \coloneqq \inf_{K_A, L_B, V_{AB} \in \mathrm{Herm}} \left\{ \begin{array}{c} \mathrm{Tr}[K_A \otimes L_B] : \\ T_B(V_{AB} \pm \sigma_{AB}) \geq 0, \\ K_A \otimes L_B \pm V_{AB} \geq 0 \end{array} \right\}. \tag{2}$$

We also denote this quantity by $\gamma(A;B)_\sigma$. We also define

$$C_\gamma(\sigma_{AB}) \coloneqq C_\gamma(A;B)_\sigma \coloneqq \log_2 \gamma(\sigma_{AB}). \tag{3}$$

For a bipartite state ρ_{AB}, we then define

$$\boldsymbol{\Gamma}(\rho_{AB}) \coloneqq \inf_{\substack{\sigma_{AB}\geq 0: \\ \gamma(\sigma_{AB})\leq 1}} \boldsymbol{D}(\rho_{AB}\|\sigma_{AB}), \tag{4}$$

which follows an approach for defining resource measures proposed originally by [27]–[29] in the context of entanglement theory. Also, as a consequence of the scaling and zero-value properties listed above, we conclude that

$$\boldsymbol{\Gamma}(\rho_{AB}) \leq \boldsymbol{D}(\rho_{AB}\|\rho_{AB}/\gamma(\rho_{AB})) = C_\gamma(\rho_{AB}). \tag{5}$$

Our goal is to show that $\boldsymbol{\Gamma}(\rho_{AB})$ is an upper bound on the distillable randomness of a bipartite state ρ_{AB}. We do so in Section V after proving various properties of γ, C_γ, and $\boldsymbol{\Gamma}$, in Section III-A.

A. Properties of Classical Correlation Measures

In this section, we establish a number of properties of the quantities γ, C_γ, and $\boldsymbol{\Gamma}$, proposed in Section III. In particular, we prove that these measures satisfy several properties expected for a measure of classical correlations of a bipartite state, including

1) symmetry under exchange of A and B (Proposition 1),
2) data processing under local channels (Proposition 2),
3) local isometric invariance (Corollary 3),
4) non-negativity, faithfulness for product states (Proposition 4),
5) dimension bound (Proposition 5),
6) scale invariance (Proposition 6),
7) classical communication bound (Proposition 7),
8) subadditivity (Proposition 8), and
9) continuity near maximally classically correlated states (Proposition 10).

Proposition 1 (Exchange Symmetry): Let σ_{AB} be a bipartite positive semi-definite operator, and let ρ_{AB} be a state. Then

$$C_\gamma(A;B)_\sigma = C_\gamma(B;A)_\sigma, \tag{6}$$
$$\boldsymbol{\Gamma}(A;B)_\rho = \boldsymbol{\Gamma}(B;A)_\rho. \tag{7}$$

Proof. This follows by inspecting the definitions and using the fact that, for a Hermitian operator W_{AB}, the inequality $T_B(W_{AB}) \geq 0$ holds if and only if $T_A(W_{AB}) \geq 0$ does. ■

Proposition 2 (Data-Processing under Local Channels): Let σ_{AB} be a bipartite positive semi-definite operator, and let $\mathcal{N}_{A\to A'}$ and $\mathcal{M}_{B\to B'}$ be quantum channels. Then

$$C_\gamma(A;B)_\sigma \geq C_\gamma(A';B')_\omega, \tag{8}$$
$$\boldsymbol{\Gamma}(A;B)_\sigma \geq \boldsymbol{\Gamma}(A';B')_\omega, \tag{9}$$

where $\omega_{A'B'} \coloneqq (\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(\rho_{AB})$.

Proof. See Appendix B. ■

Corollary 3 (Local Isometric Invariance): Let σ_{AB} be a bipartite positive semi-definite operator, $\mathcal{U}_{A\to A'}$ an isometric channel, and $\mathcal{V}_{B\to B'}$ an isometric channel. Then

$$C_\gamma(A;B)_\sigma = C_\gamma(A';B')_\omega, \tag{10}$$

$$\mathbf{\Gamma}(A;B)_\sigma = \mathbf{\Gamma}(A';B')_\omega, \tag{11}$$

where $\omega_{A'B'} := (\mathcal{U}_{A\to A'} \otimes \mathcal{V}_{B\to B'})(\sigma_{AB})$.

Proof. The inequality $C_\gamma(A;B)_\sigma \geq C_\gamma(A';B')_\omega$ follows by a direct application of Proposition 2. Let us define the channels

$$\begin{aligned}\mathcal{L}_{A'\to A}(\cdot) &:= (\mathcal{U}_{A\to A'})^\dagger(\cdot) + \operatorname{Tr}[(\mathrm{id}_{A'} - (\mathcal{U}_{A\to A'})^\dagger)(\cdot)]\tau_A,\\ \mathcal{M}_{B'\to B}(\cdot) &:= (\mathcal{V}_{B\to B'})^\dagger(\cdot) + \operatorname{Tr}[(\mathrm{id}_{B'} - (\mathcal{V}_{B\to B'})^\dagger)(\cdot)]\eta_B,\end{aligned} \tag{12}$$

where τ_A and η_B are arbitrary quantum states. Consider that $\mathcal{L}_{A'\to A} \circ \mathcal{U}_{A\to A'} = \mathrm{id}_A$ and $\mathcal{M}_{B'\to B} \circ \mathcal{V}_{B\to B'} = \mathrm{id}_B$. Then the opposite inequality also follows by applying Proposition 2:

$$C_\gamma(A';B')_\omega \geq C_\gamma(A;B)_\zeta = C_\gamma(A;B)_\sigma, \tag{13}$$

where $\zeta_{AB} := (\mathcal{L}_{A'\to A} \otimes \mathcal{M}_{B'\to B})(\omega_{A'B'}) = \sigma_{AB}$. The same line of reasoning establishes (11). ■

Proposition 4 (Non-Negativity and Faithfulness): Let ρ_{AB} be a bipartite state. Then $C_\gamma(A;B)_\rho$ and $\mathbf{\Gamma}(A;B)_\rho$ are non-negative, i.e.

$$C_\gamma(A;B)_\rho \geq 0, \qquad \mathbf{\Gamma}(A;B)_\rho \geq 0, \tag{14}$$

and equal to zero if $\rho_{AB} = \sigma_A \otimes \tau_B$ is a product state. Furthermore, C_γ is faithful, i.e., $C_\gamma(\rho_{AB}) = 0$ implies that ρ_{AB} is a product state, and $\mathbf{\Gamma}$ is faithful provided that the underlying divergence $\boldsymbol{D}$ is itself faithful (positive definite) and also lower semicontinuous in its second argument.

Proof. See Appendix C. ■

Proposition 5 (Dimension Bound): Let ρ_{AB} be a bipartite state. Then

$$\mathbf{\Gamma}(A;B)_\rho \leq C_\gamma(A;B)_\rho \leq \log_2 \min\{d_A, d_B\}, \tag{15}$$

where d_A and d_B denote the dimensions of the local systems.

Proof. See Appendix D. ■

Proposition 6 (Scale Invariance): Let σ_{AB} be a positive semi-definite, bipartite operator, and let $c > 0$. Then

$$\gamma(c\sigma_{AB}) = c\gamma(\sigma_{AB}). \tag{16}$$

Proof. Let K_A, L_B, and V_{AB} be arbitrary choices for the optimization problem for $\gamma(\sigma_{AB})$. Then cK_A, L_B, and cV_{AB} are particular choices for which the objective function evaluates to $\operatorname{Tr}[cK_A \otimes L_B] = c\operatorname{Tr}[K_A \otimes L_B]$ and such that the constraints for $c\sigma_{AB}$ hold. It follows then that $\gamma(c\sigma_{AB}) \leq c\gamma(\sigma_{AB})$. To see the opposite inequality, consider applying this again to find that $\gamma(c^{-1}c\sigma_{AB}) \leq c^{-1}\gamma(c\sigma_{AB})$, which is equivalent to $c\gamma(\sigma_{AB}) \leq \gamma(c\sigma_{AB})$. ■

Proposition 7 (Classical Communication Bound): Let ρ_{XAB} be a tripartite state, for which system X is classical, i.e.,

$$\rho_{XAB} := \sum_x p(x)|x\rangle\langle x|_X \otimes \rho^x_{AB}, \tag{17}$$

where $\{p(x)\}_x$ is a probability distribution and $\{\rho^x_{AB}\}_x$ is a set of states. Then

$$C_\gamma(AX;B)_\rho \leq \log_2 d_X + C_\gamma(A;BX)_\rho, \tag{18}$$

$$\mathbf{\Gamma}(AX;B)_\rho \leq \log_2 d_X + \mathbf{\Gamma}(A;BX)_\rho. \tag{19}$$

Proof. See Appendix E. ■

Proposition 8 (Subadditivity): Let $\rho_{A_1A_2B_1B_2} := \sigma_{A_1B_1} \otimes \tau_{A_2B_2}$, where $\sigma_{A_1B_1}$ and $\tau_{A_2B_2}$ are bipartite operators. Then the following subadditivity inequality holds

$$C_\gamma(A_1A_2;B_1B_2)_\rho \leq C_\gamma(A_1;B_1)_\sigma + C_\gamma(A_2;B_2)_\tau. \tag{20}$$

If the underlying generalized divergence $\boldsymbol{D}$ is subadditive, then $\mathbf{\Gamma}$ is subadditive for a state $\rho_{A_1A_2B_1B_2} = \sigma_{A_1B_1} \otimes \tau_{A_2B_2}$:

$$\mathbf{\Gamma}(A_1A_2;B_1B_2)_\rho \leq \mathbf{\Gamma}(A_1;B_1)_\sigma + \mathbf{\Gamma}(A_2;B_2)_\tau. \tag{21}$$

Proof. See Appendix F. ■

Lemma 9: The following bound holds:

$$\sup_{\sigma_{AB}\geq 0:\gamma(\sigma_{AB})\leq 1} F\left(\overline{\Phi}^d_{AB}, \sigma_{AB}\right) \leq \frac{1}{d}, \tag{22}$$

where $\overline{\Phi}^d_{AB}$ is the maximally classical correlated state from (1) and $F(\omega,\tau) := \|\sqrt{\omega}\sqrt{\tau}\|_1^2$ is the fidelity of states ω and τ [30].

Proof. See Appendix G. ■

Recall that the sandwiched Rényi relative entropy of a state ρ and a positive semi-definite operator σ is defined for $\alpha \in (0,1)\cup(1,\infty)$ as [31], [32]

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{2\alpha}{\alpha-1}\log_2\left\|\sigma^{(1-\alpha)/2\alpha}\rho^{1/2}\right\|_{2\alpha}. \tag{23}$$

It converges to the quantum relative entropy [33]

$$D(\rho\|\sigma) := \operatorname{Tr}[\rho(\log_2\rho - \log_2\sigma)] \tag{24}$$

in the limit $\alpha \to 1$, and it is a generalized divergence for $\alpha \geq 1/2$. See [26] for further properties.

Proposition 10: Let ω_{AB} be a state satisfying

$$F\left(\overline{\Phi}^d_{AB}, \omega_{AB}\right) \geq 1-\varepsilon, \tag{25}$$

for $\varepsilon \in [0,1]$. Then, for all $\alpha > 1$,

$$\log_2 d \leq \widetilde{\Gamma}_\alpha(\omega_{AB}) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right), \tag{26}$$

where $\widetilde{\Gamma}_\alpha(\omega_{AB})$ is defined from (4) with the underlying divergence taken to be the sandwiched Rényi relative entropy $\widetilde{D}_\alpha$.

Proof. See Appendix H. ■

IV. Distillable Randomness of a Bipartite State

Let ρ_{AB} be a bipartite state. A protocol for randomness distillation assisted by 1W-LOCC begins with a quantum channel of the form $\mathcal{E}_{A\to LM}$, where the output systems are classical. The system L is communicated to Bob over a noiseless classical channel. After that, he acts with a quantum channel $\mathcal{D}_{BL\to M'}$. The state at the end of the protocol is thus

$$\omega_{MM'} := (\mathcal{D}_{BL\to M'} \circ \mathcal{E}_{A\to LM})(\rho_{AB}). \tag{27}$$

A (d,ε) protocol for randomness distillation satisfies

$$p_{\mathrm{err}}((\mathcal{E},\mathcal{D});\rho_{AB}) := 1 - F(\omega_{MM'}, \overline{\Phi}^d_{MM'}) \leq \varepsilon. \tag{28}$$

The one-shot distillable randomness of ρ_{AB} is defined as

$R^\varepsilon(\rho_{AB}) :=$

$$\sup_{\substack{\mathcal{E}_{A\to LM},\\ \mathcal{D}_{BL\to M'}}} \left\{\log_2 d - \log_2 d_L : p_{\text{err}}((\mathcal{E},\mathcal{D});\rho_{AB}) \leq \varepsilon\right\}. \quad (29)$$

In words, it is the largest net rate at which maximally correlated randomness can be generated. Indeed, we need to subtract off the classical communication used in the protocol; otherwise, the quantity would be trivially equal to $+\infty$.

The distillable randomness of ρ_{AB} is defined as

$$R(\rho_{AB}) \coloneqq \inf_{\varepsilon\in(0,1]} \liminf_{n\to\infty} \frac{1}{n} R^{\varepsilon}(\rho_{AB}^{\otimes n}), \quad (30)$$

and the strong converse distillable randomness as

$$\widetilde{R}(\rho_{AB}) \coloneqq \sup_{\varepsilon\in[0,1)} \limsup_{n\to\infty} \frac{1}{n} R^{\varepsilon}(\rho_{AB}^{\otimes n}). \quad (31)$$

We can generalize these definitions to allow for assistance by arbitrary LOCC (see [34] for a detailed account of LOCC). A protocol for randomness distillation assisted by LOCC begins with Alice performing the channel $\mathcal{E}^{(1)}_{A\to L_1A_1}$, where the system L_1 is classical and communicated to Bob. Bob then performs the channel $\mathcal{D}^{(2)}_{L_1B\to L_2B_2}$, where the system L_2 is classical and communicated to Alice. This process continues for k rounds; we denote Alice's other channels by $\{\mathcal{E}^{(i)}_{L_{i-1}A_{i-2}\to L_iA_i}\}_i$, for $i\in\{3,5,\ldots\}$, and Bob's other channels by $\{\mathcal{D}^{(i)}_{L_{i-1}B_{i-2}\to L_iB_i}\}_i$, for $i\in\{4,6,\ldots\}$. The last two channels in the protocol, without loss of generality, are then $\mathcal{E}^{(k)}_{L_{k-1}A_{k-2}\to L_kM}$ for Alice and $\mathcal{D}^{(k+1)}_{L_kB_{k-1}\to M'}$ for Bob. The state at the end of the protocol is thus

$$\omega_{MM'} \coloneqq (\mathcal{D}^{(k+1)}\circ\mathcal{E}^{(k)}\circ\cdots\circ\mathcal{D}^{(2)}\circ\mathcal{E}^{(1)})(\rho_{AB}). \quad (32)$$

A (d,ε) protocol for LOCC-assisted randomness distillation satisfies

$$p_{\text{err}}(\mathcal{P}^{(k)};\rho_{AB}) \coloneqq 1 - F(\omega_{MM'},\overline{\Phi}^{d}_{MM'}) \leq \varepsilon, \quad (33)$$

where $\mathcal{P}^{(k)}$ is a shorthand for the whole protocol, i.e., $\mathcal{P}^{(k)} = \{\mathcal{E}^{(1)},\mathcal{D}^{(2)},\ldots,\mathcal{E}^{(k)},\mathcal{D}^{(k+1)}\}$. The one-shot distillable randomness of ρ_{AB}, assisted by LOCC, is defined as

$$R^{\varepsilon}_{\leftrightarrow}(\rho_{AB}) \coloneqq \sup_{k\in\mathbb{N}}\sup_{\mathcal{P}^{(k)}} \left\{\log_2 d - \sum_{i=1}^{k}\log_2 d_{L_i} : p_{\text{err}}(\mathcal{P}^{(k)};\rho_{AB}) \leq \varepsilon\right\}. \quad (34)$$

The LOCC-assisted distillable randomness of a bipartite state ρ_{AB} is defined as

$$R_{\leftrightarrow}(\rho_{AB}) \coloneqq \inf_{\varepsilon\in(0,1]} \liminf_{n\to\infty} \frac{1}{n} R^{\varepsilon}_{\leftrightarrow}(\rho_{AB}^{\otimes n}), \quad (35)$$

and the strong converse LOCC-assisted distillable randomness as

$$\widetilde{R}_{\leftrightarrow}(\rho_{AB}) \coloneqq \sup_{\varepsilon\in[0,1)} \limsup_{n\to\infty} \frac{1}{n} R^{\varepsilon}_{\leftrightarrow}(\rho_{AB}^{\otimes n}). \quad (36)$$

The following inequalities hold by definition:

$$R^{\varepsilon}(\rho_{AB}) \leq R^{\varepsilon}_{\leftrightarrow}(\rho_{AB}), \qquad R(\rho_{AB}) \leq R_{\leftrightarrow}(\rho_{AB}), \quad (37)$$

$$\widetilde{R}(\rho_{AB}) \leq \widetilde{R}_{\leftrightarrow}(\rho_{AB}), \qquad R(\rho_{AB}) \leq \widetilde{R}(\rho_{AB}), \quad (38)$$

$$R_{\leftrightarrow}(\rho_{AB}) \leq \widetilde{R}_{\leftrightarrow}(\rho_{AB}). \quad (39)$$

We provide upper bounds for $R^{\varepsilon}_{\leftrightarrow}(\rho_{AB})$ and $\widetilde{R}_{\leftrightarrow}(\rho_{AB})$ in the next section.

V. Upper Bounds on the Distillable Randomness

Theorem 11: The following bound holds for all $\alpha > 1$:

$$R^{\varepsilon}_{\leftrightarrow}(\rho_{AB}) \leq \widetilde{\Gamma}_{\alpha}(\rho_{AB}) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right), \quad (40)$$

where $\widetilde{\Gamma}_{\alpha}(\rho_{AB})$ is defined from (4) and (23).

Proof. Let us begin by proving the bound

$$R^{\varepsilon}(\rho_{AB}) \leq \widetilde{\Gamma}_{\alpha}(\rho_{AB}) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right), \quad (41)$$

and then we discuss how to generalize the proof afterward. Consider an arbitrary protocol for randomness distillation assisted by 1W-LOCC. Since the final state $\omega_{MM'}$ satisfies (28), we apply Proposition 10 to conclude that

$$\log_2 d \leq \widetilde{\Gamma}_{\alpha}(M;M')_{\omega} + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right). \quad (42)$$

Next we apply data processing under local channels (Proposition 2) to conclude that

$$\widetilde{\Gamma}_{\alpha}(M;M')_{\omega} \leq \widetilde{\Gamma}_{\alpha}(M;BL)_{\mathcal{E}(\rho)}. \quad (43)$$

We then apply Propositions 1 and 7 to find that

$$\widetilde{\Gamma}_{\alpha}(M;BL)_{\mathcal{E}(\rho)} \leq \log_2 d_L + \widetilde{\Gamma}_{\alpha}(LM;B)_{\mathcal{E}(\rho)}. \quad (44)$$

Next we apply data processing under local channels again (Proposition 2) to conclude that

$$\widetilde{\Gamma}_{\alpha}(LM;B)_{\mathcal{E}(\rho)} \leq \widetilde{\Gamma}_{\alpha}(A;B)_{\rho}. \quad (45)$$

Putting everything together, we finally conclude that

$$\log_2 d - \log_2 d_L \leq \widetilde{\Gamma}_{\alpha}(A;B)_{\rho} + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right). \quad (46)$$

Since this holds for an arbitrary randomness distillation protocol, we conclude the desired bound.

To extend this to $R^{\varepsilon}_{\leftrightarrow}(\rho_{AB})$, we can iterate the same reasoning, going backward through a protocol of the form discussed around (32), using Propositions 1, 2, and 7 to arrive at the following upper bound for a k-round protocol:

$$\log_2 d - \sum_{i=1}^{k}\log_2 d_{L_i} \leq \widetilde{\Gamma}_{\alpha}(A;B)_{\rho} + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right). \quad (47)$$

Since this upper bound is independent of the number k of rounds, we conclude the upper bound in (40). ■

Theorem 12: The following upper bound holds

$$R_{\leftrightarrow}(\rho_{AB}) \leq \widetilde{R}_{\leftrightarrow}(\rho_{AB}) \leq \Gamma(A;B)_{\rho}, \quad (48)$$

where $\Gamma(A;B)_{\rho}$ is defined from (4) and (24).

Proof. Consider that, for all $\varepsilon\in(0,1]$ and $\alpha > 1$, the following bound holds from Theorem 11:

$$\frac{1}{n}R^{\varepsilon}_{\leftrightarrow}(\rho_{AB}^{\otimes n}) \leq \frac{1}{n}\widetilde{\Gamma}_{\alpha}(A^n;B^n)_{\rho^{\otimes n}} + \frac{\alpha}{n(\alpha-1)}\log_2\left(\frac{1}{1-\varepsilon}\right)$$

$$\leq \widetilde{\Gamma}_\alpha(A;B)_\rho + \frac{\alpha}{n(\alpha-1)}\log_2\!\left(\frac{1}{1-\varepsilon}\right), \quad (49)$$

where we applied subadditivity of $\widetilde{\Gamma}_\alpha$, which follows from Proposition 8 and the fact that $\widetilde{D}_\alpha$ is additive. Taking the limit as $n \to \infty$, we find that

$$\limsup_{n\to\infty} \frac{1}{n} R^{\varepsilon}_{\leftrightarrow}(\rho_{AB}^{\otimes n}) \leq \widetilde{\Gamma}_\alpha(A;B)_\rho. \quad (50)$$

Since this holds for all $\alpha > 1$, we conclude that

$$\limsup_{n\to\infty} \frac{1}{n} R^{\varepsilon}_{\leftrightarrow}(\rho_{AB}^{\otimes n}) \leq \inf_{\alpha>1} \widetilde{\Gamma}_\alpha(A;B)_\rho = \Gamma(A;B)_\rho. \quad (51)$$

The upper bound holds for all $\varepsilon \in [0,1)$, and so we conclude the desired inequalities in (48). ■

VI. Semi-Definite Restrictions

The upper bounds in Theorems 11 and 12 are not efficiently computable, because the $\mathbf{\Gamma}$-measure in (4) involves a bilinear optimization. One could attempt to use the approach from [35] to evaluate γ, but this does not remove all obstacles to having an efficient method for computing $\mathbf{\Gamma}$. Here we consider instead a semi-definite restriction of the γ-measure in (2), which leads to alternative upper bounds on $R^{\varepsilon}_{\leftrightarrow}(\rho_{AB})$ and $\widetilde{R}_{\leftrightarrow}(\rho_{AB})$. Indeed, for a given bipartite operator σ_{AB} and state ρ_A, we define the following semi-definite restriction of $\gamma(\sigma_{AB})$, taken by fixing $K_A = \rho_A$ in (2):

$$\beta(\sigma_{AB}, \rho_A) \coloneqq \inf_{L_B, V_{AB} \in \text{Herm}} \left\{ \begin{array}{c} \operatorname{Tr}[\rho_A \otimes L_B] : \\ T_B(V_{AB} \pm \sigma_{AB}) \geq 0, \\ \rho_A \otimes L_B \pm V_{AB} \geq 0 \end{array} \right\}. \quad (52)$$

It is then clear that $\gamma(\sigma_{AB}) \leq \beta(\sigma_{AB}, \rho_A)$ $\forall \rho_A$, as well as that $\mathbf{\Gamma}(\rho_{AB}) \leq \mathbf{\Upsilon}^A(\rho_{AB})$, where

$$\mathbf{\Upsilon}^A(\rho_{AB}) \coloneqq \inf_{\substack{\sigma_{AB} \geq 0: \\ \beta(\sigma_{AB}, \rho_A) \leq 1}} \boldsymbol{D}(\rho_{AB} \| \sigma_{AB}). \quad (53)$$

Note that one could also restrict the optimization of L_B by fixing $L_B = \rho_B$ and obtain the following quantities:

$$\beta(\sigma_{AB}, \rho_B) \coloneqq \inf_{K_A, V_{AB} \in \text{Herm}} \left\{ \begin{array}{c} \operatorname{Tr}[K_A \otimes \rho_B] : \\ T_B(V_{AB} \pm \sigma_{AB}) \geq 0, \\ K_A \otimes \rho_B \pm V_{AB} \geq 0 \end{array} \right\},$$

$$\mathbf{\Upsilon}^B(\rho_{AB}) \coloneqq \inf_{\substack{\sigma_{AB} \geq 0: \\ \beta(\sigma_{AB}, \rho_B) \leq 1}} \boldsymbol{D}(\rho_{AB} \| \sigma_{AB}). \quad (54)$$

So then it follows that

$$\mathbf{\Gamma}(\rho_{AB}) \leq \min\{\mathbf{\Upsilon}^A(\rho_{AB}), \mathbf{\Upsilon}^B(\rho_{AB})\}, \quad (55)$$

as well as

$$R^{\varepsilon}_{\leftrightarrow}(\rho_{AB}) \leq \min\{\widetilde{\Upsilon}^A_\alpha(\rho_{AB}), \widetilde{\Upsilon}^B_\alpha(\rho_{AB})\} + \alpha' \log_2\!\left(\frac{1}{1-\varepsilon}\right),$$

$$\widetilde{R}_{\leftrightarrow}(\rho_{AB}) \leq \min\{\Upsilon^A(A;B)_\rho, \Upsilon^B(A;B)_\rho\}, \quad (56)$$

where $\alpha' \equiv \frac{\alpha}{\alpha-1}$, as an immediate consequence of (55) and Theorems 11 and 12. To compute the upper bound in the second line of (56) efficiently, one can make use of the relative entropy optimization method from [36]. We remark here that the quantities in (52)–(54) are related to those defined previously in [14]–[16], and this point is discussed further in Appendix I.

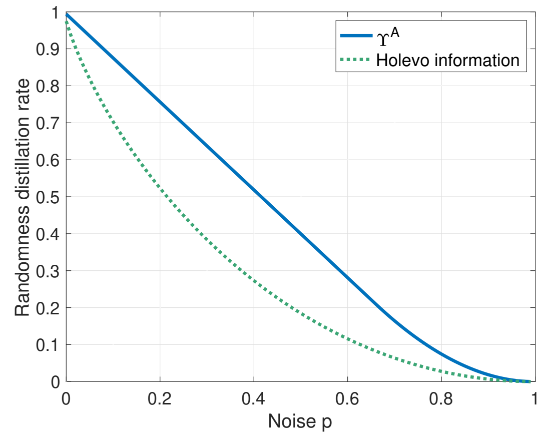

Fig. 1. Lower and upper bounds on the LOCC-assisted distillable randomness of a qubit-qubit isotropic state, as a function of the parameter $p \in [0,1]$.

As an example, in Figure 1, we plot the upper bound in (56) for an isotropic state, defined for $p \in [0,1]$ as $(1-p)\,\Phi^d_{AB} + pI_{AB}/d^2$, where $\Phi^d_{AB} \coloneqq \frac{1}{d}\sum_{i,j} |i\rangle\!\langle j|_A \otimes |i\rangle\!\langle j|_B$. For comparison, we also plot the Holevo information lower bound from [1]. The code for generating this figure is available with the arXiv posting of our paper. We note the similarity with Figure 6 of [17], which is for the dynamical case. Clearly, there is a gap between the lower and upper bounds, and a pertinent question is to close this gap, just as is the case for Figure 6 of [17].

VII. Conclusion

In this paper, we returned to the problem of distillable randomness of a bipartite state, providing a number of upper bounds on this quantity that are applicable in both the non-asymptotic and asymptotic regimes. To do so, we introduced a measure of classical correlations contained in a bipartite state. The main measure that we used to provide an upper bound is not clearly efficiently computable; however, we considered a semi-definite restriction that serves as an upper bound.

Going forward from here, it is open to establish tighter upper bounds on the distillable randomness. In future work, we plan to apply the recent lower bound on entanglement cost from [37, Eq. (13)], along with the identity in [38, Theorem 1] that relates entanglement cost and 1W-LOCC distillable randomness. It is also open to generalize these methods to the multipartite case (here, see [39]–[41]).

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References

[1] I. Devetak and A. Winter, "Distilling common randomness from bipartite quantum states," *IEEE Transactions on Information Theory*, vol. 50, no. 12, pp. 3183–3196, Dec. 2004, arXiv:quant-ph/0304196.

[2] R. Ahlswede and I. Csiszár, "Common randomness in information theory and cryptography—part I: Secret sharing," *IEEE Transactions on Information Theory*, vol. 39, no. 4, pp. 1121–1132, July 1993.

[3] ——, "Common randomness in information theory and cryptography—part II: CR capacity," *IEEE Transactions on Information Theory*, vol. 44, no. 1, pp. 225–240, Jan. 1998.

[4] L. Henderson and V. Vedral, "Classical, quantum and total correlations," *Journal of Physics A: Mathematical and General*, vol. 34, no. 35, pp. 6899–6905, Sep. 2001, arXiv:quant-ph/0105028.

[5] H. Ollivier and W. H. Zurek, "Quantum discord: A measure of the quantumness of correlations," *Physical Review Letters*, vol. 88, no. 1, p. 017901, Dec. 2001, arXiv:quant-ph/0105072.

[6] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, "The classical-quantum boundary for correlations: Discord and related measures," *Reviews of Modern Physics*, vol. 84, pp. 1655–1707, Nov. 2012, arXiv:1112.6238.

[7] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, "Thermodynamical approach to quantifying quantum correlations," *Physical Review Letters*, vol. 89, no. 18, p. 180402, Oct. 2002, arXiv:quant-ph/0112074.

[8] I. Devetak, "Distillation of local purity from quantum states," *Physical Review A*, vol. 71, no. 6, p. 062303, June 2005, arXiv:quant-ph/0406234.

[9] I. Devetak, A. W. Harrow, and A. Winter, "A resource framework for quantum Shannon theory," *IEEE Transactions on Information Theory*, vol. 54, no. 10, pp. 4587–4618, October 2008, arXiv:quant-ph/0512015.

[10] H. Krovi and I. Devetak, "Local purity distillation with bounded classical communication," *Physical Review A*, vol. 76, no. 1, p. 012321, July 2007, arXiv:0705.4089.

[11] G. Manzano, F. Plastina, and R. Zambrini, "Optimal work extraction and thermodynamics of quantum measurements and correlations," *Physical Review Letters*, vol. 121, no. 12, p. 120602, Sep. 2018, arXiv:1805.08184.

[12] B. Morris, L. Lami, and G. Adesso, "Assisted work distillation," *Physical Review Letters*, vol. 122, no. 13, p. 130601, Apr. 2019, arXiv:1811.12329.

[13] S. Chakraborty, A. Nema, and F. Buscemi, "One-shot purity distillation with local noisy operations and one-way classical communication," Aug. 2022, arXiv:2208.05628.

[14] X. Wang, W. Xie, and R. Duan, "Semidefinite programming strong converse bounds for classical capacity," *IEEE Transactions on Information Theory*, vol. 64, no. 1, pp. 640–653, Jan. 2018, arXiv:1610.06381.

[15] X. Wang, K. Fang, and M. Tomamichel, "On converse bounds for classical communication over quantum channels," *IEEE Transactions on Information Theory*, vol. 65, no. 7, pp. 4609–4619, Jul. 2019, arXiv:1709.05258.

[16] K. Fang and H. Fawzi, "Geometric Rényi divergence and its applications in quantum channel capacities," *Communications in Mathematical Physics*, vol. 384, pp. 1615–1677, Jun. 2021, arXiv:1909.05758.

[17] D. Ding, S. Khatri, Y. Quek, P. W. Shor, X. Wang, and M. M. Wilde, "Bounding the forward classical capacity of bipartite quantum channels," *IEEE Transactions on Information Theory (to appear)*, 2023, arXiv:2010.01058.

[18] R. König, R. Renner, A. Bariska, and U. Maurer, "Small accessible quantum information does not imply security," *Physical Review Letters*, vol. 98, no. 14, p. 140502, Apr. 2007, arXiv:quant-ph/0512021.

[19] K. Horodecki, M. Studziński, R. P. Kostecki, O. Sakarya, and D. Yang, "Upper bounds on the leakage of private data and an operational approach to Markovianity," *Physical Review A*, vol. 104, no. 5, p. 052422, Nov. 2021, arXiv:2107.10737.

[20] Z.-W. Liu, K. Bu, and R. Takagi, "One-Shot Operational Quantum Resource Theory," *Physical Review Letters*, vol. 123, no. 2, p. 020401, Jul. 2019, arXiv:1904.05840.

[21] K. Fang and Z.-W. Liu, "No-go theorems for quantum resource purification," *Physical Review Letters*, vol. 125, no. 6, p. 060405, Aug. 2020, arXiv:1909.02540.

[22] B. Regula, K. Bu, R. Takagi, and Z.-W. Liu, "Benchmarking one-shot distillation in general quantum resource theories," *Physical Review A*, vol. 101, no. 6, p. 062315, Jun. 2020, arXiv:1909.11677.

[23] F. Leditzky, M. M. Wilde, and N. Datta, "Strong converse theorems using Rényi entropies," *Journal of Mathematical Physics*, vol. 57, no. 8, p. 082202, August 2016, arXiv:1506.02635.

[24] X. Wang and M. M. Wilde, "Resource theory of asymmetric distinguishability," *Physical Review Research*, vol. 1, no. 3, p. 033170, Dec. 2019, arXiv:1905.11629.

[25] M. M. Wilde, *Quantum Information Theory*, 2nd ed. Cambridge University Press, 2017, arXiv:1106.1445.

[26] S. Khatri and M. M. Wilde, *Principles of Quantum Communication Theory: A Modern Approach*, Nov. 2020, arXiv:2011.04672v1.

[27] E. M. Rains, "Bound on distillable entanglement," *Physical Review A*, vol. 60, no. 1, pp. 179–184, Jul. 1999, arXiv:quant-ph/9809082.

[28] ——, "A semidefinite program for distillable entanglement," *IEEE Transactions on Information Theory*, vol. 47, no. 7, pp. 2921–2933, Nov. 2001, arXiv:quant-ph/0008047.

[29] K. Audenaert, B. De Moor, K. G. H. Vollbrecht, and R. F. Werner, "Asymptotic relative entropy of entanglement for orthogonally invariant states," *Physical Review A*, vol. 66, no. 3, p. 032310, Sep. 2002, arXiv:quant-ph/0204143.

[30] A. Uhlmann, "The "transition probability" in the state space of a *-algebra," *Reports on Mathematical Physics*, vol. 9, pp. 273–279, 1976.

[31] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, "On quantum Rényi entropies: A new generalization and some properties," *Journal of Mathematical Physics*, vol. 54, no. 12, p. 122203, Dec. 2013, arXiv:1306.3142.

[32] M. M. Wilde, A. Winter, and D. Yang, "Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy," *Communincations in Mathematical Physics*, vol. 331, no. 2, pp. 593–622, Oct. 2014, arXiv:1306.1586.

[33] H. Umegaki, "Conditional expectations in an operator algebra IV (entropy and information)," *Kodai Mathematical Seminar Reports*, vol. 14, pp. 59–85, 1962.

[34] E. Chitambar, D. Leung, L. Mančinska, M. Ozols, and A. Winter, "Everything you always wanted to know about LOCC (but were afraid to ask)," *Communications in Mathematical Physics*, vol. 328, no. 1, pp. 303–326, May 2014, arXiv:1210.4583.

[35] S. Huber, R. König, and M. Tomamichel, "Jointly constrained semidefinite bilinear programming with an application to Dobrushin curves," *IEEE Transactions on Information Theory*, vol. 66, no. 5, pp. 2934–2950, May 2020, arXiv:1808.03182.

[36] H. Fawzi and O. Fawzi, "Efficient optimization of the quantum relative entropy," *Journal of Physics A: Mathematical and Theoretical*, vol. 51, no. 15, p. 154003, Mar. 2018, arXiv:1705.06671.

[37] L. Lami and B. Regula, "No second law of entanglement manipulation after all," Nov. 2021, arXiv:2111.02438.

[38] M. Koashi and A. Winter, "Monogamy of quantum entanglement and other correlations," *Physical Review A*, vol. 69, no. 2, p. 022309, Feb. 2004, arXiv:quant-ph/0310037.

[39] J. A. Smolin, F. Verstraete, and A. Winter, "Entanglement of assistance and multipartite state distillation," *Physical Review A*, vol. 72, no. 5, p. 052317, Nov. 2005, arXiv:quant-ph/0505038.

[40] P. Vrana and M. Christandl, "Distillation of Greenberger–Horne–Zeilinger states by combinatorial methods," *IEEE Transactions on Information Theory*, vol. 65, no. 9, pp. 5945–5958, Sep. 2019, arXiv:1805.09096.

[41] F. Salek and A. Winter, "Multi-user distillation of common randomness and entanglement from quantum states," Aug. 2020, arXiv:2008.04964.

[42] G. Vidal and R. F. Werner, "Computable measure of entanglement," *Physical Review A*, vol. 65, no. 3, p. 032314, Feb. 2002, arXiv:quant-ph/0102117.

[43] J. Eisert and M. M. Wilde, "A smallest computable entanglement monotone," in *Proceedings of the 2022 International Symposium on Information Theory*, Espoo, Finland, June 2022, pp. 2439–2444, arXiv:2201.00835.

[44] M. M. Wilde *et al.*, "Bounding the randomness distribution capacity of quantum states and channels," unpublished notes, Sep. 2020, available upon request.

Appendix A
Notation

Here we provide some further notation needed to understand the proofs in the following appendices. We denote the unnormalized maximally entangled operator by

$$\Gamma_{RA} \coloneqq |\Gamma\rangle\langle\Gamma|_{RA}, \qquad |\Gamma\rangle_{RA} \coloneqq \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_A, \tag{57}$$

where $R \simeq A$ with dimension d and $\{|i\rangle_R\}_{i=0}^{d-1}$ and $\{|i\rangle_A\}_{i=0}^{d-1}$ are orthonormal bases. The notation $R \simeq A$ means that the systems R and A are isomorphic. The Choi operator of a quantum channel $\mathcal{N}_{A\to B}$ is denoted by

$$\Gamma^{\mathcal{N}}_{RB} \coloneqq \mathcal{N}_{A\to B}(\Gamma_{RA}). \tag{58}$$

Appendix B
Proof of Proposition 2 (Data Processing under Local Channels)

We prove that $\gamma(\rho_{AB}) \geq \gamma(\omega_{A'B'})$, which is equivalent to the inequality (8) in Proposition 2. Let K_A, L_B, and V_{AB} be arbitrary Hermitian operators satisfying $T_B(V_{AB} \pm \rho_{AB}) \geq 0$ and $K_A \otimes L_B \pm V_{AB} \geq 0$. Consider that the map $T_{B'} \circ \mathcal{M}_{B\to B'} \circ T_B$ is completely positive, which follows because

$$\begin{aligned}
&(T_{B'} \circ \mathcal{M}_{B\to B'} \circ T_B)(\Gamma_{\hat{B}B}) \\
&= (T_{B'} \circ \mathcal{M}_{B\to B'} \circ T_{\hat{B}})(\Gamma_{\hat{B}B}) &(59)\\
&= ((T_{\hat{B}} \otimes T_{B'}) \circ \mathcal{M}_{B\to B'})(\Gamma_{\hat{B}B}) &(60)\\
&= T_{\hat{B}B'}(\mathcal{M}_{B\to B'}(\Gamma_{\hat{B}B})) \geq 0. &(61)
\end{aligned}$$

The last inequality follows because $\mathcal{M}_{B\to B'}$ is completely positive and $T_{\hat{B}B'}$ is a positive map, which in this case is acting on both systems $\hat{B}B'$ and thus preserves positivity. We also present an alternative proof that $T_{B'} \circ \mathcal{M}_{B\to B'} \circ T_B$ is completely positive if $\mathcal{M}_{B\to B'}$ is. Consider the following chain of equalities for a Kraus decomposition of $\mathcal{M}_{B\to B'}$ as $\mathcal{M}_{B\to B'}(\cdot) = \sum_i M_i(\cdot)M_i^\dagger$:

$$\begin{aligned}
&(T_{B'} \circ \mathcal{M}_{B\to B'} \circ T_B)(\cdot) \\
&= \sum_{\ell,m} \sum_{j,k} \sum_i |\ell\rangle\langle m| M_i |j\rangle\langle k| (\cdot) |j\rangle\langle k| M_i^\dagger |\ell\rangle\langle m| &(62)\\
&= \sum_{\ell,m} \sum_{j,k} \sum_i |\ell\rangle\langle k| M_i^\dagger |\ell\rangle\langle k| (\cdot) |j\rangle\langle m| M_i |j\rangle\langle m| &(63)\\
&= \sum_i \sum_{\ell,k} |\ell\rangle\langle k| M_i^\dagger |\ell\rangle\langle k| (\cdot) \sum_{j,m} |j\rangle\langle m| M_i |j\rangle\langle m| &(64)\\
&= \sum_i T(M_i^\dagger)(\cdot)T(M_i) &(65)\\
&= \sum_i E_i(\cdot)E_i^\dagger, &(66)
\end{aligned}$$

where $E_i \coloneqq T(M_i^\dagger)$. Thus, $\{E_i\}_i$ is a set of Kraus operators for $T_{B'} \circ \mathcal{M}_{B\to B'} \circ T_B$, which implies that this map is completely positive.

Since $\mathcal{N}_{A\to A'}$ and $T_{B'} \circ \mathcal{M}_{B\to B'} \circ T_B$ are completely positive, it follows that

$$(\mathcal{N}_{A\to A'} \otimes (T_{B'} \circ \mathcal{M}_{B\to B'} \circ T_B))(T_B(V_{AB} \pm \rho_{AB})) \geq 0. \tag{67}$$

Now consider that

$$\begin{aligned}
&(\mathcal{N}_{A\to A'} \otimes (T_{B'} \circ \mathcal{M}_{B\to B'} \circ T_B))(T_B(V_{AB} \pm \rho_{AB})) \\
&= (\mathcal{N}_{A\to A'} \otimes (T_{B'} \circ \mathcal{M}_{B\to B'}))(V_{AB} \pm \rho_{AB}) &(68)\\
&= T_{B'}(\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(V_{AB} \pm \rho_{AB}) &(69)\\
&= T_{B'}(V'_{A'B'} \pm \omega_{A'B'}), &(70)
\end{aligned}$$

where

$$V'_{A'B'} \coloneqq (\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(V_{AB}). \tag{71}$$

So it follows that

$$T_{B'}(V'_{A'B'} \pm \omega_{A'B'}) \geq 0. \tag{72}$$

Since $\mathcal{N}_{A\to A'}$ and $\mathcal{M}_{B\to B'}$ are completely positive, it follows that

$$(\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(K_A \otimes L_B \pm V_{AB}) \geq 0, \tag{73}$$

which is equivalent to

$$K'_{A'} \otimes L'_{B'} \pm V'_{A'B'} \geq 0, \tag{74}$$

where

$$K'_{A'} \coloneqq \mathcal{N}_{A\to A'}(K_A), \qquad L'_{B'} \coloneqq \mathcal{M}_{B\to B'}(L_B). \tag{75}$$

So we have shown that

$$T_B(V_{AB} \pm \rho_{AB}) \geq 0 \implies T_{B'}(V'_{A'B'} \pm \omega_{A'B'}) \geq 0, \tag{76}$$

$$K_A \otimes L_B \pm V_{AB} \geq 0 \implies K'_{A'} \otimes L'_{B'} \pm V'_{A'B'} \geq 0. \tag{77}$$

Also, since $\mathcal{N}_{A\to A'}$ and $\mathcal{M}_{B\to B'}$ are trace preserving, it follows that

$$\mathrm{Tr}[K_A \otimes L_B] = \mathrm{Tr}[K'_{A'} \otimes L'_{B'}]. \tag{78}$$

Putting together (76)–(78), we conclude that

$$\begin{aligned}
&\inf_{K_A, L_B, V_{AB}} \left\{ \begin{array}{c} \mathrm{Tr}[K_A \otimes L_B] : \\ T_B(V_{AB} \pm \rho_{AB}) \geq 0, \\ K_A \otimes L_B \pm V_{AB} \geq 0 \end{array} \right\} \\
&\geq \inf_{K'_{A'}, L'_{B'}, V'_{A'B'}} \left\{ \begin{array}{c} \mathrm{Tr}[K'_{A'} \otimes L'_{B'}] : \\ T_{B'}(V'_{A'B'} \pm \omega_{A'B'}) \geq 0, \\ K'_{A'} \otimes L'_{B'} \pm V'_{A'B'} \geq 0 \end{array} \right\}. &(79)
\end{aligned}$$

This concludes the proof of (8).

To see (9), let σ_{AB} be an arbitrary positive semi-definite operator satisfying $\gamma(\sigma_{AB}) \leq 1$. Then it follows from (8) that $\tau_{A'B'} \coloneqq (\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(\sigma_{AB})$ satisfies

$$\gamma(\tau_{A'B'}) \leq \gamma(\sigma_{AB}) \leq 1. \tag{80}$$

So then, defining $\omega_{A'B'} \coloneqq (\mathcal{N}_{A\to A'} \otimes \mathcal{M}_{B\to B'})(\rho_{AB})$, we find that

$$\begin{aligned}
\boldsymbol{D}(\rho_{AB} \| \sigma_{AB}) &\geq \boldsymbol{D}(\omega_{A'B'} \| \tau_{A'B'}) &(81)\\
&\geq \inf_{\substack{\tau_{A'B'} \geq 0, \\ \gamma(\tau_{A'B'}) \leq 1}} \boldsymbol{D}(\omega_{A'B'} \| \tau_{A'B'}) &(82)\\
&= \boldsymbol{\Gamma}(\omega_{A'B'}). &(83)
\end{aligned}$$

The first inequality follows from the data-processing inequality for $\boldsymbol{D}$, and the second from (80). The equality follows from the definition in (4). Since the inequality holds for all $\sigma_{AB} \geq 0$ satisfying $\gamma(\sigma_{AB}) \leq 1$, we conclude the desired inequality in (9) after taking the infimum.

Appendix C
Proof of Proposition 4 (Non-Negativity and Faithfulness)

First, it follows that $C_\gamma(A;B)_\rho$ takes its minimal value on a product state, and it is equal to the same value for all product states. This is because one can transition from an arbitrary state to a product state by performing local channels that trace out the input and replace with a state. Indeed, let $\mathcal{R}^\sigma_A(\cdot) := \mathrm{Tr}_A[\cdot]\sigma_A$ and $\mathcal{R}^\tau_B(\cdot) := \mathrm{Tr}_B[\cdot]\tau_B$ be local replacer channels. Then by applying inequality (8) in Proposition 2, we conclude that

$$C_\gamma(A;B)_\rho \geq C_\gamma(A;B)_\omega. \tag{84}$$

By the same procedure, one can transition from an arbitrary product state to another arbitrary product state by means of local channels. So the claim stated above follows. The same argument, but using (9), implies that $\boldsymbol{\Gamma}(\rho_{AB})$ takes its minimal value on product states.

We now prove that this minimal value is zero. By definition, for a product state $\omega_{AB} = \sigma_A \otimes \tau_B$,

$$\gamma(A;B)_\omega = \inf_{\substack{K_A, L_B,\\ V_{AB} \in \mathrm{Herm}}} \left\{ \begin{array}{c} \mathrm{Tr}[K_A \otimes L_B] : \\ T_B(V_{AB} \pm \sigma_A \otimes \tau_B) \geq 0, \\ K_A \otimes L_B \pm V_{AB} \geq 0 \end{array} \right\}. \tag{85}$$

Applying some of the constraints, we conclude that

$$T_B(\sigma_A \otimes \tau_B) \leq T_B(V_{AB}), \tag{86}$$
$$V_{AB} \leq K_A \otimes L_B. \tag{87}$$

Now taking a trace over these constraints, we conclude that

$$1 = \mathrm{Tr}[\sigma_A \otimes \tau_B] = \mathrm{Tr}[T_B(\sigma_A \otimes \tau_B)] \tag{88}$$
$$\leq \mathrm{Tr}[T_B(V_{AB})] = \mathrm{Tr}[V_{AB}] \leq \mathrm{Tr}[K_A \otimes L_B]. \tag{89}$$

So this establishes that $\gamma(A;B)_\omega \geq 1$ for every product state ω_{AB} (and also $\gamma(A;B)_\rho \geq 1$ for every state ρ_{AB}, by combining the observation in the first paragraph with $\gamma(A;B)_\omega \geq 1$ for every product state ω_{AB}).

To see the opposite inequality for a product state ω_{AB}, let us make the choice $V_{AB} = T_B(\sigma_A \otimes \tau_B)$, $K_A = \sigma_A$, and $L_B = \tau_B$. For this choice, all constraints are satisfied, in part because the partial transpose map is a positive map when acting on a product state. So it follows that $\gamma(A;B)_\omega \leq 1$, and combining with what we previously showed, we conclude that $\gamma(A;B)_\omega = 1$ for every product state.

Now we turn to $\boldsymbol{\Gamma}(\rho_{AB})$. Consider that, for an arbitrary positive semi-definite operator σ_{AB}, the condition $\gamma(\sigma_{AB}) \leq 1$ implies that $\mathrm{Tr}[\sigma_{AB}] \leq 1$ because the following holds for arbitrary V_{AB}, K_A, and L_B satisfying the constraints in (2):

$$\mathrm{Tr}[\sigma_{AB}] = \mathrm{Tr}[T_B(\sigma_{AB})] \leq \mathrm{Tr}[T_B(V_{AB})] \tag{90}$$
$$= \mathrm{Tr}[V_{AB}] \leq \mathrm{Tr}[K_A \otimes L_B]. \tag{91}$$

Then taking an infimum over all V_{AB}, K_A, and L_B satisfying (2) and applying the assumption $\gamma(\sigma_{AB}) \leq 1$, we conclude that $\mathrm{Tr}[\sigma_{AB}] \leq 1$. Now applying a trace channel to $\boldsymbol{D}(\rho_{AB}\|\sigma_{AB})$ and the non-negative property of a generalized divergence (i.e., $\boldsymbol{D}(1\|c) \geq 0, \quad \forall c \in (0,1]$), we conclude that $\boldsymbol{\Gamma}(\rho_{AB}) \geq 0$ for every state ρ_{AB}.

If the state of interest is a product state (i.e., $\omega_{AB} = \sigma_A \otimes \tau_B$), then it follows that $\gamma(\omega_{AB}) \leq 1$, as argued above, so that we can choose $\sigma_{AB} = \omega_{AB}$. With this choice it follows that $\boldsymbol{\Gamma}(\omega_{AB}) \leq \boldsymbol{D}(\omega_{AB}\|\omega_{AB}) = 0$, with the latter equality following from the zero-value property of generalized divergences. Combining with the previous inequality, we conclude that $\boldsymbol{\Gamma}(\omega_{AB}) = 0$ for every product state ω_{AB}.

Let us now turn to the proof that C_γ is faithful, i.e. that $C_\gamma(\omega_{AB}) = 0$ implies $\omega_{AB} = \rho_A \otimes \tau_B$ is a product state, and that $\boldsymbol{\Gamma}$ is also faithful provided that the underlying divergence $\boldsymbol{D}$ is lower semicontinuous in its second argument. First, observe that $C_\gamma = \boldsymbol{\Gamma}$ for the particular choice of divergence

$$\boldsymbol{D}(\rho\|\sigma) = \begin{cases} -\log c & \sigma = c\rho, \\ +\infty & \text{otherwise}, \end{cases} . \tag{92}$$

This observation is related to the inequality in (5). Indeed, when performing the optimization for $\boldsymbol{\Gamma}$, consider that choosing σ_{AB} to be any operator other than $c\rho_{AB}$ leads to a value of $+\infty$. This then forces σ_{AB} to be equal to $c\rho_{AB}$ for some $c > 0$, and the objective function for $\boldsymbol{\Gamma}(\rho_{AB})$ reduces to

$$\begin{aligned} \inf_{c>0, \gamma(c\rho_{AB}) \leq 1} \boldsymbol{D}(\rho_{AB}\|c\rho_{AB}) &= \inf_{c>0, c\gamma(\rho_{AB}) \leq 1} -\log_2 c \\ &= \inf_{c>0, c \leq \frac{1}{\gamma(\rho_{AB})}} -\log_2 c \\ &= \log_2 \gamma(\rho_{AB}), \end{aligned} \tag{93}$$

following from the scaling property of $\boldsymbol{D}$ and the scale invariance of $\gamma(\rho_{AB})$ (Proposition 6). Since the divergence in (92) happens to be lower semicontinuous in its second argument, it suffices to prove the claim for $\boldsymbol{\Gamma}$.

Thus, let ω_{AB} be a state such that $\boldsymbol{\Gamma}(\omega_{AB}) = 0$. Due to (4), for all $\delta > 0$ we can find $\sigma_{AB} \geq 0$ such that $\mathrm{Tr}[\sigma_{AB}] \leq \gamma(\sigma_{AB}) \leq 1$ and

$$\delta > \boldsymbol{D}(\omega_{AB}\|\sigma_{AB}) \geq \boldsymbol{D}(1\|\mathrm{Tr}[\sigma_{AB}]) = -\log_2 \mathrm{Tr}[\sigma_{AB}], \tag{94}$$

where we also used the data processing inequality and the scaling property for $\boldsymbol{D}$. Combining these two inequalities yields

$$\gamma(\sigma_{AB}) \leq 1 < 2^\delta \,\mathrm{Tr}\,\sigma_{AB}. \tag{95}$$

By definition of γ, we now find a Hermitian operator V_{AB} and positive semi-definite K_A and L_B such that $T_B(V_{AB} \pm \sigma_{AB}) \geq 0$, $K_A \otimes L_B \pm V_{AB} \geq 0$, and $\mathrm{Tr}[K_A \otimes L_B] \leq 1$. This implies that

$$1 \geq \mathrm{Tr}[K_A \otimes L_B] \tag{96}$$
$$\geq \mathrm{Tr}[V_{AB}] \tag{97}$$
$$= \mathrm{Tr}[T_B(V_{AB})] \tag{98}$$
$$\geq \mathrm{Tr}[T_B(\sigma_{AB})] \tag{99}$$

$$\geq 2^{-\delta}, \tag{100}$$

and thus in turn that

$$\begin{aligned} &\|K_A \otimes L_B - \sigma_{AB}\|_1 \\ &\leq \|K_A \otimes L_B - V_{AB}\|_1 + \|V_{AB} - \sigma_{AB}\|_1 && (101) \\ &= \mathrm{Tr}[K_A \otimes L_B - V_{AB}] + \|V_{AB} - \sigma_{AB}\|_1 && (102) \\ &\leq 1 - 2^{-\delta} + \min\{d_A, d_B\} \|T_B(V_{AB} - \sigma_{AB})\|_1 && (103) \\ &= 1 - 2^{-\delta} + \min\{d_A, d_B\} \mathrm{Tr}[T_B(V_{AB} - \sigma_{AB})] && (104) \\ &\leq (1 + \min\{d_A, d_B\}) (1 - 2^{-\delta}). && (105) \end{aligned}$$

In the above calculation, we used the fact that $\|X_{AB}\|_1 \leq \min\{d_A, d_B\} \|T_B(X_{AB})\|_1$ for every Hermitian operator X_{AB} (see [42, proof of Proposition 7]), as can be seen immediately by writing a spectral decomposition for $T_B(X_{AB})$ and leveraging the fact that $\|T_B(\psi_{AB})\|_1 \leq \min\{d_A, d_B\}$ for all pure states ψ_{AB} [42, Proposition 8].

Since $\delta > 0$ is arbitrary, we have shown that we can construct a sequence of subnormalized states $\sigma_{AB}(n)$ with the property that (a) $\lim_{n\to\infty} \boldsymbol{D}(\omega_{AB}\|\sigma_{AB}(n)) = 0$ and (b) $\lim_{n\to\infty} \inf_{K_A, L_B} \|\sigma_{AB}(n) - K_A \otimes L_B\|_1 = 0$. Since the set of subnormalized states is compact, up to extracting a subsequence we can assume that $\lim_{n\to\infty} \sigma_{AB}(n) = \sigma_{AB}$; due to (b) and to the closedness of the set of tensor product operators, we have that $\sigma_{AB} = \rho_A \otimes \tau_B$ is itself a product subnormalized state. But then (a) together with the lower semicontinuity of $\boldsymbol{D}$ imply that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \boldsymbol{D}(\omega_{AB}\|\sigma_{AB}(n)) && (106) \\ &\geq \boldsymbol{D}(\omega_{AB}\|\sigma_{AB}) && (107) \\ &= \boldsymbol{D}(\omega_{AB}\|\rho_A \otimes \tau_B). && (108) \end{aligned}$$

By faithfulness of $\boldsymbol{D}$, this is only possible if $\omega_{AB} = \rho_A \otimes \tau_B$, which concludes the proof.

Appendix D
Proof of Proposition 5 (Dimension Bound)

The first inequality in (15) follows from recalling (5). So we prove the second inequality in (15). Let us set $K_A = \rho_A$, $L_B = I_B$, and $V_{AB} = \rho_A \otimes I_B$. For these choices, we have that

$$\mathrm{Tr}[K_A \otimes L_B] = d_B. \tag{109}$$

We now need to argue that these choices are feasible, i.e., that they satisfy

$$\begin{aligned} T_B(V_{AB} \pm \rho_{AB}) &\geq 0, && (110) \\ K_A \otimes L_B \pm V_{AB} &\geq 0. && (111) \end{aligned}$$

The second set of inequalities is trivially satisfied. For the first, consider that we need to show that

$$\begin{aligned} T_B(V_{AB} + \rho_{AB}) &= \rho_A \otimes I_B + T_B(\rho_{AB}) && (112) \\ &= \mathcal{P}_B^+(\rho_{AB}) && (113) \\ &\geq 0, && (114) \\ T_B(V_{AB} - \rho_{AB}) &= \rho_A \otimes I_B - T_B(\rho_{AB}) && (115) \\ &= \mathcal{P}_B^-(\rho_{AB}) && (116) \\ &\geq 0, && (117) \end{aligned}$$

where the linear maps $\mathcal{P}_B^+$ and $\mathcal{P}_B^-$ are defined as

$$\begin{aligned} \mathcal{P}_B^+(\cdot) &\coloneqq \mathrm{Tr}_B[\cdot] I_B + T_B(\cdot), && (118) \\ \mathcal{P}_B^-(\cdot) &\coloneqq \mathrm{Tr}_B[\cdot] I_B - T_B(\cdot). && (119) \end{aligned}$$

The Choi operators of these maps are given by

$$\begin{aligned} \mathcal{P}_B^+(\Gamma_{RB}) &= I_{RB} + F_{RB} = 2\Pi_{RB}^{\mathcal{S}}, && (120) \\ \mathcal{P}_B^-(\Gamma_{RB}) &= I_{RB} - F_{RB} = 2\Pi_{RB}^{\mathcal{A}}, && (121) \end{aligned}$$

where F_{RB} is the unitary swap operator and $\Pi_{RB}^{\mathcal{S}}$ and $\Pi_{RB}^{\mathcal{A}}$ are the projections onto the symmetric and antisymmetric subspaces, respectively. Since these operators are positive semi-definite and they are the Choi operators of $\mathcal{P}_B^+$ and $\mathcal{P}_B^-$, it follows that $\mathcal{P}_B^+$ and $\mathcal{P}_B^-$ are completely positive maps. Thus, the constraints in (110) hold, and we conclude the upper bound $C_\gamma(A;B)_\rho \leq \log_2 d_B$.

The proof of the other upper bound is similar, and we show it for completeness. Pick $K_A = I_A$, $L_B = \rho_B$, and $V_{AB} = I_A \otimes \rho_B$. For these choices, we have that

$$\mathrm{Tr}[K_A \otimes L_B] = d_A. \tag{122}$$

We need to argue that these choices are feasible. The constraint in (111) holds trivially. The first constraints in (110) become

$$\begin{aligned} T_B(V_{AB} + \rho_{AB}) &= I_A \otimes T_B(\rho_B) + T_B(\rho_{AB}) \geq 0, && (123) \\ T_B(V_{AB} - \rho_{AB}) &= I_A \otimes T_B(\rho_B) - T_B(\rho_{AB}) \geq 0. && (124) \end{aligned}$$

The inequalities above are equivalent to the following inequalities because they are related by taking a full transpose:

$$\begin{aligned} I_A \otimes \rho_B + T_A(\rho_{AB}) &\geq 0, && (125) \\ I_A \otimes \rho_B - T_A(\rho_{AB}) &\geq 0. && (126) \end{aligned}$$

Then we conclude that the constraints are satisfied because $\mathcal{P}_A^+(\rho_{AB}) = I_A \otimes \rho_B + T_A(\rho_{AB})$ and $\mathcal{P}_A^-(\rho_{AB}) = I_A \otimes \rho_B - T_A(\rho_{AB})$, and we already proved that $\mathcal{P}_A^+$ and $\mathcal{P}_A^-$ are completely positive.

Appendix E
Proof of Proposition 7 (Classical Communication Bound)

Let K_A, L_{BX}, and V_{ABX} be arbitrary operators for the optimization problem for $C_\gamma(A;BX)_\rho$, which satisfy

$$\begin{aligned} T_{BX}(V_{ABX} \pm \rho_{ABX}) &\geq 0, && (127) \\ K_A \otimes L_{BX} \pm V_{ABX} &\geq 0. && (128) \end{aligned}$$

Pick

$$\begin{aligned} K'_{AX} &\coloneqq K_A \otimes I_X, && (129) \\ L'_B &\coloneqq \mathrm{Tr}_X[L_{BX}], && (130) \\ V'_{ABX} &\coloneqq \overline{\Delta}_X(V_{ABX}), && (131) \end{aligned}$$

where $\overline{\Delta}_X(\cdot) \coloneqq \sum_x |x\rangle\!\langle x|(\cdot)|x\rangle\!\langle x|$ is the completely dephasing channel. Then we find that

$$\mathrm{Tr}[K'_{AX} \otimes L'_B] = \mathrm{Tr}[K_A \otimes I_X \otimes \mathrm{Tr}_X[L_{BX}]] \tag{132}$$

$$= \mathrm{Tr}[K_A \otimes L_{BX}]\,\mathrm{Tr}[I_X] \tag{133}$$
$$= \mathrm{Tr}[K_A \otimes L_{BX}] d_X. \tag{134}$$

We then need to show that

$$T_B(V'_{ABX} \pm \rho_{ABX}) \geq 0, \tag{135}$$
$$K'_{AX} \otimes L'_B \pm V'_{ABX} \geq 0. \tag{136}$$

Since $\overline{\Delta}_X$ is a completely positive map, we see that

$$K_A \otimes L_{BX} \pm V_{ABX} \geq 0 \tag{137}$$
$$\Longrightarrow \quad K_A \otimes \overline{\Delta}_X(L_{BX}) \pm \overline{\Delta}_X(V_{ABX}) \geq 0 \tag{138}$$
$$\Longleftrightarrow \quad K_A \otimes \overline{\Delta}_X(L_{BX}) \pm V'_{ABX} \geq 0. \tag{139}$$

We also have that $\overline{\Delta}_X(L_{BX}) \leq L_B \otimes I_X$, which follows because

$$\overline{\Delta}_X(L_{BX}) = \sum_x L_B^x \otimes |x\rangle\langle x|_X \tag{140}$$
$$\leq \sum_x L_B^x \otimes I_X \tag{141}$$
$$= L_B \otimes I_X, \tag{142}$$

where

$$L_B^x \coloneqq {}_X\langle x|L_{BX}|x\rangle_X. \tag{143}$$

The inequality $\overline{\Delta}_X(L_{BX}) \leq L_B \otimes I_X$ implies that

$$K_A \otimes L_B \otimes I_X \pm V'_{ABX} \geq 0 \tag{144}$$
$$\Longleftrightarrow \quad K'_{AX} \otimes L'_B \pm V'_{ABX} \geq 0. \tag{145}$$

Now consider that

$$T_{BX}(V_{ABX} \pm \rho_{ABX}) \geq 0 \tag{146}$$
$$\Longleftrightarrow \quad \overline{\Delta}_X(T_{BX}(V_{ABX} \pm \rho_{ABX})) \geq 0 \tag{147}$$
$$\Longleftrightarrow \quad T_B(\overline{\Delta}_X(V_{ABX}) \pm \overline{\Delta}_X(\rho_{ABX})) \geq 0 \tag{148}$$
$$\Longleftrightarrow \quad T_B(V'_{ABX} \pm \rho_{ABX}) \geq 0. \tag{149}$$

In the above, we applied the equality $\overline{\Delta}_X \circ T_X = \overline{\Delta}_X$. Since K'_{AX}, L'_B, and V'_{ABX} are specific choices that satisfy the constraints for $C_\gamma(AX;B)_\rho$, we conclude the desired inequality after minimizing over all such operators and noticing that the objective function for $C_\gamma(AX;B)_\rho$ is $\mathrm{Tr}[K'_{AX} \otimes L'_B]$, which satisfies (132)–(134). This establishes (18).

To see (19), consider that

$$\log_2 d_X + \mathbf{\Gamma}(A;BX)_\rho$$
$$= \log_2 d_X + \inf_{\substack{\sigma_{ABX}\geq 0,\\ \gamma(A;BX)_\sigma \leq 1}} \boldsymbol{D}(\rho_{ABX}\|\sigma_{ABX}) \tag{150}$$
$$= \inf_{\substack{\sigma_{ABX}\geq 0,\\ \gamma(A;BX)_\sigma \leq 1}} \boldsymbol{D}(\rho_{ABX}\|\sigma_{ABX}/d_X) \tag{151}$$
$$= \inf_{\substack{\sigma'_{ABX}\geq 0,\\ \gamma(A;BX)_{d_X\sigma'} \leq 1}} \boldsymbol{D}(\rho_{ABX}\|\sigma'_{ABX}) \tag{152}$$
$$= \inf_{\substack{\sigma'_{ABX}\geq 0,\\ d_X\,\gamma(A;BX)_{\sigma'} \leq 1}} \boldsymbol{D}(\rho_{ABX}\|\sigma'_{ABX}) \tag{153}$$
$$\geq \inf_{\substack{\sigma'_{ABX}\geq 0,\\ \gamma(AX;B)_{\sigma'} \leq 1}} \boldsymbol{D}(\rho_{ABX}\|\sigma'_{ABX}) \tag{154}$$
$$= \mathbf{\Gamma}(AX;B)_\rho. \tag{155}$$

The fourth equality follows from Proposition 6, and the subsequent inequality from (18).

Appendix F
Proof of Proposition 8 (Subadditivity)

Let $K^1_{A_1}$, $L^1_{B_1}$, and $V^1_{A_1B_1}$ be feasible choices for $C_\gamma(A_1;B_1)_\sigma$ (satisfying the constraints in (2)), and let $K^2_{A_2}$, $L^2_{B_2}$, and $V^2_{A_2B_2}$ be feasible choices for $C_\gamma(A_2;B_2)_\tau$. Then it follows that $K^1_{A_1}\otimes K^2_{A_2}$, $L^1_{B_1}\otimes L^2_{B_2}$, and $V^1_{A_1B_1}\otimes V^2_{A_2B_2}$ are feasible choices $C_\gamma(A_1A_2;B_1B_2)_\rho$. This latter statement is a consequence of the general fact that if A, B, C, and D are Hermitian operators satisfying $A \pm B \geq 0$ and $C \pm D \geq 0$, then $A \otimes C \pm B \otimes D \geq 0$. To see this, consider that the original four operator inequalities imply the four operator inequalities $(A \pm B)\otimes(C \pm D) \geq 0$, and then summing these four different operator inequalities in various ways leads to $A \otimes C \pm B \otimes D \geq 0$.

To see the inequality in (21), let $\omega_{A_1B_1} \geq 0$ satisfy $\gamma(A_1;B_1)_\omega \leq 1$ and let $\theta_{A_2B_2} \geq 0$ satisfy $\gamma(A_2;B_2)_\theta \leq 1$. Then, by applying (20), we have that $\gamma(A_1A_2;B_1B_2)_{\omega\otimes\theta} \leq \gamma(A_1;B_1)_\omega \cdot \gamma(A_2;B_2)_\theta \leq 1$, so that

$$\mathbf{\Gamma}(A_1A_2;B_1B_2)_\rho \tag{156}$$
$$= \inf_{\substack{\zeta_{A_1A_2B_1B_2}\geq 0,\\ \gamma(\zeta_{A_1A_2B_1B_2})\leq 1}} \boldsymbol{D}(\rho_{A_1A_2B_1B_2}\|\zeta_{A_1A_2B_1B_2}) \tag{157}$$
$$= \inf_{\substack{\zeta_{A_1A_2B_1B_2}\geq 0,\\ \gamma(\zeta_{A_1A_2B_1B_2})\leq 1}} \boldsymbol{D}(\sigma_{A_1B_1}\otimes\tau_{A_2B_2}\|\zeta_{A_1A_2B_1B_2}) \tag{158}$$
$$\leq \boldsymbol{D}(\sigma_{A_1B_1}\otimes\tau_{A_2B_2}\|\omega_{A_1B_1}\otimes\theta_{A_2B_2}) \tag{159}$$
$$\leq \boldsymbol{D}(\sigma_{A_1B_1}\|\omega_{A_1B_1}) + \boldsymbol{D}(\tau_{A_2B_2}\|\theta_{A_2B_2}). \tag{160}$$

The last inequality follows from the hypothesis that $\boldsymbol{D}$ is subadditive. Since the inequality holds for all $\omega_{A_1B_1}$ and $\theta_{A_2B_2}$ satisfying the constraints for $\mathbf{\Gamma}(A_1;B_1)_\sigma$ and $\mathbf{\Gamma}(A_2;B_2)_\tau$, respectively, we conclude the desired inequality in (21).

Appendix G
Proof of Lemma 9

Let σ_{AB} satisfy $\sigma_{AB} \geq 0$ and let K_A, L_B, and V_{AB} be Hermitian operators satisfying the constraints for $\gamma(\sigma_{AB})$. Consider that

$$F\Big(\overline{\Phi}^d_{AB}, \sigma_{AB}\Big) \leq F\Big(\overline{\Phi}^d_{AB}, \left(\overline{\Delta}_A \otimes \overline{\Delta}_B\right)(\sigma_{AB})\Big), \tag{161}$$

where $\overline{\Delta}$ is a local dephasing channel, defined as

$$\overline{\Delta}(\cdot) \coloneqq \sum_{m=0}^{d-1} |m\rangle\langle m|(\cdot)|m\rangle\langle m|. \tag{162}$$

We then find that

$$\left(\overline{\Delta}_A \otimes \overline{\Delta}_B\right)(\sigma_{AB})$$
$$= \sum_{\ell,m=0}^{d-1} |\ell m\rangle\langle \ell m|_{AB}\sigma_{AB}|\ell m\rangle\langle \ell m|_{AB} \tag{163}$$

$$= \sum_{\ell,m=0}^{d-1} \operatorname{Tr}\left[|\ell\rangle\langle\ell|_A \otimes |m\rangle\langle m|_B \, \sigma_{AB}\right] |\ell\rangle\langle\ell|_A \otimes |m\rangle\langle m|_B. \tag{164}$$

Then the fidelity $F\Big(\overline{\Phi}^d_{AB}, \left(\overline{\Delta}_A \otimes \overline{\Delta}_B\right)(\sigma_{AB})\Big)$ becomes the classical fidelity, i.e.,

$$F\Big(\overline{\Phi}^d_{AB}, \left(\overline{\Delta}_A \otimes \overline{\Delta}_B\right)(\sigma_{AB})\Big) = \left[\sum_{m=0}^{d-1} \sqrt{\frac{1}{d} \operatorname{Tr}[|m\rangle\langle m| \otimes |m\rangle\langle m|\sigma_{AB}]}\right]^2. \tag{165}$$

Then we use the fact that

$$\begin{aligned}
&\operatorname{Tr}[|m\rangle\langle m|_A \otimes |m\rangle\langle m|_B \sigma_{AB}] \\
&= \operatorname{Tr}[|m\rangle\langle m|_A \otimes |m\rangle\langle m|_B T_B(\sigma_{AB})] \qquad (166)\\
&\leq \operatorname{Tr}[|m\rangle\langle m|_A \otimes |m\rangle\langle m|_B T_B(V_{AB})] \qquad (167)\\
&= \operatorname{Tr}[|m\rangle\langle m|_A \otimes |m\rangle\langle m|_B V_{AB}] \qquad (168)\\
&\leq \operatorname{Tr}[|m\rangle\langle m|_A \otimes |m\rangle\langle m|_B K_A \otimes L_B] \qquad (169)\\
&= \langle m|K_A|m\rangle\langle m|L_B|m\rangle. \qquad (170)
\end{aligned}$$

so that the classical fidelity is less than

$$\begin{aligned}
&\left[\sum_m \sqrt{\frac{1}{d}\langle m|K_A|m\rangle\langle m|L_B|m\rangle}\right]^2 \\
&\leq \frac{1}{d} \sum_m \langle m|K_A|m\rangle \sum_m \langle m|L_B|m\rangle \qquad (171)\\
&= \frac{1}{d} \operatorname{Tr}[K_A] \operatorname{Tr}[L_B] \qquad (172)\\
&= \frac{1}{d} \operatorname{Tr}[K_A \otimes L_B]. \qquad (173)
\end{aligned}$$

We applied the Cauchy–Schwarz inequality. Now leveraging the assumption that $\gamma(\sigma_{AB}) \leq 1$ and taking an infimum over all K_A, L_B, and V_{AB} satisfying the constraints for $\gamma(\sigma_{AB})$, we conclude that

$$F\Big(\overline{\Phi}^d_{AB}, \sigma_{AB}\Big) \leq \frac{1}{d}. \tag{174}$$

We conclude the statement of the lemma because we have proven that this inequality holds for all σ_{AB} satisfying $\sigma_{AB} \geq 0$ and $\gamma(\sigma_{AB}) \leq 1$.

Appendix H
Proof of Proposition 10

Recall the following inequality from [24, Lemma 1]:

$$\widetilde{D}_\alpha(\rho_0\|\sigma) - \widetilde{D}_\beta(\rho_1\|\sigma) \geq \frac{\alpha}{\alpha-1} \log_2 F(\rho_0, \rho_1), \tag{175}$$

where ρ_0 and ρ_1 are states, σ is a positive semi-definite operator, $\alpha > 1$, and β satisfies $\frac{1}{2\alpha} + \frac{1}{2\beta} = 1$, so that $\beta \in (1/2, 1)$. By the same argument given in [43, Lemma 14], this implies that

$$\begin{aligned}
\widetilde{\Gamma}_\alpha(\omega_{AB}) - \widetilde{\Gamma}_\beta\Big(\overline{\Phi}^d_{AB}\Big) &\geq \frac{\alpha}{\alpha-1} \log_2 F\Big(\overline{\Phi}^d_{AB}, \omega_{AB}\Big) \qquad (176)\\
&\geq \frac{\alpha}{\alpha-1} \log_2(1-\varepsilon). \qquad (177)
\end{aligned}$$

We can rewrite this as

$$\begin{aligned}
\widetilde{\Gamma}_\alpha(\omega_{AB}) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{1-\varepsilon}\right) &\geq \widetilde{\Gamma}_\beta\Big(\overline{\Phi}^d_{AB}\Big) \qquad (178)\\
&\geq \widetilde{\Gamma}_{1/2}\Big(\overline{\Phi}^d_{AB}\Big) \qquad (179)\\
&\geq \log_2 d, \qquad (180)
\end{aligned}$$

where the second inequality follows from β-monotonicity of the sandwiched Renyi relative entropy (see [26, Prop. 4.29]), and the third from Lemma 9.

Appendix I
Relation to β- and $\mathbf{\Upsilon}$-Measures of [14]–[16]

In this appendix, we discuss the relation of the γ, C_γ, and $\mathbf{\Gamma}$ measures introduced in the main text, with the β, C_β, and $\mathbf{\Upsilon}$ measures from [14]–[16].

Indeed, by fixing the operator K_A in (2) to be the marginal operator σ_A (where σ_{AB} is a bipartite positive semi-definite operator), we arrive at the following quantity:

$$\beta(\sigma_{AB}) := \inf_{L_B, V_{AB} \in \text{Herm}} \left\{ \begin{array}{c} \operatorname{Tr}[\sigma_A \otimes L_B] : \\ T_B(V_{AB} \pm \sigma_{AB}) \geq 0, \\ \sigma_A \otimes L_B \pm V_{AB} \geq 0 \end{array} \right\}. \tag{181}$$

This is precisely the static version of the β measure from [14, Eq. (45)]. We also define

$$C_\beta(\sigma_{AB}) := C_\beta(A;B)_\sigma := \log_2 \beta(\sigma_{AB}). \tag{182}$$

For a bipartite state ρ_{AB}, we then define

$$\mathbf{\Upsilon}(\rho_{AB}) := \inf_{\substack{\sigma_{AB} \geq 0: \\ \beta(\sigma_{AB}) \leq 1}} \boldsymbol{D}(\rho_{AB}\|\sigma_{AB}), \tag{183}$$

which is the static version of the $\mathbf{\Upsilon}$ measure from [15, Eq. (49)]. The following inequalities clearly hold, by applying definitions:

$$\gamma(\sigma_{AB}) \leq \beta(\sigma_{AB}), \qquad \mathbf{\Gamma}(\rho_{AB}) \leq \mathbf{\Upsilon}(\rho_{AB}). \tag{184}$$

Thus, β, C_β, and $\mathbf{\Upsilon}$ can be understood as static counterparts of the corresponding measures of classical correlations of a channel, from [14]–[16]. We wrote them down explicitly above to make the connection with prior literature. However, in the main text, we focused exclusively on γ, C_γ, and $\mathbf{\Gamma}$ because these quantities lead to tighter upper bounds on the distillable randomness. The semi-definite restrictions of these quantities in Section VI are closely related as well with β, C_β, and $\mathbf{\Upsilon}$, and as discussed there, they lead to computationally efficient upper bounds on distillable randomness.

We finally note that the static β- and $\mathbf{\Upsilon}$-measures for a bipartite state and the dynamic ones for point-to-point channels are further generalized by the corresponding measures for a bipartite channel, as proposed in [17]. As such, the γ, C_γ, and $\mathbf{\Gamma}$ measures can be generalized to bipartite channels, as considered in [44], and will be the subject of a future publication.