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Scaling Invariance of the $k[S]$-Hierarchy and Its Strict Version

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Abstract—Let $LT_{\mathbb{N}}(R)$ denote the algebra of $\mathbb{N} \times \mathbb{N}$-matrices with coefficients from the commutative k-algebra R, $k = \mathbb{R}$ or $\mathbb{C}$, that possess only a finite number of nonzero diagonals above the central diagonal. In a previous paper we discussed integrable deformations inside $LT_{\mathbb{N}}(R)$ of various commutative subalgebras of $LT_{\mathbb{N}}(k)$ that contain S^n, where S is the $\mathbb{N} \times \mathbb{N}$-matrix corresponding to the shift operator. Here we focus on two deformations of $k[S]$, called the $k[S]$-hierarchy and its strict version and we discuss the scaling invariance that they possess. To do so, it is necessary to discuss both deformations from a wider perspective and consider them in a presetting instead of the usual setting. In this more general set-up we will present two k-subalgebras of R that are stable under the basic derivations of R and such that these derivations commute on these k-subalgebras. This we apply at the introduction of the minimal realizations of both deformations, we show how these realizations relate to solutions in different settings and use them to show that both hierarchies possess invariant scaling transformations.

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1. INTRODUCTION

We introduce here two deformations in $LT_{\mathbb{N}}(R)$ of the algebra $k[S]$. The evolution equations that these deformations have to satisfy determine two integrable hierarchies: the $k[S]$-hierarchy and its strict version. Our final aim is to discuss invariant scaling transformations for each hierarchy. Thereto we need the solutions of both the $k[S]$-hierarchy and its strict version that have a minimal dependence between the different coefficients, the so-called minimal realizations. The introduction of these basic realizations requires that we treat both integrable hierarchies from a wider perspective and drop the commutativity assumption on the basic derivations $\{\partial_i\}$ and work from a presetting instead of a setting. Nevertheless we will present a number of k-subalgebras of R that are stable under the $\{\partial_i\}$ and such that the $\{\partial_i\}$ commute on these k-subalgebras. This is used at the introduction of the minimal realizations to show that they are defined in a setting. Further we show how the minimal realizations relate to solutions in different settings. Finally, we conclude by presenting the invariant scaling transformations for both hierarchies.

The contents of the various sections is as follows: we start in Section 2 with a description of the properties of the central algebra in which our deformations take place. Section 3 describes the $k[S]$-hierarchy and its strict version in the wider perspective. It discusses the properties that survive and shows that the coefficients of each solution are contained in a k-subalgebra that is stable under the basic derivations and on which these derivations commute. In Section 3 we treat the minimal realizations of both hierarchies and their relation with solutions in other settings. In the last section we show that also these systems of differential-difference equations possess certain invariant scaling transformations like their differential counterparts described in [1, 2].

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2. THE ALGEBRA $LT_{\mathbb{N}}(R)$

We start by recalling a number of basic notations in the algebra $LT_{\mathbb{N}}(R)$, where the central deformations of this paper take place. Each $A \in M_{\mathbb{N}}(R)$ will be denoted as $A = (a_{ij})$ or as $A = (a_{(i,j)})$ if confusion in the labeling might occur. On the space $M_{\mathbb{N}}(R)$ we use the ordering of columns and rows as in the finite dimensional case. Any $A \in M_{\mathbb{N}}(R)$ corresponds to an R-linear map. Consider thereto the space of all $1 \times \mathbb{N}$-matrices with coefficients from R

$$V = R^{\mathbb{N}} = \{\vec{x} = (x_n) = \begin{pmatrix} x_0 & \dots & x_n & x_{n+1} & \dots \end{pmatrix}^T | x_n \in R\}$$

and its subspace

$$V_{\text{fin}} = \{\vec{x} = (x_n) \in V | x_n \neq 0 \text{ for only a finite number of } n\}.$$

Define for each $i \in \mathbb{N}$ the vector $\vec{e}(i)$ in V_{fin} by requiring its ith coordinate to be equal to one and its remaining coordinates to be zero. Then, V_{fin} is a free R-module with basis the $\{\vec{e}(i) | i \in \mathbb{N}\}$. On V_{fin} we can define an R-linear action M_A of a matrix $A \in M_{\mathbb{N}}(R)$ by

$$M_A(\vec{x}) := \vec{x}A.$$

Hence, the matrix A determines the R-linear map $M_A \in \text{Hom}_R(V_{\text{fin}}, V)$.

Inside $M_{\mathbb{N}}(R)$ there are some classes of basic matrices with their own notation: first of all there are the basic matrices $E_{(i,j)}$, i and $j \in \mathbb{N}$, whose matrix entries, in Kronecker notation, are given by

$$(E_{(i,j)})_{mn} = \delta_{im}\delta_{jn}.$$

It is convenient to use the notation $A = \sum_{i,j} a_{(i,j)} E_{(i,j)}$ for an $A = (a_{(i,j)}) \in LT_{\mathbb{N}}(R)$. The second class of matrices for which we introduce a special notation are the diagonal matrices. Let $\{d(s) | s \in \mathbb{N}\}$ be a set of elements in R. Then, the diagonal matrix $\text{diag}(d(s))$ in $M_{\mathbb{N}}(R)$ is given by

$$\text{diag}(d(s)) := \sum_{s \in \mathbb{N}} d(s) E_{(s,s)} = \begin{pmatrix} d(0) & 0 & 0 & \dots \\ 0 & d(1) & 0 & \ddots \\ 0 & 0 & d(2) & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The algebra of all diagonal matrices in $M_{\mathbb{N}}(R)$ is denoted by

$$\mathcal{D}_{\mathbb{N}}(R) = \{d = \text{diag}(d(s)) | \, d(s) \in R, \text{ for all } s \in \mathbb{N}\}$$

and its group of units by $\mathcal{D}_{\mathbb{N}}(R)^*$, i.e., all $\text{diag}(d(s))$ with $d(s) \in R^*$ for all $s \in \mathbb{N}$. One has a diagonal embedding i_1 from R into $\mathcal{D}_{\mathbb{N}}(R)$ by taking all diagonal coefficients of $i_1(r)$ equal to r, i.e.,

$$i_1(r) = \begin{pmatrix} r & 0 & 0 & \dots \\ 0 & r & 0 & \ddots \\ 0 & 0 & r & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

A central role in this paper is played by the shift matrix S, its transpose S^T and their powers, where S is the matrix corresponding to the operator $M_S : V \to V$ defined by

$$M_S(\begin{pmatrix} x_0 & x_1 & x_2 & \dots \end{pmatrix}) = \begin{pmatrix} 0 & x_0 & x_1 & x_2 & \dots \end{pmatrix}.$$

Note that by induction w.r.t. $k, k \geqslant 1$, one shows the relations

$$S^k (S^T)^k = \text{Id} \quad \text{and} \quad (S^T)^k S^k = \sum_{i \geqslant k} E_{(i,i)}. \tag{1}$$

Besides the expression in the basic matrices it is also convenient to have at one's disposal the decomposition of a matrix $A = (a_{ij}) \in M_{\mathbb{N}}(R)$ in its diagonals. If $m \geqslant 0$, then the mth diagonal of A is by definition the matrix

$$d_m(A)S^m = \operatorname{diag}(a_{(s,s+m)})S^m = \sum_{s \geqslant 0} a_{(s,s+m)}E_{(s,s+m)}$$

and those diagonals are called *positive*. Similarly, for $m \leqslant 0$, the m-th diagonal of A is defined as the matrix

$$(S^T)^{-m}d_m(A) = (S^T)^{-m}\operatorname{diag}(a_{(s-m,s)}) = \sum_{s \geqslant 0} a_{(s-m,s)}E_{(s-m,s)}$$

and they are called *negative*. So each matrix $A \in M_{\mathbb{N}}(R)$ decomposes uniquely as

$$A = \sum_{m \geqslant 0} d_m(A)S^m + \sum_{m<0} (S^T)^{-m}d_m(A). \tag{2}$$

One shows with induction w.r.t. $m \geqslant 1$ that the powers of S and S^T interact with the diagonal matrices as follows:

$$S^m\operatorname{diag}(a(s)) = \operatorname{diag}(a(s+m))S^m \tag{3}$$

and

$$\operatorname{diag}(b(s))(S^T)^m = (S^T)^m\operatorname{diag}(b(s+m)). \tag{4}$$

Let $LT_{\mathbb{N}}(R)$ be the collection of all matrices that have only a finite number of nonzero positive diagonals. The multiplication rules (1), (3), and (4) yield the following result

Lemma 1. *If $A \in LT_{\mathbb{N}}(R)$ is equal to its ℓ-th diagonal and $B \in LT_{\mathbb{N}}(R)$ is equal to its n-th diagonal, then AB is equal to its $\ell + n$-th diagonal. In particular, $LT_{\mathbb{N}}(R)$ is an algebra w.r.t. matrix multiplication.*

We use the decomposition (2) to assign a degree to elements of $LT_{\mathbb{N}}(R)$. For a nonzero $A \in LT_{\mathbb{N}}(R)$ the degree is equal to m if its highest nonzero diagonal is the m-th and the degree of the zero element is by definition $-\infty$.

Lemma 2. *The centralizer in $LT_{\mathbb{N}}(R)$ of the matrix S consists of the* $\left\{\sum_{j \geqslant 0} i_1(r_j)S^j | r_j \in R\right\}$.

Proof. Let $A = (a_{(i,j)})$ belong to $LT_{\mathbb{N}}(R)$. Then, we have on one hand

$$AS = \begin{pmatrix} 0 & a_{(0,0)} & \cdots & a_{(0,n)} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & a_{(n,0)} & \cdots & a_{(n,n)} & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix} \tag{5}$$

and on the other

$$SA = \begin{pmatrix} a_{(1,0)} & a_{(1,1)} & \cdots & a_{(1,n+1)} & \cdots \\ a_{(2,0)} & a_{(2,1)} & \cdots & a_{(2,n+1)} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ a_{(n+1,0)} & a_{(n+1,1)} & \cdots & a_{(n+1,n+1)} & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix}. \tag{6}$$

If the expressions (5) and (6) are equal then induction w.r.t. i shows first of all that A is uppertriangular, i.e., for all $j < i$, $a_{(i,j)} = 0$. For the remaining coefficients the identity $AS = SA$ yields then that $a_{(i,j)} = a_{(i+1,j+1)}$ for all $i \leqslant j$. This proves the claim. □

A consequence of Lemma 1 is the following property of $k[S] = \{\sum_{i=0}^{N} k_i S^i | k_i \in k\}$:

Corollary 1. *The algebra $k[S]$ is a maximal commutative subalgebra of $LT_{\mathbb{N}}(k)$.*

The algebra $LT_{\mathbb{N}}(R)$ is a Lie algebra with the commutator as a bracket. Each hierarchy corresponds to a different decomposition of $LT_{\mathbb{N}}(R)$ in the direct sum of two Lie subalgebras. For the $k[S]$-hierarchy one splits A as $A = \pi_{\geqslant 0}(A) + \pi_{<0}(A)$, where

$$\pi_{\geqslant 0}(A) = \sum_{m \geqslant 0} d_m(A) S^m \quad \text{and} \quad \pi_{<0}(A) = \sum_{m<0} (S^T)^{-m} d_m(A)$$

and at the strict version one splits A as $A = \pi_{>0}(A) + \pi_{\leqslant 0}(A)$, where

$$A = \pi_{>0}(A) = \sum_{m>0} d_m(A) S^m \quad \text{and} \quad \pi_{\leqslant 0}(A) = \sum_{m \leqslant 0} (S^T)^{-m} d_m(A).$$

The first way of splitting elements of $LT_{\mathbb{N}}(R)$ yields the Lie algebra decomposition

$$LT_{\mathbb{N}}(R) = \pi_{\geqslant 0}(LT_{\mathbb{N}}(R)) \oplus \pi_{<0}(LT_{\mathbb{N}}(R)),$$

where the Lie subalgebra $\pi_{\geqslant 0}(LT_{\mathbb{N}}(R))$ equals $\{A \in LT_{\mathbb{N}}(R) | \pi_{\geqslant 0}(A) = A\}$ and similarly $\pi_{<0}(LT_{\mathbb{N}}(R)) = \{A \in LT_{\mathbb{N}}(R) | \pi_{<0}(A) = A\}$. The second decomposition leads to

$$LT_{\mathbb{N}}(R) = \pi_{>0}(LT_{\mathbb{N}}(R)) \oplus \pi_{\leqslant 0}(LT_{\mathbb{N}}(R)),$$

with the Lie subalgebra $\pi_{>0}(LT_{\mathbb{N}}(R))$ equal to $\{A \in LT_{\mathbb{N}}(R) | \pi_{>0}(A) = A\}$ and its complement $\pi_{\leqslant 0}(LT_{\mathbb{N}}(R))$ is equal to $\{A \in LT_{\mathbb{N}}(R) | \pi_{\leqslant 0}(A) = A\}$.

To both Lie subalgebras $\pi_{<0}(LT_{\mathbb{N}}(R))$ and $\pi_{\leqslant 0}(LT_{\mathbb{N}}(R))$ we associate a group. Consider the set

$$\mathcal{U}_-(R) := \{\mathrm{Id} + Y | Y \in \pi_{<0}(LT_{\mathbb{N}}(R))\} =: \mathcal{U}_-,$$

where the notation $\mathcal{U}_-(R)$ is used if the source of the coefficients needs to be stressed. Recursively one proves that each element of $\mathcal{U}_-$ has an inverse w.r.t. multiplication. Hence, $\mathcal{U}_-$ is a multiplicative group. On $\pi_{<0}(LT_{\mathbb{N}}(R))$ the exponential map is well-defined, it yields elements in $\mathcal{U}_-$ and this map is a bijection. Therefore, we see $\mathcal{U}_-$ as the group corresponding to $\pi_{<0}(LT_{\mathbb{N}}(R))$. If the exponential map is well-defined on $\pi_{\leqslant 0}(LT_{\mathbb{N}}(R))$, then the resulting elements belong to the group

$$\mathcal{P}_- := \left\{ \sum_{m \leqslant 0} (S^T)^{-m} d_m(A) | d_0(A) = \mathrm{diag}(d(s)),\ \text{all } d(s) \in R^* \right\} =: \mathcal{P}_-(R)$$

and we see $\mathcal{P}_-$ as the group associated with $\pi_{\leqslant 0}(LT_{\mathbb{N}}(R))$.

Remark 1. Besides the decomposition (2) of a matrix $A \in M_{\mathbb{N}}(R)$ we still meet a different one. For conjugating S^m resp. $(S^T)^m, m \geqslant 0$, with an $t = \mathrm{diag}(t(s)) \in \mathcal{D}_{\mathbb{N}}(k)^*$ yields the mth power of a twisted shift matrix $S(t)$ resp. of a twisted transpose $S^T(t)$

$$(S(t))^m := t S^m t^{-1} = \left(\mathrm{diag} \left(\frac{t(s)}{t(s+1)} \right) S \right)^m = \mathrm{diag} \left(\frac{t(s)}{t(s+m)} \right) S^m, \tag{7}$$

$$(S^T(t))^m := t (S^T)^m t^{-1} = (t^{-1} S^m t)^T = (S^T)^m \mathrm{diag} \left(\frac{t(s+m)}{t(s)} \right). \tag{8}$$

For example, if $a \in k^*$ and $t(s) = a^{-s}$ for all $s \in \mathbb{N}$, then $S(t) = aS$ and $S^T(t) = a^{-1} S$. From the formulas (7) and (8) you see that one can write any positive diagonal of A as a product of a diagonal matrix and a power of $S(t)$ and each negative diagonal of A as the product of a power of $S^T(t)$ and a diagonal matrix. Thus, any $A \in M_{\mathbb{N}}(R)$ decomposes also as

$$A = \sum_{m \geqslant 0} d_m(t, A) S(t)^m + \sum_{m<0} S^T(t)^{-m} d_m(t, A),$$

with all $d_m(t, A) \in \mathcal{D}_{\mathbb{N}}(R)$.

3. THE RELEVANT HIERARCHIES

In this section we discuss the two deformations of $k[S]$ that we consider and the evolution equations we want the deformations of S to satisfy. At the first deformation each $\sum_{i\geqslant 0} k_i S^i$ in $k[S]$ is deformed into $\sum_{i\geqslant 0} k_i \mathcal{L}^i$, where $\mathcal{L} \in LT_{\mathbb{N}}(R)$ is an element of the form

$$\mathcal{L} = S + \sum_{i\leqslant 0} (S^T)^i \ell_i, \quad \ell_i \in \mathcal{D}_{\mathbb{N}}(R). \tag{9}$$

One directly checks that any element USU^{-1}, with $U \in \mathcal{U}_-$, has this form and we call USU^{-1} a $\mathcal{U}_-$-*deformation* of S. We call U also the *dressing matrix* of USU^{-1}. At the second deformation we transform each matrix $\sum_{i\geqslant 0} k_i S^i \in k[S]$ into $\sum_{i\geqslant 0} k_i \mathcal{M}^i$, where $\mathcal{M} \in LT_{\mathbb{N}}(R)$ is an element of the form

$$\mathcal{M} = m_1 S + \sum_{i\leqslant 0} (S^T)^{-i} m_i, \quad m_i \in \mathcal{D}_{\mathbb{N}}(R), \quad m_1 \in \mathcal{D}_{\mathbb{N}}(R)^*. \tag{10}$$

Also in this case one easily verifies that any matrix PSP^{-1}, with $P \in \mathcal{P}_-$, possesses the form (10) and therefore it is called a $\mathcal{P}_-$-*deformation* of S. Likewise we call P also the *dressing matrix* of the deformation PSP^{-1}. Moreover, we have

Lemma 3. *Reversely there holds for the deformations (9) and (10)*

(a) *Any $\mathcal{L}$ of the form (9) can uniquely be written in the form $\mathcal{L} = USU^{-1}$ with $U \in \mathcal{U}_-$, i.e. $\mathcal{L}$ is a $\mathcal{U}_-$-deformation of S.*

(b) *Any $\mathcal{M}$ of the form (10) can be written in the form $\mathcal{M} = dUSU^{-1}d^{-1}$, where $U \in \mathcal{U}_-$ is unique and $d \in \mathcal{D}_{\mathbb{N}}(R)^*$ is determined up to a factor from $i_1(R^*)$. In particular, $\mathcal{M}$ is a $\mathcal{P}_-$-deformation of S.*

Proof. We start with a proof of statement (a). So, given an $\mathcal{L}$ of the form (9), we have to find an $U = \mathrm{Id} + \sum_{j\geqslant 1} (S^T)^j u_j, u_j \in \mathcal{D}_{\mathbb{N}}(R)$, such that $\mathcal{L}U = US$. For any $U \in \mathcal{U}_-$ the matrix $\mathcal{L}U - US$ has no strict positive diagonals. So, it suffices to show that for all $n \geqslant 0$ the equations

$$d_{-k}(\mathcal{L}U) = d_{-k}(US), \quad 0 \leqslant k \leqslant n,$$

determine the $\{u_1, \cdots, u_{n+1}\}$ uniquely. Hereby we use the relations in (1), (3), and (4). By applying (1) and (3) one gets for US the expression

$$US = S + \sum_{j\geqslant 1} (S^T)^j u_j S = S + \sum_{j\geqslant 1} (S^T)^j S \mathrm{diag}(1, u_j(0), u_j(1), \cdots)$$
$$= S + \sum_{j\geqslant 1} (S^T)^{j-1} \mathrm{diag}(0, u_j(0), u_j(1), \cdots).$$

From this expression we conclude for each $n \geqslant 0$ that

$$d_{-n}(US) = \mathrm{diag}(0, u_{n+1}(0), u_{n+1}(1), \cdots). \tag{11}$$

Now we apply the first relation in (1) and repeatedly relation (4) to the product $\mathcal{L}U$ and get

$$\mathcal{L}U = S + \sum_{i\geqslant 0} (S^T)^i \ell_i + S \sum_{j\geqslant 1} (S^T)^j u_j + \sum_{i\geqslant 0; j\geqslant 1} (S^T)^i \ell_i (S^T)^j u_j$$
$$= S + \sum_{i\geqslant 0} (S^T)^i \ell_i + \sum_{j\geqslant 1} (S^T)^{j-1} u_j + \sum_{i\geqslant 0; j\geqslant 1} (S^T)^{i+j} \mathrm{diag}(\ell_i(s+j)) u_j.$$

Thus, we get $d_0(\mathcal{L}U) = \ell_0 + u_1$ and for the remaining diagonal components of $\mathcal{L}U$

$$d_{-n}(\mathcal{L}U) = \ell_n + u_{n+1} + \sum_{k=1}^{n} \mathrm{diag}(\ell_{k-1}(s+n+1-k)) u_{n+1-k}, n \geqslant 1. \tag{12}$$

The equality $d_0(\mathcal{L}U) = d_0(US)$ is in terms of the diagonal components of $\mathcal{L}$ and U the identity

$$\ell_0 + u_1 = \operatorname{diag}(0, u_1(0), u_1(1), \cdots).$$

The $(0,0)$-entry of this matrix identity yields $u_1(0) = -\ell_0(0)$, the $(1,1)$-entry gives $u_1(1) = u_1(0) - \ell_0(1)$ and continuing in this fashion, one gets that any $u_1(s)$ is a linear combination of the matrix coefficients of ℓ_0. By induction w.r.t. n we may assume that the equations

$$d_{-k}(\mathcal{L}U) = d_{-k}(US), \quad 0 \leqslant k \leqslant n-1,$$

determine the $\{u_1, \cdots, u_n\}$ and each $u_k(s), s \in \mathbb{N}$ and $0 \leqslant k \leqslant n-1$, is a polynomial expression in the matrix coefficients of the $\{\ell_k, 0 \leqslant k \leqslant n-1\}$. By combining the expressions (11) and (12), we get for $n \geqslant 1$ from $d_{-n}(US) = d_{-n}(\mathcal{L}U)$ the relation

$$u_{n+1} = \operatorname{diag}(0, u_{n+1}(0), u_{n+1}(1), \cdots) - \ell_n - \sum_{k=1}^{n} \operatorname{diag}(\ell_{k-1}(s+n+1-k)) u_{n+1-k}.$$

Again we look successively at the diagonal entries of this matrix identity, starting with the $(0,0)$-entry and recalling that all $u_{n+1-k}(s)$ are known. This yields us

$$u_{n+1}(0) = -\ell_n(0) - \sum_{k=1}^{n} \ell_{k-1}(n+1-k) u_{n+1-k}(0).$$

Next we consider the $(1,1)$-entry and that gives us

$$u_{n+1}(1) = u_{n+1}(0) - \ell_n(1) - \sum_{k=1}^{n} \ell_{k-1}(n+2-k) u_{n+1-k}(1).$$

Continuing in this fashion, one gets that any $u_{n+1}(s)$ is a polynomial expression in the matrix coefficients of $\ell_0, \cdots, \ell_n$. This proves the claim in statement (a).

The proof of statement (b) can be reduced to that of (a) by the following observation: take an arbitrary element $k \in \mathcal{D}_{\mathbb{N}}(R)$ and an $d = \operatorname{diag}(d(s)) \in \mathcal{D}_{\mathbb{N}}(R)^*$. Then, there holds $d^{-1}kSd = \operatorname{diag}(\frac{d(s+1)}{d(s)})kS$. Given any $\mathcal{M}$ of the form (10), choose a $d \in \mathcal{D}_{\mathbb{N}}(R)^*$ such that for all $s \in \mathbb{N}$ the element $\frac{d(s+1)}{d(s)}$ equals $m_1(s)^{-1}$. Then, the matrix $d^{-1}\mathcal{M}d$ has the form (9) and, hence there is a unique $U \in \mathcal{U}_-$ such that $d^{-1}\mathcal{M}d = USU^{-1}$. Then, $\mathcal{M}$ equals PSP^{-1} with $P = dU \in \mathcal{P}_-$. In this case d is not unique, because any element $i_1(a)$, with $a \in R^*$, is in the center of $LT_{\mathbb{N}}(R)$ and there also holds $\mathcal{M} = i_1(a)dUSU^{-1}d^{-1}i_1(a^{-1})$. □

Next we discuss the evolution equations that an $\mathcal{U}_-$-deformation $\mathcal{L}$ of S has to satisfy and those for a $\mathcal{P}_-$-deformation $\mathcal{M}$ of S. Thereto we assume for both hierarchies that the algebra R is equipped with a set of k-linear derivations $\{\partial_i : R \to R | i \geqslant 1\}$, but only this time we do not require that these derivations commute. By letting each ∂_i act coefficient wise on the matrices in $LT_{\mathbb{N}}(R)$, we get a set of derivations of $LT_{\mathbb{N}}(R)$ also denoted by $\{\partial_i\}$. The data $(R, \{\partial_i | i \geqslant 1\})$ we call a *presetting* for both deformations. If all the $\{\partial_i\}$ mutually commute, then we speak of a *setting*.

For each $\mathcal{U}_-$-deformation $\mathcal{L}$ of S and all $i \geqslant 1$ we consider the cut-off's $\mathcal{B}_i(S) := \pi_{\geqslant 0}(\mathcal{L}^i)$ and the system of Lax equations

$$\partial_i(\mathcal{L}) = [\mathcal{B}_i(S), \mathcal{L}] = -[\pi_{<0}(\mathcal{L}^i), \mathcal{L}]. \tag{13}$$

This set of Lax equations is called the $k[S]$-hierarchy after the commutative algebra that gets deformed. Any $\mathcal{U}_-$-deformation $\mathcal{L}$ of S in $LT_{\mathbb{N}}(R)$ satisfying the equations (13) is called a *solution* of the $k[S]$-hierarchy in the presetting $(R, \{\partial_i\})$. The hierarchy has at least one solution, the trivial one $\mathcal{L} = S$.

We treat the evolution equations for a $\mathcal{P}_-$-deformation $\mathcal{M}$ in a similar fashion. In that case we consider the strict cut-off's

$$\mathcal{C}_i(S) := \pi_{>0}(\mathcal{M}^i), \quad i \geqslant 1,$$

and the system of Lax equations

$$\partial_i(\mathcal{M}) = [\mathcal{C}_i(S), \mathcal{M}] = -[\pi_{\leqslant 0}(\mathcal{M}^i), \mathcal{M}]. \tag{14}$$

This system is called the *strict* $k[S]$-*hierarchy* ([3]). Any $\mathcal{P}_-$-deformation $\mathcal{M}$ of S in $LT_{\mathbb{N}}(R)$ that satisfies all the equations (14) is called a *solution* of the strict $k[S]$-hierarchy in the presetting $(R, \{\partial_i\})$. Also the strict $k[S]$-hierarchy has at least one solution, namely $\mathcal{M} = S$, and it is likewise called the *trivial* one.

Remark 2. Since the matrix coefficients of each $\mathcal{B}_i(S)$ are polynomials in the $\{l_{s,s+r}\}$ and those of $\mathcal{C}_i(S)$ are polynomials in the $\{m_{s,s+r}\}$, the Lax equations (13) show that the action of each ∂_i on the coefficients of $\mathcal{L}$ expresses each of them in a polynomial expression of the coefficients of $\mathcal{L}$ and, similarly, the Lax equations (14) make that each $\partial_i(m_{s,s+r})$ is a polynomial expression in the matrix coefficients of $\mathcal{M}$.

Remark 3. Given a presetting $(R, \{\partial_i\})$ and an $t \in \mathcal{D}_{\mathbb{N}}(k)^*$ as in Remark 1, one can equally well study deformations of the maximal commutative subalgebra $k[S(t)]$ by substituting an $\mathcal{U}_-$-deformation $\mathcal{L}(t)$ of $S(t)$ in $k[S(t)]$ or a $\mathcal{P}_-$-deformation $\mathcal{M}(t)$ of $S(t)$ and require that they satisfy the Lax equations

$$\partial_i(\mathcal{L}(t)) = [\pi_{\geqslant 0}(\mathcal{L}(t)^i), \mathcal{L}(t)] = [\mathcal{B}_i(S(t)), \mathcal{L}(t)] \tag{15}$$

respectively

$$\partial_i(\mathcal{M}(t)) = [\pi_{>0}(\mathcal{M}(t)^i), \mathcal{M}(t)] = [\mathcal{C}_i(S(t)), \mathcal{M}(t)]. \tag{16}$$

The equations (15) respectively (16) are then the *Lax equations of the* $k[S(t)]$-*hierarchy* respectively the *Lax equations of the strict* $k[S(t)]$-*hierarchy*.

To see the common property of the hierarchies introduced in Remark 3 we look at the $k[S(t)]$-hierarchy and its strict version as an infinite set of evolution equations for a chain of matrices in $\mathcal{D}_{\mathbb{N}}(R)$. More precisely, for the $k[S(t)]$-hierarchy one considers chains $(l_0, \cdots, l_j, \cdots)$ of matrices $l_j \in \mathcal{D}_{\mathbb{N}}(R)$, one forms the matrix

$$\mathcal{L}(t) = S(t) + \sum_{j \leqslant 0} (S^T(t))^{-j} l_j$$

and looks for chains that satisfy for all $i \geqslant 1$ and all $j \leqslant 0$

$$\partial_i(l_j) = d_j(t^{-1}[\pi_{\geqslant 0}(\mathcal{L}(t)^i), \mathcal{L}(t)]t). \tag{17}$$

By Lemma 1 these equations are equivalent to the Lax equations for $\mathcal{L}(t)$ and we call it the *diagonal form* of the $k[S(t)]$-hierarchy. Since the relations (1), (3) and (4) hold with S replaced by $S(t)$ and S^T by $S^T(t)$, the equations (17) are independent of t and equal to those for $t = \mathrm{Id}$, i.e. the diagonal form of the $k[S]$-hierarchy. Similarly, for the strict $k[S(t)]$-hierarchy one considers chains $(m_1, \cdots, m_j, \cdots)$ of matrices $m_j \in \mathcal{D}_{\mathbb{N}}(R)$ and with $m_1 \in \mathcal{D}_{\mathbb{N}}(R)^*$. One forms

$$\mathcal{M}(t) = m_1 S(t) + \sum_{j \leqslant 0} (S^T(t))^{-j} m_j$$

and one searches for chains satisfying for all $i \geqslant 1$ and all $j \leqslant 1$

$$\partial_i(m_j) = d_j(t^{-1}[\pi_{>0}(\mathcal{M}(t)^i), \mathcal{M}(t)]t). \tag{18}$$

By Lemma 1 these equations are equivalent to the Lax equations for $\mathcal{M}(t)$ and we call it the *diagonal form* of the strict $k[S(t)]$-hierarchy. By the same argument as for the $k[S]$-hierarchy, the equations (18) are also independent of t and equal to the diagonal form of the strict $k[S]$-hierarchy. By combining these two results with the fact that conjugation with $t \in \mathcal{D}_{\mathbb{N}}(k)^*$ normalizes the groups $\mathcal{U}_-$ and $\mathcal{P}_-$, we obtain

Proposition 1. *Let $(R, \{\partial_i\})$ be a presetting. For each $t \in \mathcal{D}_{\mathbb{N}}(k)^*$, the map $\mathcal{L} \to t\mathcal{L}t^{-1}$ is a bijection between the solutions of the $k[S]$-hierarchy and the solutions of the $k[S(t)]$-hierarchy, all in the presetting $(R, \{\partial_i\})$. Similarly, the map $\mathcal{M} \to t\mathcal{M}t^{-1}$ is a bijection between the solutions of the strict $k[S]$-hierarchy and the solutions of the strict $k[S(t)]$-hierarchy, all in the same presetting.*

Thanks to Proposition 1 it suffices to focus on the $k[S]$-hierarchy and its strict version. The Lax equations (13) and (14) imply other useful relations for these hierarchies.

Proposition 2. *Both sets of Lax equations (13) and (14) are so-called compatible systems, i.e. the projections* $\{\mathcal{B}_i(S) := \pi_{\geqslant 0}(\mathcal{L}^i) | i \geqslant 1\}$ *satisfy the zero curvature relations*

$$\partial_{i_1}(\mathcal{B}_{i_2}(S)) - \partial_{i_2}(\mathcal{B}_{i_1}(S)) - [\mathcal{B}_{i_1}(S), \mathcal{B}_{i_2}(S)] = 0 \tag{19}$$

and the projections $\{\mathcal{C}_j(S) := \pi_{>0}(\mathcal{M}^j) | j \geqslant 1\}$ *satisfy the zero curvature relations*

$$\partial_{j_1}(\mathcal{C}_{j_2}(S)) - \partial_{j_2}(\mathcal{C}_{j_1}(S)) - [\mathcal{C}_{j_1}(S), \mathcal{C}_{j_2}(S)] = 0. \tag{20}$$

Proof. The idea of the proof is to show that the left hand side of the identities (19) resp. (20) belongs to

$$\pi_{\geqslant 0}(LT_{\mathbb{N}}(R)) \cap \pi_{<0}(LT_{\mathbb{N}}(R) \text{ resp. } \pi_{>0}(LT_{\mathbb{N}}(R)) \cap \pi_{\leqslant 0}(LT_{\mathbb{N}}(R))$$

and thus has to be zero. We give the proof for the $\{\mathcal{B}_i(S)\}$, that for the $\{\mathcal{C}_j(S)\}$ is similar and is left to the reader. The inclusion in $\pi_{\geqslant 0}(LT_{\mathbb{N}}(R))$ is clear as both $\mathcal{B}_i(S)$ and $\partial_n(\mathcal{B}_i(S))$ belong to the Lie subalgebra $\pi_{\geqslant 0}(LT_{\mathbb{N}}(R))$. To show the other one, we use the Lax equations (13). Recall that similar Lax equations hold for all the $\{\mathcal{L}^N | N \geqslant 1\}$

$$\partial_i(\mathcal{L}^N) = [\pi_{\geqslant 0}(\mathcal{L}^i), \mathcal{L}^N].$$

Now we substitute $\mathcal{B}_{i_k}(S) = \mathcal{L}^{i_k} - \pi_{<0}(\mathcal{L}^{i_k}), k = 1, 2$, and get for

$$\begin{aligned} \partial_{i_1}(\mathcal{B}_{i_2}(S)) - \partial_{i_2}(\mathcal{B}_{i_1}(S)) &= \partial_{i_1}(\mathcal{L}^{i_2}) - \partial_{i_1}(\pi_{<0}(\mathcal{L}^{i_2})) - \partial_{i_2}(\mathcal{L}^{i_1}) + \partial_{i_2}(\pi_{<0}(\mathcal{L}^{i_1})) \\ &= [\mathcal{B}_{i_1}(S), \mathcal{L}^{i_2}] - [\mathcal{B}_{i_2}(S), \mathcal{L}^{i_1}] - \partial_{i_1}(\pi_{<0}(\mathcal{L}^{i_2})) + \partial_{i_2}(\pi_{<0}(\mathcal{L}^{i_1})) \end{aligned}$$

and for

$$\begin{aligned} &[\mathcal{B}_{i_1}(S), \mathcal{B}_{i_2}(S)] = [\mathcal{L}^{i_1} - \pi_{t,<0}(\mathcal{L}^{i_1}), \mathcal{L}^{i_2} - \pi_{t,<0}(\mathcal{L}^{i_2})] \\ &= -[\pi_{<0}(\mathcal{L}^{i_1}), \mathcal{L}^{i_2}] + [\pi_{<0}(\mathcal{L}^{i_2}), \mathcal{L}^{i_1}] + [\pi_{<0}(\mathcal{L}^{i_1}), \pi_{<0}(\mathcal{L}^{i_2})]. \end{aligned}$$

Taking into account the second identity in (13), we see that the left hand side of (19) is equal to

$$-\partial_{i_1}(\pi_{<0}(\mathcal{L}^{i_2})) + \partial_{i_2}(\pi_{<0}(\mathcal{L}^{i_1})) - [\pi_{<0}(\mathcal{L}^{i_1}), \pi_{<0}(\mathcal{L}^{i_2})].$$

This element belongs to the Lie subalgebra $\pi_{<0}(LT_{\mathbb{N}}(R))$ and that proves the claim. □

Remark 4. Note from the proof of Proposition 2 that it never uses the fact that we require the diagonal part of the leading diagonal of $\mathcal{M}$ to be invertible in $\mathcal{D}_{\mathbb{N}}(R)$. Hence, for any $\mathcal{M}$ of the form (10) that satisfies the Lax equations (14), the zero curvature relations (20) hold for the projections $\{\pi_{>0}(\mathcal{M}^j)\}$.

Reversely, we have

Proposition 3. *Given a presetting* $(R, \{\partial_i\})$ *to realize both hierarchies. Suppose we have in* $LT_{\mathbb{N}}(R)$ *a* U_-*-deformation* $\mathcal{L}$ *of* S *and a* P_-*-deformation* $\mathcal{M}$ *of* S*. Then, there holds:*

(a) *Assume that the projections* $\{\mathcal{B}_i(S) := \pi_{\geqslant 0}(\mathcal{L}^i) | i \geqslant 1\}$ *satisfy the zero curvature relations (19), then* $\mathcal{L}$ *is a solution of the* $k[S]$*-hierarchy.*

(b) *Similarly, if the projections* $\{\mathcal{C}_j(S) := \pi_{>0}(\mathcal{M}^j) | j \geqslant 1\}$ *satisfy the zero curvature relations (20), then* $\mathcal{M}$ *is a solution of the strict* $k[S]$*-hierarchy.*

Proof. Again we prove the statement for $\mathcal{L}$, that for $\mathcal{M}$ is shown in a similar way. As a first step we show that the zero curvature conditions imply that for each fixed $i \geqslant 1$ and all $N \geqslant 1$ the degree of

$$\partial_i(\mathcal{L}^N) - [\mathcal{B}_i(S), \mathcal{L}^N]$$

has an upper bound $i - 1$. Substitute namely $\mathcal{L}^N = \mathcal{B}_N(S) + \pi_{<0}(\mathcal{L}^N)$ in this expression and use the zero curvature relation for $i_1 = i$ and $i_2 = N$. Then, we get

$$\begin{aligned} &\partial_i(\mathcal{B}_N(S)) + \partial_i(\pi_{<0}(\mathcal{L}^N)) - [\mathcal{B}_i(S), \mathcal{B}_N(S)] - [\mathcal{B}_i(S), \pi_{<0}(\mathcal{L}^N)] \\ &= \partial_N(\mathcal{B}_i(S)) + \partial_i(\pi_{<0}(\mathcal{L}^N)) - [\mathcal{B}_i(S), \pi_{<0}(\mathcal{L}^N)] \end{aligned}$$

and this last expression has degree $i-1$ or less. So, assume that there is one Lax equation (13) that does not hold. Then, there is an $i_0 \geqslant 1$ such that

$$\partial_{i_0}(\mathcal{L}) - [\mathcal{B}_{i_0}(S), \mathcal{L}] = \sum_{i \leqslant k(i_0)} (S^T)^{-i}\alpha_i, \quad \text{with} \quad \alpha_{k(i_0)} \neq 0 \quad \text{and} \quad \alpha_i \in \mathcal{D}_{\mathbb{N}}(R).$$

As both $\partial_{i_0}(\mathcal{L})$ and $[\mathcal{B}_{i_0}(S), \mathcal{L}]$ have only negative diagonals, we know that $k(i_0) \leqslant 0$. Further, we can say that for all $N \geqslant 1$

$$\partial_{i_0}(\mathcal{L}^{N+1}) - [\mathcal{B}_{i_0}(S), \mathcal{L}^{N+1}] = \sum_{j=0}^{N} \mathcal{L}^j(\partial_{i_0}(\mathcal{L}) - [\mathcal{B}_{i_0}(S), \mathcal{L}])\mathcal{L}^{N-j}.$$

If we write $\mathcal{L} = S + \sum_{j\leqslant 0}(S^T)^{-j}d_j(\mathcal{L})$, then the highest possible nonzero diagonal that can appear in $\partial_{i_0}(\mathcal{L}^{N+1}) - [\mathcal{B}_{i_0}(S), \mathcal{L}^{N+1}]$ is the $N + k(i_0)$-st and it is equal to

$$\sum_{j=0}^{N} (S)^j (S^T)^{-k(i_0)}\alpha_{k(i_0)} S^{N-j}.$$

Note that thanks to Lemma 1 we have for all $j, 0 \leqslant j \leqslant N$, that each factor

$$S^j(S^T)^{-k(i_0)}\alpha_{k(i_0)}S^{(N-j)} = A_j S^{N+k(i_0)}$$

with $A_j \in \mathcal{D}_{\mathbb{N}}(R)$. Each A_j has an expression into the coefficients of $\alpha_{k(i_0)}$. For all $j < -k(i_0)$, we can apply formula (4) and get

$$A_j = (S^T)^{(-k(i_0)-j)}\alpha_{k(i_0)}S^{(-k(i_0)-j)} = \operatorname{diag}(0, \cdots, 0, \alpha_{k(i_0)}(0), \alpha_{k(i_0)}(1), \cdots),$$

where the first $-k(i_0) - j$ diagonal entries of A_j before $\alpha_{k(i_0)}(0)$ are zero. For all j, with $N \geqslant j \geqslant -k(i_0)$ we can apply formula (3) and get

$$\begin{aligned} A_j S^{N+k(i_0)} &= S^{(j+k(i_0))}\alpha_{k(i_0)}S^{N-i}S^{N+k(i_0)} \\ &= \operatorname{diag}(\alpha_{k(i_0)}(j+k(i_0)), \alpha_{k(i_0)}(j+k(i_0)+1), \cdots)S^{N+k(i_0)}. \end{aligned} \tag{21}$$

We have shown above that $\sum_{j=0}^{N} A_j = 0$ for all $N \geqslant i_0 - k(i_0)$ and that is equivalent with $\sum_{j=0}^{i_0-k(i_0)} A_j = 0$ and $A_j = 0$ for all $j > i_0 - k(i_0)$. This last result implies that $\alpha_{k(i_0)}$ has the form $\operatorname{diag}(\alpha_{k(i_0)}(0), \cdots, \alpha_{k(i_0)}(N_0), 0, \cdots)$ with $\alpha_{k(i_0)}(N_0) \neq 0$ and $N_0 \leqslant i_0 - k(i_0)$. Then, there follows from expression (21) that $A_j = 0$ for all $j > N_0$ so that the only relevant A_j are those with $j \leqslant N_0$ and there should hold $\sum_{j=0}^{N_0} A_j = 0$. Consider now the $N_0 + 1 - k(i_0) \times N_0 + 1 - k(i_0)$-matrix L that has the first $N_0 + 1 - k(i_0)$ diagonal entries of A_j as its jth row. The matrix L has the following form

$$L = \begin{pmatrix} * & \cdots & \cdots & \alpha_{k(i_0)}(N_0-1) & \alpha_{k(i_0)}(N_0) \\ \vdots & & \iddots & \alpha_{k(i_0)}(N_0) & 0 \\ \alpha_{k(i_0)}(N_0-2) & \alpha_{k(i_0)}(N_0-1) & \iddots & \iddots & \vdots \\ \alpha_{k(i_0)}(N_0-1) & \alpha_{k(i_0)}(N_0) & 0 & & \\ \alpha_{k(i_0)}(N_0) & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

The condition $\sum_{j=0}^{N_0} A_j = 0$ is now the same as demanding that the sum of the entries in each column of L is equal to zero and the form of L shows that this is equivalent to all diagonal entries of $\alpha_{k(i_0)}$ being zero. This contradicts our assumption that one of the Lax equations (13) would not hold. So, the Lax equations for $\mathcal{L}$ and its powers hold. This concludes the proof of this Proposition. □

Because of the equivalence between the Lax equations (13) for $\mathcal{L}$ and the zero curvature relations (19) for the $\{\mathcal{B}_i(S)|i \geqslant 1\}$, we call this last set of equations also the *zero curvature form* of the $k[S]$-hierarchy. Similarly, the zero curvature relations (20) for the $\{\mathcal{C}_j(S)|j \geqslant 1\}$ are called the *zero curvature form* of the strict $k[S]$-hierarchy.

Given a presetting $(R, \{\partial_i\})$, we have a look now at the subset R_c of R, where all the $\{\partial_i\}$ commute, i.e.,

$$R_c = \{r \in R | \partial_{i_1}\partial_{i_2}(r) = \partial_{i_2}\partial_{i_1}(r) \text{ for all } i_1, i_2 \geqslant 1\}.$$

As in [1] one shows that this set has the property

Lemma 4. *R_c is a k-subalgebra of R.*

Let $\mathcal{L} = (l_{s,s+r})$ resp. $\mathcal{M} = (m_{s,s+r})$ be solutions in the presetting $(R, \{\partial_i\})$ of respectively the $k[S]$-hierarchy and its strict version. We denote the k-subalgebra of R generated by the matrix coefficients $\{l_{s,s+r}\}$ of $\mathcal{L}$ by $R(\mathcal{L})$ and likewise we write $R(\mathcal{M})$ for the k-subalgebra of R generated by the matrix coefficients $\{m_{s,s+r}\}$ of $\mathcal{M}$. From the Lax equations (13) and (14) follows that each ∂_i maps $R(\mathcal{L})$ into $R(\mathcal{L})$ and $R(\mathcal{M})$ into $R(\mathcal{M})$. The restriction of ∂_i to $R(\mathcal{L})$ resp. $R(\mathcal{M})$ is denoted by $\partial_{i,\mathcal{L}}$ resp. $\partial_{i,\mathcal{M}}$. We claim now that all $\{\partial_{i,\mathcal{L}}\}$ commute on $R(\mathcal{L})$ and all $\{\partial_{i,\mathcal{M}}\}$ on $R(\mathcal{M})$. The proof in both cases is the same and we give the argument for the $k[S]$-hierarchy. As R_c is a k-subalgebra according to Lemma 4 and since the coefficients $l_{s,s+r}$ generate $R(\mathcal{L})$, it suffices to show for all $i_1 \geqslant 1$ and $i_2 \geqslant 1$ that

$$\partial_{i_1}\partial_{i_2}(\mathcal{L}) - \partial_{i_2}\partial_{i_1}(\mathcal{L}) = 0. \tag{22}$$

Using the Lax equations for $\mathcal{L}$, we get

$$\partial_{i_1}\partial_{i_2}(\mathcal{L}) = \partial_{i_1}([\mathcal{B}_{i_2}(S), \mathcal{L}]) = [\partial_{i_1}(\mathcal{B}_{i_2}(S)), \mathcal{L}] + [\mathcal{B}_{i_2}(S), [\mathcal{B}_{i_1}(S), \mathcal{L}]]$$

and likewise

$$\partial_{i_2}\partial_{i_1}(\mathcal{L}) = [\partial_{i_2}(\mathcal{B}_{i_1}(S)), \mathcal{L}] + [\mathcal{B}_{i_1}(S), [\mathcal{B}_{i_2}(S), \mathcal{L}]].$$

Since $\mathrm{ad}([\mathcal{B}_{i_1}(S), \mathcal{B}_{i_2}(S)]) = [\mathrm{ad}(\mathcal{B}_{i_1}(S)), \mathrm{ad}(\mathcal{B}_{i_2}(S))]$ we see that the left hand side of equation (22) is equal to

$$[\partial_{i_1}(\mathcal{B}_{i_2}(S)) - \partial_{i_2}(\mathcal{B}_{i_1}(S)) - [\mathcal{B}_{i_1}(S), \mathcal{B}_{i_2}(S)], \mathcal{L}]$$

and, because $\mathcal{L}$ is a solution of the $k[S]$-hierarchy, the left factor of this commutator is zero by Proposition 2. Thus, we have proved the following results

Proposition 4. *Let $\mathcal{L}$ be a solution of the $k[S]$-hierarchy and $\mathcal{M}$ a solution of the strict $k[S]$-hierarchy, both in the presetting $(R, \{\partial_i\})$. Then, the data $(R(\mathcal{L}), \{\partial_{i,\mathcal{L}}\})$ are a setting for the $k[S]$-hierarchy and the data $(R(\mathcal{M}), \{\partial_{i,\mathcal{M}}\})$ are a setting for the strict $k[S]$-hierarchy.*

As a consequence of this Proposition one can come up with a k-subalgebra of R such that all the $\{\partial_i\}$ map this k-subalgebra to itself and the derivations $\{\partial_i\}$ mutually commute on it. Let namely R_{sol} be the subalgebra of R consisting of all polynomial expressions in elements of all the $R(\mathcal{L})$ with $\mathcal{L}$ any solution of the $k[S]$-hierarchy in the presetting $(R, \{\partial_i\})$ and likewise R_{ssol} that of all polynomial expressions in elements of all the $R(\mathcal{M})$ with $\mathcal{M}$ a solution of the strict $k[S]$-hierarchy in the same presetting. Then, one can conclude from Lemma 4 and Proposition 4 that

Corollary 2. *All the $\{\partial_i\}$ map R_{sol} to itself and the same holds for R_{ssol}. The data $(R_{sol}, \{\partial_i | R_{sol}\})$ are a setting for the $k[S]$-hierarchy and the $(R_{ssol}, \{\partial_i | R_{ssol}\})$ for its strict version.*

Hence, by replacing the algebra R in a presetting by R_{sol} resp. R_{ssol}, one does not loose relevance for the hierarchies under consideration and one ends up with a setting instead of a presetting. Therefore, we worked with settings in [3].

4. THE MINIMAL REALIZATIONS

In this section we want to discuss a minimal realization of the equations (13) resp. (14), in the sense that there are a minimal number of relations between the coefficients of the potential solution $\mathcal{L}$ resp. $\mathcal{M}$. We start with the description of a proper coefficient algebra $\tilde{R}_{\geqslant 0}$ for the case of the $k[S]$-hierarchy. We choose the algebra

$$\tilde{R}_{\geqslant 0} := k[\tilde{l}_{s,s+r} | r \leqslant 0, s \in \mathbb{N}]$$

of all polynomials in the unknown $\{\tilde{l}_{s,s+r} | r \leqslant 0, s \in \mathbb{N}\}$ with coefficients from k. Define for each $r \leqslant 0$ the diagonal matrix $\tilde{l}_r$ in $\mathcal{D}_{\mathbb{N}}(\tilde{R}_{\geqslant 0})$ by

$$\tilde{l}_r = \mathrm{diag}(\tilde{l}_r(s)), \quad \text{with} \quad \tilde{l}_r(s) = \tilde{l}_{s,s+r}, \quad \text{for all} \quad s \in \mathbb{N},$$

and let $\tilde{\mathcal{L}}$ be the operator in $LT_{\mathbb{N}}(R)(\tilde{R}_{\geqslant 0})$ defined by

$$\tilde{\mathcal{L}} = S + \sum_{r \leqslant 0} (S^T)^{-r} \tilde{l}_r = S + \sum_{r \leqslant 0} (S^T)^{-r} \mathrm{diag}(\tilde{l}_{s,s+r}). \tag{23}$$

Then, $\tilde{\mathcal{L}}$ has the right form (9). Now we still have to define the commuting derivations $\{\tilde{\partial}_i, i \geqslant 1\}$ of $\tilde{R}_{\geqslant 0}$. Keep in mind that any k-linear derivation D of a polynomial ring $k[x_\sigma, \sigma \in \Sigma]$ in any number Σ of variables $\{x_\sigma, \sigma \in \Sigma\}$, is uniquely determined by prescribing the images $D(x_\sigma)$ of all the $\{x_\sigma\}$ thanks to the derivation property: $D(fg) = D(f)g + fD(g)$, for all f and $g \in k[x_s]$. So each $\tilde{\partial}_i$ is fully determined by its action on the coefficients of $\tilde{\mathcal{L}}$:

$$\tilde{\partial}_i(\tilde{\mathcal{L}}) := [\tilde{\mathcal{L}}, \pi_{<0}(\tilde{\mathcal{L}}^i)] = [\pi_{\geqslant 0}(\tilde{\mathcal{L}}^i), \tilde{\mathcal{L}}] = [\tilde{\mathcal{B}}_i(S), \tilde{\mathcal{L}}],$$

where the second identity follows from the fact that $\tilde{\mathcal{L}}$ commutes with all its powers. Hence, $\tilde{\mathcal{L}}$ satisfies the Lax equations (13) in the presetting ($\tilde{R}_{\geqslant 0}$,$\{\tilde{\partial}_i\}$). By definition $\tilde{R}_{\geqslant 0} = \tilde{R}_{\geqslant 0}(\tilde{\mathcal{L}})$ so that by Proposition 4 our presetting is even a setting.

The setting ($\tilde{R}_{\geqslant 0}$,$\{\tilde{\partial}_i\}$) together with the solution $\tilde{\mathcal{L}}$ we call a *minimal realization* of the $k[S]$-hierarchy. It possesses homogeneity properties if we put a suitable multiplicative grading on the monomials in the unknown of $\tilde{R}_{\geqslant 0}$. We prescribe it on their building blocks by

$$\deg(\tilde{l}_{s,s+r}) = 1 - r, s \in \mathbb{N}, r \leqslant 0$$

and that gives a decomposition

$$\tilde{R}_{\geqslant 0} = \oplus_{m \geqslant 0} \tilde{R}^{(m)}_{\geqslant 0}, \quad \text{where } \tilde{R}^{(m)}_{\geqslant 0} \text{ is the span of the monomials of degree } m.$$

A diagonal matrix $d = \mathrm{diag}(d(s)) \in \mathcal{D}_{\mathbb{N}}(\tilde{R}_{\geqslant 0})$ is called *homogeneous* of degree m if all $d(s), s \in \mathbb{N}$, belong to $\tilde{R}^{(m)}_{\geqslant 0}$. The space of all homogeneous diagonal matrices of degree m is denoted by $\mathcal{D}_{\mathbb{N}}(\tilde{R}^{(m)}_{\geqslant 0})$. Similarly, an element $p = \sum_{j \leqslant N} p_j S^j, p_j \in \mathcal{D}_{\mathbb{N}}(\tilde{R}_{\geqslant 0})$ is called homogeneous of degree k if all $p_j \in \mathcal{D}_{\mathbb{N}}(\tilde{R}^{(k-j)}_{\geqslant 0})$. The matrix $\tilde{\mathcal{L}}$ for example is then homogeneous of degree one. The multiplication on $LT_{\mathbb{N}}(\tilde{R}_{\geqslant 0})$ is such that the product of homogeneous matrices is again homogeneous and the degrees add up. In particular, all the matrices $\tilde{\mathcal{L}}^i$, $\tilde{\mathcal{B}}_i(S)$ and $\pi_{<0}(\tilde{\mathcal{L}}^i)$ are homogeneous of degree i and each derivation $\tilde{\partial}_i$ maps $\tilde{\mathcal{L}}$ to a homogeneous matrix of degree $i+1$.

Next we describe in an algebraic way how solutions of the $k[S]$-hierarchy in other settings are related to this minimal realization. Consider any setting (R, $\{\partial_i\}$) for this hierarchy and a potential solution $\mathcal{L} \in LT_{\mathbb{N}}(R)$ of the form (9) with each $d_r(\mathcal{L}) = \mathrm{diag}(l_{s,s+r})$. Each such a matrix $\mathcal{L}$ determines uniquely a k-algebra morphism $i_{\mathcal{L}} : \tilde{R}_{\geqslant 0} \to R$ by substituting $l_{s,s+r}$ for each $\tilde{l}_{s,s+r}, r \leqslant 0$ and $s \in \mathbb{N}$ in the decomposition (23). The map $i_{\mathcal{L}}$ extends to a k-algebra morphism from the matrices in $LT_{\mathbb{N}}(\tilde{R}_{\geqslant 0})$ to those in $LT_{\mathbb{N}}(R)$ such that

$$i_{\mathcal{L}}(\tilde{\mathcal{L}}) = \mathcal{L} \quad \text{and} \quad i_{\mathcal{L}}(\tilde{\mathcal{B}}_i(S)) = \mathcal{B}_i(S).$$

Assume now that $\mathcal{L}$ is a solution of the Lax equations of the $k[S]$-hierarchy, then we have for all $i \geqslant 1$ that

$$\partial_i(\mathcal{L}) = \partial_i \circ i_{\mathcal{L}}(\tilde{\mathcal{L}}) = [\mathcal{B}_i(S), \mathcal{L}] = [i_{\mathcal{L}}(\tilde{\mathcal{B}}_i(S)), i_{\mathcal{L}}(\tilde{\mathcal{L}})] = i_{\mathcal{L}}([\tilde{\mathcal{B}}_i(S), \tilde{\mathcal{L}}]) = i_{\mathcal{L}} \circ \tilde{\partial}_i(\tilde{\mathcal{L}}).$$

Thus, the k-linear maps $\partial_r \circ i_{\mathcal{L}}$ and $i_{\mathcal{L}} \circ \tilde{\partial}_i$ are equal on the coefficients of $\tilde{\mathcal{L}}$ and that implies

$$\partial_i \circ i_{\mathcal{L}} = i_{\mathcal{L}} \circ \tilde{\partial}_i, \quad \text{for all} \quad i \geqslant 1. \tag{24}$$

On the other hand, if the compatibilities (24) hold for the map $i_{\mathcal{L}}$, then one applies these identities to $\tilde{\mathcal{L}}$ and, as $i_{\mathcal{L}}$ is a k-algebra morphism, we obtain the Lax equations for $\mathcal{L}$. So we may conclude

Lemma 5. *The relations (24) for the map $i_{\mathcal{L}}$ are equivalent to $\mathcal{L}$ being a solution of the $k[S]$-hierarchy w.r.t. the setting* (R, $\{\partial_i\}$).

We continue with the description of a proper coefficient algebra $\tilde{R}_{>0}$ for the case of the strict $k[S]$-hierarchy So, as in the case of the $k[S]$-hierarchy, the unknown will correspond to the nontrivial coefficients $\{\tilde{m}_{s,s+r}|r \leqslant 1, s \in \mathbb{N}\}$ of a matrix

$$\tilde{\mathcal{M}} = \tilde{m}_1 S + \sum_{r \leqslant 0} (S^T)^{-r} \tilde{m}_r, \quad \text{where each} \quad \tilde{m}_r = \mathrm{diag}(\tilde{m}_{s,s+r}). \tag{25}$$

Let $\tilde{R}$ be the k-algebra $k[\tilde{m}_{s,s+r}|r \leqslant 1, s \in \mathbb{N}]$. Then, $\tilde{\mathcal{M}}$ belongs to $LT_{\mathbb{N}}(R)(\tilde{R})$ and is of the form (10), but its leading diagonal component is not invertible in $\mathcal{D}_{\mathbb{N}}(\tilde{R})$. This we repair later on. First we define for each $i \geqslant 1$ a k-linear derivation $\tilde{\partial}_i$ of $\tilde{R}$ by prescribing how $\tilde{\partial}_i$ acts on $\tilde{\mathcal{M}}$:

$$\tilde{\partial}_i(\tilde{\mathcal{M}}) := [\tilde{\mathcal{M}}, \pi_{\leqslant 0}(\tilde{\mathcal{M}}^i)] = [\pi_{>0}(\tilde{\mathcal{M}}^i), \tilde{\mathcal{M}}] = [\tilde{\mathcal{C}}_i(S), \tilde{\mathcal{M}}].$$

By definition, $\tilde{\mathcal{M}}$ satisfies the Lax equations (14) and by Remark 4 the projections $\{\tilde{\mathcal{C}}_i(S)\}$ satisfy the zero curvature relations (20) and as you can see from the proof of Proposition 4 these relations suffice to show that the $\{\tilde{\partial}_i\}$ commute on $\tilde{R}$. We overcome the problem that $d_1(\tilde{\mathcal{M}})$ is not invertible in $\mathcal{D}_{\mathbb{N}}(\tilde{R})$ by extending $\tilde{R}$ and the $\{\tilde{\partial}_i\}$. Let S be the multiplicative subset of $\tilde{R}$ generated by the $\{\tilde{m}_{s,s+1}|s \in \mathbb{N}\}$. Then, we choose for $\tilde{R}_{>0}$ the localization of $\tilde{R}$ w.r.t. S, i.e., the algebra $\tilde{R}_{>0} := S^{-1}\tilde{R}$. Then, $\tilde{\mathcal{M}}$ is a $\mathcal{P}_-$-deformation of S in $LT_{\mathbb{N}}(\tilde{R}_{>0})$. Recall that any k-linear derivation D of a polynomial ring $k[x_\sigma, \sigma \in \Sigma]$ extends uniquely to a k-linear derivation D of its quotient field $k(x_\sigma)$ by putting

$$D\left(\frac{f}{g}\right) := \frac{D(f)}{g} - \frac{fD(g)}{g^2}.$$

This we apply to the derivations $\{\tilde{\partial}_i\}$ and we use the same notations for their extensions. The restriction of the $\{\tilde{\partial}_i\}$ to the localization $S^{-1}\tilde{R}$ yields a collection of k-linear derivations of this algebra. Since the $\{\tilde{\partial}_i\}$ mutually commute on $\tilde{R}$, we have for each $g \in S$

$$\tilde{\partial}_{i_1}\tilde{\partial}_{i_2}(1/g) = \frac{2}{g^3}\tilde{\partial}_{i_1}(g)\tilde{\partial}_{i_2}(g) - \frac{1}{g^2}\tilde{\partial}_{i_1}\tilde{\partial}_{i_2}(g) = \tilde{\partial}_{i_2}\tilde{\partial}_{i_1}(1/g).$$

Hence, by Lemma 4, the $\{\tilde{\partial}_i\}$ also commute on $S^{-1}\tilde{R}$, in other words $(S^{-1}\tilde{R}, \{\tilde{\partial}_i\})$ is a setting for the strict $k[S]$-hierarchy and $\tilde{\mathcal{M}}$ is a solution in this setting.

The setting $(\tilde{R}_{>0}$,$\{\tilde{\partial}_i\})$ together with the solution $\tilde{\mathcal{M}}$ we call a *minimal realization* of the strict $k[S]$-hierarchy. It possesses homogeneity properties if we put a suitable multiplicative grading on the monomials in the unknown of $\tilde{R}_{>0}$. We prescribe it on their building blocks by

$$\deg(\tilde{m}_{s,s+r}) = 2 - r \quad \text{and} \quad \deg(\tilde{m}_{s,s+1}^{-1}) = -1, \quad r \leqslant 1, s \in \mathbb{N},$$

and that gives a decomposition

$$\tilde{R}_{>0} = \oplus_{m \in \mathbb{Z}} \tilde{R}_{>0}^{(m)}, \quad \text{where } \tilde{R}_{\geqslant 0}^{(m)} \text{ is the span of the monomials of degree } m.$$

A diagonal matrix $d = \mathrm{diag}(d(s)) \in \mathcal{D}_{\mathbb{N}}(\tilde{R}_{>0})$ is called *homogeneous* of degree m if all $d(s), s \in \mathbb{N}$, belong to $\tilde{R}_{>0}^{(m)}$. The space of all homogeneous diagonal matrices of degree m is denoted by $\mathcal{D}_{\mathbb{N}}(\tilde{R}_{>0}^{(m)})$. Similarly, an element $p = \sum_{j \leqslant N} p_j S^j, p_j \in \mathcal{D}_{\mathbb{N}}(\tilde{R}_{>0})$ is called homogeneous of degree k if all $p_j \in \mathcal{D}_{\mathbb{N}}(\tilde{R}_{>0}^{(k-j)})$. The matrix $\tilde{\mathcal{M}}$ for example is then homogeneous of degree two. The multiplication on $LT_{\mathbb{N}}(\tilde{R}_{>0})$ is such that the product of homogeneous matrices is again homogeneous and the degrees add up. In particular, all the matrices $\tilde{\mathcal{M}}^i$, $\tilde{\mathcal{C}}_i(S)$ and $\pi_{\leqslant 0}(\tilde{\mathcal{M}}^i)$ are homogeneous of degree $2i$ and each derivation $\tilde{\partial}_i$ maps $\tilde{\mathcal{M}}$ to a homogeneous matrix of degree $2i + 2$.

Next we describe in a purely algebraic way how solutions of the strict $k[S]$-hierarchy in other settings are related to this minimal realization. Consider any setting $(R, \{\partial_i\})$ for this hierarchy and a potential solution $\mathcal{M} \in LT_{\mathbb{N}}(R)$ of the form (10). Each such a matrix $\mathcal{M} = (m_{s,s+r})$ determines uniquely a k-algebra morphism $i_{\mathcal{M}} : \tilde{R}_{>0} \to R$ by substituting $m_{s,s+r}$ for each $\tilde{m}_{s,s+r}, r \leqslant 1$ and $s \in \mathbb{N}$ in the decomposition (25). The map $i_{\mathcal{M}}$ extends to a k-algebra morphism from the matrices $LT_{\mathbb{N}}(\tilde{R}_{>0})$ to $LT_{\mathbb{N}}(R)$ such that

$$i_{\mathcal{M}}(\tilde{\mathcal{M}}) = \mathcal{M} \quad \text{and} \quad i_{\mathcal{M}}(\tilde{\mathcal{C}}_i(S)) = \mathcal{C}_i(S).$$

Assume now that $\mathcal{M}$ is a solution of the Lax equations of the strict $k[S]$-hierarchy, then we have for all $i \geqslant 1$ that

$$\partial_i(\mathcal{M}) = \partial_i \circ i_{\mathcal{M}}(\tilde{\mathcal{M}}) = [\mathcal{C}_i(S), \mathcal{M}] = [i_{\mathcal{M}}(\tilde{\mathcal{C}}_i(S)), i_{\mathcal{M}}(\tilde{\mathcal{M}})] = i_{\mathcal{M}}([\tilde{\mathcal{C}}_i(S), \tilde{\mathcal{M}}]) = i_{\mathcal{M}} \circ \tilde{\partial}_i(\tilde{\mathcal{M}}).$$

Thus, the k-linear maps $\partial_i \circ i_{\mathcal{M}}$ and $i_{\mathcal{M}} \circ \tilde{\partial}_i$ are equal on the coefficients of $\tilde{\mathcal{M}}$ and that implies

$$\partial_i \circ i_{\mathcal{M}} = i_{\mathcal{M}} \circ \tilde{\partial}_i, \quad \text{for all} \quad i \geqslant 1. \tag{26}$$

On the other hand, if the compatibilities (26) hold for the map $i_{\mathcal{M}}$, then one applies these identities to $\tilde{\mathcal{M}}$ and, as $i_{\mathcal{M}}$ is a k-algebra morphism, we obtain the Lax equations for $\mathcal{M}$. So we may conclude

Lemma 6. *The relations (26) for the map $i_{\mathcal{M}}$ are equivalent to $\mathcal{M}$ being a solution of the strict $k[S]$-hierarchy w.r.t. the setting $(R, \{\partial_i\})$.*

5. SCALING TRANSFORMATIONS

We can scale in principle the shift matrix S, the derivations $\{\partial_i\}$ from the setting $(R, \{\partial_i\})$ and the dependent variables

$$\{l_{s,s+r} | r \leqslant 0, s \in \mathbb{N}\} \quad \text{and} \quad \{m_{s,s+r} | r \leqslant 1, s \in \mathbb{N}\}.$$

So, for the $k[S]$-hierarchy we look at scaling transformations of the form

$$S = \alpha \hat{S}, \partial_i = \beta_i \hat{\partial}_i \quad \text{and} \quad l_{s,s+r} = \gamma_r \hat{l}_{s,s+r}, \tag{27}$$

with $\alpha \in k^*, \{\beta_i \in k^* | i \geqslant 1\}$ and $\{\gamma_r \in k^* | r \leqslant 0\}$ and for the strict $k[S]$-hierarchy they look like

$$S = \alpha \hat{S}, \partial_i = \beta_i \hat{\partial}_i, m_{s,s+r} = \varepsilon_r \hat{m}_{s,s+r}, \tag{28}$$

with α and the $\{\beta_i\}$ as above and $\{\varepsilon_r \in k^* | r \leqslant 1\}$. The first two equations in (27) and (28) determine the matrix $\hat{S}$ and the set of commuting derivations $\{\hat{\partial}_i\}$ unambiguously. However, it may be problematic to define the third part of the scaling transformations (27) and (28) on the whole of R. Therefore we pass to the minimal realizations of both hierarchies and we start with that of the $k[S]$-hierarchy: the solution $\tilde{\mathcal{L}}$ in the setting $(\tilde{R}_{\geqslant 0}, \{\tilde{\partial}_i\})$. The third set of equations in

$$S = \alpha \hat{S}, \partial_i = \beta_i \hat{\partial}_i, \quad \text{and} \quad \tilde{l}_{s,s+r} = \gamma_r \hat{\tilde{l}}_{s,s+r}. \tag{29}$$

determines a k-linear isomorphism between $\tilde{R}_{\geqslant 0}$ and itself. Now we substitute the first and the third equation of (29) in $\tilde{\mathcal{L}}$ and get

$$\tilde{\mathcal{L}} = \alpha \hat{S} + \sum_{r \leqslant 0} \alpha^{-r} (\hat{S}^T)^{-r} \gamma_r \mathrm{diag}(\hat{\tilde{l}}_{s,s+r}).$$

Hence, if we take from now on $\gamma_r = \alpha^{1+r}$ for all $r \leqslant 0$, then $\tilde{\mathcal{L}} = \alpha_1 \hat{\tilde{\mathcal{L}}}$, where the matrix $\hat{\tilde{\mathcal{L}}} = \hat{S} + \sum_{r \leqslant 0} (\hat{S}^T)^{-r} \mathrm{diag}(\hat{\tilde{l}}_{s,s+r})$ is an $\mathcal{U}_-$-deformation of $\hat{S}$. The relation $\tilde{\mathcal{L}} = \alpha_1 \hat{\tilde{\mathcal{L}}}$ implies $\tilde{B}_i(S) = \alpha^i \hat{\tilde{B}}_i(\hat{S})$ for all $i \geqslant 1$. Hence, under the present scaling transformation the Lax equations for $\tilde{\mathcal{L}}$ become

$$\tilde{\partial}_i(\tilde{\mathcal{L}}) = \beta_i \alpha \hat{\tilde{\partial}}_i(\hat{\tilde{\mathcal{L}}}) = [\tilde{B}_i, \tilde{\mathcal{L}}] = \alpha^{1+i} [\hat{\tilde{B}}_i(\hat{S}), \hat{\tilde{\mathcal{L}}}]. \tag{30}$$

If we choose, moreover, all $\beta_i = \alpha^i$, then the equations (30) show that $\hat{\tilde{\mathcal{L}}}$ is a solution of the $k[\hat{S}]$-hierarchy in the setting $(\hat{\tilde{R}}, \hat{\tilde{\partial}}_i)$ and by the diagonal equivalence of the Lax equations of the $k[\hat{S}]$-hierarchy and those of the $k[\hat{S}]$-hierarchy the operator $S + \sum_{r \leqslant 0} (S^T)^{-r} \mathrm{diag}(\hat{\tilde{l}}_{s,s+r})$ is a solution of the $k[S]$-hierarchy in the setting $(\hat{\tilde{R}}, \hat{\tilde{\partial}}_i)$. Combining the considerations above for the minimal realization with how this solution relates to solutions in other settings leads to the following scaling invariance for solutions of the $k[S]$-hierarchy.

Theorem 1. *Let $\mathcal{L} = S + \sum_{r\leqslant 0}(S^T)^{-r} diag(l_{s,s+r})$ solve the $k[S]$-hierarchy in the setting $(R, \{\partial_i\})$. For $\alpha \in k^*$, we consider the scaling transformation (27) with $\beta_i = \alpha^i$, $i \geqslant 1$ and $\gamma_r = \alpha^{r+1}, r \leqslant 0$. Then, substitution of this transformation to $\mathcal{L}$ yields an operator $S + \sum_{r\leqslant 0}(S^T)^{-r} \times diag(\hat{l}_{s,s+r})$ that is a solution of the $k[S]$-hierarchy in the setting $(R, \{\hat{\partial}_i\})$.*

We proceed in the strict $k[S]$-case in a similar way and start with the solution $\tilde{\mathcal{M}}$ in the setting $(\tilde{R}_{>0},\{\tilde{\partial}_i\})$. The transformations we are interested in, are

$$S = \alpha\hat{S}, \partial_i = \beta_i\hat{\partial}_i, \tilde{m}_{s,s+r} = \varepsilon_r\hat{\tilde{m}}_{s,s+r} \quad \text{and} \quad \tilde{m}^{-1}_{s,s+1} = \varepsilon_1^{-1}\hat{\tilde{m}}^{-1}_{s,s+1}. \tag{31}$$

The last two sets of equations determine again a k-linear isomorphism between $\tilde{R}_{>0}$ and itself. Now we substitute the first and the third equation of (31) in $\tilde{\mathcal{M}}$ and get

$$\tilde{\mathcal{M}} = \varepsilon_1 \mathrm{diag}(\hat{\tilde{m}}_{s,s+1})\alpha\hat{S} + \sum_{r\leqslant 0}\varepsilon_r\alpha^{-r}(\hat{S}^T)^{-r}\mathrm{diag}(\hat{\tilde{m}}_{s,s+r}).$$

Hence, if we take from now on $\varepsilon_r = \varepsilon_1\alpha^{1+r}$ for all $r \leqslant 0$, then $\tilde{\mathcal{M}} = \alpha\varepsilon_1\hat{\tilde{\mathcal{M}}}$, where the matrix $\hat{\tilde{\mathcal{M}}} = \mathrm{diag}(\hat{\tilde{m}}_{s,s+1})\hat{S} + \sum_{r\leqslant 0}(\hat{S}^T)^{-r}\mathrm{diag}(\hat{\tilde{m}}_{s,s+r})$ is an $\mathcal{P}_-$-deformation of $\hat{S}$. The relation $\tilde{\mathcal{M}} = \varepsilon_1\alpha\hat{\tilde{\mathcal{M}}}$ implies $\tilde{C}_i(S) = \varepsilon_1^i\alpha^i\hat{\tilde{C}}_i(\hat{S})$ for all $i \geqslant 1$. Hence, under the present scaling transformation the Lax equations for $\tilde{\mathcal{M}}$ become

$$\tilde{\partial}_i(\tilde{\mathcal{M}}) = \beta_i\varepsilon_1\alpha\hat{\tilde{\partial}}_i(\hat{\tilde{\mathcal{M}}}) = [\tilde{C}_i(S), \tilde{\mathcal{M}}] = \varepsilon_1^{i+1}\alpha^{1+i}[\hat{\tilde{C}}_i(\hat{S}), \hat{\tilde{\mathcal{M}}}]. \tag{32}$$

If we choose, moreover, all $\beta_i = \varepsilon_1^i\alpha^i$, then the equations (32) show that $\hat{\tilde{\mathcal{M}}}$ is a solution of the $k[\hat{S}]$-hierarchy in the setting $(\hat{\tilde{R}}, \hat{\tilde{\partial}}_i)$ and by the diagonal equivalence of the Lax equations of the strict $k[\hat{S}]$-hierarchy and those of the strict $k[S]$-hierarchy the operator $\mathrm{diag}(\hat{\tilde{m}}_{s,s+1})S + \sum_{r\leqslant 0}(S^T)^{-r}\mathrm{diag}(\hat{\tilde{m}}_{s,s+r})$ is a solution of the strict $k[S]$-hierarchy in the setting $(\hat{\tilde{R}}, \hat{\tilde{\partial}}_i)$. Combining the considerations above for the minimal realization of the strict $k[S]$-hierarchy with how this solution relates to solutions in other settings leads to the following scaling invariance for solutions of the strict $k[S]$-hierarchy

Theorem 2. *Let $\mathcal{M} = diag(m_{s,s+1})S + \sum_{r\leqslant 0}(S^T)^{-r} diag(m_{s,s+r})$ solve the $k[S]$-hierarchy in the setting $(R, \{\partial_i\})$. For $\alpha \in k^*$ and $\varepsilon_1 \in k^*$, we consider the scaling transformation (27) with $\beta_i = \varepsilon_1^i\alpha^i$, $i \geqslant 1$ and $\varepsilon_r = \varepsilon_1\alpha^{1+r}, r \leqslant 0$. Then, substitution of this transformation into $\mathcal{M}$ yields an operator*

$$\mathrm{diag}(\hat{m}_{s,s+1})S + \sum_{r\leqslant 0}(S^T)^{-r}\mathrm{diag}(\hat{m}_{s,s+r})$$

that is a solution of the strict $k[S]$-hierarchy in the setting $(R, \{\hat{\partial}_i\})$.

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