Intergenerational sharing of unhedgeable inflation risk

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Abstract

We explore how members of a collective pension scheme can share inflation risks in the absence of suitable financial market instruments. Using intergenerational risk-sharing arrangements, risks can be allocated better across the scheme’s participants than would be the case in a strictly individual- or cohort-based pension scheme, as these can only lay off risks via existing financial market instruments. Hence, intergenerational sharing of these risks enhances welfare. In view of the sizes of their funded pension sectors, this would be particularly beneficial for the Netherlands and the U.K.

Keywords:
Pension funds
Intergenerational risk-sharing
Unhedgeable inflation risk
Incomplete markets
Welfare loss

1. Introduction

One of the major debating points when it comes to the reform of collective pension arrangements is how to allocate investment risk to specific age-cohorts to better exploit the risk-bearing capacity of the individual age-cohorts. However, pension fund participants cannot be protected against all financial risks using the available market instruments. For example, inflation risk and long-term interest rate risk are largely unhedgeable financial risks, as the financial instruments necessary to (fully) hedge these risks are illiquid or non-existent in the market. Chen et al. (2020) indicate that for retirees the welfare loss from unhedgeable inflation risk can vary between 1%-8% of certainty-equivalent consumption depending on the participants’ risk preferences. While inflation has undershot its target of below, but close to, 2% in the eurozone for a long time, recently inflation has jumped to levels that had become hard to imagine after the worldwide shift to independent central banks following inflation-targeting policies. The rise in inflation is likely a combination of ultraloose monetary policies, pent-up demand following corona and restricted supply due to corona lockdowns and the war in Ukraine. The future path of inflation is unclear. What is clear, though, is that inflation uncertainty has increased substantially and that the materialization of inflation risks can have large consequences for the returns on savings and disposable incomes, especially for the poorer parts of the population.

The current paper focuses on the question how participants of a collective pension scheme can share inflation risks that cannot be hedged, or only be hedged incompletely, using existing market instruments. In comparison with an individual-based pension scheme such risk-sharing arrangements within a collective scheme allow for a better allocation of risks, thereby raising participants’ welfare, in particular by effectively mitigating financial market incompleteness.1

1 Mandatory participation in such collective schemes may be essential to reap the benefits of intergenerational risk-sharing, as individuals may decide to walk away from the arrangement when they have to pay. Beetsma et al. (2012) and Romp and Beetsma (2020) demonstrate that only in the presence of risk aversion it is possible to maintain a collective scheme based on voluntary participation.

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Many countries feature collective funded pension schemes.\textsuperscript{2} These schemes typically aim to index benefits to wage inflation, price inflation or a combination of the two. However, such indexation tends to be conditional on the financial situation of the pension fund. For example, Wang and Lu (2019) and Zhu et al. (2021) derive optimal designs for this type of pension scheme, where both contributions and pension benefits are adjusted based on the funding level. However, even in cases of unconditional indexation, protection against inflation risk cannot be perfect, if there is no external backstop guaranteeing the pensions. Hence, the participants in these schemes are generally exposed to inflation risk. The obvious remedy would be for the fund to hold financial instruments that hedge against inflation risk. However, in at least two dimensions financial markets tend to be incomplete with regard to inflation risks. First, there may be no swap contracts or index-linked bonds for the country-specific consumer price index (CPI). This is the case, for example, for the Netherlands, where inflation is at best hedged using foreign indexed debt or European inflation swaps, which are only partially correlated with Dutch inflation (Chen et al., 2020).\textsuperscript{3} Moreover, the liquidity and outstanding volumes of foreign index-linked bonds and inflation swaps are limited. Therefore, the investor is still exposed to real interest rate risk, because with limited availability of index-linked bonds the duration of the bond portfolio can still not be matched with the liabilities (Booth and Yakoubov, 2000). Even in countries that issue indexed debt and even if liquidity is high,\textsuperscript{4} CPI risks can only be imperfectly hedged due to limited supply, frequent mispricing (Fleckenstein et al., 2014) and frictions caused by standard contract maturities. Second, even if it were possible to perfectly hedge CPI inflation, workers and retirees would still be exposed to the inflation risks associated with their own specific consumption bundles, to the extent that these bundles differ from the bundle consumed by the average population member.

This raises the question to what extent an “internal” market for inflation risk hedging among pension fund participants can substitute for the relevant missing instruments in the financial market. Such an internal market should a priori have the potential to generate welfare gains, as the indexation of working cohorts’ pension entitlements to their nominal salary already provides substantial protection due to the correlation between nominal wage growth and inflation. This suggests that there is room for a mutually beneficial trade: the working generation receives a risk premium from retirees who buy the inflation risk protection from them and/or receive similar protection from future workers when they are retired themselves. In this paper we design such intergenerational risk-sharing arrangements, which are mutually welfare improving by reducing consumption uncertainty. Part of their appeal is that they are easy to implement. To this end we develop a simple two overlapping-generations model in which we embed such intergenerational risk-sharing arrangement. The model is deliberately kept simple to bring out the mechanisms at work as intuitively as possible. We do consider a number of extensions to the baseline set-up, though.

Most closely related to this paper is Chen et al. (2020), which only identifies and quantifies the problem of having incomplete markets in terms of inflation risk, but does not design or analyse concrete risk-sharing arrangements. In this paper, we thus go beyond this earlier work by exploring how to design risk-sharing arrangements within collective funded pension arrangements that address the aforementioned market incompleteness. Related literature on funded pensions has studied the sharing of risks that are also traded on the market. Hence, this literature considers pension fund arrangements that can be replicated in financial markets. However, if pension contracts can be designed to share non-traded risks, pension funds have an edge as the only currently-existing institutions to provide protection against those risks.

Kleinow (2009) demonstrates that risk-neutral valuation only leads to meaningful results if a self-financing portfolio is available, but this is not the case with unhedgeable risk, such as inflation risk. Hence, other valuation techniques apply. Similar to Henderson (2002) we apply utility maximization to value an incomplete market setting. This way, the optimal portfolio weights assigned to the different asset classes depend on the risk appetite of the investor. When markets are incomplete, larger differences in risk appetite amplify the range of feasible prices for non-tradeable products. Empirical evidence suggests that the elderly are fundamentally more risk averse, see, for example, Halek and Eisenhauer (2001); Albert and Duffy (2012) and Dohmen et al. (2017), which would make our proposed risk-sharing arrangement even more attractive.

There exist a number of contributions that apply valuation based on mean-variance hedging in an incomplete market. For example, Thomson (2005) and Barigou and Dhaene (2019) show that fair valuation decomposes into a diversifiable component that is hedgeable and an unhedgeable component that cannot be diversified. Yao et al. (2013) provide analytical investment policies for a pension fund under hedgeable inflation risk based on the mean-variance criterion. However, our paper considers unhedgeable inflation risk. Brennan and Xia (2002) obtain analytical investment policies for a lifecycle investor with hedgeable inflation risk and De Jong (2008) applies this to a pension fund. We include unhedgeable CPI inflation risk similar to Brennan and Xia (2002), but in contrast to their model our model abstracts from interest rate risk, while it adds an additional unhedgeable inflation risk factor that results from workers and pensioners having different consumption bundles.

There are several papers that investigate other sources of unhedgeable risk that are relevant for pension schemes. Young (2003) applies utility indifference pricing to unhedgeable mortality risk. Shen et al. (2019) consider market incompleteness for hedging long-term liabilities by solving a min-max optimization.

The remainder of this paper is organized as follows. Section 2 develops our two overlapping generations framework. Section 3 presents the baseline results, while Section 4 presents the results assuming more realistic wage dynamics. Finally, Section 5 concludes the main text of the paper. The Appendix describes in detail the solution method of our model.

2. A two overlapping generations model

This section presents a simple two overlapping generations framework with unhedgeable inflation risk. Individuals consume and contribute savings to their pension fund, which they draw on once they have retired. Pension savings are invested in various risky assets. Hence, in our model we abstract from pay-as-you-go (PAYG) pension arrangements. The model allows to compare an arrangement with intergenerational risk-sharing to one without intergenerational risk-sharing (“autarky”).

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\textsuperscript{2} For example, Austria, Belgium, Czech Republic, Finland, France, Greece, Hungary, Iceland, Japan, Korea, Portugal, Slovenia, Spain, Switzerland, Turkey, the United States, Canada, Luxembourg and the Netherlands.

\textsuperscript{3} Other examples of countries that aim at indexing pension benefits to inflation in the absence of suitable financial instruments to protect against inflation risk are Austria, Belgium, Czech Republic, Finland, Hungary, Korea, Luxembourg, Portugal, Slovenia, Switzerland and Turkey.

\textsuperscript{4} Only France, the U.S. and the U.K. have liquid index-linked bonds markets. Other countries issuing index-linked bonds are Australia, Brazil, Canada, Germany, Greece, Iceland, Italy, Japan Spain and Sweden (Swinkels, 2012).
The economy  The CPI is given by $\Pi_t$, which evolves as

$$
d\Pi_t = \pi_t dt + \sigma_t dZ_{h.t} + \sigma_d dZ_{u.t},
$$

where $\sigma_t dZ_{h.t}$ denotes the hedgeable component of inflation risk and $\sigma_d dZ_{u.t}$ denotes the unhedgeable component. Both $Z_{h.t}$ and $Z_{u.t}$ are Brownian processes. For example, in the case of the Netherlands, eurozone inflation is hedgeable, while the Netherlands-specific deviation from eurozone inflation cannot be hedged in financial markets. We assume that $dZ_{h.t}$ and $dZ_{u.t}$ are independent of each other, so $dZ_{u.t}$ is unhedgeable indeed.

Stock prices are also lognormally distributed with returns given by

$$
dS_t = \mu_S dt + \sigma_S dZ_{S,t}.
$$

Stocks are expected to offer a certain level of protection against inflation, since nominal revenues would generally increase in tandem with inflation. However, the presence of inflation also tends to introduce higher volatility and a higher risk premium in the stock market (Campbell and Vuolteenaho, 2004). They show that historical data suggest that periods of high inflation have been associated with lower real returns on equities. Hence, we assume that the Brownian process $Z_{S,t}$ is independent of the inflation risk driven by the hedgeable and unhedgeable components $Z_{h,t}$ and $Z_{u,t}$, respectively.

We do not consider interest rate risk and therefore assume that the bond market has a constant nominal return that is independent of the inflation trend.

$$
dB_{N,t} = rd.t.
$$

We also consider a real bond

$$
dB_{R,t} = rd.t + \sigma_h dZ_{h.t}.
$$

Hence, the nominal return on the real bond moves with the hedgeable component of actual inflation. Similar to Koijen et al. (2010) we assume that the risk premium for inflation risk is zero.\footnote{Campbell and Viceira (2002) and Sangvinatsos and Wachter (2005) argue that identifying the inflation risk premium is cumbersome with the available data.}

Demography and wage income  We refer to generation “$\tau$” as the generation that enters the pension fund as participants at time $t = \tau$. The generation retires at time $t + T^R$ and dies at time $t + T^D$. Hence, during the period $t$ to $t + T^R$, it earns a wage income, while during the period $t + T^R$ to $t + T^D$, it is retired. The number of working years before retirement is half of the total time that each generation spends in the pension scheme, so $T^D = 2 * T^R$. We assume that the nominal wage rate $H_{\tau,t}$ of generation $\tau$ at time $t$ is linked to the price index\footnote{This assumption is relaxed in Section 4 by assuming more realistic wage dynamics.}

$$
H_{\tau,t} = \begin{cases} 
\Pi_{\tau}, & t \in [\tau, \tau + T^R], \\
0, & t \notin [\tau, \tau + T^R].
\end{cases}
$$

We normalize the wage rate of generation 0 at time $t = 0$ to 1, i.e. $H_{0,0} = 1$. Moreover, all generations are of equal size, which we normalize to 1. Every $T^R$ years a new generation enters the pension scheme, the working generation retires and the retired generation dies. For simplicity, we abstract from demographic or political risk affecting the intergenerational risk-sharing arrangement.

Preferences  We assume that agents have a constant relative risk aversion (CRRA) level of $\gamma > 1$, i.e. period utility is given by $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$. Since the individual is concerned about her purchasing power, utility is obtained from nominal consumption divided by the price index.

Lifetime utility of an individual from generation $\tau$ is given by

$$
U_{\tau} = E_t \left[ \int_t^{T^D} e^{-\rho(t-u)} u \left( \frac{C_{\tau,t} \Pi_{\tau,t}}{\Pi_{\tau,t}} \right) du \right]. \tag{6}
$$

Parameter $\rho$ denotes the subjective discount rate and $C_{\tau,t}$ denotes consumption in nominal terms of generation $\tau$ at time $t$. Consumption in real terms is given by $\frac{C_{\tau,t}}{\Pi_{\tau,t}}$, where $\Pi_{\tau,t}$ can be the overall price index (i.e. $\Pi_{\tau,t} = \Pi_t \forall \tau$) or it can be a generation-specific price index, as defined later in equation (17).

Using (6), the corresponding certainty equivalent consumption level is obtained by

$$
U_{\tau} = \int_t^{T^D} e^{-\rho(t-u)} u \left( c_{e\tau} \right) du 
\iff c_{e\tau} = \left( \frac{(1-\gamma) \rho U_{\tau}}{1 - e^{-\rho T^\tau}} \right)^{1/(1-\gamma)}. \tag{7}
$$

This is the constant consumption level at each moment that would provide the individual with expected utility equal to the expected utility in (6) that is derived from the set of future stochastic scenarios and resulting policy settings.

Individual retirement plan (“autarky”)  “Autarky” is defined as the situation in which different generations do not engage in a risk-sharing arrangement. When the generation enters the individual retirement plan at the start of its life, no wealth has been accumulated yet, i.e. $W_{t,0} = 0$. All savings take place via the pension plan.\footnote{Governments often make pension savings relatively attractive by providing certain tax advantages, such as the possibility to deduct pension contributions from income before taxes, while the pension benefits received later are taxed at a lower rate than the income tax rate at the time the contributions were made. We abstract from explicitly modelling such tax advantages.}

Hence, the contribution (in nominal terms) at moment $t$ to the retirement plan of generation $\tau$ is the nominal wage ($H_{\tau,t}$) minus consumption ($C_{\tau,t}$). The individual optimizes its contribution $H_{\tau,t} - C_{\tau,t}$, which is added to its accumulated nominal pension wealth, which evolves as

$$
dW_{t,\tau} = H_{t,\tau} - C_{t,\tau} dt + \theta_{t,\tau}^S dS_t + \theta_{t,\tau}^R dB_{R,t} + \left(1 - \theta_{t,\tau}^S - \theta_{t,\tau}^R\right) dB_{N,t}.
$$

$$
= \begin{cases} 
H_{\tau,t} - C_{\tau,t} dt + \theta_{\tau,t}^S dS_t + \theta_{\tau,t}^R dB_{R,t} & t \in [\tau, \tau + T^R], \\
\theta_{\tau,t}^R \sigma_h dZ_{h,t} & t \notin [\tau, \tau + T^R].
\end{cases} \tag{8}
$$

$$
= \begin{cases} 
H_{\tau,t} - C_{\tau,t} dt + \theta_{\tau,t}^S dS_t + \theta_{\tau,t}^R dB_{R,t} & t \in [\tau, \tau + T^R], \\
\theta_{\tau,t}^R \sigma_h dZ_{h,t} & t \notin [\tau, \tau + T^R].
\end{cases} \tag{9}
$$

where $\theta_{t,\tau}^S$ is the fraction of wealth invested in stocks and $\theta_{t,\tau}^R$ the fraction invested in real bonds. The remainder is invested in nominal bonds. We restrict these allocations to $\theta_{t,\tau}^S, \theta_{t,\tau}^R \in [0,1]$ in line with the portfolios of real-life institutional investors.

From the moment of retirement $\tau + T^R$ of generation $\tau$, accumulated wealth is used to finance consumption. The remaining
pension wealth continues to be invested in financial instruments during retirement.  

In the remainder capital letters denote nominal amounts, while lowercase letters denote real amounts. For example, $h_{\tau,t}$, $w_{\tau,t}$ and $c_{\tau,t}$ are the real counterparts of the nominal amounts $H_{\tau,t}$, $W_{\tau,t}$ and $C_{\tau,t}$, respectively. The real wealth dynamics are given by

\[
dw_{\tau,t} = d(W_{\tau,t}/\Pi_t) \tag{10} 
\]
\[
= W_{\tau,t} d(1/\Pi_t) + (1/\Pi_t) dW_{\tau,t} + d(W_{\tau,t}/\Pi_t) \tag{11} 
\]
\[
\iff \frac{dw_{\tau,t}}{w_{\tau,t}} = \left[ r - \pi + \theta_{\tau,t}^{S} (\mu_{\tau} - r) + (1 - \theta_{\tau,t}^{R}) \sigma_{h}^{2} + \sigma_{u}^{2} \right] dt + \ldots \theta_{\tau,t}^{S} \sigma_{d} dz_{S,t} - (1 - \theta_{\tau,t}^{R}) \sigma_{h} dz_{h,t} - \sigma_{d} dz_{u,t} + \frac{h_{\tau,t} - c_{\tau,t}}{w_{\tau,t}} dt, \forall t \in (\tau + T)^D \tag{12}
\]

where $h_{\tau,t} = 0$, $\forall t \in (\tau + T)^D$. Appendix A.1 describes how to solve the model numerically under autarky regarding the consumption - savings decision and the decision to allocate the savings over the different assets.

**Intergenerational risk-sharing (IGR)** Suppose the different generations participating in the pension fund have a risk-sharing arrangement for the unhedgeable component of inflation risk, where the working generation transfers an amount $f_{\tau,t}$ to the retirees. This can be an infinite horizon risk-sharing arrangement, where each generation shares risks with another generation. It could also be a one-off arrangement, in which current workers and retirees agree on a risk-sharing arrangement for as long as both are alive. In the one-off arrangement these generations can negotiate a risk premium that the working generation receives for insuring the retirees against unhedgeable inflation risk. Also in the infinite horizon setting a risk premium can be considered. However, current workers will retire at some moment and themselves benefit from the insurance provided by the new generation of workers. Hence, in the infinite horizon case, a risk premium paid by the retired generation may not always be necessary to benefit from the risk-sharing arrangement.

The real wealth dynamics of retirees from generation $\tau$ become:

\[
\frac{dw_{\tau,t}}{w_{\tau,t}} = \left[ r - \pi + \theta_{\tau,t}^{S} (\mu_{\tau} - r) + (1 - \theta_{\tau,t}^{R}) \sigma_{h}^{2} + \sigma_{u}^{2} \right] dt + \ldots \theta_{\tau,t}^{S} \sigma_{d} dz_{S,t} - (1 - \theta_{\tau,t}^{R}) \sigma_{h} dz_{h,t} - \sigma_{d} dz_{u,t} + \frac{f_{\tau,t}}{w_{\tau,t}} dt, \forall t \in (\tau + T)^D \tag{13}
\]

We assume the following functional form for the transfer from the generation that enters the pension scheme at $\tau + T$ to generation $\tau$:

\[
df_{\tau,t} = -w_{\tau,t} \theta^{\mu} \mu_{\tau} dt + \theta^{\mu} w_{\tau,t} \sigma_{d} dz_{u,t}, \forall t \in (\tau + T)^D \tag{14}
\]

where $\theta^{\mu} \in [0,1]$ denotes the constant weight that is applied in the agreement. The contract consists of a fixed leg and a floating leg. The fixed leg is given by the constant parameter $\mu_{\tau} \geq 0$, which can be considered a risk premium that the retired generation pays to the working generation. The floating leg is given by the unhedgeable risk $(\sigma_{h} dz_{u,t})$. If $\theta^{\mu} = 0$, we have $df_{\tau,t} = 0$, so no transfer takes place. If $\theta^{\mu} = 1$, all unhedgeable inflation risk is transferred from the retired generation to the working generation, as this implies $df_{\tau,t} = w_{\tau,t} (-\mu_{\tau} dt + \sigma_{d} dz_{u,t})$.

Hence, under the above risk-sharing arrangement retirees’ real wealth evolves as

\[
dw_{\tau,t} = w_{\tau,t} \left[ r - \pi + \theta_{\tau,t}^{S} (\mu_{\tau} - r) + (1 - \theta_{\tau,t}^{R}) \sigma_{h}^{2} + \sigma_{u}^{2} \right] dt + \ldots w_{\tau,t} \left( \theta_{\tau,t}^{S} \sigma_{d} dz_{S,t} - (1 - \theta_{\tau,t}^{R}) \sigma_{h} dz_{h,t} + \sigma_{d} dz_{u,t} \right) - c_{\tau,t} dt, \tag{15}
\]

while that of the working generation evolves as:

\[
dw_{\tau,t} = w_{\tau,t} \left[ r - \pi + \theta_{\tau,t}^{S} (\mu_{\tau} - r) + (1 - \theta_{\tau,t}^{R}) \sigma_{h}^{2} + \sigma_{u}^{2} \right] dt + \ldots w_{\tau,t} \left( \theta_{\tau,t}^{S} \sigma_{d} dz_{S,t} - (1 - \theta_{\tau,t}^{R}) \sigma_{h} dz_{h,t} - \sigma_{d} dz_{u,t} \right) + \frac{f_{\tau,t} - c_{\tau,t}}{w_{\tau,t}} dt, \tag{16}
\]

Appendix A.2 describes how to solve the model numerically with intergenerational risk-sharing, such that participants of the pension fund optimize the consumption - savings decision and the decision to allocate the savings over the different assets.

**Age-dependent inflation** We now consider the case in which different generations face a different price index to evaluate their purchasing power. In particular, retirees typically have a different consumption basket than the working generation. For example, Stewart (2008) shows that the CPIE for elderly and the CPI for workers have moved differently over time. Munnell and Chen (2015) argue that the reference index that is applied to adjust the Social Security in the U.S. should be linked to an index that gives more weight to costs related to medical care, as this is more relevant for the elderly. Fig. 1 shows for the U.S. the CPIs of the elderly (CPI-E) and the workers (CPI-W). Indeed, the trend rise in CPI-E exceeds that of CPI-W. Important for our analysis is that the shocks to the two indices are imperfectly correlated. Chen et al. (2020) estimate that the standard deviation of the quarterly difference between the CPI of workers and elderly is 0.30%, based on data from the Bureau of Labor Statistics (2018). This is more than one-third of the standard deviation of each of CPI-W and CPI-E.

Nominal spending of generation $t$ at time $\tau$ is denoted by

\[
c_{\tau,t} = \left\{ \begin{array}{ll}
P_{\tau,t} & \forall t \in (\tau, \tau + T)^R, \\
P_{R,t} & \forall t \in (\tau + T, \tau + T)^D \end{array} \right. \tag{17}
\]

We follow the approach by Corsetti et al. (2008) in setting the prices of different consumption baskets. In their setup, there are two countries and the price index of the consumption basket is a geometrically-weighted average of the price indices of the individual countries’ production. Here, the CPI is a geometrically weighted average of the prices of the workers’ and retirees’ consumption baskets with weights $(T^D/T) / T^D$ and $T^R/T^D$, respectively, since all generations are equally sized:

\[
\Pi_t = \frac{\Pi_{\tau,t}^T}{\Pi_{R,t}^T} = \frac{T^R}{T^D} \tag{18}
\]
The difference between worker and retiree inflation is given by
\[ d \log \left( \frac{\Pi_{W,t}}{\Pi_{R,t}} \right) = \sigma_R dZ_{R,t}. \]  
\[ (19) \]

We estimate a standard deviation of this difference of 0.60% per annum, similar to the quarterly estimate by Chen et al. (2020).

When worker inflation is lower than CPI inflation, retiree inflation exceeds CPI inflation, and vice versa. Hence, the idiosyncratic component of the age-specific inflation rates, which cannot be hedged with existing financial market instruments, can potentially be offset through a risk-sharing arrangement among the generations. To accommodate this possibility we consider the following functional form for the transfer from the generation that enters at \( T + T^k \) to generation \( t \), which generalises (14):

\[ df_{t,T} = w_{\pi,t,T} \theta^u \left( -\mu_{tp} dt + \sigma_a dZ_{u,t} - \frac{T^R}{T^D} \sigma_R dZ_{R,t} \right), \]

\[ \forall t \in \{ T + T^k, T + T^D \}, \]

where the term \( \frac{T^D}{T^R} \sigma_R dZ_{R,t} \) is an accurate approximation for the term \( d \left( \frac{\Pi_{R,T}}{\Pi_{W,T}} \right)^T \), which refers to the risk from age-dependent inflation that the wealth of retirees is exposed to.\(^{11}\)

The wealth dynamics of the retirees become now

\[ dw_{t,T} = w_{t,T} \left( \frac{\tau - \pi + \theta^u_{t,T} (\mu_S - r) + \left( 1 - \theta^R_{t,T} \right) \sigma_h^2 + \sigma_a^2 - \theta^u \mu_{tp} }{\sigma_R} \right) dt + c_{t,T} dt + 
\]

\[ \ldots w_{t,T} \left[ \theta^u_{t,T} \sigma_a dZ_{u,t} - \left( 1 - \theta^R_{t,T} \right) \sigma_a dZ_{h,t} + (\theta^u - 1) \sigma_a dZ_{u,t} + \frac{T^R}{T^D} \theta^u \sigma_R dZ_{R,t} \right] \]

\[ (21) \]

while the wealth dynamics of the workers become

\[ dw_{t,T} = w_{t,T} \left( \frac{\tau - \pi + \theta^u_{t,T} (\mu_S - r) + \left( 1 - \theta^R_{t,T} \right) \sigma_h^2 + \sigma_a^2 - \theta^u \mu_{tp} }{\sigma_R} \right) dt + c_{t,T} dt + 
\]

\[ \ldots w_{t,T} \left[ \theta^u_{t,T} \sigma_a dZ_{u,t} - \left( 1 - \theta^R_{t,T} \right) \sigma_a dZ_{h,t} + (\theta^u - 1) \sigma_a dZ_{u,t} + \frac{T^R}{T^D} \theta^u \sigma_R dZ_{R,t} \right] \]

\[ (22) \]

3. Results

3.1. Social welfare evaluation

The individual generation members optimise their investment allocations and consumption for given parameters of the IGR contract. A social planner in turn selects the parameters of the IGR contract. Working generations prefer receiving a high risk premium, while retired generations prefer paying a low risk premium within the risk-sharing arrangement. Also the optimal size of the risk-sharing arrangement is generation dependent. In setting the parameters of the IGR contract, the social planner makes a trade-off between the welfare gains of the different generations, resulting into a social optimum. In practice, the social planner can be represented, for example, by the pension fund board with representatives of all generations.

The social welfare function (SW) takes into account the lifetime utility of multiple generations. This includes the generation that retires at the time the intergenerational risk-sharing arrangement is introduced, i.e. the retired generation with lifetime utility denoted by \( U_{T-T^k} \), the generation that enters the pension scheme at the time the intergenerational risk-sharing arrangement is introduced, i.e. the working generation with lifetime utility denoted by \( U_t \), and all future generations. Hence, when evaluating the welfare gain of the first generation of retirees that participates in the arrangement, we compare lifetime welfare when that generation participates in the risk-sharing arrangement only during retirement with lifetime welfare when it does not participate at any moment of its life. Formally, social welfare at time \( t \) is defined as

\[ SW_t = U_t - T^k + \sum_{n \in N} e^{-\zeta(T^k T^k)} U_{t+(n-1)T^k}. \]

The discounting parameter \( \zeta > 0 \) determines how much weight is given to the future generations relative to the current generations. The future generations born at time \( s \), which are generations \( t \) to infinity, are weighted by \( e^{-\zeta(s-t)} \). Hence, this weight converges to

\[ \frac{T^D}{T^R} = t_{max} \]
zero as \( s \) goes to infinity. The corresponding certainty equivalent consumption level (CEC) is derived as follows:

\[
SW_t = \int_0^T e^{-\rho s} u(cec_t) ds + \sum_{n \in \mathbb{N}} \left[ e^{-\gamma n T} \int_0^T e^{-\rho s} u(cec_t) ds \right]
\]

\( \leftrightarrow u(cec_t) = \rho SW_t \frac{1 - e^{-\gamma T}}{1 - e^{-\rho T}} \) (25)

\( \leftrightarrow cec_t = \left[ (1 - \gamma) \rho SW_t \frac{1 - e^{-\gamma T}}{1 - e^{-\rho T}} \right]^{1/(1-\gamma)} \) (26)

This is the constant consumption level at each moment of each cohort that would yield the same level of social welfare as the level derived from the set of future stochastic scenarios and resulting policy settings using (24). Since there is no analytical or closed-form solution for \( SW_t \), the latter is obtained through simulations.

3.2. Parameterisation

Table 1 reports the calibration of the parameters. We set the parameter for the expected hedgeable inflation component at \( \pi = 2\% \), in line with ECB’s inflation target. The volatility of unhedgeable inflation is estimated by Chen et al. (2020) at 0.72% per annum, so we set the volatility of the unhedgeable inflation component at \( \sigma_{\pi} = 0.72\% \). We set the nominal interest rate to \( r = 2\% \), in line with the euro swap rates with tenor 10 to 20 years as of 22 July 2022. This way, we have a zero real interest rate on average \( (r - \pi) = 0 \). The parameters \( \sigma_{\pi}, \sigma_{\pi} \) and \( \mu_{\pi} \) are chosen in line with the estimations by Brennan and Xia (2002). By employing this parameterization, we generate economic scenarios characterized by an average CPI inflation rate of 2\%. However, it is important to note that within the inflation distribution, the lowest quintile exhibits negative values. Consequently, our welfare analysis effectively encompasses deflation scenarios as well.\(^{12}\)

The total time an individual spends in the pension scheme is set to \( T^D = 60 \) years. The number of working years before retirement is half of the total time spent in the pension scheme, so \( T^R = 30 \) years. In line with estimates in the literature (e.g., Beetsma and Schotman (2001) and Meyer and Meyer (2005)) and parameterizations that are typically applied in the literature with similar lifecycle optimization problems (e.g., Brennan and Xia (2002), Gollier (2008), Koijen et al. (2010) and Cui et al. (2011)), the constant relative risk aversion is set to \( \gamma = 5 \) in the benchmark calculations, but will be varied in Section 3.7.1. We assume that the subjective discount rate of the agents, \( \rho \), and the subjective discount rate of the social planner, \( \zeta \), are both equal to the nominal interest rate, i.e., \( \rho = \zeta = r \). Our baseline analysis assumes that working and retired generations feature the same consumption package and thus the same inflation rate, i.e. \( \sigma_{\pi} = 0\% \). This allows us to explore the cost of age-dependent inflation with standard deviation \( \sigma_{\pi} = 0.60\% \) per annum in Section 3.6 onward. We derive the results using 10,000 simulation paths of the economy.

3.3. Autarky

By applying the solution method described in Appendix A.1, we obtain the optimal savings and investment allocations under autarky. Fig. 2 shows the corresponding real wealth and consumption evolution over 60 years based on the optimal allocations. Wealth accumulates through saving out of wage income during the first 30 years and the returns on those savings. The last 30 years of life wealth gradually decreases towards zero. Obviously, due to the structure with two overlapping generations there is little flexibility to produce realistic consumption patterns, but that is also not the focus of this paper. The median consumption level is about 0.6 in the beginning and increases to 0.9 close to the end of life. As individuals become older the bandwith around their median consumption widens. In the final period of their life the 5th and 95th percentiles of the 10,000 simulation paths of consumption are about 0.4, respectively 1.9. The rise in average consumption over lifetime is driven by the fact that consumption becomes more uncertain with age and the individual is risk averse, leading her to build up precautionary savings that are released as she grows older. The certainty equivalent consumption level is 0.635. This implies that the optimal portfolio allocations and consumption result in a lifetime certainty equivalent consumption that is slightly lower than two thirds of annual real wage income.

At the start of the career the optimal fraction of wealth invested in equity is at its upper bound of 100\%, while the optimal fraction in real bonds is at its lower bound of 0\%. In line with standard lifecycle portfolio theory, the (relatively) safe human capital is largest at the start of the career and decreases with age. As a consequence, the financial wealth is invested less risky later in life. Hence, the optimal fraction invested in equity decreases with age, while the optimal fraction invested in real bonds increases with age.

3.4. One-off risk-sharing arrangement

Now we consider a situation in which workers cover (part of) the unhedgeable inflation of the retirees. In itself this is welfare improving for the retirees and welfare deteriorating for workers. However, to compensate the latter group, they receive a risk premium from the retirees in return \( \mu_{\pi} > 0 \). This reduces the appetite of the retirees to shift unhedgeable risk to workers, but makes the latter more willing to do so.

We vary both parameters \( \mu_{\pi} \) and \( \theta^\pi \) to investigate the corresponding welfare effects for both generations in Fig. 3. The left-hand graph represents the working generation, while the right-hand graph represents the generation that is retired when the IGR arrangement is introduced. For \( \theta^\pi = 0 \) there is no IGR, so we are back at the autarky case and the welfare gain is 0\% irrespec-
Table 1
Parameters.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>π</td>
<td>0.02</td>
<td>expected hedgeable inflation</td>
</tr>
<tr>
<td>σ₀</td>
<td>0.027</td>
<td>volatility hedgeable inflation</td>
</tr>
<tr>
<td>σ₁</td>
<td>0.0072</td>
<td>volatility unhedgeable inflation</td>
</tr>
<tr>
<td>μᵤ</td>
<td>[0, γσ₁²]</td>
<td>risk premium IGR contract</td>
</tr>
<tr>
<td>μₛ</td>
<td>0.074</td>
<td>expected equity return</td>
</tr>
<tr>
<td>σ₂</td>
<td>0.158</td>
<td>volatility equity return</td>
</tr>
<tr>
<td>r</td>
<td>0.02</td>
<td>nominal interest rate</td>
</tr>
<tr>
<td>γ</td>
<td>30</td>
<td>number of years working before retirement</td>
</tr>
<tr>
<td>γ</td>
<td>60</td>
<td>number of years in the pension scheme</td>
</tr>
<tr>
<td>γ</td>
<td>5</td>
<td>constant relative risk aversion</td>
</tr>
<tr>
<td>ρ</td>
<td>0.02</td>
<td>subjective discount rate</td>
</tr>
<tr>
<td>ζ</td>
<td>0.02</td>
<td>social planner’s discount rate</td>
</tr>
<tr>
<td>σₓ</td>
<td>0.006</td>
<td>standard deviation of difference between price index of retirees and workers</td>
</tr>
</tbody>
</table>

Fig. 3. Welfare gain for different combinations of the one-off IGR contract (θᵤ) and the risk premium (μᵦᵣ). The left-hand panel shows the optimal allocation for workers, while the right-hand panel shows the optimal allocation for the retirees.

Fig. 4. Optimal value of the one-off IGR contract (θᵤ) given the risk premium (μᵦᵣ). The left-hand panel shows the optimal allocation for workers, while the right-hand panel shows the optimal allocation for the retirees.

Fig. 5. Welfare gain working generation (in %) for different combinations of the one-off IGR contract (θᵤ) and the risk premium (μᵦᵣ). The left-hand panel shows the optimal allocation for workers, while the right-hand panel shows the optimal allocation for the retirees.

Workers prefer a higher risk premium, while retirees prefer a zero risk premium. The question is whether there is a set of parameter combinations for which both groups simultaneously enjoy a welfare gain. In the left-hand panel of Fig. 5 we show for each generation the area for which it experiences a positive welfare gain, while the right-hand panel depicts the area for which both generations enjoy positive welfare gains. The areas are obtained for stepsizes of 10 percentage-points for the allocation of the IGR contract θᵤ and stepsizes of 0.005 percentage-points for the risk premium μᵦᵣ. For θᵤ ∈ [10%, 90%] and for μᵦᵣ ∈ [0.005%, 0.025%] there exist IGR arrangements that make both workers and retirees better off. For example, when θᵤ = 60% and μᵦᵣ = 0.02%, the working generation has a welfare gain of 0.07%, which is the maximum...
gain for the working generation without the retirees having a loss. For \( \theta^u = 60\% \) and \( \mu_{rp} = 0.01\% \), the retirees have their maximum welfare gain of 0.03\%, without the working generation having a loss.

However, it is up to the social planner to make a tradeoff between the welfare gains of the different generations. As described in Section 3.1, the social planner takes into account the welfare of current and future generations, but puts more weight on the current generations by discounting the future generations with a higher discount rate. The left-hand panel of Fig. 6 shows the social planner’s welfare gains. The right-hand panel shows the socially optimal values of \( \theta^u \) for given risk premia. They rise with the risk premium and range from \( \theta^u = 30\% \) to \( \theta^u = 80\% \). When the risk premium becomes too large (\( \geq 0.03\% \)), the social optimum is slightly welfare deteriorating for the retirees.

3.5. Infinite horizon risk-sharing

During working life a generation (partially) covers the unhedgeable inflation risk of the current retired. Subsequently, when the current workers have retired, the next generation of workers (partially) covers the unhedgeable inflation risk of the former. This goes on forever and hence is referred to as infinite horizon risk-sharing. We would expect all current and future generations to benefit from such an arrangement.

Fig. 7 shows the welfare effects of varying the parameters \( \mu_{rp} \) and \( \theta^u \). The left-hand graph represents the working generation, while the right-hand graph represents the generation that is retired when the IGR arrangement is introduced. With \( \theta^u = \mu_{rp} = 0 \) there is no IGR, so the welfare gain is 0% irrespective of the risk premium. For other allocations of the IGR contract (\( \theta^u > 0 \)), we again observe that there are welfare gains for both generations currently alive. Again a risk-sharing arrangement with a larger risk premium is welfare deteriorating for the retired generation, while it is welfare improving for the current working generation and future generations, ceteris paribus.\(^{13} \)

Fig. 8 depicts the optimal allocations of the IGR contract (\( \theta^u \)) given the risk premium (\( \mu_{rp} \)) for current working and future generations (left-hand panel) and for the current retired generation (right-hand panel). For the former group the optimal value of \( \theta^u \) increases with the risk premium, while for the latter group it falls with the risk premium.

The left-hand panel of Fig. 9 depicts the areas with positive welfare gains for the currently-working and future generations and for the currently-retired generation, while the right-hand panel of Fig. 9 depicts the area for which all generations on net benefit from the IGR arrangement. The results indicate that for \( \theta^u \in [10\%, 100\%] \) and for \( \mu_{rp} \in [0\%, 0.025\%] \) there exist IGR arrangements that make all generations better off. This range is wider than under the setting with one-off risk-sharing, as the welfare gains under the infinite horizon risk-sharing setting are larger as well. In effect, risks can be spread over a larger set of generations. When \( \theta^u = 60\% \) and \( \mu_{rp} = 0.02\% \), the working generation has a welfare gain of 0.09\%, which is the maximum gain for the working generation without the retirees having a loss. For \( \theta^u = 70\% \) and \( \mu_{rp} = 0\% \), the retirees have their maximum welfare of 0.07\%, without the working generation having a loss.

\(^{13}\) The evaluation for future generations is (essentially) the same as for working generations. They both face an entire lifetime in the pension scheme with the IGR arrangement and the same economy.
However, it is up to a social planner to make a tradeoff between the welfare gains of the different generations in order to determine the social optimum. In the left-hand panel of Fig. 10 the social planner’s welfare gains from the IGR arrangement are shown. The right-hand panel shows the socially optimal allocation of the IGR contract given the risk premium. The socially optimal value of \( \theta^u \) increases with the risk premium and varies between \( \theta^u = 50\% \) and \( \theta^u = 60\% \). Again, when the risk premium becomes too large (\( \geq 0.03\% \)), the social optimum is slightly welfare deteriorating for the retirees.

### 3.6. Age-dependent inflation

We extend the analysis by assuming age-dependent inflation by setting parameter \( \sigma_R \) to 0.60%. This way we have two sources of unhedgeable inflation risk, for which we investigate the welfare gains from the different risk-sharing arrangements.

#### 3.6.1. Autarky

The certainty-equivalent consumption level with identical consumption baskets of workers and retirees was 0.635 in autarky (see Section 3.3). If we allow for age-dependent inflation, the certainty-equivalent consumption level is marginally lower. The welfare loss associated with age-dependent inflation is -0.04% and that from both sources of unhedgeable risk together is -0.07%, in comparison with an economy without unhedgeable risk (i.e. \( \sigma_R = \sigma_G = 0\% \)).

Fig. 11 shows the welfare effects over the remaining lifetime under different settings for unhedgeable inflation relative to the complete market setting, by considering only CPI inflation containing an unhedgeable component (solid black line), only the age-dependent inflation rate containing an unhedgeable component (dashed red line) and both sources of unhedgeable inflation risk being present (dashed blue line). Welfare is lowest in the latter case. The welfare loss from the unhedgeable component of CPI inflation risk increases with age, because it is more difficult to smooth the price fluctuations when one grows older and the working generation is partially hedged through their salary income which is assumed to be a constant real wage. Moreover, the working generation is currently exposed to the price index for workers, but also exposed to the price index for retirees in the future, while the retired generation is affected by the price index for retirees only. Being exposed to both price indices dampens the risk exposure, as these indices move in opposite directions (see (18)). At the retirement age (i.e. \( T^R = 30 \)), the welfare loss from both sources of unhedgeable inflation risk is 0.14% and the welfare loss ten years before passing away (i.e. \( T^D = 10 = 50 \)) is 0.27%.

#### 3.6.2. Risk-sharing

In Sections 3.4 and 3.5 we evaluated risk-sharing when both workers and retirees featured the same consumption bundle. The welfare gains from our risk-sharing arrangement are larger in the presence of age-dependent inflation, as there is an additional source of unhedgeable risk to be shared – see Fig. 12, which shows the welfare gains from IGR under both the one-off and infinite-horizon arrangement. Varying the risk premium \( (\mu_R) \) has a relatively small welfare effect for both the working and retired generations, while the allocation of the IGR contract \( (\theta^u) \) has a relatively substantial effect. When the allocation to the IGR contract is zero \( (\theta^u = 0) \) there is no welfare gain from risk-sharing. For a strictly positive allocation to the IGR contract \( (\theta^u > 0) \) we see that the workers’ welfare gain from the risk-sharing arrangement becomes larger at first, but falls in \( \theta^u \) for values of \( \theta^u \) close to 1, which implies that too much risk is shifted from retirees to workers compared to what is optimal for workers.

The largest additional welfare gain from age-dependent inflation when compared to the case without age-dependent inflation is obtained at around \( \theta^u = 30\% \) to \( \theta^u = 40\% \) for the working generation and around \( \theta^u = 90\% \) for the retired generation.

The results from the one-off risk-sharing arrangement (left panel) and infinite horizon risk-sharing arrangement (middle panel) are quite similar, as shown in Fig. 12. The additional welfare gains from IGR are slightly larger for the working generation under infinite horizon risk-sharing, as they now also benefit from protection against age-dependent inflation when they have retired themselves.

In order to measure the adequacy of the risk-sharing arrangements, we can relate the welfare gains of the risk-sharing arrangement to the welfare loss of the unhedgeable risks: i.e. \( \left( \frac{\text{gain}}{\text{loss}} - 1 \right) \times 100\% \). The loss from both sources of risk combined is -0.07% for the working generation and -0.14% for the retired generation. Fig. 13 shows that an adequate policy for intergenerational risk-sharing can more than cover the welfare loss from unhedgeable risk for the working generation and future generations (left-hand panel: one-off risk-sharing; middle panel: infinite horizon risk-sharing). Concretely, the subsidy \( (\mu_R) \) from the retired to the working generation allows the latter to invest more wealth over a longer period, i.e. effectively constituting an “inverse PAYG transfer”. Hence, better allocating risks among generations not only covers the loss of unhedgeable risk, but also better allocates risks over generations’ lives. However, for the retired generation (right panel) up to 62% of the welfare loss can be covered, which holds for the case without a risk premium \( (\mu_R = 0) \) and all unhedgeable risk is shared \( (\theta^u = 1) \).

#### 3.6.3. Varying risk-sharing by source of unhedgeable inflation risk

So far, parameter \( \theta^u \) captured the common degree of sharing of both sources of unhedgeable inflation risk. Now we make the arrangement more sophisticated by allowing for different degrees of sharing of the two types of unhedgeable inflation risk. This yields the following transfer from generation \( \tau + T^R \) (workers) to generation \( \tau \) (retirees):

\[
\begin{align*}
\text{df}_{\tau,t} &= W_{\tau,t} \left( - (\theta^{u,1} + \theta^{u,2}) \mu_R dt + \theta^{u,1} \sigma_G dZ_{\tau,t} + \frac{T^R - \tau}{T^D} \theta^{u,2} \sigma_R dZ_{R,t} \right),
\forall t \in (\tau + T^R, \tau + T^D),
\end{align*}
\]

where parameter \( \theta^{u,1} \) captures the degree to which the unhedgeable component of CPI inflation is shared, while parameter \( \theta^{u,2} \) denotes the degree of intergenerational sharing of age-specific inflation risk.

We start with the situation without a risk premium, i.e. \( \mu_R = 0 \), which is shown in the top line of Fig. 14. The welfare gain of the working generation is increasing in the parameters \( \theta^{u,1} \) and \( \theta^{u,2} \).
Fig. 8. Optimal allocation of the infinite horizon IGR contract ($\theta^u$) given the risk premium ($\mu_{rp}$). The left-hand panel shows the optimal allocation for the current working generation and future generations, while the right-hand panel shows the optimal allocation for the current retired generation.

Fig. 9. Welfare gain for different allocations of the infinite horizon IGR contract ($\theta^u$) and different risk premia ($\mu_{rp}$). The left panel presents the positive welfare gains of the current working generation and future generations and of the current retired generation. The right panel presents the welfare gains for contract parameter combinations for which these gains are strictly positive for all current and future generations. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

Fig. 10. Social welfare evaluation of the infinite horizon IGR contract. The left-hand panel shows the social welfare gain relative to no IGR for different allocations of the IGR contract ($\theta^u$) and different risk premia ($\mu_{rp}$). The right-hand panel shows the socially optimal allocation of the IGR contract ($\theta^u$) given the risk premium ($\mu_{rp}$).

for small values of these parameters, but decreasing for larger values. The welfare gains are more sensitive to the degree of sharing of CPI inflation ($\theta^u_1$) than to the degree of sharing of age-specific inflation risk ($\theta^u_2$). This is in line with the fact that the loss from unhedgeable CPI inflation risk is larger than the loss associated with age-specific inflation risk.

Introducing a risk premium raises the benefit of the risk-sharing arrangement for workers (see the second line with $\mu_{rp} = 0.01\%$). This effect rises with the size of the risk premium (see the bottom line with $\mu_{rp} = 0.02\%$). This positive welfare effect for the workers is particularly strong with one-off risk-sharing, as with infinite horizon risk-sharing workers will have to pay the risk premium when they have retired themselves. The opposite holds for the current retirees: a larger risk premium lowers their welfare benefit from risk-sharing.
3.7. Sensitivity analysis

3.7.1. Varying risk aversion

This subsection investigates the welfare losses from unhedgeable inflation risk without and with risk-sharing when we vary the degree of relative risk aversion. Table 2 shows the results for a CRRA ranging from $\gamma = 3$ to $\gamma = 10$, where $\gamma = 3$ is at the lower end of the estimates in the literature (e.g., Beetsma and Schotman (2001)) and $\gamma = 10$ towards the higher end (see also Meyer and Meyer (2005)). The lifetime loss from the absence of risk-sharing ranges from about 0.02% for relative risk aversion level $\gamma = 3$ to 0.24% for $\gamma = 10$. For individuals at the start of their retirement the welfare loss ranges from 0.07% for $\gamma = 3$ to 0.35% for $\gamma = 10$. While not large, these effects are also not negligible.\footnote{The constant relative risk aversion utility function constrains the coefficient of relative risk aversion to equal the reciprocal of the coefficient of intertemporal substitution. The limited flexibility prevents accounting for the equity premium puzzle that is generally observed in equity markets. It seems likely that utility formulations consistent with the observed equity premium puzzle would also produce a higher loss of unhedgeable inflation risk in our context. Our assessment of these losses and of the gains from risk-sharing instruments thus seems to be on the conservative side.}

The risk-sharing arrangements for the different degrees of relative risk aversion are shown in Fig. 15. From the top right ($\gamma = 3$) to the bottom right ($\gamma = 10$), the welfare gains from risk-sharing increase for the retired generation, because they benefit more from risk-sharing when they are more risk averse. The same holds for the working generation with infinite horizon risk-sharing (top middle to bottom middle graphs), because they also benefit from risk-sharing when they have retired themselves. For the one-off risk-sharing arrangement (left graphs), the working generation only profits from the risk premium, which results in a larger welfare gain when risk aversion is lower (top left to bottom left graphs).

3.7.2. Higher levels of inflation and volatility

After many years of being low and stable, inflation has now risen to levels rarely seen over the past decades. While inflation is projected to gradually fall towards its target, the uncertainty around the expected inflation path is much higher than before. Hence, we investigate the risk-sharing arrangements in a setting with twice the baseline inflation volatility, so $\sigma_i = 5.4\%$, $\sigma_r = 1.44\%$ and $\sigma_R = 1.2\%$. In order to maintain the ratio of expected inflation over volatility, we also double the expected inflation rate to $\bar{\pi} = 4\%$. For consistency, we do the same for the nominal interest rate and the subjective discount rates, i.e. $\bar{\delta} = \rho = \zeta = r$. By comparing Fig. 16 with Fig. 11, we observe that the welfare losses from unhedgeable risk have more than doubled. For example, we observe that for a generation that enters the pension fund, the welfare loss increases from 0.07% to 0.31%, while for a generation at retirement age (i.e. at age $T = 30$) it increases from 0.14% to 0.48% and ten years before passing away (i.e. at age $T^D - 10 = 50$) it increases from 0.27% to 0.69%.

The welfare effects from risk-sharing in the setting with higher inflation volatility are shown in Fig. 17 for different allocations of the IGR contract and different risk premia. The results with higher levels of inflation and volatility indicate larger welfare gains, but also higher risk premia to achieve these welfare gains. For the retirees the welfare gains increase by up to 0.26%-point. The retirees’ optimum, for example, is attained for $\theta^D = 100\%$ and $\mu_{RP} = 0$, where the retirees’ welfare gain from IGR amounts to 0.35%, which is about four times the welfare gain under the initial setting for the inflation parameters (see the right-hand panel of Fig. 7). For the working generation the optimal welfare gains from the infinite horizon risk-sharing arrangement increase as well by up to 0.16%-point. The corresponding optimum is achieved at $\theta^W = 60\%$ and $\mu_{RP} = 0.08\%$ with a welfare gain of 0.26% (see the middle panel of Fig. 17). For the one-off risk-sharing arrangement, the welfare gain of the working generation only increases with a strictly positive risk premium, where the corresponding optimum is achieved at $\theta^W = 80\%$ and $\mu_{RP} = 0.08\%$. The corresponding welfare gain of 0.26% (see the left panel of Fig. 17) is 0.18%-point larger than what is obtained in the baseline setting with lower inflation volatility.

3.7.3. Three overlapping generations

In reality, working life is generally longer than the time spent in retirement. Hence, we now investigate a pension scheme with three overlapping generations, where generations work for $T^R = 40$ years and the retirement period is $T^D - T^R = 20$ years. In this setting every 20 years a new generation enters the pension scheme. Hence, there are 2 working generations and 1 retired generation at each moment in time. The intergenerational contract is between the retired generation and the youngest working generation only. The oldest working generation does not participate in the intergenerational risk-sharing arrangement, because there is little benefit from absorbing the unhedgeable risk of the retired generation so close before one’s own retirement. Also, since the remaining human capital is smaller, the risk absorption capacity is smaller.

Lifetime certainty equivalent consumption increases from 0.635 to 0.813 compared to the benchmark setting, because wage income will be received over an extra ten years. The welfare effects of the different settings for unhedgeable inflation are shown in Fig. 18. The welfare loss at retirement age increases from 0.14% to 0.21% as retirement starts at a later age. The lifetime welfare loss from unhedgeable inflation risk shrinks from 0.07% to 0.06%, as the partial hedge from salary income is extended by ten more years - see Fig. 18.

Fig. 19 depicts the welfare gains from risk-sharing in this modified setting. The left-hand graph and the middle graph show the welfare gains of the working generation under, respectively, the one-off risk-sharing arrangement and the infinite horizon risk-sharing arrangement. Since the lifetime welfare loss from unhedgeable risk is slightly lower, the welfare gains from risk-sharing are also slightly lower than for the setting with only two overlapping generations, but the general patterns are similar.
Fig. 12. The panels show the additional welfare gain for different allocations of the IGR contract ($\theta_u$) and different risk premia ($\mu_{rp}$) comparing a setting with age-dependent inflation and a setting without age-dependent inflation.

Fig. 13. The extent of welfare loss that is covered by the welfare gain from intergenerational risk-sharing for different allocations of the IGR contract ($\theta_u$) and different risk premia ($\mu_{rp}$).

Table 2

<table>
<thead>
<tr>
<th>Parameter setting</th>
<th>$\sigma_u$ (in %)</th>
<th>$\sigma_R$ (in %)</th>
<th>Lifetime welfare effect (in %)</th>
<th>$\gamma = 3$</th>
<th>$\gamma = 5$</th>
<th>$\gamma = 7$</th>
<th>$\gamma = 10$</th>
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<tbody>
<tr>
<td>No unhedgeable inflation risk (benchmark)</td>
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<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Unhedgeable inflation risk overall price index</td>
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<td>-0.04</td>
<td>-0.07</td>
<td>-0.15</td>
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<tr>
<td>Age-dependent inflation risk</td>
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<td>-0.07</td>
<td>-0.11</td>
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</tr>
<tr>
<td>Both sources of unhedgeable inflation risk</td>
<td>0.72</td>
<td>0.60</td>
<td>-0.02</td>
<td>-0.07</td>
<td>-0.13</td>
<td>-0.24</td>
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</table>

<table>
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<tr>
<th>Welfare effect during retirement (in %)</th>
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<th>$\gamma = 5$</th>
<th>$\gamma = 7$</th>
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<tr>
<td>No unhedgeable inflation risk (benchmark)</td>
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<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Unhedgeable inflation risk overall price index</td>
<td>0.72</td>
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<tr>
<td>Age-dependent inflation risk</td>
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<tr>
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<td>0.72</td>
<td>0.60</td>
<td>-0.07</td>
<td>-0.14</td>
</tr>
</tbody>
</table>

Note: the welfare effect is relative to the setting without unhedgeable inflation risk, i.e., $\sigma_u = \sigma_R = 0$. The welfare effects during retirement are evaluated at the moment of retirement.
3.8. Age-dependent degree of risk-sharing

In order to investigate whether the risk-sharing arrangement can be improved further, we make parameter $\theta_u$ age-dependent. Therefore, we consider the following form:

\[
\theta_{t, t} = \theta_{\text{start}} + \frac{T^D - T^R}{T^D - T^R} \cdot \frac{t - T^R}{T^D - T^R} \quad \forall t \in (T + T^R, T + T^D),
\]

(28)

Hence, at the retirement age $t = T + T^R$ the degree of risk-sharing is $\theta_{T + T^R} = \theta_{\text{start}}$ and at the age of death $t = T + T^D$ the degree of risk-sharing is $\theta_{T + T^D} = \theta_{\text{end}}$. During retirement the allocation goes linearly from $\theta_{\text{start}}$ to $\theta_{\text{end}}$.

We start with the case of a zero risk premium, i.e. $\mu_{rp} = 0$, which is shown in the top line of Fig. 20, followed by the introduction of a gradually increasing risk premium that benefits the working generation (second to fourth line in the figure). The results do not indicate that $\theta_{\text{start}}$ and $\theta_{\text{end}}$ should differ much from each other to obtain higher welfare gains. We can argue that when approaching retirement it is optimal for the working cohort to have relatively less risk-sharing with the retirees. Since the allocation $\theta_u$ represents the fraction of un hedgeable risk of
Fig. 15. Welfare gain from intergenerational risk-sharing with different relative risk aversion parameters. Top graphs: $\gamma = 3$; second line graphs: $\gamma = 5$; third line graphs: $\gamma = 7$; bottom graphs: $\gamma = 10$. 
the retired generation that is shared with the working generation, this already holds with a constant degree of risk-sharing, as the wealth of retirees that is subject to unhedgeable risk decreases and reaches zero at the age of death, while the wealth of the working generation increases. Table 3 shows that in most cases the retirees prefer a slightly increasing allocation over time, as the last earned nominal wages become less of an implicit hedge of retirees at older ages. For working generations a relatively stable degree of risk-sharing would be optimal depending on the risk premium.

For a given risk premium ($\mu_{rp}$) we can determine the optimal certainty equivalent social welfare measured by (26) for each intergenerational risk-sharing arrangement. Table 4 shows the corresponding welfare gains compared to autarky. The welfare gains with an age-dependent degree of risk-sharing do not indicate much improvement compared to the benchmark case with constant degree of risk-sharing. However, varying risk-sharing by source of risk (Section 3.6.3) can further improve the welfare gains. Because with a positive risk premium, the social welfare gains become larger with infinite horizon risk-sharing and almost four times as large with the one-off risk-sharing arrangement compared to the benchmark case without varying the degree of risk-sharing by source of risk. In the socially optimal IGR arrangements all of the age-dependent inflation risk is shared, while about half of the unhedgeable inflation risk of the overall price index is shared.

4. Stochastic real wage with productivity shocks

In this subsection we drop the assumption of a constant real wage, i.e. $h_{t,\tau} = 1$, allowing for a more realistic setting in which the nominal wage is not perfectly linked to the price index. Following Cocco et al. (2005) we assume that the real wage evolves as:

$$\begin{align*}
h_{t,\tau} &= \begin{cases} 
\exp\left( \nu_{t,\tau} + \kappa_{t,\tau} + \nu_{t,\tau} \right), & t \in [\tau, \tau + T_R^\tau] , \\
0, & t \notin [\tau, \tau + T_R^\tau].
\end{cases}
\end{align*}
$$

(29)

Again $\tau$ refers to the generation and $t$ to the period in time. Further, $\nu_{t,\tau}$ refers to the deterministic age component of the real wage profile, which typically increases with age due to career making. The processes $\kappa_{t,\tau}$ and $\nu_{t,\tau}$ represent an idiosyncratic shock to generation $\tau$ and a shock for all generations in period $t$, respectively. We assume that the deterministic part is given by a polynomial function of age $t - \tau$:

$$g_{t,\tau} = \theta_0 + \theta_1 (t - \tau) + \theta_2 (t - \tau)^2 + \theta_3 (t - \tau)^3.$$  

(30)

We obtain the following estimates using data for the Netherlands on average income per age cohort of the working population (CBS, 2019)

$$\hat{\theta}_0 = 3.3634, \quad \hat{\theta}_1 = 0.0674, \quad \hat{\theta}_2 = -0.0271, \quad \hat{\theta}_3 = 0.0025.$$  

(31)
Fig. 17. Welfare gain for different allocations of the IGR contract ($\theta^w$) and different risk premia ($\mu_{rp}$) in the setting with a higher average level of inflation and its volatility.

Table 4
Social welfare gain at the optimal degree of risk-sharing in percent increase in certainty-equivalent consumption for the different risk-sharing arrangements compared to autarky.

<table>
<thead>
<tr>
<th>One-off risk-sharing arrangement</th>
<th>$\mu_{rp} = 0$</th>
<th>$\mu_{rp} = 0.01%$</th>
<th>$\mu_{rp} = 0.02%$</th>
<th>$\mu_{rp} = 0.03%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark (Section 3.6.2)</td>
<td>0.013</td>
<td>0.025</td>
<td>0.042</td>
<td>0.061</td>
</tr>
<tr>
<td>Varying risk-sharing by source (Section 3.6.3)</td>
<td>0.013</td>
<td>0.052</td>
<td>0.104</td>
<td>0.159</td>
</tr>
<tr>
<td>Age-dependent degree of risk-sharing (Section 3.8)</td>
<td>0.013</td>
<td>0.025</td>
<td>0.042</td>
<td>0.062</td>
</tr>
<tr>
<td>Infinite horizon risk-sharing</td>
<td>$\mu_{rp} = 0$</td>
<td>$\mu_{rp} = 0.01%$</td>
<td>$\mu_{rp} = 0.02%$</td>
<td>$\mu_{rp} = 0.03%$</td>
</tr>
<tr>
<td>Benchmark (Section 3.6.2)</td>
<td>0.052</td>
<td>0.055</td>
<td>0.057</td>
<td>0.059</td>
</tr>
<tr>
<td>Varying risk-sharing by source (Section 3.6.3)</td>
<td>0.052</td>
<td>0.058</td>
<td>0.063</td>
<td>0.068</td>
</tr>
<tr>
<td>Age-dependent degree of risk-sharing (Section 3.8)</td>
<td>0.053</td>
<td>0.055</td>
<td>0.057</td>
<td>0.059</td>
</tr>
</tbody>
</table>

For the persistent shock of real labour income, we assume that $\nu_t$ represents the part of unexpected inflation that is not captured by nominal wage arrangements:

$$\nu_t = (1 - \eta) \left( \sigma_h dZ_{h,t} + \sigma_u dZ_{u,t} \right) + \nu_{t-1}, \text{ with } \eta \in [0, 1]$$

and $v_0 = 0$. \hspace{1cm} (33)

For $\eta = 1$ there is no persistent risk, but only idiosyncratic real wage risk, i.e., $h_{t,t} = \exp (\theta_{t-1} + \nu_{t-1})$. For $\eta < 1$, there is nominal rigidity in wages, similar to the staggered nominal-wage contract first formulated by Taylor (1979), as part of the unexpected inflation (i.e. $\sigma_h dZ_{h,t} + \sigma_u dZ_{u,t}$) is not captured in the nominal wage contracts. Collective labour contracts are typically revised every two to four years and, therefore, unexpected inflation is only gradually made up for to protect the real wage. Nominal rigidity increases the market incompleteness of unhedgeable inflation risk, as the nominal wages become less of an implicit hedge.

4.1. Autarky

Fig. 22 assumes persistent risk ($\eta = 0$). The welfare effects are similar to those presented in Fig. 11; the welfare losses from both sources of unhedgeable risk combined are 0.07% over the entire lifetime and 0.14% at the retirement age (i.e. $T^R = 30$). For other degrees of persistence, for example $\eta = 0.5$ or $\eta = 1$, the corresponding welfare losses are almost the same.

4.2. Risk-sharing

Fig. 23 depicts the welfare gains for different risk-sharing arrangements. The left-hand graph and the middle graph show the welfare gains of the working generation under, respectively, the
one-off risk-sharing arrangement and the infinite horizon risk-sharing arrangement. In general the patterns are similar, but there are two main differences. First, introducing a risk premium has a stronger positive welfare effect for the working generation under the one-off risk-sharing arrangement, as they only receive the risk premium, while under the infinite horizon arrangement they also have to pay the risk premium when they have retired themselves. Second, the welfare gains are larger under the infinite horizon risk-sharing arrangement, because in that setting the working generation also benefits from the protection against unhedgeable risk when they have retired. The right-hand graph shows the welfare gain from risk-sharing of the current retired generation. Again, the risk premium ($\mu_{R}$) is welfare deteriorating for the retired generation.

The risk-sharing arrangement is set up such that workers cover part of the unhedgeable inflation of the retirees. However, the former should also have the capacity to do so. As shown in Fig. 21, the real wage is lowest early in working life. This might limit the degree to which the risks of the retirees can be hedged. Therefore, Fig. 23 only presents the results for the allocation of the IGR contract $\theta^u \in [0.0, 0.8]$, because for values $\theta^u > 0.8$ the wealth of the youngest workers could become negative, a situation not admitted in our setup as consumption cannot fall below zero. Hence, the optimal value of $\theta^u$ is lower than what we have obtained in Section 3. For workers the optimal one-off risk-sharing allocation is in the range $\theta^w \in [0.1, 0.6]$ and the optimal infinite horizon risk-sharing allocation is in the range $\theta^w \in [0.4, 0.5]$, depending on the risk premium.

The labour income of workers can absorb the unhedgeable inflation risk of the retirees to a lesser extent when there are persistent labour income shocks and when wage income is relatively low at the start of the working career. Hence, the welfare gains from the IGR contract are smaller in the presence of this more realistic representation of the real wage process than in the stylised case of a constant real wage considered in Section 3.6.2.15

5. Conclusion

We investigated several intergenerational risk-sharing arrangements in a collective pension scheme aimed at reducing welfare losses from unhedgeable inflation risks. There were two sources of unhedgeable inflation risk: CPI inflation containing an unhedgeable component and age-dependent consumption bundles. Workers are better capable of absorbing both these risks, as their wage income is a natural hedge against inflation. Workers can benefit from the risk-sharing arrangement by receiving a risk premium from the retirees and through an infinite horizon risk-sharing contract by benefitting from shedding unhedgeable inflation risk when they have become retirees themselves. The welfare benefits from risk-sharing are larger for higher levels of risk aversion. For a realistic real wage process the welfare benefits from the risk-sharing arrangement are lower, because the amount of risk that can be shifted is limited by the earnings of individuals in their early working life.

One might view the welfare gains from risk-sharing on the order of 0.2% of lifetime welfare for our standard value of constant relative risk aversion of 5 as rather limited. However, relative to the underlying risk driver, the welfare gains are of the same order of magnitude as what is found in the literature. Consider, for example, Cui et al. (2011). There, the welfare gain in terms of lifetime certainty equivalent consumption is +2.3%, which is larger for two reasons. First, all investment returns are shared intergenerationally, while we only consider the component related to unhedgeable inflation rate risk. Second, they assume a collective buffer that allows generations to relax their borrowing constraint. It is important to notice that with a zero or low risk premium the gains from risk-sharing over the remaining lifetime are substantially larger for the retired. In addition, a number of amendments to our current framework could well make the gains from sharing unhedgeable inflation risk larger. First, we resorted to a standard relative risk aversion utility formulation, which is difficult to reconcile with the size of the observed risk premium on risky assets, such as stocks. Other, non-standard utility formulations may yield larger benefits from sharing unhedgeable inflation risks. Second, inflation volatility may be higher than assumed, as recent experience suggests. The costs of inflation volatility may also manifest themselves in ways that have not been captured in our simple model setup. An example is a contract that takes time to adjust in response to changes in inflation. Third, including a collective buffer in our setup and allowing for borrowing or short-selling would also raise the welfare gains from intergenerational risk-sharing. Finally, in practice it is not possible to perfectly hedge euro-area inflation risk, for example because the range of available instruments is limited and there are transaction costs involved in any hedging activity. These imperfections raise the variance of the unhedgeable inflation component. Future research will include more demographic and other detail in the design of our model. With such a more realistic setup, the model can, for example, be calibrated to the Dutch second-pillar pension scheme and more pre-

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Fig. 19. Welfare gain for different allocations of the IGR contract ($\theta^u$) and different risk premia ($\mu_{R}$) in the 3 overlapping generation setting.
Fig. 20. Welfare gain from intergenerational risk-sharing for different degrees of risk-sharing over one's lifetime. Top line: $\mu_{rp} = 0\%$; second line: $\mu_{rp} = 0.01\%$; third line: $\mu_{rp} = 0.02\%$; bottom line: $\mu_{rp} = 0.03\%$. 
cise estimates of the gains from intergenerational risk-sharing can be obtained.

A vulnerability of intergenerational risk-sharing arrangements is the risk of discontinuity: when the ex-post transfer from one generation to another becomes too large, members of the former may try to walk away from the arrangement (see Beetsma et al. (2012)). Our model takes this implicitly into account, as with the assumed utility function marginal utility tends to infinity when consumption falls to zero. Therefore, social welfare maximisation prevents such contracts from being optimal when there is even only a small probability of IGR transfers becoming so large that consumption gets very close to zero. In practice, policy makers could prevent this by limiting the intergenerational transfers. For example, in the new Dutch pension contract the solidarity buffer that can be organized for intergenerational risk-sharing may not exceed 15% of the total assets of the pension fund. However, with a buffer of this size, there is still plenty of leeway to achieve intergenerational sharing of unhedgeable inflation risk.16

Declaration of competing interest

There is no competing interest.

Data availability

Data will be made available on request.

Appendix A

A.1. Solving method autarky

The Bellman equation is given by

$$V(w_{t, t}, t) = \max_{\theta_{t, t}, c_{t, t}} u(c_{t, t})dt + e^{-\rho dt}E_t \left[V(w_{t+dt, t+dt}, t+dt)\right].$$

(34)

The corresponding first-order conditions with respect to the portfolio allocations are

16 In our baseline analysis with transferring 100% of the unhedgeable risk from retires to workers, the intergenerational transfers are in 90% of the simulations in the range of \([-1.2\%, +1.2\%]\) of the total assets of retirees having just retired. Since the wealth of the retirees decreases during retirement and the wealth of the working generation accumulates, this range shrinks to \([-0.07\%, +0.07\%]\) of the total assets of the pension fund during the last year of the IGR arrangement.
\[
0 = \frac{\partial}{\partial \theta} e^{-\rho dt} E_t \left[ V \left( w_{t,t+dt} - t dt \right) \right]
\]
\[
\Leftrightarrow 0 = \frac{\partial}{\partial \theta} E_t \left[ V \left( (w_{t,t} - c_{t,t}) \left[ 1 + (r - \bar{\pi} + \theta^R_{t,t} (\mu S - r) \right. \right. \right.
\]
\[
+ \left. \left. \left. (1 - \theta^R_{t,t}) \sigma_h^2 + \sigma_u^2 \right) dt + \ldots \right) \sigma_d Z_{S,t} + \left( 1 - \theta^R_{t,t} \right) \sigma_h dZ_{h,t} - \sigma_u dZ_{u,t} \right] + h_{t,t} dt, t + dt \right] \right]
\]
\[
\Leftrightarrow 0 = \frac{\partial}{\partial \theta} E_t \left[ V \left( (w_{t,t} - c_{t,t}) \left[ 1 + (r - \bar{\pi} + h_{t,t} \right) dt + \ldots \theta^R_{t,t} \sigma_u dZ_{u,t} \right] \right] \right] \right] \right] \right]
\]
\[
\Leftrightarrow 0 = \left[ E_t \left( (w_{t,t} - c_{t,t}) dX_{S,t} + V_w \left( w_{t,t+dt}, t + dt \right) \right) \right] \right] \right] \right] \right] \right] \right] \right]
\]
\[
\text{with } dX_{S,t} = (\mu S - r) dt + \sigma_{S} dZ_{S,t}
\]
\[
dX_{t,t} = \sigma^2 dt - \sigma dZ_{t,t}, \ i \in \{h, u\}.
\]

The first-order condition with respect to the optimal consumption is
\[
c_{t,t} = \frac{\Pi_{t+dt}}{\Pi_{t,t} + dt} e^{-\rho dt} E_t \left[ \left[ 1 + (r - \bar{\pi} + h_{t,t}) dt + \theta^R_{t,t} \right] dX_{S,t} \right]
\]
\[
+ \left( 1 - \theta^R_{t,t} \right) dX_{u,t} + V_w \left( w_{t,t+dt}, t + dt \right) \right]
\]
\[
(41)
\]

The envelope theorem yields
\[
V_w \left( w_{t,t+dt}, t + dt \right) = \frac{\Pi_{t+dt}}{\Pi_{t,t} + dt} c_{t,t+dt} \]
\[
(42)
\]

implying the following first-order conditions
\[
0 = \left( w_{t,t} - c_{t,t} \right) E_t \left[ dX_{S,t} \right] \]
\[
0 = \left( w_{t,t} - c_{t,t} \right) E_t \left[ dX_{u,t} \right] \]
\[
(43)
\]
\[
(44)
\]

This system of first-order conditions we solve numerically through backward recursion, using endogenous gridpoints (Carroll, 2006). The step-by-step procedure is as follows:

**Step 1** We generate a grid for wealth after consumption, which we refer to as savings: i.e. \( S_{t,t} = w_{t,t} - c_{t,t} \). This grid is designed to have finer granularity at lower values and gradually coarser granularity at higher values:
\[
\left\{ \exp \left( \frac{0}{n_s} \log (S_{\text{max}}) \right), \exp \left( \frac{1}{n_s} \log (S_{\text{max}}) \right), \ldots \right\}
\]
\[
\exp \left( \frac{n_s}{n_s} \log (S_{\text{max}}) \right) \right] - 1.
\]

Hence, the grid starts at zero and ends at \( S_{\text{max}} \). We assume \( n_s = 101 \) gridpoints.

**Step 2** In the final period it is optimal to consume all wealth, as we assume no bequest motive or after-death utility. We indicate the optimal policies by an asterisk (*), so we write \( c_{t,t}^* = w_{t,t}^* \) for \( t = T + 1 \). Since there are no savings in the final period, there is no portfolio available for optimization.

**Step 3a** In the next-to-final period, so for \( t = T + 1 \), we use a numerical solver to find the optimal portfolio \( \theta_{t,t}^* \) and optimal consumption \( c_{t,t}^* \) from (43)-(45). We do this for each grid point \( S_{t,t} \), as defined in step 1. To solve the first-order conditions, we apply linear interpolation to the optimal consumption obtained from step 2, i.e. \( c_{t,t+1}^* = w_{t,t+1}^* \).

**Step 3b** At \( t = T + 1 \), we determine an endogenous grid. For each grid point \( S_{t,t} \), we have \( w_{t,t}^* \left( S_{t,t} \right) = c_{t,t}^* \left( S_{t,t} \right) + S_{t,t} \). This way, we have optimal consumption \( c_{t,t}^* \) for each grid point on a new (endogenous) grid for wealth \( w_{t,t}^* \).

**Step 4a** For all other periods, so for \( t = T + 1 \) down to \( t = \tau \), we also use a numerical solver to find the optimal portfolio \( \theta_{t,t}^* \) and optimal consumption \( c_{t,t}^* \) from (43)-(45). We do this for each grid point \( S_{t,t} \), as defined in step 1, and to solve the first-order conditions we apply linear interpolation to the optimal next-period policies \( \theta_{t,t+1}^* \) and \( c_{t,t+1}^* \), that are derived for each grid point in the previous iteration from the backward recursion.

**Step 4b** For each period we determine an endogenous grid. For each grid point \( S_{t,t} \) we have \( w_{t,t}^* \left( S_{t,t} \right) = c_{t,t}^* \left( S_{t,t} \right) + S_{t,t} \). This way, we obtain optimal consumption \( c_{t,t}^* \) for each grid point on a new (endogenous) grid for wealth \( w_{t,t}^* \).

**Step 5** We repeat steps 4a and 4b down to \( t = \tau \).
Forward simulation  Once the model has been solved numerically using backward recursion, we can do the welfare analysis by simulating 10,000 economic scenarios, starting at period $t = T$ up to $t = 0$. Each period, we first determine consumption $c_{t,T}$ given wealth $w_{t,T}$ by interpolating the optimal consumption from the wealth grid in steps 3b and 4b. Second, the corresponding savings become $w_{t,T} - c_{t,T}$, which can then be used to determine the portfolio allocations by interpolating the optimal allocation from the saving grid from steps 3a and 4a.

A.2. Solving method IGR

The Bellman equation is again given by

$$V(w_{t,t},t) = \max_{\theta_{t,t},c_{t,t}} u(c_{t,t})dt + e^{-\rho dt} E_t \left[ V(w_{t,t+dt},t+dt) \right].$$

(46)

The first-order conditions with respect to the portfolio allocations are

$$0 = \frac{\partial}{\partial \theta} e^{-\rho dt} E_t \left[ V(w_{t,t+dt},t+dt) \right]$$

(47)

$$\implies 0 = \frac{\partial}{\partial \theta} E_t \left[ \left( w_{t,t} - c_{t,t} \right) \left( 1 + (r - \pi + h_{t,t})dt + \sum_{k=1}^{K} e_{t,k} dX_{s,t} + \left( 1 - \alpha_{t,k} \right) dX_{h,t} + dX_{a,t} \right) 
+ d f_{t,t} - df_{t,T} + dt \right]$$

(48)

$$\implies 0 = \left[ E_t \left[ \left( w_{t,t} - c_{t,t} \right) dX_{s,t} V_w(w_{t,t+dt},t+dt) \right] 
- E_t \left[ \left( w_{t,t} - c_{t,t} \right) dX_{h,t} V_w(w_{t,t+dt},t+dt) \right] \right].$$

(49)

The remainder of the solution method is equivalent to first-order conditions and the step-by-step backward recursion that is described for the autarky case in Appendix A.1.

References


