Conformal symmetry and holographic cosmology

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Chapter 1

Conformal invariance

1.1. Fundamentals

1.1.1. Conformal symmetry

Let $M$ be a smooth manifold and let $g$ and $h$ be two Riemannian metrics on $M$. We say that $g$ and $h$ are conformally equivalent if there exists a smooth scalar function $\Omega : M \rightarrow \mathbb{R}$ such that $g(x) = \Omega^2(x)h(x)$. This is an equivalence relation in space of Riemannian structures on $M$. Its classes of equivalence are called conformal structures on $M$ and the pair $(M, [g])$ is a conformal manifold.

Morphisms in the category of conformal manifolds are smooth maps that preserve their conformal structures. To be precise a conformal map $F : (M, [g]) \rightarrow (N, [h])$ between conformal manifolds $(M, [g])$ and $(N, [h])$ is a smooth function $F : M \rightarrow N$ such that

$$F^* h = \Omega^2 g,$$

where $F^* h(X, Y) = h(F_* X, F_* Y)$ is a pull-back of $h$ via $F$ and $X, Y$ are vector fields on $M$. In a local system of coordinates $x^\mu$ on $M$ this equation reads

$$h_{ij}(F(x))\partial_\mu F^i(x)\partial_\nu F^j(x) = \Omega^2(x)g_{\mu\nu}(x)$$

at each point $x \in M$.

As an important example, consider a $d$-dimensional unit sphere $S^d \subseteq \mathbb{R}^{d+1}$, with the natural metric following from this embedding. Let $x_1, \ldots, x_{d+1}$ denote the standard coordinates on $S^d$ satisfying $\sum_{j=1}^{d+1} x_j^2 = 1$ and denote $N = (0, \ldots, 0, 1)$. Define a stereographic projection $X : S^d \setminus \{N\} \rightarrow \mathbb{R}^d$ by

$$(X_1, \ldots, X_d) = \frac{1}{1 - x_{d+1}}(x_1, \ldots, x_d).$$

(1.1.3)
The stereographic projection is a diffeomorphism and the pull-back of the metric on $S^d$ via its inverse leads to the induced metric on $\mathbb{R}^d$ to be

$$d s^2 = \frac{4}{\left(1 + \sum_{j=1}^d X_j^2\right)^2} \sum_{j=1}^d d X_j^2. \quad (1.1.4)$$

This metric is conformally equivalent to the flat metric $\sum_{j=1}^d d X_j^2$. Therefore the standard metric on a sphere is conformally equivalent to the flat metric on $\mathbb{R}^d$.

In this work we are interested in flat Euclidean field theories, i.e., field theories living on $\mathbb{R}^d$ with a metric $\delta_{\mu\nu}$. Therefore, to simplify further discussion and bring it closer to the physical point of view, we will study conformal maps between open subsets of $\mathbb{R}^d$ with a constant, flat metric $\delta_{\mu\nu}$ and small perturbations around it. It can be shown, [4], that our results remain valid for a general metric. The generalisation amounts to a change of coordinate derivatives into covariant ones.

One can classify all conformal maps $F : \mathbb{R}^d \supseteq M \to \mathbb{R}^d$ by looking at vector fields that generate them. In other words we expand (1.1.1) locally in an infinitesimal parameter $\epsilon$,

$$x'^\mu = F^\mu(x) = x^\mu + \epsilon \xi^\mu + O(\epsilon^2), \quad \Omega(x) = 1 + \epsilon \omega(x) + O(\epsilon^2) \quad (1.1.5)$$

and use the fact that the variation of a metric under the change of coordinates is

$$\delta g_{\mu\nu} = -\epsilon (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu). \quad (1.1.6)$$

Therefore, the equation (1.1.1) leads to

$$-(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) = 2 \omega g_{\mu\nu}. \quad (1.1.7)$$

By taking a trace of this equation one finds that $\omega$ is not arbitrary and must satisfy

$$\omega = -\frac{1}{d} (\partial \cdot \xi). \quad (1.1.8)$$

By applying some derivatives and contractions to (1.1.7) one retrieves a simple differential equation

$$\partial_\mu \partial_\nu \omega = 0. \quad (1.1.9)$$

See [5] for details. The most general solution is

$$\omega = \alpha + \beta_\mu x^\mu \quad (1.1.10)$$

for some number $\alpha$ and a vector $\beta^\mu$. This means that the infinitesimal diffeomorphism $\xi^\mu$ is at most quadratic in $x^\mu$. The detailed analysis leads to the conclusion,
that the most general form of the infinitesimal conformal transformation is a combination of the following maps

(i) translations: \( x^\mu \mapsto x^\mu + \epsilon \xi^\mu \),

(ii) rotations: \( x^\mu \mapsto x^\mu + \epsilon \omega^\mu_\alpha x^\alpha \),

(iii) dilatations or scalings: \( x^\mu \mapsto x^\mu + \epsilon \lambda x^\mu \),

(iv) special conformal: \( x^\mu \mapsto x^\mu + \epsilon \left[ 2(b \cdot x) x^\mu - x^2 b^\mu \right] \),

where \( \lambda \in \mathbb{R} \) is a real number, \( \xi \) and \( b \) are arbitrary vectors and \( \omega \) is an antisymmetric matrix of an infinitesimal rotation.

The first two transformations: translations and rotations are isometries. As such that have \( \Omega = 1 \) in (1.1.1). The dilatation is the expected scaling transformation. Conformal invariance implies scaling invariance as discussed in the introduction. It is more interesting that also the fourth class of transformation appears: the special conformal transformations. They are indeed in many respects very different than maps (i) – (iii), for example they are quadratic, rather than linear in \( x \). As we will see they are quite essential in the further analysis.

It is also possible to integrate the transformations in (1.1.11) or check, that the following maps reduce to them under when the transformation parameters are expanded in \( \epsilon \),

(i) translations: \( x^\mu \mapsto x^\mu + a^\mu \),

(ii) rotations: \( x^\mu \mapsto \Lambda^\mu_\alpha x^\alpha \),

(iii) dilatations or scalings: \( x^\mu \mapsto \lambda x^\mu \),

(iv) special conformal: \( x^\mu \mapsto \frac{x^\mu - x^2 b^\mu}{1 - 2b \cdot x + b^2 x^2} \),

where \( a \) and \( b \) are vectors, \( \lambda \in \mathbb{R}_+ \) is a positive real number and \( \Lambda \) is a matrix of a rotation. Note that parameters \( b \) and \( \lambda \) here are different than ones in (1.1.11). Parameters in (1.1.11) are obtained by the first order expansion \( \epsilon \) of the parameters in (1.1.12),

\[
\begin{align*}
    a^\mu &\mapsto \epsilon \xi^\mu, \\
    \Lambda^\mu_\alpha &\mapsto \delta^\mu_\alpha + \frac{1}{2} \epsilon \omega^\mu_\alpha, \\
    \lambda &\mapsto 1 + \epsilon \lambda, \\
    b^\mu &\mapsto \epsilon b^\mu.
\end{align*}
\]

From the context it should be clear whether we consider the generators of the transformations or finite transformations.

The special conformal transformations, unlike the maps (i) – (iii) in (1.1.12) are not defined in the entire space \( \mathbb{R}^d \). Indeed, for \( x = b/b^2 \) the denominator of (iv) vanishes. There are two ways to solve the problem. The first one is to ignore it and stick with the infinitesimal form of the transformations. In physical context this is a reasonable procedure, since all transformation properties are local and therefore the local analysis is sufficient. The mathematical solution is to embed \( \mathbb{R}^d \) into some compact manifold \( M \) in such a way that all transformations (1.1.12)
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are well-defined as functions \( \mathbb{R}^d \rightarrow M \) and extend to conformal diffeomorphisms \( M \rightarrow M \). Such a construction can be found in [4].

An important property of the special conformal transformations (1.1.12) is that they can be written as a composition of two inversions \( I^\mu(x) \) and a translation by a vector \( b, \ i.e., \)

\[
I^\mu(x) = \frac{x^\mu}{x^2}, \quad \frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} + b^\mu. \tag{1.1.14}
\]

Therefore, in order to analyse the implications of the action of the special conformal transformation, in many cases it is enough to analyse the action of inversions. Also note that the distance of two point after the inversion is

\[
|I(x) - I(y)| = \frac{|x - y|}{xy}. \tag{1.1.15}
\]

1.1.2. Conformal group

We would like to understand the group structure generated by the conformal transformations. Clearly, the \( d \)-dimensional Poincaré group consisting of translations and rotations as well as the multiplicative group \( \mathbb{R}_+ \) generated by dilatations are subgroups of the conformal group.

As usual in such cases, instead of looking at the whole group, one can look at its Lie algebra, \( i.e., \) the algebra of vector fields/differential operators coming from the \( \epsilon \) part of the infinitesimal transformations (1.1.11). The corresponding vector fields are

(i) translations: \( P_\mu = \partial_\mu, \)

(ii) rotations: \( L_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu, \)

(iii) dilatations: \( D = x^\alpha \partial_\alpha, \)

(iv) special conformal: \( K_\mu = 2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu. \tag{1.1.16}\)

The commutation relations of (1.1.16) can be easily worked out,

\[
[D, D] = 0, \quad [P_\mu, P_\nu] = 0, \quad [P_\mu, K_\nu] = 2(\delta_{\mu\nu} D + L_{\mu\nu}),
\]

\[
[D, P_\mu] = -P_\mu, \quad [P_\mu, K_\nu] = \delta_{\mu\nu} P_\mu - \delta_{\mu \nu} P_\nu,
\]

\[
[D, K_\mu] = K_\mu, \quad [P_\rho, L_{\mu\nu}] = \delta_{\rho\nu} P_\mu - \delta_{\rho \mu} P_\nu,
\]

\[
[D, L_{\mu\nu}] = 0, \quad [K_\mu, K_\nu] = 0,
\]

\[
[K_\rho, L_{\mu\nu}] = \delta_{\rho\nu} K_\mu - \delta_{\rho \mu} K_\nu,
\]

\[
[L_{\mu\nu} L_{\rho\sigma}] = \delta_{\mu\rho} L_{\nu\sigma} + \delta_{\nu\sigma} L_{\mu\rho} - \delta_{\nu\rho} L_{\mu\sigma} - \delta_{\mu\sigma} L_{\nu\rho}. \tag{1.1.17}
\]

In order to see the group structure behind the commutation relations (1.1.17) it is convenient to define a new set generators \( J_{AB} \) as follows

\[
J_{\mu\nu} = L_{\mu\nu}, \quad J_{-1\mu} = \frac{1}{2}(P_\mu - K_\mu),
\]

\[
J_{-10} = D, \quad J_{0\mu} = \frac{1}{2}(P_\mu + K_\mu). \tag{1.1.18}
\]
where $J_{AB} = -J_{BA}$ and $A, B \in \{-1, 0, 1, \ldots, d\}$. One can work out that the new generators satisfy

$$[J_{AB}, J_{CD}] = \eta_{AC}J_{BD} + \eta_{BD}J_{AC} - \eta_{AD}J_{BC} - \eta_{BC}J_{AD}$$  \hspace{1cm} (1.1.19)$$

where $\eta = \text{diag}(-1, 1, 1, \ldots, 1)$. These are commutation relations of the group $SO(d + 1, 1)$. Therefore, we have proved that the conformal group in $d$ Euclidean dimensions is locally isomorphic to the symmetry group $SO(d + 1, 1)$ of the $D$-dimensional hyperbolic space as we will discuss in section 5.1.3.

### 1.1.3. Conformal fields

A (classical) field theory is called a \textit{conformal field theory} if all fields transform in some representation of the conformal group. Let us be more precise and give a general definition valid for a curved manifold. It will be useful when we couple the theory to gravity. If $M$ is a Riemannian manifold than the conformal group $G$ is defined by (1.1.1). Let $g \in G$ be a conformal transformation. By $x^\mu$ we denote coordinates on $M$ in some chart and by $x'^\mu = g^\mu(x)$ the coordinates after the transformation $g$ is applied. Consider a field $\phi$ on $M$ with values in a vector space $V$. The field $\phi$ is conformal if there exists a family of representations $R^\phi(x)$ of the conformal group $G$ at each $x \in M$ such that components of the field in coordinates $x^\mu$ and $x'^\mu$ are related by

$$\phi'(x') = R^\phi(x)(g)\phi(x).$$  \hspace{1cm} (1.1.20)$$

This definition is rather imprecise, since it does not say anything about the continuity/smoothness and is local in nature. From the mathematical point of view $\phi$ is a section of an associated vector bundle over $M$ with the structure group being the conformal group. For the details, see [6, 7, 8, 9]. We will disregard mathematical subtleties here.

Infinitesimally, under the action of the infinitesimal symmetry $\delta g$, both space-time coordinates and field transform,

$$x'^\mu = x^\mu + \delta g x'^\mu, \quad \phi'(x') = \phi(x) + \delta g R^\phi(x).$$  \hspace{1cm} (1.1.21)$$

Then the infinitesimal transformation of the field $\phi$ at the same point $x$ is

$$\phi'(x) = \phi(x) - \partial_\mu \phi(x) \delta g x'^\mu + \delta g R^\phi(x)\phi(x).$$  \hspace{1cm} (1.1.22)$$

We define an \textit{infinitesimal transformation} $\delta g \phi$ of the field and a \textit{generator} $G^\phi_g$ of the infinitesimal transformation $\delta g$ as

$$\delta g \phi(x) = \phi'(x) - \phi(x) = -G^\phi_g \phi(x) \delta g.$$  \hspace{1cm} (1.1.23)$$
From (1.1.22) we find
\[ G^\phi_g(\phi(x))\delta g = \partial_\mu \phi(x) \delta_g x'^\mu - \delta_g R^\phi(x). \] (1.1.24)

The first term in this expression is due to the change of coordinates only and is given exactly by the \( \epsilon \) part of the infinitesimal transformations (1.1.11). The second term is determined by the choice of the representation on the target space \( V \). Finally note that we can write the finite transformation \( R^\phi \) in (1.1.20) as
\[ R^\phi(x)(g) = e^{G^\phi_g(x)\delta g}. \] (1.1.25)

For example, the generator of the translations for any field is just
\[ P^\phi_\mu = \partial_\mu. \] (1.1.26)

This follows directly from (1.1.24) and (1.1.11) since for the translations the representation \( R^\phi \) is trivial for any field, so \( \delta R^\phi / \delta g = 0 \).

Since the conformal group has a group of regular rotations of \( \mathbb{R}^d \) as its subgroup, the conformal fields carry a representation of rotations. In other words the conformal fields have a determined value of spin. We define a spin operator \( S_{\mu\nu} \) by the transformation properties of \( \phi \) at \( x = 0 \),
\[ L^\phi_{\mu\nu}(0) = S_{\mu\nu}(0), \] (1.1.27)
where \( S_{\mu\nu} \) is a representation matrix of rotations on the target vector space \( V \). In other words \( S_{\mu\nu} \) is one of the standard representations of the group of rotations, for example for vectors
\[ (S_{\mu\nu})^{\alpha\beta} = \delta^\beta_\mu \delta^\alpha_\nu - \delta^\alpha_\mu \delta^\beta_\nu. \] (1.1.28)
This follows from the transformation property (1.1.24) applied to rotations at \( x = 0 \) and the form of infinitesimal rotations (1.1.11). From this expression one can easily get representations for other tensors. Given a representation \( S_V \) on a vector space \( V \), we have
\[ S_{V \otimes V} = 1 \otimes S_V + S_V \otimes 1, \quad S_{V^*} = -S_V, \] (1.1.29)
where \( V^* \) denotes a dual of \( V \). We will discuss representations for fermions in sections 2.7.5 and 4.3.4.

The action of the operator \( L^\phi_{\mu\nu} \) can be extended to all points by means of the formula (1.1.25). Any generator \( G_g(0) \) defined at \( x = 0 \) can be extended to \( G_g(x) \) via,
\[ (G_g^\phi)(x) = e^{x^\alpha P_\alpha} G_g(0) e^{-x^\alpha P_\alpha} \phi(x), \] (1.1.30)
where we used the fact that $[G_g(0), P_\mu] = 0$. Then the right hand side can be evaluated by means of the Hausdorff formula

$$e^{x^\alpha P_\alpha} A e^{-x^\alpha P_\alpha} = A - x^\alpha [A, P_\alpha] + \frac{1}{2} x^\alpha x^\beta [[A, P_\alpha], P_\beta]$$

$$- \frac{1}{3!} x^\alpha x^\beta x^\gamma [[A, P_\alpha], [P_\beta], P_\gamma] + \ldots$$  \hspace{1cm} (1.1.31)

In this way, using the commutation relations (1.1.17) we find

$$L^\phi_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu + S_{\mu\nu}.$$  \hspace{1cm} (1.1.32)

Finally, we can analyse the consequences of the dilatations and special conformal transformations. As for the rotations, let us start with the rigid transformations at $x = 0$. Since, according to (1.1.17), $D$ commutes with all generators of rotations, then by Schur’s lemma in an irreducible representation of rotations the representation matrix for dilatations is diagonal and takes form $\Delta 1$, where $\Delta \in \mathbb{R}$ is a real number called a conformal dimension of field $\phi$ and $1$ is the identity matrix. Similarly, $[D, K_\mu] = K_\mu$, which implies that at $x = 0$ we have $K_\mu \phi(0) = 0$. By means of (1.1.31) one finds the following actions of the generators of the conformal group

(i) translations: $P^\phi_\mu \phi(x) = \partial_\mu \phi(x),$

(ii) rotations: $L^\phi_{\mu\nu} \phi(x) = (x_\nu \partial_\mu - x_\mu \partial_\nu) \phi(x) + S_{\mu\nu} \phi(x),$

(iii) dilatations: $D^\phi \phi(x) = (x_\alpha \partial_\alpha + \Delta) \phi(x),$

(iv) special conf.: $K^\phi_\mu \phi(x) = (2 x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu + 2 \Delta x_\mu) \phi(x) - 2 x^\alpha S_{\mu\alpha} \phi(x).$  \hspace{1cm} (1.1.33)

Note that the representation of the conformal group in $d \geq 3$ is uniquely determined by specification of the representation of the rotations $S_{\mu\nu}$, i.e., the spin of the representation and the conformal dimension $\Delta$.

Observe that for a conformal field $\phi$ we have,

$$K^\phi \phi(0) = 0.$$  \hspace{1cm} (1.1.34)

This can be viewed as a definition of the conformal field. Indeed, assume that some field $\phi$ with the spin operator $S_{\mu\nu}$ satisfies (1.1.34). Using the Jacobi identity one can show that this implies

$$D^\phi \phi(0) = \Delta \phi(0)$$  \hspace{1cm} (1.1.35)

for some number $\Delta$. Then, the use of the Hausdorff formula (1.1.31) leads to the action (1.1.33).

Furthermore, observe that if a general field $\phi$ is an eigenfunction of $D^\phi$ with
1. Conformal invariance

Let us assume that the eigenvalue $\Delta$, then by means of the commutation relations (1.1.17),

\[
D^\phi P^\phi_\mu \phi(x) = (\Delta + 1)P^\phi_\mu \phi(x),
\]

\[
D^\phi K^\phi_\mu \phi(x) = (\Delta - 1)K^\phi_\mu \phi(x),
\]

\[
D^\phi L^\phi_{\mu\nu} \phi(x) = \Delta L^\phi_{\mu\nu} \phi(x).
\]

Therefore $P^\phi_\mu$ increases the conformal dimension by 1, $K^\phi_\mu$ decreases by 1 and $L^\phi_{\mu\nu}$ does not change the conformal dimension of $\phi$.

Finally, the infinitesimal variations (1.1.24) can be integrated out to the global transformation rule. For a conformal map $x \mapsto x'$ denote

\[
J^\mu_\alpha = \frac{x'^\mu}{x^\alpha}, \quad J = \det(J^\mu_\alpha) = \Omega^d,
\]

where $\Omega$ is defined in (1.1.1). The finite conformal transformations for the tensor field $\phi_{\mu_1 \ldots \mu_m}$ with all indices up and of conformal dimension $\Delta$ are

\[
\phi'_{\mu_1 \ldots \mu_m}(x') = J^{-\Delta} J^\mu_{\alpha_1} \ldots J^\mu_{\alpha_m} \phi_{\alpha_1 \ldots \alpha_m}(x).
\]

For forms with indices down one must replace $J$ by $J^{-1}$. The values of the Jacobian appearing in the transformation property above for various conformal maps are

(i) translations: $J = 1$,

(ii) rotations: $J = 1$,

(iii) dilatations: $J = \lambda^d$,

(iv) special conformal: $J = (1 + 2b \cdot x + b^2 x^2)^{-d}$,

inversions (1.1.14): $J = \lambda^{2d}$.

In particular for any $\lambda \in \mathbb{R}^+$,

\[
\phi'_{\nu_1 \ldots \nu_n}(\lambda x) = \lambda^{-\Delta + m - n} \phi_{\nu_1 \ldots \nu_n}(x).
\]

The last entry in the list above shows the Jacobian for the inversion defined in (1.1.14).

1.1.4. Conserved currents

Assume a classical field theory is given on a flat background $\mathbb{R}^d$ by specifying its action $S$. By Noether theorem, each continuous symmetry of the action implies an existence of a conserved current. In particular, all theories we consider are transnationally invariant, which implies that the theory is invariant under $x^\mu \mapsto x^\mu + \epsilon \xi^\mu$, with a constant vector $\xi^\mu$. The set of conserved currents is indexed by a choice of the vector $\xi^\mu$, $j^\mu_\xi = \xi^\nu \Theta^\mu_\nu$, where

\[
\Theta^\mu_\nu = \frac{\delta L}{\delta (\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu L
\]
is a canonical stress-energy tensor and \( L \) is a Lagrangian which integrates to the action \( S \).

The canonical stress-energy tensor is not necessarily symmetric and therefore is not a stress-energy tensor that appears in Einstein equations of General Relativity. However, it is always possible to obtain a symmetric stress-energy tensor by an addition of a total derivative. Such a procedure does not change the conserved charges, which are obtained by the integration of the stress-energy tensor over appropriate hyperplanes. The symmetrized stress-energy tensor, called a Belinfante stress-energy tensor is

\[
T_{\mu\nu}^B = \Theta_{\mu\nu} + \frac{1}{4} \partial_\alpha (B^{\alpha \mu \nu} + B^{\nu \alpha \mu} + B^{\mu \nu \alpha}),
\]

where

\[
B^{\alpha \mu \nu} = \frac{\delta L}{\delta (\partial_\mu \phi^I)} S_{\nu \alpha} \phi.
\]

Using the fact that \( S_{\mu \nu} = -S_{\nu \mu} \) one can show that the Belinfante tensor is indeed symmetric.

Now we want to analyse the conservation laws following from scaling and special conformal transformations. The Noether theorem states, that the conserved current \( j^\mu \) associated with an infinitesimal symmetry \( \delta g \) is [5],

\[
j^\mu = \Theta^\mu_\alpha \delta g^\alpha - \frac{\delta L}{\delta (\partial_\mu \phi)} \delta g R^\phi(x) \tag{1.1.46}
\]

where \( R^\phi \) is defined by the transformation property (1.1.20). For special conformal transformations we find a set of currents \( j^\mu_b \) indexed by a choice of the vector \( b \) in (1.1.12). Therefore we can define the set of special conformal currents \( j^\mu_b \) by \( j^\mu_b = b^\nu j^\nu_b \). Using (1.1.46) and the transformations we found in the previous section we obtain [10],

\[
j^\mu_D = \Theta^\mu_\alpha \delta g^\alpha + l^\mu,
\]

\[
j^\mu_\nu = \Theta^\mu_\alpha (2x_\nu x^\alpha - x^2 \delta^\mu_\nu) - 2x_\nu k^\mu - 2l^\mu_\nu,
\]

for some functions of fields \( k_\mu \) and \( l_{\mu \nu} \). They can be evaluated explicitly, but they depend on the Lorentz structure of the fields. For a tensor field \( \phi_{\mu_1 \cdots \nu_n} \) we have

\[
k^\mu = (\Delta - m + n) \frac{\delta L}{\delta (\partial_\mu \phi)} \phi,
\]

\[
l^\mu_\nu = - \frac{\delta L}{\delta (\partial_\mu \phi)} S^\mu_\nu \phi.
\]

Applying the derivative to the currents, \( 0 = \partial_\mu j^\mu_D = \partial_\mu j^\mu_D \) we find that the conservation laws imply

\[
\Theta^\mu_\mu = -\partial_\alpha k^\alpha, \quad \partial_\alpha l^\alpha_\nu = -k_\nu.
\]

\[
\tag{1.1.51}
\]
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These conditions give \( \Theta_{\mu} = \partial_\alpha \partial_\beta l^{\alpha \beta} \) and the following tensor

\[
T^{\mu \nu} = T_B^{\mu \nu} + \frac{1}{d-2} \left[ \delta^{\alpha \mu} \delta^{\beta \gamma} \delta^{\nu \delta} + \delta^{\alpha \nu} \delta^{\beta \gamma} \delta^{\mu \delta} - \delta^{\alpha \beta} \delta^{\mu \gamma} \delta^{\nu \delta} - \delta^{\mu \nu} \delta^{\alpha \gamma} \delta^{\beta \delta} \right] \partial_\alpha \partial_\beta l^{\gamma \delta},
\]

(1.1.52)

is a conserved, symmetric and traceless tensor that has the same charges as the canonical stress-energy tensor. By \( T_B^{\mu \nu} \) we denoted the Belinfante tensor (1.1.44).

From now on we will refer to \( T^{\mu \nu} \) as the stress-energy tensor of a conformal field theory. It satisfies

\[
T^{\mu \nu} = T^{\nu \mu}, \quad \partial_\mu T^{\mu \nu} = 0, \quad T = T^{\mu \mu} = 0.
\]

(1.1.53)

Having such a stress-energy tensor in the theory is equivalent to the conformal symmetry.

Note that if we assume the scaling symmetry only, then the conservation of the dilatation current would imply the first equality in (1.1.51) only. This equation is not enough to improve the canonical stress-energy tensor to the traceless one via (1.1.52). The classical theory invariant under scalings but not necessarily under the special conformal transformations is called scale-invariant.

The existence of scale invariance leads also to the conclusion that the Lagrangian of a conformally invariant theory cannot contain dimensionful coupling constants. Indeed, apart from its conformal dimension, any field has also its physical dimension in units of the length \( L \). Assume a dimension of a coupling constant to be \( L^\alpha \) and note that the physical dimension can be also assigned to derivatives as follows

<table>
<thead>
<tr>
<th></th>
<th>Conformal dim.</th>
<th>Physical dim.</th>
</tr>
</thead>
<tbody>
<tr>
<td>fields</td>
<td>( \Delta )</td>
<td>( L^\Delta )</td>
</tr>
<tr>
<td>derivatives</td>
<td>+1</td>
<td>( L )</td>
</tr>
<tr>
<td>couplings</td>
<td>0</td>
<td>( L^\alpha )</td>
</tr>
</tbody>
</table>

For the action to be well-defined, the Lagrangian must have both conformal and physical dimension equal to \( d \). Therefore if a term in a Lagrangian contains a coupling constant, it must be dimensionless, \( \alpha = 0 \).

1.1.5. Weyl invariance

In the previous section we have shown that each conformal field theory possess a stress-energy tensor that is symmetric, conserved and traceless and has the same conserved charges as the canonical stress-energy tensor. There is another possibility
for a construction of such a tensor by coupling the theory to a background metric and taking a functional derivative

\[ T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}} \]  

(1.1.54)

or equivalently

\[ T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}. \]  

(1.1.55)

To motivate it, note that for a symmetric and conserved stress-energy tensor we can write by Noether theorem

\[ 0 = \delta \xi S = \frac{1}{2} \int d^d x \sqrt{g} T^{\mu\nu} \left( \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \right), \]

\[ = -\frac{1}{2} \int d^d x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu}, \]  

(1.1.56)

where in the second line we used (1.1.6). By applying the conformal transformations (1.1.40) and taking the flat space limit one shows that \( T_{\mu\nu} \) is a conformal primary field of the conformal weight \( \Delta = d \), where \( d \) is the dimension of spacetime.

Such a procedure follows yet from a different perspective, when the conserved currents can be obtained by gauging the symmetry and looking at the response of the action in the first order of perturbation of the background gauge field. For a stress-energy tensor, the symmetry to be gauged is a rotational symmetry (Lorentz symmetry) and the gauge field is a Cartan metric connection. By definition, the gauged system is invariant under arbitrary diffeomorphisms,

\[ x \mapsto x', \quad g_{\mu\nu} \mapsto \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}. \]

(1.1.57)

The initial Lorentz invariance is now encoded as the local Lorentz symmetry, which requires that under an infinitesimal diffeomorphism \( x^\mu \mapsto x^\mu + \xi^\mu(x) \) the action is invariant.

A similar procedure can be applied to the scaling symmetry. By coupling the theory to gravity this symmetry extends to the local invariance under

\[ g_{\mu\nu}(x) \mapsto e^{2\sigma(x)} g_{\mu\nu}(x), \]

(1.1.58)

for arbitrary function \( \sigma(x) \). Infinitesimally,

\[ \delta_\sigma g_{\mu\nu} = 2\sigma g_{\mu\nu}. \]  

(1.1.59)

Such an invariance is called a Weyl invariance. Weyl invariance follows from the full conformal symmetry since the variation of the action is

\[ 0 = \delta S = 2 \int d^d x \sqrt{g} T^{\mu\nu}_\mu \sigma, \]  

(1.1.60)
1. Conformal invariance

which is equivalent to the tracelessness of the stress-energy tensor. The converse
is also true: a local Lorentz and Weyl invariant theory is conformally invariant.

When a flat theory is extended to the theory valid for more general back-
grounds, one can extend it in a Weyl invariant way. This may require addition to
the action of some terms that vanish in a flat space limit. Such terms must be
invariant under diffeomorphisms, and therefore must be built up from Riemann
tensors, covariant derivatives and other generally covariant objects. As an example,
consider a free scalar field on a flat background

\[ S = \frac{1}{2} \int d^d x \partial_\mu \phi \partial^\mu \phi. \]  (1.1.61)

If the conformal dimension \( \Delta \) of the field \( \phi \) is \( \frac{d}{2} - 1 \), then this action is invariant
under conformal transformations (1.1.12). The canonical stress-energy tensor is

\[ \Theta_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \delta_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi, \]  (1.1.62)

and while it is symmetric, it is not traceless. By means of the procedure described
in section 1.1.4, one can modify it by the addition of an appropriate full derivative
term in order to find

\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \delta_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi + \frac{d-2}{4(d-1)} (\delta_{\mu\nu} \delta^{\alpha\beta} - \delta_\mu^{\alpha} \delta_\nu^{\beta}) \partial_\alpha \partial_\beta \phi^2. \]  (1.1.63)

This field is traceless when the equations of motion \( \square \phi = 0 \) are used.

Now we would like to couple the system (1.1.61) to the metric in a Weyl
invariant way, which limits to (1.1.61) upon substitution \( g_{\mu\nu} = \delta_{\mu\nu} \). The correct action is

\[ S = \frac{1}{2} \int d^d x \sqrt{g} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{d-2}{4(d-1)} R \phi^2 \right], \]  (1.1.64)

where \( R \) is a Ricci scalar for a background metric \( g_{\mu\nu} \). The Ricci scalar vanishes for
the flat space and this action reproduces (1.1.61). The additional factor is necessary
for the Weyl invariance, as one can see from the following Weyl transformations

\[ \delta_\sigma \sqrt{g} = d \sigma \sqrt{g}, \quad \delta_\sigma R = -2 \sigma R - 2(d-1) \Box \sigma. \]  (1.1.65)

Applying the transformation both to the background metric and the dynamical
field one finds indeed \( \delta_\sigma S = 0 \).

Finally note that the stress-energy stress-energy tensor of the Weyl invariant
theory (1.1.64) is

\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi + \frac{d-2}{4(d-1)} \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu + R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \phi^2 \]  (1.1.66)
1.2. Conformal quantum field theory

1.2.1. Definitions

A quantum field theory can be defined either by a set of quantum operators acting on a Hilbert space of states or by a set of time-ordered correlation functions. Since we want to work mostly in Euclidean signature, it will be much more convenient to use the formulation with correlation functions. Note that by means of the reconstruction theorems one can reinterpret all postulates in terms of quantum fields. The details of the mathematical constructions can be found in [11, 12, 13].

A conformal field theory is a quantum field theory which satisfies the following properties:

1. There exists a set of quantum fields \( \{A_j\} \), which in general is infinite and contains all derivatives of all fields.

2. There exists a subset \( \{O_j\} \subseteq \{A_j\} \) of primary fields that transform covariantly under the action of the conformal group. To be more precise, each field \( O_j \) carries a representation of the conformal group, i.e., it has a definite conformal dimension \( \Delta_j \) and a representation of the group of rotations \( S_{\mu\nu} \). The transformation property for the correlation functions follows from the transformation property of the fields (1.1.20). For scalar conformal primaries one has

\[
\langle O_1(x'_1) \ldots O_n(x'_n) \rangle = \left| \frac{\partial x'_1}{\partial x_1} \right|^{-\Delta_1/d} \ldots \left| \frac{\partial x'_n}{\partial x_n} \right|^{-\Delta_n/d} \langle O_1(x_1) \ldots O_n(x_n) \rangle.
\]

(1.2.1)

For fields of general spin one adds the appropriate transformation matrices as in (1.1.40). Every field in \( \{A_j\} \) can be expressed as linear combinations of the primary fields and their derivatives, called descendant fields.

3. The theory can be coupled to the background matrix \( g_{\mu\nu} \) and has a well-defined symmetric stress-energy tensor, traceless and conserved on-shell, that generates conformal transformations.

and its trace is

\[
T = T^\mu_\mu = \frac{d-2}{2} \left[ \phi \Box \phi - \frac{d-2}{4(d-1)} R \phi^2 \right],
\]

which vanishes by equations of motion. As we can see, (1.1.66) reduces to (1.1.63) on flat background \( g_{\mu\nu} = \delta_{\mu\nu} \). The additional term in the action (1.1.64) is chosen in such a way that it reproduces the correction in (1.1.63) when the derivative with respect to the metric is taken.
4. The vacuum state $|0\rangle$ is invariant under conformal transformations.

Every conformal field theory is Weyl invariant, \textit{i.e.} under the Weyl transformation of the metric $g_{\mu\nu}(x) \mapsto e^{2\sigma(x)}g_{\mu\nu}(x)$,

$$\langle O_1(x_1)\ldots O_n(x_n) \rangle_{g_{\mu\nu}} = e^{-\sigma(x_1)\Delta_1 - \ldots - \sigma(x_n)\Delta_n} \langle O_1(x_1)\ldots O_n(x_n) \rangle_{g_{\mu\nu}},$$

(1.2.2)

for fields $O_j, j = 1, \ldots, n$ of any tensor structure and conformal dimensions $\Delta_j$.

Due to the regularisation issues this statement is valid at non-coincident point only, \textit{i.e.} for $x_i \neq x_j, i \neq j$. The reason is that in order to define a quantum field theory one must use a regularisation scheme that may break some of the symmetries of the theory. After the regulator is removed, the broken symmetries may but do not have to be restored. Throughout this work we will use various types of the dimensional regularisations, which maintain Lorentz invariance, but break Weyl invariance. As we will see in specific examples, the Weyl invariance sometimes is not restored after the regulator is removed. In such cases the violation of Weyl invariance is local, \textit{i.e.}, it affects the correlation functions at coincident points only. Therefore, the correlation functions remain Weyl invariant at non-coincident points, while the generating functional as a functional of a metric fails to be Weyl invariant. Such a behaviour is called a \textit{trace} or \textit{Weyl anomaly}.

In this work we assume that the conformal field theories we consider are invariant under the action of the entire conformal group. However, one can consider quantum field theories invariant under Poincaré group and scalings only. Such field theories are called \textit{scale invariant}. Every conformal field theory is by definition scale invariant, but the opposite is true only for $d = 2$ [14, 10]. In other dimensions scale invariance does not imply conformal invariance. For counterexamples and the detailed discussion see [15, 16, 17].

Alternatively, one can define a theory by a set of axioms to be satisfied by quantum operators in Lorentzian signature. An element $g$ of the Poincaré group can be uniquely decomposed as a pair $(a, \Lambda)$, where $a$ is a vector corresponding to the translation $x \mapsto x + a$ and $\Lambda$ is a matrix of rotation. In a standard quantum field theory one has given a unitary representation $(a, \Lambda) \mapsto U(a, \Lambda)$ of the Poincaré group on the Hilbert space of states such that

$$U(a, \Lambda)O(x)U^{-1}(a, \Lambda) = L(\Lambda^{-1})O(\Lambda x + a),$$

(1.2.3)

where $L(\Lambda^{-1})$ is a representation matrix for rotations. In conformal field theories we require an invariance under the full conformal group. Therefore, we need a representation $g \mapsto U(g)$ of the conformal group on the Hilbert space of states such that the field operators transform accordingly. For dilatations $x \mapsto \lambda x$ the transformation property following from (1.1.40) is

$$U(\lambda)O_{\mu_1 \ldots \mu_m}(x)U^{-1}(\lambda) = \lambda^{\Delta - m + n}O^{\mu_1 \ldots \mu_m}_{\nu_1 \ldots \nu_n}(\lambda x).$$

(1.2.4)
1.2. Conformal quantum field theory

Infinitesimally, this leads to the adjoint representation of the generators of the conformal group on the space of quantum fields. Exact expressions are very similar to (1.1.33),

(i) translations: \[ P_\mu \mathcal{O}(x) = \partial_\mu \mathcal{O}(x), \]
(ii) rotations: \[ R_{\mu\nu} \mathcal{O}(x) = (x_\nu \partial_\mu - x_\mu \partial_\nu) \mathcal{O}(x) + S_{\mu\nu} \mathcal{O}(x), \]
(iii) dilatations: \[ D_\alpha \mathcal{O}(x) = \left( x_\alpha \partial_\alpha + \Delta \right) \mathcal{O}(x), \]
(iv) special conf.: \[ K_\mu \mathcal{O}(x) = \left( 2x_\mu x_\alpha \partial_\alpha - x_\alpha^2 \partial_\mu + 2\Delta x_\mu \right) \mathcal{O}(x) - 2x_\alpha S_{\mu\alpha} \mathcal{O}(x). \]

In particular the defining property of the conformal primary field is a quantum version of (1.1.34),
\[ [K_\mu^\mathcal{O}, \mathcal{O}(0)] = 0. \]

1.2.2. Operator-state correspondence

In any quantum field theory the Hilbert space of states can be generated by the action of smeared field operators on the vacuum state \(|0\rangle\),
\[ |\psi\rangle = f_0|0\rangle + \int d^d x_1 f_1(x_1)\mathcal{O}(x_1)|0\rangle + \int d^d x_1 d^d x_2 f_2(x_1, x_2)\mathcal{O}(x_1)\mathcal{O}(x_2)|0\rangle + \ldots \] (1.2.7)

Here \(f_j\) is the set of smoothing functions that vanish at infinity faster than any polynomial. For the state \(|\psi\rangle\) to be well-defined, only a finite number of the smoothing functions should be non-zero. In principle, by changing smoothing functions, one can generate a class of different states. In a conformal field theory the situation is more treatable. There is a one-to-one correspondence between states and local operators in a CFT. In other words for each state \(|\psi\rangle\) there exists a unique local operator \(\mathcal{O}_\psi\) such that
\[ |\psi\rangle = \mathcal{O}_\psi(0)|0\rangle. \]

In order to prove the operator-state correspondence, consider a general state \(|\psi\rangle\), (1.2.7). By linearity, we can consider a single term, and for simplicity consider only the second term on the right hand side and assume \(\mathcal{O}\) is a scalar. By \(U(\lambda)\) denote the unitary action of the scaling on the Hilbert space as in (1.2.4). By substituting \(x_1 = \lambda y\) under the integral in (1.2.7) we have
\[ |\psi\rangle = \int d^d y \lambda^d f_1(\lambda y) \lambda^{-\Delta} U(\lambda) \mathcal{O}(y)|0\rangle. \] (1.2.9)

Now take \(\lambda \to \infty\) and notice that by the assumption on the support of the smoothing functions
\[ \lim_{\lambda \to \infty} f_1(\lambda y) = \frac{c_1}{\lambda^d} \delta(y), \]
(1.2.10)
where $c_1$ is some constant and $\delta(y)$ is Dirac delta. Therefore

$$|\psi\rangle = c_1 \lim_{\lambda \to \infty} \lambda^{-\Delta} U(\lambda) \mathcal{O}(0)|0\rangle,$$

(1.2.11)

which finishes the proof.

Let us elaborate on the correspondence. First observe that the identity operator is an operator of conformal dimension zero and via the operator-state correspondence it is mapped to the vacuum state $|0\rangle$. Next, if we assume that $\mathcal{O}$ is a primary field of dimension $\Delta$, then the generated state $|\Delta\rangle = \mathcal{O}(0)|0\rangle$ is an eigenstate of the dilatation operator,

$$D^\mathcal{O}|\Delta\rangle = \Delta|\Delta\rangle,$$

(1.2.12)

where we use the definition of $|\Delta\rangle$ and (1.2.5).

### 1.2.3. Operator product expansion

A direct consequence of the operator-state correspondence is the existence of the operator product expansion. Consider a complete basis $|n\rangle$ in the Hilbert space of states. By the operator-state correspondence, each state in the basis corresponds to a local operator in the theory $\mathcal{O}_n$. Operator product expansion is a statement that a product of any two local operators $\mathcal{O}$ and $\mathcal{O}'$ can be expanded as

$$\mathcal{O}(x)\mathcal{O}'(0) = \sum_n c_n(x)\mathcal{O}_n(0),$$

(1.2.13)

under the expectation value. The coefficients $c_n(x)$ are functions depending on the operators $\mathcal{O}$, $\mathcal{O}'$ and a choice of the basis. Indeed, if we consider the state $\mathcal{O}'$ corresponding to the operator $\mathcal{O}'$, we can write

$$\mathcal{O}(x)|\mathcal{O}'\rangle = \sum_n c_n(x)|\mathcal{O}_n\rangle.$$  

(1.2.14)

Then the second use of the operator-state correspondence leads to (1.2.13).

As we know operators in a CFT are ordered into conformal families. Therefore, one can group operators $\mathcal{O}_n$ into a set of primaries $\tilde{\mathcal{O}}_k$ and their descendants. Then the OPE (1.2.13) can be rewritten as

$$\mathcal{O}(x)\mathcal{O}'(0) = \sum_k \left[ c_{k0}(x) \tilde{\mathcal{O}}_k(0) + c_{k1}^{\mu_1}(x) \partial_{\mu_1} \tilde{\mathcal{O}}_k(0) + c_{k2}^{\mu_1\mu_2}(x) \partial_{\mu_1} \partial_{\mu_2} \tilde{\mathcal{O}}_k(0) + \ldots \right],$$

(1.2.15)

where the sum is taken over conformal families. If $\mathcal{O}$ and $\mathcal{O}'$ are conformal primaries of dimensions $\Delta$ and $\Delta'$ respectively, then

$$c_{k\mu_1\ldots\mu_j}^{\mu_1\ldots\mu_j}(\lambda x) = \lambda^{\Delta_k-\Delta_1-\Delta_2+j} c_{k\mu_1\ldots\mu_j}(x).$$

(1.2.16)

This means that the expansion under the sum in (1.2.15) is in fact a Taylor expansion. This observation will allow to calculate the OPE coefficients for specific examples. We will present such an example in section 1.4.1.
1.2.4. Unitarity bounds

So far conformal dimensions were arbitrary real numbers. In a quantum field theory there exists a lower bound on all conformal dimensions, known as a unitarity bound. Any theory with operators of conformal dimensions violating the unitarity bound would be non-unitary in Lorentzian signature and non-positive in Euclidean one.

From section 1.2.2 we know that the identity operator corresponds to the vacuum state of a CFT. This is the ‘lowest energy state’ with respect to the ‘hamiltonian’ $D$, which exponentiates to the ‘evolution operator’ $U(\lambda)$. In other words $\Delta > 0$ for all states in the theory.

Using the operator-state correspondence (1.2.11) we can derive more strict positivity conditions. To do so, introduce polar coordinates in $\mathbb{R}^d$ by writing any point as $(r, e)$, where $e \in S^{d-1}$ is a point on a unit sphere. Define a new ‘radial’ coordinate $\tau = \log r$, which we can view as a diffeomorphism between $\mathbb{R}^d \backslash \{0\}$ and the cylinder $\mathbb{R} \times S^{d-1}$ with $\tau$ parametrising $\mathbb{R}$ and $e$ parametrising the sphere. Note that the metric induced on the cylinder is

$$ds^2 = e^{2\tau}(d\tau^2 + d\Omega_{d-1}^2), \quad (1.2.17)$$

where $d\Omega_{d-1}^2$ is a standard metric on a unit sphere $S^{d-1}$. This means that the conformal field theory on the cylinder parametrised by $\tau$ and $e$ is Weyl equivalent to the flat space theory. The evolution in $\tau$ corresponds to the evolution in the radial direction $r$ and therefore it is generated by the dilatation operator $D$. If one quantises the theory on the cylinder, (so called radial quantisation) then the usual reflection positivity of the Euclidean theory requires

$$O^\dagger(\tau, e) = O(-\tau, e), \quad (1.2.18)$$

so that the norm of the state (1.2.7) is non-negative, $\langle \psi | \psi \rangle \geq 0$. This means that in the radial quantisation,

$$O^\dagger(r, e) = O^\dagger(e^\tau, e) = O(e^{-\tau}, e) = r^{2\Delta}O(r^{-1}, e), \quad (1.2.19)$$

where $\Delta$ is the conformal dimension of $O$. Therefore the state conjugate to $|\psi\rangle$ is the state corresponding to

$$O^\dagger(x) = x^{2\Delta}O(I(x)), \quad (1.2.20)$$

where $I$ is the inversion defined in (1.1.14). From this we can establish conjugation rules for the operators in the conformal algebra. Using the relation (1.1.14) and the fact that $I^{-1} = I$ we find

$$P^\dagger_\mu = IP_\mu I = K_\mu. \quad (1.2.21)$$
The procedure for the extraction of unitarity bounds is straightforward but complex in execution. One considers an expression

\[ A_{\mu_1...\mu_n\nu_1...\nu_n} = \langle O | K_{\mu_1} \cdots K_{\mu_n} P_{\nu_1} \cdots P_{\nu_n} | O \rangle, \]

which is positively definite due to the reflection positivity. By permuting all \( K_\mu \) with \( P_\nu \) one arrives at the expectation value with insertions of \( D \) and \( L_{\mu\nu} \), due to the commutation relations (1.1.17). By means of the representation theory one is able to extract the bound on the lowest eigenvalue of such operators. In such a way the following unitarity bounds for fields of spin \( s \) can be obtained

\[ \Delta \geq \frac{d}{2} - 1, \quad s = 0, \]
\[ \Delta \geq \frac{d-1}{2}, \quad s = \frac{1}{2}, \]
\[ \Delta \geq d + s - 2, \quad s \geq 1. \]

In general one can derive stringent bounds for fields in any representation of the group of rotations. For the detail of the procedure, see [18, 19].

As we have seen in the example in section 1.1.5 a free scalar field has its conformal dimension saturating the bound, \( \Delta = \frac{d}{2} - 1 \). As it was shown in [20], in \( d = 4 \), if the field transforming in the representation \((s, 0)\) or \((0, s)\) of the complexification of the Lorentz group \( sl_2\mathbb{C} \oplus sl_2\mathbb{C} \) saturates the unitarity bound, it is a free field with free field correlation functions.

### 1.3. Ward identities

Ward identities are quantum laws of conservation. The equations of conservation such as \( \partial_\mu j^\mu = 0 \) do not hold in quantum case in general. The reason is that in a classical theory \( \partial_\mu j^\mu = 0 \) holds on shell only. The quantum laws of conservation are stated in terms of Ward identities. These identities express the \( n \)-point functions with the insertion of a divergence of a conserved current in terms of \((n - 1)\)-point correlation functions. In this section we will study the Ward identities for all conformal symmetries.

#### 1.3.1. Canonical Ward identities

Invariance of the field theory under conformal transformations is expressed via (1.2.1) and its obvious generalisation to higher spin fields. Assume now that the correlation functions in the field theory are given by path integrals

\[ \langle O_1(x_1) \cdots O_n(x_n) \rangle = \int D\Phi \, O_1(x_1) \cdots O_n(x_n) e^{-S}, \]
1.3. Ward identities

where $S$ is the action. In such case the infinitesimal version of the invariance means that

$$0 = \delta_{g} \langle O_1(x_1) \ldots O_n(x_n) \rangle$$

$$= - \langle \delta_{g} S O_1(x_1) \ldots O_n(x_n) \rangle + \sum_{j=1}^{n} \langle O_1(x_1) \ldots \delta_{g} O_j(x_j) \ldots O_n(x_n) \rangle. \quad (1.3.2)$$

We assume here that the integration measure in the Feynman integral is invariant under $g$. If this is not the case, anomalies appear. If $g$ is a classical symmetry of the system, then by Noether theorem

$$\delta_{g} S = - \int d^d x \ g(x) \partial_{\mu} j^\mu(x) \quad (1.3.3)$$

where $j^\mu$ is a Noether current. Using Dirac deltas we can write (1.3.2) as

$$\partial_{\mu} \langle j^\mu(x) O_1(x_1) \ldots O_n(x_n) \rangle = - \sum_{j=1}^{n} \delta(x - x_j) \langle O_1(x_1) \ldots \delta_{g} O_j(x) \ldots O_n(x_n) \rangle. \quad (1.3.4)$$

This form of the Ward identity is called local. It expresses the $n$-point function with the insertion of a divergence of a conserved current in terms of $(n - 1)$-point correlation functions. One can integrate both sides of (1.3.4) over $x$ to obtain the global Ward identity,

$$0 = \sum_{j=1}^{n} \langle O_1(x_1) \ldots \delta_{g} O_j(x_j) \ldots O_n(x_n) \rangle$$

$$= \sum_{j=1}^{n} G_{g}(x_j) \langle O_1(x_1) \ldots O_n(x_n) \rangle, \quad (1.3.5)$$

where $G_{g}$ is defined in (1.1.23).

In this section we will analyse the basic consequences of global Ward identities. This is due to the fact that it is much more convenient to analyse the local Ward identities in the context of background fields. We will analyse their properties in sections 1.3.3 and 1.3.4.

Let us see what is implied by the global Ward identities for the conformal symmetries (1.1.12). We can directly use (1.3.5) with the infinitesimal transformations of the fields given by (1.1.33) due to the definition (1.1.23). For example, for translations the global Ward identity is

$$0 = \sum_{j=1}^{n} \frac{\partial}{\partial x_j^\mu} \langle O_1(x_1) \ldots O_n(x_n) \rangle, \quad (1.3.6)$$
which implies that the correlation function depends on the differences \( x_i - x_j \) only. Similarly, the rotational invariance implies that the correlation function depends on the distances \( |x_i - x_j| \) only.

Let now \( O_1, \ldots, O_n \) denote a set of conformal primaries of dimensions \( \Delta_1, \ldots, \Delta_n \) and of arbitrary Lorentz structure. The dilatation Ward identity in position space is especially simple and reads

\[
0 = \left[ \sum_{j=1}^{n} \Delta_j + \sum_{j=1}^{n} x_j^\alpha \frac{\partial}{\partial x_j^\alpha} \right] \langle O_1(x_1) \ldots O_n(x_n) \rangle. \tag{1.3.7}
\]

This identity is easy to solve. Notice that for any constant \( c \), the monomial \( f(x) = cx^{-\alpha} \) is the most general solution to \((x\partial + \alpha)f(x) = 0\). This means that (1.3.7) fixes the degree of the correlation function to be \( \Delta_t = \sum_{j=1}^{n} \Delta_j \). One can write the form of the \( n \)-point function as

\[
\langle O_1(x_1) \ldots O_n(x_n) \rangle = x_{12}^{-\Delta_t} F \left( \frac{x_{ij}}{x_{12}} \right), \tag{1.3.8}
\]

where \( F \) is a function of dimensionless quantities. The distance \( x_{12} \) can be replaced by any other distance or combination of distances in such a way that the total degree remains equal to \( \Delta_t \).

By taking \( G_g \) to be \( K_\mu \) in (1.3.5) we obtain the Ward identity associated with special conformal transformations. For the \( n \)-point function of scalar operators \( O_1, O_2, \ldots, O_n \) it reads

\[
0 = \left[ \sum_{j=1}^{n} \left( 2\Delta_j x_j^\kappa + 2x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} - x_j^2 \frac{\partial}{\partial x_j^\kappa} \right) \right] \langle O_1(x_1) \ldots O_n(x_n) \rangle, \tag{1.3.9}
\]

where \( \kappa \) is a free Lorentz index. For tensor operators one needs to add an additional term to the equation. This term depends on the Lorentz structure, and to write it down, we assume that the tensor \( O_j \) has \( r_j \) Lorentz indices, \( i.e., O_j = O_j^{\mu_1 \ldots \mu_{r_j}} \), for \( j = 1, 2, \ldots, n \). In this case, the contribution

\[
2 \sum_{j=1}^{n} \sum_{k=1}^{r_j} \left[ (x_j)_{\alpha j k} \delta^{\kappa j k} - x_j^{\mu j k} \delta^{\kappa}_{\alpha j k} \right] \times \langle O_1^{\mu_1 \ldots \mu_{r_1} \alpha_1} (x_1) \ldots O_j^{\mu_1 \ldots \alpha_1 \ldots \mu_{r_j}} (x_j) \ldots O_n^{\mu_1 \ldots \mu_{r_n} \alpha_n} (x_n) \rangle \tag{1.3.10}
\]

must then be added to the right-hand side of (1.3.9). The Ward identities associated to the special conformal transformation impose very strong conditions on the correlation functions. We will solve them in section 1.4 in position space and in section 2.4 it will be our starting point for the analysis in momentum space.
1.3. Ward identities

Local Ward identities deliver a different kind of information. In classical field theory divergences of conserved currents vanish on-shell. In quantum field theory they do not vanish, but the expectation values of such operators can be simplified via (1.3.4). We could study the consequences of local Ward identities right now, but it will be much more convenient to discuss them in the context of background fields. The main reason is that the formalism utilising background fields is much more convenient, but the currents it produces usually differ slightly from the canonical ones.

1.3.2. Generating functional and coupling to gravity

In previous section, in order to write down the global Ward identities, we worked directly with correlation functions. From now on we will assume that all correlation functions can be packaged into a generating functional. Assume \( \{ O_j \} \) is a set of conformal primaries of arbitrary Lorentz spin in a given CFT and the dynamics of the CFT is determined by the flat space action \( S_{CFT} \). The generating functional is

\[
Z[\phi^{(j)}_0] = \int D\Phi \exp \left( -S_{CFT} - \sum_j \int d^dx \phi^{(j)}_0 O_j \right), \tag{1.3.11}
\]

where \( \{ \phi^{(j)}_0 \} \) is a set smooth functions rapidly vanishing at infinity known as sources or background fields for corresponding operators. Sources are not dynamical, i.e., we do not integrate over them in the path integral.

Now by taking the functional derivatives with respect to the sources one gets correlation functions, e.g.,

\[
\langle O_1(x_1) O_2(x_2) \rangle = \left. \frac{-\delta}{\delta \phi^{(1)}_0(x_1)} \frac{-\delta}{\delta \phi^{(2)}_0(x_2)} Z[\phi^{(j)}_0] \right|_{\text{All } \phi^{(j)}_0 = 0}, \tag{1.3.12}
\]

We would like to express the invariance of the correlation functions using the generating functional. In this case we must find out the action of the symmetries on the background fields in the same way as we found the action of the conformal and Weyl transformations on the metric in section 1.1.5. From there we know that a CFT can be coupled to gravity and the stress-energy tensor can be obtained by a differentiation with respect to the metric. In other words, the source for stress-energy tensor is the metric. This means that the source for \( T^{\mu\nu} \) can be removed from the explicit list in (1.3.11). Instead, the CFT action becomes a functional of the metric. In a similar fashion we can gauge every other symmetry group \( G \) leading to the conserved current. Then, by taking the derivatives with respect to the corresponding gauge connection, one obtains the correlation functions of the currents. Therefore we assume that every other current corresponding to a global
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Symmetry can be covariantly coupled to a background gauge field via the change of derivatives into covariant derivatives,

$$\partial_\mu \mapsto D^I_\mu = \delta^I_\mu - i A^a_\mu (T^a_R)_{IJ}. \quad (1.3.13)$$

Here we assume that each operator $O$ in the theory transforms in some representation $R_O$ given by a set of matrices $T^a_R$, $a = 1, \ldots, \text{dim} \, G$. Our conventions follow [21], for generators of the group $G$ we have

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}, \quad [T^a, T^b] = i f^{abc} T^c. \quad (1.3.14)$$

where $f^{abc}$ are structure constants of the group and in the adjoint representation we have

$$(T^a_A)^{bc} = -i f^{abc}. \quad (1.3.15)$$

In the remaining part of the thesis we will be interested in correlation functions involving stress-energy tensor, conserved currents and scalar primary operators only, therefore we will limit ourselves to the following form of the generating functional

$$Z[\phi^I_0, A^a_\mu, g^{\mu \nu}] = \int D\Phi \exp \left( -S_{CFT} [A^a_\mu, g^{\mu \nu}] - \sum_j \int d^d x \sqrt{g} \phi^{(j)}_0 O_j \right). \quad (1.3.16)$$

Now the 1-point functions with sources turned on are given by the following functional derivatives,

$$\langle T_{\mu \nu}(x) \rangle = -\frac{2}{\sqrt{g(x)}} \frac{\delta}{\delta g^{\mu \nu}(x)} Z,$$  

$$\langle J^{\mu a}(x) \rangle = -\frac{1}{\sqrt{g(x)}} \frac{\delta}{\delta A^a_\mu(x)} Z,$$  

$$\langle O_j(x) \rangle = -\frac{1}{\sqrt{g(x)}} \frac{\delta}{\delta \phi^{(j)}_0(x)} Z. \quad (1.3.17)$$

By taking more functional derivatives we can obtain higher-point correlation functions, e.g.,

$$\langle T_{\mu_1 \nu_1}(x_1) T_{\mu_2 \nu_2}(x_2) T_{\mu_3 \nu_3}(x_3) \rangle =$$  

$$= -\frac{2}{\sqrt{g(x_1)}} \frac{\delta}{\delta g^{\mu_1 \nu_1}(x_1)} \frac{2}{\sqrt{g(x_2)}} \frac{\delta}{\delta g^{\mu_2 \nu_2}(x_2)} \frac{2}{\sqrt{g(x_3)}} \frac{\delta}{\delta g^{\mu_3 \nu_3}(x_3)} Z[g^{\mu \nu}]$$  

$$+ 2 \frac{\delta}{\delta g^{\mu_1 \nu_1}(x_1)} \frac{\delta}{\delta g^{\mu_2 \nu_2}(x_2)} T_{\mu_3 \nu_3}(x_3) + 2 \frac{\delta}{\delta g^{\mu_1 \nu_1}(x_1)} \frac{\delta}{\delta g^{\mu_2 \nu_2}(x_2)} T_{\mu_3 \nu_3}(x_3 T_{\mu_1 \nu_1}(x_1))$$  

$$+ 2 \frac{\delta}{\delta g^{\mu_1 \nu_1}(x_1)} T_{\mu_3 \nu_3}(x_3) \text{ } T_{\mu_2 \nu_2}(x_2). \quad (1.3.20)$$
Note that here we define the 3-point function of the stress-energy tensor to be
the correlator of three separate stress-energy tensor insertions (and similarly for
other correlators involving conserved currents), rather than the correlator obtained
by functionally differentiating the generating functional with respect to the metric
three times. While the latter definition is used in [22, 23, 24, 25], our definition here
is simpler for direct QFT computations. To convert between the two definitions
simply requires the addition or subtraction of the semi-local terms in the formula
above.

Furthermore note that we always include the source terms in the definition of
the stress-energy tensor. We have

\[ T_{\mu\nu}(x) = T_{\text{CFT}\mu\nu}(x) - g_{\mu\nu} \sum_j \phi_j^{(0)}(x)O_j(x), \]

where \( T_{\text{CFT}\mu\nu} \) is the stress-energy tensor following form the action \( S_{\text{CFT}} \).

Now one would expect that the invariance of the correlation functions under a
given symmetry group \( G \) is

\[ \delta_g Z[g_{\mu\nu}, A_\mu, \phi_0^{(j)}] = 0, \]

for any \( g \in G \), provided we know the transformation properties of the background
fields. We will give these in the following section, however, before we do it, we
must point out that (1.3.22) may not be valid for all symmetries in quantum case.
The reason is that in order to define a quantum field theory one must usually
use a regularisation scheme that may break some of the symmetries of the theory.
After the regulator is removed, the broken symmetries may but do not have to
be restored. Throughout this work we will use various types of the dimensional
regularisations, which maintain Poincaré invariance. Therefore (1.3.22) is valid for
translations and rotations. In general, (1.3.22) receives an anomalous contribution
on its right hand side. Since in case of the conformal symmetry, dilatations and
special conformal transformations can be realised by Weyl transformation, the
anomaly manifests itself when the Weyl transformation \( g_{\mu\nu} \mapsto e^{2\sigma} g_{\mu\nu} \) is taken,

\[ \delta_\sigma Z[g_{\mu\nu}, A_\mu, \phi_0^{(j)}] = \mathcal{W}_\sigma[g_{\mu\nu}, A_\mu, \phi_0^{(j)}], \]

where \( \mathcal{W} \) is a computable, theory dependent functional. In flat space theory one
can also find this anomaly by considering the scalings. It turns out that the the
integration measure in the path integral is not invariant under the scalings, which
leads to the Weyl anomaly.

### 1.3.3. Transverse Ward identities

By coupling the system to the background fields, we can obtain the conserved
currents by a functional differentiation with respect to the background field, rather
than by considering the action of the symmetry on the dynamical fields. We have three local symmetries to consider,

1. Diffeomorphism symmetry that follows from the gauging of the Lorentz symmetry. This symmetry is related to the divergence of the stress-energy tensor $\partial_\mu T^{\mu\nu}$.

2. Other gauge symmetries that follow from global symmetries of the flat space theory. These symmetries are related to the divergence of Noether currents $\partial_\mu j^\mu$.

3. Weyl symmetry. This symmetry is related to the trace of the stress-energy tensor $T_{\mu\nu}$.

In this section we will discuss the first two symmetries and the Weyl symmetry will be analysed in the following section.

Under a diffeomorphism $\xi^\mu$ the sources transform as

$$\delta g^{\mu\nu} = - (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu), \quad (1.3.24)$$
$$\delta A^a_\mu = \xi^\nu \nabla_\nu A^a_\mu + \nabla_\mu \xi^\nu \cdot A^a_\nu, \quad (1.3.25)$$
$$\delta \phi^I_0 = \xi^\nu \partial_\nu \phi^I_0, \quad (1.3.26)$$

where $\nabla$ is a Levi-Civita connection.

Under a gauge symmetry transformation with parameter $\alpha^a$ the sources transform as

$$\delta g^{\mu\nu} = 0, \quad (1.3.27)$$
$$\delta A^a_\mu = - D^a_c \alpha^c = - \partial_\mu \alpha^a - f^{abc} A^b_\mu \alpha^c, \quad (1.3.28)$$
$$\delta \phi^I_0 = - i \alpha^a (T^a_R)^{IJ} \phi^J_0, \quad (1.3.29)$$

where $T^a_R$ are matrices of a representation $R$ and $f^{abc}$ are structure constants of the group $G$. The gauge field transforms in the adjoint representation while $\phi^I$ may transform in any representation $R$. The covariant derivative is $D^{IJ}_\mu = \delta^{IJ}_\mu \partial_\mu - i A^a_\mu (T^a_R)^{IJ}$.

Ward identities follow from the requirement that the generating functional (1.3.16) is invariant under the variations

$$\delta \xi = \int d^d x \left[ - (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu) \frac{\delta}{\delta g^{\mu\nu}} + (\xi^\nu \nabla_\nu A^a_\mu + \nabla_\mu \xi^\nu \cdot A^a_\nu) \frac{\delta}{\delta A^a_\mu} + \xi^\mu \partial_\mu \phi^I_0 \frac{\delta}{\delta \phi^I_0} \right], \quad (1.3.30)$$
$$\delta \alpha = - \int d^d x \left[ (\partial_\mu \alpha^a - f^{abc} A^b_\mu \alpha^c) \frac{\delta}{\delta A^a_\mu} + i \alpha^a (T^a_R)^{IJ} \phi^J_0 \frac{\delta}{\delta \phi^I_0} \right], \quad (1.3.31)$$
so that the transverse Ward identities are
\[
\delta_\xi Z = 0, \quad \delta_\alpha Z = 0. \tag{1.3.32}
\]

Using definitions (1.3.17) - (1.3.19) these lead to the following equations with sources turned on,
\[
0 = D_{\mu}^{ac}(J^{\mu a}) - i(T_R^a)^{IJ}\phi_0^I\langle O^J \rangle
= \nabla_\mu(J^{\mu a}) + f^{abc}A_b^\mu(J^{\mu c}) - i(T_R^a)^{IJ}\phi_0^I\langle O^J \rangle, \tag{1.3.33}
\]
\[
0 = \nabla_\mu(T_{\mu \nu}) + \nabla_\nu A_\mu^a(J^{\mu a}) - \nabla_\mu A_\nu^a(J^{\mu a}) + \partial_\nu\phi_0^I\langle O^I \rangle - A_\nu^a\nabla_\mu(J^{\mu a})
= \nabla_\mu(T_{\mu \nu}) - F_{\mu \nu}^a(J^{\mu a}) + D_{\mu}^{IJ}\phi_0^I\langle O^J \rangle, \tag{1.3.34}
\]

These equations may then be differentiated with respect to the sources to obtain the corresponding Ward identities for higher point functions.

### 1.3.4. Trace Ward identities

In section 1.1.3 we showed that the Lagrangian of a conformally invariant theory cannot contain dimensionful coupling constants. This constraint can be circumvented if we allow ‘position-dependent couplings’, i.e., background fields and if we prescribe the correct transformation properties under Weyl transformation. Assume the operator $O$ has a conformal dimension $\Delta$. For the term
\[
\int d^d x \mathcal{O}_{\nu_1 \ldots \nu_n \mu_1 \ldots \mu_m}^{\mu_1 \ldots \mu_n} \phi_{\mu_1 \ldots \mu_m}^{\nu_1 \ldots \nu_n} (1.3.35)
\]
to be invariant under the scalings, we must have
\[
\phi_{\mu_1 \ldots \mu_m}^{\nu_1 \ldots \nu_n} (\lambda x) = \lambda^{-(d-\Delta+m-n)} \phi_{\mu_1 \ldots \mu_m}^{\nu_1 \ldots \nu_n} (x). \tag{1.3.36}
\]

Therefore under the Weyl transformation parametrised by $\sigma$ a general source transforms as
\[
\delta_\sigma \phi_{\mu_1 \ldots \mu_m}^{\nu_1 \ldots \nu_n} = (d - \Delta + m - n)\phi_{\mu_1 \ldots \mu_m}^{\nu_1 \ldots \nu_n} \sigma. \tag{1.3.37}
\]

For metric, gauge field and scalar source we have familiar transformation rules,
\[
\delta_\sigma g_{\mu \nu} = 2g_{\mu \nu} \sigma, \tag{1.3.38}
\]
\[
\delta_\sigma A_\mu^a = 0, \tag{1.3.39}
\]
\[
\delta_\sigma \phi_0 = (d - \Delta)\phi_0 \sigma. \tag{1.3.40}
\]

Let us first consider the case when (1.3.16) is Weyl anomaly free, i.e., $\delta_\sigma Z = 0$. The variation of the generating functional is realised by the following operator,
\[
\delta_\sigma = \int d^d x \sigma \left[ 2g_{\mu \nu} \frac{\delta}{\delta g_{\mu \nu}} + (d - \Delta)\phi_0 \frac{\delta}{\delta \phi_0} \right]. \tag{1.3.41}
\]
In this case we find the following trace (or Weyl) Ward identity in the presence of the sources is
\[ \langle T(x) \rangle = (\Delta - d) \phi^I_0(x) \langle \mathcal{O}^I(x) \rangle, \quad (1.3.42) \]
where \( T = T^\mu_\mu \). Functionally differentiating with respect to the sources then yields trace Ward identities for \( n \)-point functions, e.g.,
\[ \langle T(x_1) O(x_2) O(x_3) \rangle = -\Delta \left[ \delta(x_1 - x_2) \langle O(x_1) O(x_3) \rangle \right. \\
\left. + \delta(x_1 - x_3) \langle O(x_1) O(x_2) \rangle \right], \quad (1.3.43) \]
\[ \langle T(x_1) T^\mu_\nu(x_2) O(x_3) \rangle = 2 \left( \frac{\delta T(x_1)}{\delta g^{\mu\nu}(x_2)} O(x_3) \right), \quad (1.3.44) \]
\[ \langle T(x_1) T^\mu_\nu(x_2) T^\rho_\sigma(x_3) \rangle = 2 \left( \frac{\delta T(x_1)}{\delta g^{\mu\nu}(x_2)} T^\rho_\sigma(x_3) \right) + 2 \left( \frac{\delta T(x_1)}{\delta g^{\rho\sigma}(x_3)} T^\mu_\nu(x_2) \right) \\
+ 2 \langle T(x_1) \delta T^\mu_\nu(x_2) \delta g^{\rho\sigma}(x_3) \rangle. \quad (1.3.45) \]

If the generating functional (1.3.16) is anomalous (1.3.23) then we have
\[ \langle T(x) \rangle = (\Delta - d) \phi^I_0(x) \langle \mathcal{O}^I(x) \rangle + A[g^{\mu\nu}, A^{(j)}, \phi^I_0], \quad (1.3.46) \]
where
\[ A[g^{\mu\nu}, A^{(j)}, \phi^I_0] = \left. \frac{\delta}{\delta \sigma} \mathcal{W}[g^{\mu\nu}, A^{(j)}, \phi^I_0] \right|_{\sigma = 0}. \quad (1.3.47) \]
is called the Weyl/scaling/trace anomaly. We will show how anomalies arise in the regularisation procedure in section 2.8.

### 1.4. Correlation functions

Conformal symmetry imposes very strong constraints on the structure of correlation functions. In particular the form of 1-, 2- and 3-point functions is uniquely fixed up to a small collection of numbers.

Due to the scaling symmetry the most general form of a 1-point function of a primary operator of dimension \( \Delta \) is
\[ \langle \mathcal{O}(x) \rangle = \frac{c_\Delta}{x^\Delta}, \quad (1.4.1) \]
where \( c_\Delta \) is a constant. By the application of the special conformal transformation,
\[ K_\mu \langle \mathcal{O}(x) \rangle = 0 \quad (1.4.2) \]
we find that \( c_\Delta = 0 \) if \( \Delta \neq 0 \). In section 1.2.4 we discussed the unitarity bounds which require that in a unitary CFT all operators, apart from the identity operator, have strictly positive conformal dimensions, therefore in a fully conformal theory
\[ \langle \mathcal{O}(x) \rangle = 0, \quad (1.4.3) \]
1.4. Correlation functions

assuming $\mathcal{O}$ is not proportional to the identity operator.

1.4.1. Scalar operators

Let us analyse the most general form of a 2-point function of two scalar operators $\mathcal{O}_1$ and $\mathcal{O}_2$ of dimensions $\Delta_1$ and $\Delta_2$. Poincaré invariance requires that the two point function takes form

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y) \rangle = F(|x-y|), \quad (1.4.4)$$

for some function $F$. Scale invariance requires that

$$F(\lambda r) = \lambda^{-\Delta_1-\Delta_2} F(r), \quad (1.4.5)$$

which uniquely solves to

$$F(r) = C_{12} r^{-\Delta_1-\Delta_2}, \quad (1.4.6)$$

where $C_{12}$ is an undetermined constant. Finally, the transformation properties under the special conformal transformations can be analysed. However, as explained in section 1.1.1, it is enough to analyse the transformations under the inversion (1.1.14). Using (1.1.15) we find that the 2-point function (1.4.4) can be non-vanishing only if $\Delta_1 = \Delta_2$. Therefore we have found,

$$\langle \mathcal{O}_1(x)\mathcal{O}_2(y) \rangle = \frac{C_{12}}{|x-y|^{2\Delta_1}} \delta_{\Delta_1\Delta_2}, \quad (1.4.7)$$

where $C_{12}$ is a constant.

A similar method can be applied to the 3-point functions, which leads to their most general form [26, 5],

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}}, \quad (1.4.8)$$

where $C_{123}$ is a single undetermined constant and $\Delta_j$, $j = 1, 2, 3$ are conformal dimensions of operators $\mathcal{O}_j$.

At this point we can return to the problem of determination of the OPE coefficients in (1.2.15). For simplicity assume all operators $\mathcal{O}_j = \mathcal{O}$, $j = 1, 2, 3$ have the same dimension $\Delta$. The OPE is then

$$\mathcal{O}(x)\mathcal{O}(0) = \frac{C_{12}}{x^{2\Delta}} + \frac{C}{x^{\Delta}} \mathcal{O}(0) + \text{descendants} + \text{other operators}. \quad (1.4.9)$$

The first term is determined by the 2-point function. We see that the constant $C_{12}$ is exactly the normalisation constant in (1.4.7). The constant $C$ is the OPE coefficient of $\mathcal{O}\mathcal{O} \mapsto \mathcal{O}$. It turns out that it is determined by the $C_{123}$ coefficient.
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in (1.4.8). If $\mathbf{x}$ is close to zero, then we have $|\mathbf{x} - \mathbf{y}| \sim y$ and we can expand the 3-point as

$$\langle \mathcal{O}(y)\mathcal{O}(x)\mathcal{O}(0) \rangle = \frac{C_{123}}{x^{\Delta} y^{2\Delta}} [1 + O(x)]. \quad (1.4.10)$$

On the other hand the OPE gives

$$\langle \mathcal{O}(y)\mathcal{O}(x)\mathcal{O}(0) \rangle = \frac{C}{x^{\Delta} y^{2\Delta}} \langle \mathcal{O}(y)\mathcal{O}(0) \rangle + \text{descendants}$$

$$= \frac{CC_{12}}{x^{\Delta} y^{2\Delta}} + \text{descendants}. \quad (1.4.11)$$

Therefore we find

$$C_{123} = CC_{12}. \quad (1.4.12)$$

In this way one can calculate any OPE coefficients, since - as we will see in the following sections - the form of all 3-point functions is fixed.

The strong constraints on 2- and 3-point functions we have obtained can be understood from the point of view of representation theory. Assume one wants to build a scalar conformal invariants from some set $\{\mathbf{x}_j\}$ of distinct points. As long as Poincaré invariance is considered only, such invariants are the distances $|\mathbf{x}_i - \mathbf{x}_j|$. For further simplicity denote

$$\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j, \quad x_{ij} = |\mathbf{x}_{ij}|. \quad (1.4.13)$$

The distances, however, are not scale invariant as they scale with $\lambda$ when all points are scaled by $\lambda$. Instead, one could try taking ratios of two distances. Such an object is scale invariant but is not invariant under inversions when (1.1.15) is used. Therefore the simplest objects that are conformally invariant are

$$u_{ijkl} = \frac{x_{ij}x_{kl}}{x_{ik}x_{jl}}, \quad v_{ijkl} = \frac{x_{ij}x_{kl}}{x_{il}x_{jk}}, \quad (1.4.14)$$

known as conformal ratios. For the conformal ratios to be well-defined and non-zero, one needs four distinct points. This leads to the conclusion that the 2- and 3-point functions in any CFT must be uniquely determined up to some number of constants. 4- and higher-point functions, on the other hand, are determined up to an arbitrary function of conformal ratios, for example

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = F(u_{1234}, v_{1234}) \prod_{1 \leq i < j \leq 4} x_{ij}^{\Delta_i - \Delta_j - \Delta_j}, \quad (1.4.15)$$

where $\Delta_i = \sum_{j=1}^{4} \Delta_j$ and $F$ is an undetermined function. In case of 4-point functions there are two independent conformal ratios, in case of $n$-point functions it follows from a simple combinatorics [5] that there are $n(n - 3)/2$ independent conformal ratios.
1.4.2. Embedding formalism

Similar considerations as in the case of correlation functions of scalar operators can be applied to general tensor operators. Historically, the form of correlation functions of scalar operators in a CFT appeared in [26] and was quickly generalised to the 3-point function of currents for \( d = 4 \) in [27]. A complete analysis of all 3-point functions of scalars and tensors of spin one and two, and in general dimension, was carried out in [22, 23]. The analysis is based on the fact that any special conformal transformation can be decomposed into translations and inversions. Therefore, in order to impose the special conformal invariance, it is enough to analyse the transformation properties of the correlation function under inversions only. Using such an approach, the analysis was extended to arbitrary operators in [28]. For a sample of more recent work on this topic see also [29, 30, 31, 32].

In this thesis we will use a different elegant approach: the embedding formalism. As we have shown in section 1.1.2, the conformal group is locally isomorphic to \( SO(d + 1, 1) \) and therefore it naturally acts via isometries on the space \( \mathbb{R}^{d+1,1} \) with the Minkowski metric. The idea is to embed the physical space \( \mathbb{R}^d \) into the embedding space \( \mathbb{R}^{d+1,1} \) in such a way that the natural action of \( SO(d + 1, 1) \) on \( \mathbb{R}^{d+1,1} \) restricts to the conformal transformations on the physical \( \mathbb{R}^d \). Such an approach dates back to Dirac, [33], and its applications to the correlation functions were developed in [34, 35, 36, 37].

Let us introduce the light-cone coordinates in \( \mathbb{R}^{d+1,1} \),

\[
X^A = (X^+, X^-, X^a) \in \mathbb{R}^{d+1,1},
\]

where \( a = 1, \ldots, d \) and the metric is

\[
\eta_{AB}dX^A dX^B = -dX^+dX^- + \delta_{ab}dX^a dX^b.
\]

Choose an embedding \( X : \mathbb{R}^d \mapsto \mathbb{R}^{d+1,1} \),

\[
X(x) = (1, x^2, x^\mu)
\]

in light-cone coordinates, known as the Poincaré section. Note that the metric induced on \( \mathbb{R}^d \) via \( X \) is a flat space metric so the embedding is isometric. Denote the image of the embedding as \( P = X(\mathbb{R}^d) \subseteq \mathbb{R}^{d+1,1} \). Notice that \( X^2(x) = 0 \) in the Minkowski metric and \( X^+(x) = 1 \) for any \( x \in \mathbb{R}^d \). This means that the physical subspace \( P \) is invariant under the natural action of the conformal group \( SO(d+1, 1) \). With little algebra one can find that this action is precisely the action (1.1.12) in the standard coordinates on \( \mathbb{R}^d \). Finally, for two points \( X_i \) and \( X_j \) in \( P \), denote \( X_{ij} = X_i - X_j \) and observe that

\[
X_{ij}^2 = -2X_i \cdot X_j = (x_i - x_j)^2 = x_{ij}^2.
\]
We used the fact that $X^2 = 0$ on the physical space $P$. The remaining piece of information is how to extend field living on $\mathbb{R}^d$ to the embedding space. It turns out that the extension is unique,

$$O(\lambda X^A) = \lambda^{-\Delta} O(X^A).$$  \hfill (1.4.20)

Let us only sketch the argument, for details see [36]. It turns out that a change of the Poincaré section (1.4.18) to a different section leads to a change of the metric on the physical space to a different metric within the same conformal class. The physical space $P$ can be viewed as a space of null rays in $\mathbb{R}^{d+1,1}$ with a conformal structure and the extension (1.4.20) is the only extension consistent with it.

The method for finding the most general form of the correlation function in the embedding space formalism is as follows. First find the most general correlation function of the extended field in the embedding space. Such a correlation function is an invariant of the Lorentz group $SO(d+1,1)$ and scales according to (1.4.20). Then substitute (1.4.18) and use (1.4.19) in order to obtain the usual position space expression. For the 2-point function of operators of dimension $\Delta$ we find

$$\langle O(X_1)O(X_2) \rangle = \frac{C_{12}}{(X_1 \cdot X_2)^\Delta}. \hfill (1.4.21)$$

Note that $X_1 \cdot X_2$ is the only Lorentz invariant built up from two points, since on $P$ $X_j^2 = 0$. Also note that we do not have translational invariance in the embedding space. Now, using (1.4.19), we recover (1.4.7).

1.4.3. Embedding formalism for tensors

In this section we will extend the embedding formalism to tensor operators. We will consider totally symmetric and traceless tensors only. This is because such tensors transform in the most common irreducible representation of the rotation group. However, it is possible to extend the discussion to other irreducible representations, [36].

In order to analyse the tensor structure, one must understand the push-forward of the tangent bundle of the physical space $\mathbb{R}^d$ via $X$ defined in (1.4.18). Since $X^2 = 0$ on the physical space $P = X(\mathbb{R}^d) \subseteq \mathbb{R}^{d+1,1}$, by differentiation we have $X \cdot X \cdot v = 0$ for any tangent vector $v$ in $\mathbb{R}^d$. Therefore for an extension of any tensor field $O_{a_1...a_n}$ to the embedding space field $O_{A_1...A_n}$ we have

$$X^{A_j} O_{A_1...A_n} (X) = 0, \quad j = 1, \ldots n \hfill (1.4.22)$$

on $P$. Even with this condition there remains a redundancy in the extension of the fields to the embedding space. Indeed, since $X^2 = 0$, one can add an arbitrary function $\Lambda_{A_2...A_n}$,

$$O_{A_1...A_n} (X) \mapsto O_{A_1...A_n} (X) + X(A_1 \Lambda_{A_2...A_n}) \hfill (1.4.23)$$
and the redefined field is still transverse, \emph{i.e.}, (1.4.22) holds. Because of this redundancy, one can always add appropriate terms to the correlation function in the embedding space so that the tensor structure depends on the differences $X_i - X_j$ only.

Now one can look for the most general Lorentz covariant expression for the 2- and 3-point function which satisfies:

(i) Lorentz covariance in the embedding space,

(ii) scaling as in (1.4.20),

(iii) transversness condition (1.4.22),

(iv) total symmetry and tracelessness in any pair of indices for each operator.

up to the redefinition (1.4.23).

Let us write the most general form of the two-point of two operators with spin one and dimension $\Delta$. According to the rules above we have

$$\langle O^A(X_1)O^B(X_2) \rangle = C_{12} \left( \eta^{AB} + \alpha \frac{X_2^A X_1^B}{X_1 \cdot X_2} \right).$$

(1.4.24)

The tracelessness condition fixes $\alpha = -1$. Note that terms proportional to $X_1^A$ or $X_2^B$ are redundant according to (1.4.23). Observe that they can be added at will so when the projection onto the physical space is carried out, one can apply the following rules

$$X_1^B \rightarrow x_1^\nu - x_2^\nu, \quad X_2^A \rightarrow x_2^\mu - x_1^\mu, \quad \eta^{AB} \rightarrow \delta^{\mu\nu}, \quad X_1 \cdot X_2 \rightarrow -\frac{1}{2} (x_1 - x_2)^2 = -\frac{1}{2} x_{12}^2.$$

(1.4.25)

Eventually we find

$$\langle O^\mu(x_1)O^{\nu}(x_2) \rangle = \frac{C_{12} I^{\mu\nu}(x_{12})}{x_{12}^2},$$

(1.4.26)

where

$$I^{\mu\nu}(x) = \delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2}$$

(1.4.27)

and $C_{12}$ is an undetermined constant.

The $I^{\mu\nu}$ tensor appearing in (1.4.26) is an important ingredient of the conformal structure in physical space. It arises as a derivative of the inversion (1.1.14),

$$\frac{\partial I^{\mu}(x)}{\partial x^\nu} = \frac{1}{x^2} I^{\mu\nu}(x)$$

(1.4.28)

and it is immediate to check that,

$$I^{\mu\nu}[I(x) - I(y)] = I_\alpha^{\mu}(x) I^{\alpha \beta}(x - y) I_\beta^{\nu}(y)$$

(1.4.29)
1. Conformal invariance

and moreover

\[ I^{\mu\alpha}(x)I^\nu_\alpha(x) = \delta^{\mu\nu}. \]  

(1.4.30)

Therefore the operator \( I^{\mu\nu} \) can be used to represent inversion on the space of fields and will eventually appear in all correlation functions of tensor operators in position space as found in [22, 23, 28].

Similar considerations can be applied to higher spin correlation functions. For a traceless spin-2 operator of dimension \( \Delta \) we find the most general form of the 2-point function,

\[
\langle O_{\mu\nu}(x_1)O_{\rho\sigma}(x_2) \rangle = C_{12} \frac{x_{12}^{2\Delta}}{x_{12}^a} \left[ I^{\mu\nu}(x_{12})I_{\nu\sigma}(x_{12}) + I^{\mu\sigma}(x_{12})I_{\nu\rho}(x_{12}) - \frac{2}{d} \delta^{\mu\nu}\delta^{\rho\sigma} \right].
\]  

(1.4.31)

The expressions (1.4.26) and (1.4.31) are symmetric and traceless but they do not vanish when the divergence is taken. Instead, for conserved currents we have

\[
\partial_\mu \langle J^\mu J^\nu \rangle = 0, \quad \partial_\mu \langle T^{\mu\nu}T^{\rho\sigma} \rangle = 0.
\]  

(1.4.32)

This follows from the transverse Ward identities (1.3.33) and (1.3.34) by a differentiation with respect to \( A_\mu \) and \( g_{\mu\nu} \) respectively and the utilisation of the fact that 1-point functions in any CFT vanish, (1.4.3). Using the expressions (1.4.26) and (1.4.31), by direct calculations one finds that \( J^\mu \) and \( T^{\mu\nu} \) are conserved for specific conformal dimensions only. The conclusion is that:

1. a conformal primary operator of spin-1 is conserved if and only if its dimension is \( \Delta = d - 1 \),

2. a conformal primary operator of spin-2 is conserved if and only if its dimension is \( \Delta = d \).

This result assures us that the dimensions of \( J^\mu \) and \( T_{\mu\nu} \) are protected in any CFT from any quantum corrections.

1.4.4. 3-point functions

For 3-point functions we follow the formalism described in the previous sections. The problem can be greatly simplified by finding all possible tensor structures that can appear in any correlation function. Consider the most general correlation function of \( n \) tensor operators \( O_j \), each being a tensor of rank \( r_j \),

\[
\langle O_{A_11}^{A_12} \cdots A_{1r_1}(X_1)O_{A_21}^{A_22} \cdots A_{2r_2}(X_2) \cdots O_{A_{n1}}^{A_{n2} \cdots A_{nr_n}(X_n)} \rangle.
\]  

(1.4.33)
1.4. Correlation functions

Then, for the conditions (i) - (iii) of listed in the previous section to be satisfied, the correlation function can depend on the following variables only, [35],

\[
H_{ij}^{A_i A_j} = X_I \cdot X_J \eta^{A_i A_j} - X_I^{A_i} X_J^{A_j},
\]
\[
V_{i,j,k}^{A_i} = \frac{1}{X_J \cdot X_K} H_{i,j,k}^{A_i A_j A_k}
\]
\[
= \frac{X_I \cdot X_J X_K^{A_i} - X_I \cdot X_K X_J^{A_i}}{X_J \cdot X_K},
\]
where \(I, J, K \in \{1, \ldots, n\}\) and \(i \in \{1, \ldots, r_I\}, j \in \{1, \ldots, r_J\}\). To make it more eligible, consider a representative example of the \(\langle T^{A_1 B_1} T^{A_2 B_2} \rangle\) correlation function, where \(T^{AB}\) is a symmetric tensor of conformal dimension \(d\) and \(O\) is a scalar of conformal dimension \(\Delta\). The most general form of the correlation function is,

\[
\langle T^{A_1 B_1} T^{A_2 B_2} \rangle = \frac{1}{(X_1 \cdot X_3)^{\frac{\Delta + 2}{2}}} (X_2 \cdot X_3)^{\frac{\Delta + 2}{2}} (X_1 \cdot X_2)^{\frac{2d-\Delta}{2}} \times
\]
\[
\left[ C_1 H^{(A_1 A_2 H^{B_1 B_2})}_{12} + C_2 H^{(A_1 A_2 V^{B_1 B_2})}_{12,23} + C_3 V^{A_1 A_2 V^{B_1 B_2}}_{12,23} \right] \]
\[
(1.4.36)
\]
where \(C_j, j = 1, 2, 3\) are three undetermined numerical constants. In the determination of the powers in the denominator we used the fact that \(H(\lambda X) = \lambda^2 H(X)\) and \(V(\lambda X) = \lambda V(X)\) for \(H\) and \(V\) functions in (1.4.34) and (1.4.35). Note also that we still did not impose tracelessness in appropriate indices.

Now one can project (1.4.36) to the physical space. This leads to the substitutions

\[
X_1 \mapsto x_1 - x_2, \quad X_2 \mapsto x_2 - x_3, \quad X_3 \mapsto x_3 - x_1,
\]
\[
A_j, B_j \mapsto \mu_j, \nu_j \quad \eta \mapsto \delta, \quad X_i \cdot X_j \mapsto -\frac{1}{2} x_{ij}^2.
\]
\[
(1.4.37)
\]
We used the freedom expressed by the equation (1.4.23) in order to obtain the result that depends on the differences of points only. Finally, if one requires that the \(T^{\mu \nu}\) operators are traceless, then the following projector

\[
\varepsilon^{\mu \nu}_{\alpha \beta} = \frac{1}{2} (\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\beta \delta^\nu_\alpha) - \frac{1}{d} \delta^{\mu \nu} \delta_{\alpha \beta}
\]
\[
(1.4.38)
\]
should be applied. After a lot of algebra one finds the following position space expression for the \(\langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} \rangle\) correlation function with \(T^{\mu \nu}\) being symmetric and traceless of conformal dimension \(d\),

\[
\langle T^{\mu_1 \nu_1} (x_1) T^{\mu_2 \nu_2} (x_2) O(x_3) \rangle = \varepsilon^{\mu_1 \nu_1}_{\alpha_1 \beta_1} \varepsilon^{\mu_2 \nu_2}_{\alpha_2 \beta_2} I^{\alpha_1 \beta_1}_{\lambda_1} (x_{13}) I^{\beta_1 \alpha_1}_{\lambda_1} (x_{13}) I^{\alpha_2 \beta_2}_{\lambda_2} (x_{23}) I^{\beta_2 \alpha_2}_{\lambda_2} (x_{23}) \times
\]
\[
\times \frac{1}{x_{12}^{2d-\Delta} x_{23}^{\Delta} x_{31}^{\Delta}} \left[ c_1 X_{12}^\lambda X_{12}^\lambda X_{12}^\lambda X_{12}^\lambda + 4c_2 \delta^\lambda_1 \delta^\lambda_2 X_{12}^\lambda X_{12}^\lambda + 2c_3 \delta^\lambda_1 \lambda_2 \delta^\lambda_1 \lambda_2 \right],
\]
\[
(1.4.39)
\]
where tensors $I^{\mu\nu}$ are defined in (1.4.27), $c_j$, $j = 1, 2, 3$ are constants proportional to $C_j$ in (1.4.36) and

$$X_1^{\mu} = \frac{x_2^{\mu}x_3^{\mu} - x_1^{\mu}x_2^{\mu}x_3^{\mu}}{x_1^{\mu}x_2^{\mu}x_3^{\mu}}.$$  

The expression (1.4.39) looks a little complicated, but the main point is that all 3-point functions in a CFT are determined up to a small number of numerical constants. In this case we found three undetermined constants. In general, using the embedding formalism one can find the number of the independent tensor structures building any correlation function exactly. If one orders spins of the operators as $s_1 \leq s_2 \leq s_3$ and defines $p = \max(0, s_1 + s_2 - s_3)$, then this number is

$$N(s_1, s_2, s_3) = \frac{(s_1 + 1)(s_1 + 2)(3s_2 - s_1 + 3)}{6} - \frac{p(p + 2)(2p + 5)}{24} - \frac{1 + (-1)^p}{16}.$$  

Each tensor structure is multiplied by a numerical constant. The number of independent constants, however, may be smaller than the number of tensors if some symmetry properties are exploited. For example in case of $s_1 = s_2 = s_3 = 2$,
1.4. Correlation functions

there is 11 tensor structures, but symmetries between them reduce the number of independent constants down to 5.

Finally, one can impose the conservation conditions on the currents and the stress-energy tensor. For non-coincident points one just has

$$\partial_{j\mu} \langle T^{\mu_1\nu_1}(x_1) T^{\mu_2\nu_2}(x_2) \mathcal{O}(x_3) \rangle = 0, \quad j = 1, 2$$

(1.4.42)

following from the Ward identity (1.3.34). When applied to (1.4.39) one finds two relations between the constants,

$$c_1 + 4c_2 - \frac{1}{2}(d - \Delta)(d - 1)(c_1 + 4c_2) - d\Delta c_2 = 0, \quad (1.4.43)$$

$$c_1 + 4c_2 + d(d - \Delta)c_2 + d(2d - \Delta)c_3 = 0, \quad (1.4.44)$$

so the $$\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle$$ correlation function is unique up to one numerical constant.

Finally, one can consider the transverse Ward identities following from (1.3.33) and (1.3.34) more carefully, including the local terms coming from 2-point functions. Such a procedure may restrict the form of the 3-point function further and express the coefficients undetermined so far in terms of normalisations of 2-point functions. We will postpone the analysis of the local terms until the discussion of the momentum space expressions for 3-point functions.

Table 1.1 lists the numbers of undetermined coefficients at each level of the analysis in generic cases in dimensions $$d \geq 3$$. 

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