Conformal symmetry and holographic cosmology

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Chapter 2

Implications of conformal invariance in momentum space

In section 1.4 we have presented a method that allows to write down the most general expressions for all 2- and 3-point functions in a conformal field theory. The method was based on the embedding formalism and the analysis was carried out in position space. The purpose of this part of the thesis is to present the analogous set of results in momentum space.

In principle, the results in momentum space can be obtained from those in position space by Fourier transform. Typically, however, the position space expressions such as (1.4.7) and (1.4.8) are only valid at separated points, and do not possess a Fourier transform prior to renormalisation. Therefore, before the Fourier transform could be applied, the regularisation procedure is necessary. In the first section of this chapter we will introduce a convenient regularisation scheme known as differential regularisation. However, even after renormalisation, it is technically rather difficult to carry out explicitly the Fourier transforms, see for example [24]. Here we will present a complete analysis from first principles of the constraints due to conformal symmetry directly in momentum space. We believe such an analysis gives considerably more insight into the results and is interesting in its own right.

A momentum space analysis is natural from the perspective of Feynman diagram computations, which are usually performed in momentum space. Furthermore, a number of recent works have exemplified the need for CFT results in momentum space. Our original motivation for analysing this question was the requirement for these results in our work on holographic cosmology [38, 39, 40, 1, 2],
and similar applications of the conformal/de Sitter symmetry in cosmology have been discussed in [41, 29, 42, 43, 44, 45, 46]. Other recent works that contain explicit computations of CFT correlation functions in momentum space include [47, 48, 49, 24, 50]. Our results may also be useful in the context of work on an $a$-theorem in diverse dimensions, see [25] for a relevant discussion in $d = 4$.

There are two main issues that complicate the analysis of the implications of conformal invariance in momentum space. While conformal transformations act naturally in position space, they lead to differential operators in momentum space. Dilatations, $\delta x^\mu = \lambda x^\mu$, being linear in $x^\mu$ lead to a Ward identity (1.3.7) that is a first-order differential equation, and as such, it is easy to solve in complete generality. Special conformal transformations however are non-linear, so after Fourier transform we obtain a Ward identity that is a second-order differential equation, which makes the analysis more complicated.

The second main issue is the complicated tensorial decomposition required for correlators involving vectors and tensors. Lorentz invariance implies that the tensor structure will be carried by tensors constructed from the momenta $p^\mu$ and the metric $\delta_{\mu\nu}$. The standard procedure consists of writing down all possible such independent tensor structures and expressing the correlators as a sum of these structures, each multiplied by scalar form factor. In the case of correlators involving conserved currents and/or stress-energy tensors one then imposes the restrictions enforced by conservation (and tracelessness of the stress-energy tensor in the case of CFTs). Recent works discussing such a tensor decomposition include [49, 25, 48, 47, 24]. This methodology is in principle straightforward, but an inefficient parametrisation can produce unwieldy expressions. Here we present a new parametrisation that appears to yield a minimal number of form factors.

In this work we assume that the underlying theory is parity-invariant. Additional parity-violating terms can appear in the tensorial decomposition of the various correlators and it would be interesting to incorporate them in our analysis.

In the course of the analysis we will show that the counting of independent OPE coefficients presented in table 1.1 is recovered by the independent momentum space analysis. The finital solution for all 3-point functions will be given in form of so called triple-$K$ integrals, containing three Bessel $K$ functions. Such integrals usually cannot be expressed by elementary functions. However, in case of correlation functions of conserved currents and stress-energy tensor in odd dimensions, these triple-$K$ integrals reduce to elementary integrals. In such cases we will find a novel way of calculating one-loop Feynman integrals, without any need for the recursion schemes typically used, e.g., [51, 52, 53].
2.1. 2-point functions in momentum space

We will start with the discussion of the form of 2-point functions in momentum space. Due to the momentum conservation any correlation function in position space contains the Dirac delta and one can defined the reduced correlation function as

\[ \langle O(p_1)O(p_2) \rangle = (2\pi)^d \delta(p_1 + p_2) \langle \langle O(p_1)O(-p_1) \rangle \rangle. \] (2.1.1)

We will often use the symbol \( \langle \langle - \rangle \rangle \) and its generalisations to any correlation functions.

2.1.1. Dimensional regularisation

The expressions for 2- and 3-point functions we have found in section 1.4 are valid at non-coincident points only. When two points coincide, the correlation function becomes singular. In Lorentzian signature correlation functions should be well-defined distributions, in particular they should have well-defined Fourier transforms. Therefore, we can analyse the coincident limit and regularise the position space correlation functions by requiring that the Fourier transform exists.

Consider the 2-point function of scalar operators first,

\[ \langle O(x)O(0) \rangle = \frac{1}{x^{2\Delta}}, \] (2.1.2)

where we put \( C_{12} = 1 \), as this is insignificant information here. The Fourier transform of this 2-point function can be done explicitly, since

\[ \int d^d x \ e^{-i p \cdot x} \frac{1}{x^{2\Delta}} = \pi^{d/2} 2^{d-2\Delta} \Gamma \left( \frac{d-2\Delta}{2} \right) p^{2\Delta-d}. \] (2.1.3)

The integral converges for \( 0 < 2\Delta < d \). By unitarity bounds \( \Delta > 0 \), but usually the upper bound on \( \Delta \) is violated. However, the right hand side of (2.1.3) is well-defined analytic function of \( \Delta \). Therefore one can extend it to any \( 2\Delta \neq d + 2n \), where \( n \) is a non-negative integer. If, however, \( 2\Delta = d + 2n \), a non-trivial regularisation is required.

The analytic continuation in \( \Delta \) is similar to the analytic continuation performed in dimensional regularisation, where the spacetime dimension \( d \) is continued away from its physical value. In standard dimensional regularisation in position space one keeps \( \Delta \) fixed and substitutes \( d \mapsto d-\epsilon \) into (2.1.3). The result can be expanded in \( \epsilon \) and the relevant part is

\[ \Gamma \left( -n - \frac{\epsilon}{2} \right) p^{2n+\epsilon} = (-p^2)^n \left[ -\frac{2}{n!\epsilon} + \frac{H_n - \gamma_E}{n!} \right] - \frac{1}{n!} p^{2n} \log p^2 + O(\epsilon) \] (2.1.4)

where \( H_n = \sum_{j=1}^{n} j^{-1} \) is \( n \)-th harmonic number and \( \gamma_E \) is the Euler gamma constant. As expected, the expansion is singular. The singularity should be subtracted.
by the addition of local counterterms to the action. For example, if one redefines
the action in (1.3.11) as follows
\[ S = S_{\text{CFT}} + \int d^{d-\epsilon} x \Lambda^{-\epsilon} \phi_0 \mathcal{O} + \frac{c_{d,\Delta}}{\epsilon} \int d^{d-\epsilon} x \Lambda^{-\epsilon} \Box^n \phi_0^2, \] (2.1.5)
then the divergent contribution in (2.1.3) cancels against the divergent counterterm
if the constant \( c_{d,\Delta} \) is chosen appropriately. After this the \( \epsilon \to 0 \) limit can be taken.
The additional term \( \Lambda^{-\epsilon} \) where \( \Lambda \) has a dimension of length is introduced since
the total dimension of spacetime was reduced by \( \epsilon \).

The described procedure leads to the interesting phenomena of the renormalisation group. Notice that (2.1.5) contains new parameters: the scale \( \Lambda \) and the
subleading corrections to \( c_{d,\Delta} \) in \( \epsilon \). One can see that both parameters are related
and they can only contribute to the finite part without the logarithm in (2.1.4). Such a freedom is called a scheme dependence and for the physical theory to be predictive, one must fix the value of \( \Lambda \) by a direct measurement of one physical
quantity. Then, the theory should not depend on \( \Lambda \) any more. We will discuss
general features of renormalisation in section 4.1.2.

**2.1.2. Differential regularisation**

There exists another elegant procedure to regularise position space expressions
in such a way that the result is automatically finite. The method of differential
regularisation was developed in [54, 55] and is based on the ‘pulling derivatives
out’ trick. Consider the Fourier transform (2.1.3). Since
\[ \frac{1}{x^{2\Delta}} = \frac{1}{(2\Delta - 2)(2\Delta - d)} \Box \frac{1}{x^{2\Delta - 2}}, \] (2.1.6)
one can pull a number of boxes so that the power of \( x \) fits into the range \( 0 < 2\Delta < d \). The Fourier transform of \( \Box \) is \( -p^2 \) and one can show that such a procedure
leads to (2.1.3) for any \( 2\Delta \neq d + 2n \).

For \( 2\Delta = d + 2n \), however, the procedure breaks down as (2.1.6) is not valid since
\[ \Box \frac{1}{x^{d-2}} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \delta(x). \] (2.1.7)
This is the reason why (2.1.3) cannot be analytically extended to these values of \( \Delta \). In this case one can exchange the expression (2.1.2) for another one that differs
only at \( x = 0 \). It is easy to see that
\[ \frac{1}{x^d} = \frac{1}{2(2 - d)} \Box \log(x^2 M^2) \] (2.1.8)
2.1. 2-point functions in momentum space

for $x \neq 0$, where $M$ is an arbitrary parameter. Now one can use (2.1.6) in order to extend it to all $\Delta > 0$. Finally the Fourier transform of (2.1.8) is finite, since

$$
\int d^d x \ e^{-ip \cdot x} \frac{\log(x^2 M^2)}{x^\alpha} = -\frac{\pi^{d/2} 2^{d-\alpha} \Gamma \left( \frac{d-\alpha}{2} \right)}{\Gamma \left( \frac{\alpha}{2} \right)} \cdot \frac{1}{p^{d-\alpha}} \log \left( \frac{p^2}{\Lambda^2} \right),
$$

(2.1.9)

with

$$
\Lambda^2 = 4M^2 \exp \left( \psi \left( \frac{d-\alpha}{2} \right) + \psi \left( \frac{\alpha}{2} \right) \right),
$$

(2.1.10)

where $\psi$ is digamma function and $\alpha \neq d + 2n$, with $n$ a non-negative integer.

As we can see, since $M$ is arbitrary, we recovered the same kind of the scheme dependence as in the dimensional regularisation. It was shown [56, 57] that dimensional regularisation is equivalent to differential regularisation up to local terms, i.e., they differ by terms that can be adjusted by fixing the regularisation scheme. The differential regularisation is convenient if one is interested in calculations of Fourier transforms, since it automatically takes care of the divergence.

### 2.1.3. Scalar 2-point function

Summarising the previous sections, the 2-point function of the scalar conformal primary operator of conformal dimension $\Delta$ is given by (2.1.3) and (2.1.9),

$$
\langle \langle O(p)O(-p) \rangle \rangle = c_O \times \begin{cases} 
p^{2\Delta-d} & \text{if } 2\Delta \neq d + 2n, 
p^{2\Delta-d} (-\log p^2 + \text{local}) & \text{if } 2\Delta = d + 2n, \end{cases}
$$

(2.1.11)

where $n$ is a non-negative integer and $c_O$ is an undetermined normalisation constant. In a unitary theory $c_O > 0$. The ‘local’ terms are terms of the form $c_M p^{2\Delta-d}$, where $c_M$ is a scheme-dependent constant depending on the regularisation scheme.

The value of $c_O$ is proportional to $C_{12}$ in the position space expression (1.4.7).

For example if $\Delta > 0$ and $2\Delta - d \neq 2n$, where $n$ is a non-negative integer, then

$$
c_O = \frac{\pi^{d/2} 2^{d-2\Delta} \Gamma \left( \frac{d-2\Delta}{2} \right)}{\Gamma(\Delta)} C_{12}.
$$

(2.1.12)

### 2.1.4. Tensorial 2-point functions

Here we will consider the most general form of the 2-point functions of conserved currents $J^\mu$ of spin-1 and the stress-energy tensor $T^{\mu\nu}$. In principle one could start with the correlation functions (1.4.26) and (1.4.31) and apply the Fourier transform. However, it is convenient to choose a different approach where the structure of the correlation functions in momentum space is much clearer.

Observe that the operator

$$
\pi_\alpha^\mu(p) = \delta_\alpha^\mu - \frac{p^\mu p_\alpha}{p^2}
$$

(2.1.13)
is a projector onto tensors transverse to $p$, i.e., $p_\mu \pi^\mu_\alpha(p) = 0$. Similarly, in $d$ dimensions, the operator

$$\Pi^{\mu\nu}_{\alpha\beta}(p) = \frac{1}{2} \left[ \pi^\mu_\alpha(p) \pi^\nu_\beta(p) + \pi^\nu_\alpha(p) \pi^\mu_\beta(p) \right] - \frac{1}{d-1} \pi^{\mu\nu}(p) \pi_{\alpha\beta}(p)$$

(2.1.14)
is a projector onto transverse to $p$, traceless, symmetric tensors of rank two. In particular

$$p_\mu \pi^\mu_\alpha(p) = 0, \quad \delta_{\mu\nu} \Pi^{\mu\nu}_{\alpha\beta}(p) = 0, \quad p_\mu \Pi^{\mu\nu}_{\alpha\beta}(p) = 0.$$  

(2.1.15)

Therefore, any transverse to $p$, traceless, symmetric tensor $t_{\mu\nu}$ of rank two may be written as $t^{\mu\nu} = \Pi^{\mu\nu}_{\alpha\beta}(p) X^{\alpha\beta}$, where $X^{\alpha\beta}$ is an arbitrary tensor. More properties of the projectors are listed in appendix 2.A.8.

Due to the Ward identities (1.3.33) and (1.3.34) the divergence of any 2-point function of conserved currents is proportional to 1-point functions. Assuming these vanish, we may write the most general decompositions

$$\langle\langle T^{\mu\nu}(p) T^{\rho\sigma}(-p) \rangle \rangle = \Pi^{\mu\nu\rho\sigma}(p) A(p) + \pi^{\mu\nu}(p) \pi^{\rho\sigma}(p) B(p),$$

(2.1.16)

$$\langle\langle J^{\mu}(p) J^{\nu}(-p) \rangle \rangle = \hat{\pi}^{\mu\nu}(p) C(p),$$

(2.1.17)

where, due to the Lorentz invariance, $A, B, C$ are arbitrary functions of the amplitude of the momentum $p$. These expressions are general, valid in any quantum field theory with $\langle J^{\mu} \rangle = \langle T^{\mu\nu} \rangle = 0$.

Let us now move to conformal theories, where 1-point functions vanish and the trace Ward identity (1.3.42) implies that $\langle\langle T^{\mu\nu} \rangle \rangle = 0$. This requires $B = 0$ in (2.1.16). Furthermore conformal dimensions of $T^{\mu\nu}$ and $J^{\mu}$ are fixed to $d$ and $d - 1$ respectively. Therefore, in position space the most general expressions for the 2-point functions following from (2.1.16) with $B = 0$ and (2.1.17) are

$$\langle T^{\mu\nu}(x) T^{\rho\sigma}(0) \rangle = \hat{\Pi}^{\mu\nu\rho\sigma}(x) C_T \frac{1}{x^{2d}},$$

(2.1.18)

$$\langle J^{\mu}(x) J^{\nu}(0) \rangle = \hat{\pi}^{\mu\nu}(x) C_J \frac{1}{x^{2(d-1)}},$$

(2.1.19)

where

$$\hat{\pi}^{\mu\nu} = \delta^{\mu\nu} - \frac{\partial^{\mu} \partial^{\nu}}{\partial^2},$$

(2.1.20)

$$\hat{\Pi}^{\mu\nu\rho\sigma} = \frac{1}{2} \left[ \hat{\pi}^{\mu\rho} \hat{\pi}^{\nu\sigma} + \hat{\pi}^{\nu\rho} \hat{\pi}^{\mu\sigma} \right] - \frac{1}{d-1} \hat{\pi}^{\mu\nu} \hat{\pi}^{\rho\sigma}$$

(2.1.21)

and $C_T$ and $C_J$ are undetermined constants. In order to Fourier transform (2.1.18) and (2.1.19) one must Fourier transform the factors $x^{-2d}$ and $x^{-2(d-1)}$ by the method presented in the previous sections. The result is

$$\langle\langle T^{\mu\nu}(p) T^{\rho\sigma}(-p) \rangle \rangle = c_T \Pi^{\mu\nu\rho\sigma}(p) \times \left\{ \begin{array}{ll} 1 & \text{if } d = 3, 5, 7, \ldots \\ p^d & \text{if } d = 4, 6, 8, \ldots \\ p^d \left( -\log p^2 + \text{local} \right) & \text{if } d = 4, 6, 8, \ldots 
\end{array} \right.$$  

(2.1.22)
\[ \langle J^\mu(p) J^\nu(-p) \rangle = c_T \pi^{\mu\nu}(p) \times \begin{cases} p^{d-2} & \text{if } d = 3, 5, 7, \ldots \\ p^{d-2} \left( -\log p^2 + \text{local} \right) & \text{if } d = 4, 6, 8, \ldots \end{cases} \tag{2.1.23} \]

where \( c_T \) and \( c_J \) are two undetermined normalisation constants. In unitary CFTs both must be positive.

In the analysis of the 2-point functions we did not use constraints following from the special conformal transformations. However, one can show that they do not impose any additional conditions on the 2-point functions we have already found. A quick argument is that all momentum space expressions possess a single undetermined constant, exactly as we found in the position space.

### 2.2. 3-point function of scalar operators

#### 2.2.1. From position to momentum space

Let us start with the analysis of the 3-point function of scalar primary operators. The position space expression is given by (1.4.8). This expression, in principle, can be Fourier transformed in order to obtain the result in momentum space. Extracting the overall Dirac delta function encoding momentum conservation, we define the reduced matrix element, denoted with double brackets,

\[ \langle O_1(p_1) O_2(p_2) O_3(p_3) \rangle = (2\pi)^d \delta(p_1 + p_2 + p_3) \langle\langle O_1(p_1) O_2(p_2) O_3(p_3) \rangle\rangle. \tag{2.2.1} \]

Assuming \( d > 3 \), since \( p_1 + p_2 + p_3 = 0 \) there are two independent momenta. We define

\[ \Delta_t = \Delta_1 + \Delta_2 + \Delta_3, \quad \delta_j = \frac{d - \Delta_t}{2} + \Delta_j, \quad j = 1, 2, 3. \tag{2.2.2} \]

A useful representation of the Fourier transform of the position space expression (1.4.8) is

\[
\langle\langle O_1(p_1) O_2(p_2) O_3(p_3) \rangle\rangle = 
C_{123} \pi^{\frac{3d}{2}} 2^{3d-\Delta_t} \prod_{j=1}^{3} \frac{\Gamma(\delta_j)}{\Gamma\left(\frac{d}{2} - \delta_j\right)} \int \frac{d^d k}{(2\pi)^d} \frac{1}{|k|^{2\Delta_1} |p_1 - k|^{2\Delta_2} |p_2 + k|^{2\Delta_3}} \times 
\frac{c_{123} \pi^d \Gamma(\Delta_t - d)}{\Gamma\left(\frac{\Delta_1 + \Delta_2 - \Delta_3}{2}\right) \Gamma\left(\frac{\Delta_2 + \Delta_3 - \Delta_1}{2}\right) \Gamma\left(\frac{\Delta_3 + \Delta_1 - \Delta_2}{2}\right)} \times 
\int_0^{\infty} dx x^{\frac{d}{2} - 1} K_{\Delta_1 - \frac{d}{2}}(p_1 x) K_{\Delta_2 - \frac{d}{2}}(p_2 x) K_{\Delta_3 - \frac{d}{2}}(p_3 x), \tag{2.2.3} \]

where \( K_{\nu}(z) \) is a Bessel \( K \) function, i.e., a modified Bessel function of the second kind. A derivation of this representation may be found in section 4.2.1. As mentioned in the introduction, we will generally refer to integrals of the form above...
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featuring three Bessel \( K \) functions and a power as triple-\( K \) integrals. This form of the 3-point function is familiar in the context of the AdS/CFT correspondence, where every bulk-to-boundary propagator for the field dual to the conformal operator \( O_j \) contains one Bessel \( K \) function [58].

The expression (2.2.3) may be severely divergent and requires a regularisation. This stems from the fact that the original position space expression (1.4.8) is valid at non-coincident points only and itself requires a regularisation. A simple solution is to analytically continue (2.2.3) to a function of \( d \) and \( \Delta_j \), with a regularisation in these parameters then yielding a finite result.

To illustrate this, consider the case of three operators of dimension one in \( d = 3 \), i.e., \( \Delta_j = 1, j = 1, 2, 3 \). In this case, the Bessel functions can be expressed in terms of elementary functions (see (2.A.24)) and the integral in (2.2.3) has a logarithmic divergence. To regularise the result, we then substitute

\[
d \mapsto d + 2\epsilon, \quad \Delta_j \mapsto \Delta_j + \epsilon. \tag{2.2.4}
\]

This regularisation scheme is extremely useful in context of triple-\( K \) integrals since it preserves the indices of Bessel functions in (2.2.3). In the present case, we find

\[
\langle \langle O(p_1)O(p_2)O(p_3) \rangle \rangle = \frac{16\pi^3 c_{123}}{\Gamma(\frac{d}{2}) p_1 p_2 p_3} \int_0^\infty dx \, x^{-1+\epsilon} e^{-x(p_1+p_2+p_3)} = \left(\frac{2\pi^3 c_{123}}{p_1 p_2 p_3}\right) + O(\epsilon). \tag{2.2.5}
\]

This result can be confirmed by direct calculation using the fact that

\[
\int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|^2 |p_1 - k|^2 |p_2 + k|^2} = \frac{1}{8p_1 p_2 p_3}, \tag{2.2.6}
\]

as follows from the substitution \( \tilde{k} = k/k^2 \).

In summary then, the Fourier transform of the position space expression (1.4.8) for the 3-point function of scalar operators in any CFT may be expressed, at least formally, and up to an overall multiplicative constant, in terms of the triple-\( K \) integral (2.2.3). In the next section we will show that this representation in terms of a triple-\( K \) integral is very natural in the context of the conformal Ward identities. In fact, we will be able to re-derive the expression (2.2.3) by solving the conformal Ward identities directly in momentum space, without any reference to position space.

2.2.2. Conformal Ward identities

The conformal Ward identities (CWIs) in position space may be found in any standard reference text, e.g., [5]. In momentum space, the Ward identities for
2.2. 3-point function of scalar operators

Scalar operators have been partially analysed in [29, 42], and we will use these results here before generalising them in the following sections. First, observe that due to Lorentz invariance any 3-point function may be expressed in terms of the magnitudes of the momenta,

$$p_j = |p_j| = \sqrt{p_j^2}, \quad j = 1, 2, 3.$$  \hspace{1cm} (2.2.7)

The expression (2.2.3) is in accord with this fact. Regarding the 3-point function as a function of the momentum magnitudes, the dilatation Ward identity then reads

$$0 = 2d + \sum_{j=1}^{3} \left( p_j \frac{\partial}{\partial p_j} - \Delta_j \right) \left\langle \left\langle O_1 (p_1) O_2 (p_2) O_3 (p_3) \right\rangle \right\rangle.$$  \hspace{1cm} (2.2.8)

Similarly, the Ward identity associated with special conformal transformations is

$$0 = \sum_{j=1}^{3} p_j^\kappa \left[ \frac{\partial^2}{\partial p_j^2} + \frac{d + 1 - 2\Delta_j}{p_j} \frac{\partial}{\partial p_j} \right] \left\langle \left\langle O_1 (p_1) O_2 (p_2) O_3 (p_3) \right\rangle \right\rangle,$$  \hspace{1cm} (2.2.9)

where $\kappa$ is a free Lorentz index. These two equations are direct Fourier transforms of the position space Ward identities (1.3.7) and (1.3.9). We will discuss them in more detail in section 2.4.1.

Choosing $p_1$ and $p_2$ as independent momenta, we may split this vector equation into two independent scalar equations

$$0 = \left[ \left( \frac{\partial^2}{\partial p_1^2} \right) - \left( \frac{\partial^2}{\partial p_2^2} \right) \right] \left\langle \left\langle O_1 (p_1) O_2 (p_2) O_3 (p_3) \right\rangle \right\rangle,$$  \hspace{1cm} (2.2.10)

$$0 = \left[ \left( \frac{\partial^2}{\partial p_2^2} \right) - \left( \frac{\partial^2}{\partial p_3^2} \right) \right] \left\langle \left\langle O_1 (p_1) O_2 (p_2) O_3 (p_3) \right\rangle \right\rangle,$$  \hspace{1cm} (2.2.11)

where $\left\langle \left\langle O_1 O_2 O_3 \right\rangle \right\rangle = \left\langle \left\langle O_1 (p_1) O_2 (p_2) O_3 (p_3) \right\rangle \right\rangle$ for short. As an immediate check, we may verify that the expression (2.2.3) satisfies (2.2.10, 2.2.11) using the well-known Bessel function relations [59]

$$\frac{\partial}{\partial a} \left[ a^\nu K_\nu (ax) \right] = -xa^\nu K_{\nu-1} (ax),$$  \hspace{1cm} (2.2.12)

$$K_{\nu-1} (x) + \frac{2\nu}{x} K_\nu (x) = K_{\nu+1} (x).$$  \hspace{1cm} (2.2.13)

As we will see shortly, equations of the form (2.2.10, 2.2.11) also arise in the case of 3-point correlators of general tensor operators.
2.2.3. **Uniqueness of the solution**

To frame our analysis purely in momentum space, we need to show that there is a unique physically acceptable solution, up to an overall multiplicative constant, of the system (2.2.8, 2.2.10, 2.2.11) of dilatation and special CWIs. To accomplish this, it suffices to transform these equations into generalised hypergeometric form by writing

\[
\langle\mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(p_3)\rangle = p_3^{\Delta_1-2d} \left(\frac{p_1^2}{p_3^2}\right)^\mu \left(\frac{p_2^2}{p_3^2}\right)^\lambda F\left(\frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2}\right),
\]

where the overall power of momenta on the right-hand side is fixed by the dilatation Ward identity (2.2.8), and we have chosen to multiply the arbitrary function \(F\) by the prefactor \(\left(\frac{p_1^2}{p_3^2}\right)^\mu \left(\frac{p_2^2}{p_3^2}\right)^\lambda\), where \(\mu\) and \(\lambda\) are arbitrary constants. Substituting this parametrisation into (2.2.10, 2.2.11) then yields a pair of differential equations satisfied by \(F\). Taking \(\mu\) and \(\lambda\) to be any of the four combinations obtainable from the values

\[
\mu = 0, \quad \Delta_1 - \frac{d}{2}, \quad \lambda = 0, \quad \Delta_2 - \frac{d}{2},
\]

these equations for \(F\) read

\[
0 = \left[\xi(1-\xi) \frac{\partial^2}{\partial \xi^2} - \eta^2 \frac{\partial^2}{\partial \eta^2} - 2\xi\eta \frac{\partial^2}{\partial \xi \partial \eta} \right. \\
+ \left. (\gamma - (\alpha + \beta + 1)\xi) \frac{\partial}{\partial \xi} - (\alpha + \beta + 1)\eta \frac{\partial}{\partial \eta} - \alpha\beta\right] F(\xi, \eta), \tag{2.2.16}
\]

\[
0 = \left[\eta(1-\eta) \frac{\partial^2}{\partial \eta^2} - \xi^2 \frac{\partial^2}{\partial \xi^2} - 2\xi\eta \frac{\partial^2}{\partial \xi \partial \eta} \right. \\
+ \left. (\gamma' - (\alpha + \beta + 1)\eta) \frac{\partial}{\partial \eta} - (\alpha + \beta + 1)\xi \frac{\partial}{\partial \xi} - \alpha\beta\right] F(\xi, \eta), \tag{2.2.17}
\]

where

\[
\xi = \frac{p_1^2}{p_3^2}, \quad \eta = \frac{p_2^2}{p_3^2}, \tag{2.2.18}
\]

and the values of the parameters \(\alpha, \beta, \gamma, \gamma'\) depend on the choice of \(\mu\) and \(\lambda\). Specifically, parametrising the four choices for \(\mu\) and \(\lambda\) by two variables \(\epsilon_1, \epsilon_2 \in \{-1, +1\}\) according to

\[
\mu = \frac{1}{2}(\Delta_1 - \frac{d}{2})(\epsilon_1 + 1), \quad \lambda = \frac{1}{2}(\Delta_2 - \frac{d}{2})(\epsilon_2 + 1),
\]

we have

\[
\alpha = \frac{1}{2} \left[\epsilon_1 \left(\Delta_1 - \frac{d}{2}\right) + \epsilon_2 \left(\Delta_2 - \frac{d}{2}\right) + \Delta_3\right], \quad \beta = \alpha - \left(\Delta_3 - \frac{d}{2}\right),
\]

\[
\gamma = 1 + \epsilon_1 \left(\Delta_1 - \frac{d}{2}\right), \quad \gamma' = 1 + \epsilon_2 \left(\Delta_2 - \frac{d}{2}\right). \tag{2.2.20}
\]
2.2. 3-point function of scalar operators

The system of equations (2.2.16, 2.2.17) defines the generalised hypergeometric function of two variables Appell $F_4$. This function has been extensively studied by mathematicians (see, e.g., [60, 61]), and its important properties are summarised in appendix 2.A.4. In particular, the system of equations (2.2.16, 2.2.17) has at most four linearly independent solutions, each of which may be expressed in terms of the $F_4$ function [61, 62]. The four possible choices for $\mu$ and $\lambda$ reproduce these four solutions exactly.

In a physical context only one linear combination of these four solutions is acceptable: all the others contain divergences for collinear momentum configurations, for example when $p_1 + p_2 = p_3$. To see this, consider the integral representation

$$F_4 \left( \alpha, \beta; \gamma, \gamma'; \frac{p_1^2}{p_3^2}, \frac{p_2^2}{p_3^2} \right) = \frac{\Gamma(\gamma)\Gamma(\gamma')}{2^{\alpha+\beta-\gamma-\gamma'}\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{p_3^{\alpha+\beta}}{p_1^{\gamma-1}p_2^{\gamma'-1}} \times$$

$$\times \int_0^\infty dx x^{\alpha+\beta-\gamma-\gamma'+1} I_{\gamma-1}(p_1 x)I_{\gamma'-1}(p_2 x)K_{\beta-\alpha}(p_3 x), \tag{2.2.21}$$

where $I_\nu(x)$ is the Bessel $I$ function. This expression is formal in the sense that the integral converges only for $\alpha, \beta, \gamma, \gamma'$ in certain ranges, see appendix 2.A.4 for details. For the remaining parameter values the integral is defined by the analytic continuation (2.2.4). Using (2.2.20), one can then write the four solutions for the 3-point functions in the form

$$p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}} \int_0^\infty dx x^{\frac{d}{2}-1} I_{\pm(\Delta_1-\frac{d}{2})}(p_1 x)I_{\pm(\Delta_2-\frac{d}{2})}(p_2 x)K_{\Delta_3-\frac{d}{2}}(p_3 x). \tag{2.2.22}$$

For large $x$ we have the asymptotic expansions

$$I_\nu(x) = \frac{1}{\sqrt{2\pi}} e^x + \ldots, \quad K_\nu(x) = \sqrt{\frac{\pi}{2}} e^{-x} + \ldots, \tag{2.2.23}$$

from which we see that the integral (2.2.22) converges at infinite $x$ only for non-triangle (i.e., unphysical) momentum configurations where $p_1 + p_2 < p_3$. Moreover, for the physical collinear momentum configuration $p_1 + p_2 = p_3$, the integral diverges for dimensions $d \geq 3$. However, the 3-point function itself is a linear combination of these four solutions and may be continued to the physical regime by choosing the linear combination for which the collinear divergences cancel. This may be accomplished by subtracting two integrals with the same asymptotics, i.e., we sum the four terms of the form (2.2.22) with signs chosen so as to obtain Bessel $K$ functions

$$K_\nu(x) = \frac{\pi}{2 \sin(\nu \pi)} [I_\nu(x) - I_{-\nu}(x)]. \tag{2.2.24}$$
Therefore we arrive at the unique solution
\[
\langle\langle O_1(p_1)O_2(p_2)O_3(p_3)\rangle\rangle = C_{123} \cdot p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}} \times
\int_0^\infty dx \ x^{\frac{d}{2}-1} K_{\Delta_1-\frac{d}{2}}(p_1 x) K_{\Delta_2-\frac{d}{2}}(p_2 x) K_{\Delta_3-\frac{d}{2}}(p_3 x),
\] (2.2.25)

where \(C_{123}\) is an overall undetermined constant. From the asymptotic expansion (2.2.23), it is clear that this triple-\(K\) integral converges at infinite \(x\) for physical momentum configurations \(p_1 + p_2 + p_3 > 0\). Depending on the values of the parameters \(\Delta_j\) and \(d\), however, the triple-\(K\) integral may still diverge at \(x = 0\). This divergence may be regularised using (2.2.4) as we will discuss in the next section.

In summary then, we have shown that the conformal Ward identities may be solved directly in momentum space leading to a unique result (2.2.25). As we will see shortly, a similar procedure also holds for tensorial correlation functions: solving the momentum space Ward identities will lead to a unique solution for 3-point correlators without any reference to the position space analysis.

### 2.2.4. Region of validity and anomalies

In this section, we now discuss the regularisation of the potential divergence of the triple-\(K\) integral at \(x = 0\). In general, assuming all parameters and variables are real, the triple-\(K\) integral (2.2.25) converges for [63]
\[
\frac{d}{2} > \sum_{j=1}^3 \left| \Delta_j - \frac{d}{2} \right| + 2, \quad p_1, p_2, p_3 > 0.
\] (2.2.26)

If the parameters in the integral do not satisfy this inequality, however, the integral may be defined via analytic continuation in \(d\) and \(\Delta_j\). If for some set of parameters the integral exhibits a singularity, then a regularisation is necessary and the scheme (2.2.4) can be used.

Let us consider a concrete example, also discussed in [29]. We set \(d = 3\) and consider three scalar operators of dimensions \(\Delta_j = 2, j = 1, 2, 3\). The triple-\(K\) integral is then logarithmically divergent and a regularisation is necessary. We obtain
\[
\langle\langle O(p_1)O(p_2)O(p_3)\rangle\rangle = C_{123} \left( \frac{\pi}{2} \right)^{3/2} \cdot \int_0^\infty dx \ x^{-1+\epsilon} e^{-x(p_1+p_2+p_3)}
\]
\[
= C_{123} \left( \frac{\pi}{2} \right)^{3/2} \Gamma(\epsilon)(p_1 + p_2 + p_3)^{-\epsilon}
\]
\[
= C_{123} \left( \frac{\pi}{2} \right)^{3/2} \left[ \frac{1}{\epsilon} - (\gamma_E + \log(p_1 + p_2 + p_3)) + O(\epsilon) \right].
\] (2.2.27)

The first two terms are proportional to the Fourier transform of \(\delta(x_1-x_3)\delta(x_2-x_3)\) and may be removed by adding local counterterms. In the regularisation
scheme (2.2.4), such a counterterm has the form

\[ S_{ct} = c_\epsilon \int d^{3+2\epsilon} x \phi_0^3 \mu^{-\epsilon}, \tag{2.2.28} \]

where \( \mu \) is a scale that we introduce (as usual) so that the action is dimensionless. By taking three functional derivatives with respect to the source, we find the contribution of this term to the 3-point function is

\[ 3! c_\epsilon \mu^{-\epsilon} \delta(x_1 - x_3) \delta(x_2 - x_3). \tag{2.2.29} \]

Therefore, in order to cancel the divergence in (2.2.27), we need to choose

\[ c_\epsilon = -\frac{C_{123}}{3! \epsilon} \left( \frac{\pi}{2} \right)^{3/2} + O(\epsilon^0). \tag{2.2.30} \]

Let us make a few comments. First, we can choose the subleading terms in \( c_\epsilon \) in such a way that the constant in (2.2.27) can attain an arbitrary value. This is just scheme dependence. Secondly, notice that the same value of \( c_\epsilon \) and the same factor of \( \mu^{-\epsilon} \) in (2.2.28) are obtained by standard dimensional regularisation \( d \mapsto d - \epsilon \). This observation will be a key point in a relation between regularisations of 2- and 3-point functions discussed in section 2.5.3.

With the renormalisation procedure carried out, we are then left with the finite result

\[ \langle\langle O(p_1)O(p_2)O(p_3)\rangle\rangle = -C_{123} \left( \frac{\pi}{2} \right)^{3/2} \log \frac{p_1 + p_2 + p_3}{\bar{\mu}}, \tag{2.2.31} \]

where

\[ \bar{\mu} = \mu \exp \left[ \frac{c_\epsilon^{(0)}}{C_{123}} \left( \frac{2}{\pi} \right)^{3/2} \right] \tag{2.2.32} \]

and \( c_\epsilon^{(0)} \) denotes the first subleading term in \( c_\epsilon \). Note that this 3-point function is anomalous, \textit{i.e.}, it does not satisfy the dilatation Ward identity (2.2.8) and the scale dependence can be extracted from the \( \mu \) dependence of the counterterm (2.2.29),

\[ \mu \frac{\partial}{\partial \mu} \langle\langle O(p_1)O(p_2)O(p_3)\rangle\rangle = C_{123} \left( \frac{\pi}{2} \right)^{3/2} + O(\epsilon). \tag{2.2.33} \]

Of course, this can also be obtained directly from (2.2.31).

The violation of scale invariance means that the trace Ward identity is anomalous,

\[ \langle T \rangle = -\phi_0 \langle O \rangle + \frac{1}{3!} C_{123} \left( \frac{\pi}{2} \right)^{3/2} \phi_0^3. \tag{2.2.34} \]

This in turn implies that the 4-point function of the stress-energy tensor with three \( \mathcal{O} \)s contains a specific ultra-local term that is not scheme-dependent because it is fixed by the anomalous Ward identity (2.2.34).
2. Implications of conformal invariance in momentum space

2.3. Decomposition of tensors

In this section, we present a natural decomposition of tensorial correlation functions. Correlation functions of conserved currents are transverse and/or traceless – up to local terms – and we would like to find a decomposition which reflects these properties. At this point, we will not yet impose conformal invariance.

The problem of decomposition has already been tackled in a number of papers, see for example [64, 25, 47, 48, 49, 24]. The usual approach consists of writing down the most general tensor structure before imposing the constraints following from symmetries and Ward identities. Here we refine this approach to take account of the permutation symmetries of operator insertions inside correlators, obtaining a convenient and natural decomposition applicable for any correlation function. In particular, our decomposition contains the minimal number of tensor structures, leading to the simplest form for the conformal Ward identities.

We remind the reader we will always be working in $d$-dimensional Euclidean field theory with a flat metric $\delta_{\mu\nu}$ for which indices are raised and lowered trivially.

2.3.1. Representations of tensor structures

As we are interested in correlation functions, we must consider tensor functions that depend on a number of momenta. Let $O_{j}^{\mu_{j}1\mu_{j}2\cdots\mu_{j}r_{j}}$, $j = 1, 2, \ldots, n$ be a given set of QFT operators. Due to the momentum conservation, the $n$-point function contains a delta function which may be written explicitly by introducing the reduced matrix element which we denote with double brackets $\langle\langle \ldots \rangle\rangle$,

$$
\langle O_{1}^{\mu_{1}1\mu_{1}2\cdots\mu_{1}r_{1}}(p_{1})O_{2}^{\mu_{2}1\mu_{2}2\cdots\mu_{2}r_{2}}(p_{2}) \cdots O_{n}^{\mu_{n}1\mu_{n}2\cdots\mu_{n}r_{n}}(p_{n}) \rangle = (2\pi)^{d} \delta \left( \sum_{k=1}^{n} p_{k}^{\mu} p_{k}^{\nu} (Z^{-1})_{kj} \right),
$$

The $n$-point function thus depends on at most $n - 1$ vectors, say $p_{1}, \ldots, p_{n-1}$. If $n - 1 < d$, then all $n - 1$ momenta are independent. If $n - 1 \geq d$, however, then only $d$ generic momenta are independent. In this case we can write [53]

$$
\delta^{\mu\nu} = \sum_{j,k=1}^{d} p_{j}^{\mu} p_{k}^{\nu} (Z^{-1})_{kj},
$$

where $Z$ is the Gram matrix, $Z = [p_{k} \cdot p_{l}]_{k,l=1}^{d}$, hence the metric $\delta^{\mu\nu}$ is no longer an independent tensor.

From now on we assume $d \geq 3$. Since we are primarily interested in 3-point functions, the degeneracy does not occur. Nevertheless, the case $d = 3$ is still special since the existence of the cross-product allows the metric tensor to be re-expressed purely in terms of the momenta. This degeneracy serves to reduce
2.3. Decomposition of tensors

the number of independent form factors for certain correlators, as we discuss in appendix 2.A.2. In the following discussion we will ignore this degeneracy however and concentrate on the general case. We will therefore choose two out of the three \( \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \) as independent momenta, and treat the metric \( \delta^{\mu\nu} \) as an independent tensor.

As an example consider a 3-point function of two transverse, traceless, symmetric rank two operators \( t^{\mu\nu} \) and a scalar operator \( \mathcal{O} \). Using the projectors (2.1.14) one can write the most general form

\[
\langle\langle t^{\mu_1\nu_1}(\mathbf{p}_1)t^{\mu_2\nu_2}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle\rangle = \Pi_{\alpha_1\beta_1}^{\mu_1\nu_1}(\mathbf{p}_1)\Pi_{\alpha_2\beta_2}^{\mu_2\nu_2}(\mathbf{p}_2)X^{\alpha_1\beta_1\alpha_2\beta_2},
\]

(2.3.3)

where \( X^{\alpha_1\beta_1\alpha_2\beta_2} \) is a general tensor of rank four built from the metric and momenta. Usually one chooses two independent momenta once and for all. On the other hand, there is no obstacle to choosing different independent momenta for different Lorentz indices. In the thesis we always choose

\[
\mathbf{p}_1, \mathbf{p}_2 \text{ for } \mu_1, \nu_1; \mathbf{p}_2, \mathbf{p}_3 \text{ for } \mu_2, \nu_2 \text{ and } \mathbf{p}_3, \mathbf{p}_1 \text{ for } \mu_3, \nu_3.
\]

(2.3.4)

Such a choice enhances the symmetry properties of the decomposition, as we will discuss at length in the next section.

Let us now enumerate all possible tensors that can appear in \( X^{\alpha_1\beta_1\alpha_2\beta_2} \). Observe that whenever a simple tensor contains at least one of the following tensors

\[
\delta^{\alpha_1\beta_1}, \delta^{\alpha_2\beta_2}, p_1^{\alpha_1}, p_1^{\beta_1}, p_2^{\alpha_2}, p_2^{\beta_2},
\]

(2.3.5)

then the contraction with the projectors in (2.3.3) vanishes. Therefore, in accordance with the choice (2.3.4), the only tensors giving a non-zero result after contraction with the projectors are

\[
\delta^{\alpha_1\alpha_2}, \delta^{\alpha_1\beta_2}, \delta^{\beta_1\alpha_2}, \delta^{\beta_1\beta_2}, p_1^{\alpha_1}, p_1^{\beta_1}, p_2^{\alpha_2}, p_2^{\beta_2}, p_3^{\alpha_3}, p_3^{\beta_3}.
\]

(2.3.6)

Since the projector (2.1.14) is symmetric in \( \mu \leftrightarrow \nu \) and \( \alpha \leftrightarrow \beta \), the most general form of our 3-point function is then

\[
\langle\langle t^{\mu_1\nu_1}(\mathbf{p}_1)t^{\mu_2\nu_2}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle\rangle = \Pi_{\alpha_1\beta_1}^{\mu_1\nu_1}(\mathbf{p}_1)\Pi_{\alpha_2\beta_2}^{\mu_2\nu_2}(\mathbf{p}_2) \left[ A_1p_2^{\alpha_1}p_2^{\beta_1}p_3^{\alpha_2}p_3^{\beta_2} + A_2p_2^{\alpha_1}p_3^{\alpha_2}\delta^{\beta_1\beta_2} + A_3\delta^{\alpha_1\alpha_2}\delta^{\beta_1\beta_2} \right],
\]

(2.3.7)

where the coefficients \( A_1, A_2 \) and \( A_3 \) are scalar functions of momenta. We will refer to the coefficients \( A_j \), and their analogous counterparts in more general correlation functions, as form factors. By Lorentz invariance, these form factors are functions of the momentum magnitudes

\[
p_j = |\mathbf{p}_j| = \sqrt{\mathbf{p}_j^2}, \quad j = 1, 2, 3,
\]

(2.3.8)
2. Implications of conformal invariance in momentum space

\( A_j = A_j(p_1, p_2, p_3) \). In particular, any scalar product of two momenta can be written as a combination of momentum magnitudes, for example

\[
p_1 \cdot p_2 = \frac{1}{2} (p_3^2 - p_1^2 - p_2^2).
\]

(2.3.9)

For brevity, we will henceforth suppress the dependence of form factors on the momentum magnitudes, writing \( A_j(p_1, p_2, p_3) \) as simply \( A_j \).

Note that the correlation function on the left-hand side of (2.3.7) is symmetric under a transposition \((p_1, \mu_1, \nu_1) \leftrightarrow (p_2, \mu_2, \nu_2)\). One can apply this symmetry to the right-hand side to find that all form factors \( A_1, A_2 \) and \( A_3 \) are symmetric under \( p_1 \leftrightarrow p_2 \). To prove this, observe that one has, for example, \( \pi^{\mu_1}_{\alpha_1}(p_1)p_3^{\alpha_1} = -\pi^{\mu_1}_{\alpha_1}(p_1)p_2^{\alpha_1} \). Therefore \( p_2 \) and \(-p_3\) can be exchanged under both \( \pi^{\mu_1}_{\alpha_1}(p_1) \) and \( \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1) \), and similarly for other momenta.

For any form factor \( A_j \) we define an associated non-negative integer \( N_j \), the tensorial dimension of \( A_j \), similar to that defined in [25]. Specifically, the tensorial dimension \( N_j \) is the number of momenta that appear in the tensorial expression multiplying \( A_j \) (excluding those in the transverse-traceless projectors) in the decomposition of the correlation function. As we will see later, this quantity will appear explicitly in the conformal Ward identities. In the example (2.3.7), we have the following tensorial dimensions: \( N_1 = 4 \), \( N_2 = 2 \) and \( N_3 = 0 \).

Decompositions for other correlation functions may be found in chapter 3. Observe that in each case the form factor \( A_1 \) stands in front of the unique tensor containing momenta only. The tensorial dimension \( N_1 \) is therefore always equal to the number of Lorentz indices in the 3-point function, and tensorial dimensions of all remaining form factors are smaller than \( N_1 \).

2.3.2. Decomposition of \( \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} t^{\mu_3 \nu_3} \rangle \)

In the previous section we introduced a natural decomposition of tensor structures. Rather than fixing two independent momenta (as is done for example in [64, 25, 47, 48, 49, 24]) we chose a different set of independent momenta for different Lorentz indices according to (2.3.4). Such a choice respects all symmetries of the correlation function, as we now discuss.

In [25], it was shown that the transverse-traceless correlation function \( \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} t^{\mu_3 \nu_3} \rangle \) can be decomposed into eight tensor structures plus their \( p_1 \leftrightarrow p_2 \) symmetric versions. In our method, however, we arrive at only five tensor structures (for the general case \( d \geq 3 \), see appendix 2.A.2 for the case \( d = 3 \)) according to the following decomposition

\[
\langle t^{\mu_1 \nu_1}(p_1)t^{\mu_2 \nu_2}(p_2)t^{\mu_3 \nu_3}(p_3) \rangle = \Pi^{\mu_1 \nu_1}_{\alpha_1 \beta_1}(p_1)\Pi^{\mu_2 \nu_2}_{\alpha_2 \beta_2}(p_2)\Pi^{\mu_3 \nu_3}_{\alpha_3 \beta_3}(p_3) \left[ A_1 p_2^{\alpha_2} p_3^{\alpha_3} p_2^{\beta_2} p_3^{\beta_3} + \right.
\]

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In particular, the form factors from (2.3.10) by substituting the mismatch follows from the choice of two independent momenta in [25] to be tensor structures between (2.3.10) and the results of [25]. As already mentioned, permutation swaps tensor structures of the first and the second term in the third line. This requires that the form factor of the second term is related to the form factor of the first term invariant, therefore the (2.3.10) symmetry down to the symmetry group $S_3$ of the set \{1, 2, 3\}, i.e., for any $\sigma \in S_3$,

$$
\langle \langle t^{\mu_1 \nu_1}(p_1)t^{\mu_2 \nu_2}(p_2)t^{\mu_3 \nu_3}(p_3) \rangle \rangle = \langle \langle t^{\mu_\sigma(1) \nu_\sigma(1)}(p_\sigma(1))t^{\mu_\sigma(2) \nu_\sigma(2)}(p_\sigma(2))t^{\mu_\sigma(3) \nu_\sigma(3)}(p_\sigma(3)) \rangle \rangle. \tag{2.3.11}
$$

In particular, the form factors $A_1$ and $A_5$ are $S_3$-invariant,

$$
A_j(p_1, p_2, p_3) = A_j(p_\sigma(1), p_\sigma(2), p_\sigma(3)), \quad j \in \{1, 5\}, \tag{2.3.12}
$$

since the tensors they multiply are $S_3$-invariant. The action of the symmetry group on the remaining terms is then clearly visible. As an example, let us concentrate on the third line of (2.3.10) with the $A_2$ form factor. The $(p_1, \mu_1, \nu_1) \leftrightarrow (p_2, \mu_2, \nu_2)$ permutation leaves the tensor in the first term invariant, therefore the $A_2$ factor exhibits the $p_1 \leftrightarrow p_2$ symmetry. On the other hand, the $(p_1, \mu_1, \nu_1) \leftrightarrow (p_3, \mu_3, \nu_3)$ permutation swaps tensor structures of the first and the second term in the third line. This requires that the form factor of the second term is related to the form factor of the first term by the $p_1 \leftrightarrow p_3$ permutation, as indicated. Working out the remaining lines of (2.3.10) one finds that both remaining factors $A_3$ and $A_4$ are $p_1 \leftrightarrow p_2$ symmetric.

Let us comment then on the apparent disagreement between the number of tensor structures between (2.3.10) and the results of [25]. As already mentioned, the mismatch follows from the choice of two independent momenta in [25] to be $p_1$ and $p_2$, in our notation. Such a choice breaks the $S_3$ symmetry down to the $(p_1, \mu_1, \nu_1) \leftrightarrow (p_2, \mu_2, \nu_2)$ symmetry. One can easily recover eight tensor structures from (2.3.10) by substituting $p_3 = -p_1 - p_2$ and writing the decomposition in
terms of $p_1$ and $p_2$ only. One finds

\[
\langle\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)t^{\mu_3\nu_3}(p_3)\rangle\rangle = \Pi_{\alpha_1\beta_1}^{\mu_1\nu_1}(p_1)\Pi_{\alpha_2\beta_2}^{\mu_2\nu_2}(p_2)\Pi_{\alpha_3\beta_3}^{\mu_3\nu_3}(p_3) \left[ \frac{1}{2} A_1 p_1^{\alpha_1} p_2^{\beta_1} p_1^{\alpha_2} p_1^{\beta_2} p_2^{\alpha_3} p_1^{\beta_3} - \frac{1}{2} A_2 \delta^{\beta_1\beta_2} p_2^{\alpha_1} p_1^{\alpha_2} p_2^{\beta_3} p_1^{\alpha_3} p_1^{\beta_1} + \frac{1}{2} A_3 \delta^{\alpha_1\alpha_2} \delta^{\beta_1\beta_2} p_1^{\alpha_1} p_1^{\beta_1} - \frac{1}{2} A_4 \delta^{\alpha_1\alpha_2} \delta^{\alpha_3\beta_3} p_2^{\beta_1} p_1^{\alpha_3} p_2^{\beta_2} p_1^{\beta_3} + \frac{1}{2} A_5 \delta^{\alpha_1\beta_2} \delta^{\alpha_2\beta_3} \delta^{\beta_1\beta_3} \right] + \text{everything with } (p_1,\alpha_1,\beta_1) \leftrightarrow (p_2,\alpha_2,\beta_2) .
\]

(2.3.13)

As we can see, the number of tensor structures increases to exactly eight, as the symmetry group is diminished.

Summarising, our decomposition method based on (2.3.4) gives the minimal number of tensor structures obeying the symmetries of the correlation function. It is an improvement over the standard method with two independent momenta fixed, since such a choice breaks symmetries and leads therefore to the larger number of tensor structures.

Finally, we should comment on the fact that we decompose the transverse-traceless part of the correlation function only. This is because the difference between the full 3-point function and its transverse-traceless part is semi-local, and hence may be entirely reconstructed from the Ward identities. We will discuss this method for recovering the full correlation function from its transverse-traceless part in the next section.

Let us note in passing that the decomposition method described here may also be used for correlation functions in non-conformal theories. For example, in cases where the stress-energy tensor is transverse but no longer traceless one should use the $\pi_\mu^\alpha$ projectors (2.1.13) in place of $\Pi^{\mu\nu}_{\alpha\beta}$ in (2.3.10). In this way one obtains ten tensor structures, five of which have nonzero trace. This decomposition is given in appendix 2.A.1.

### 2.3.3. Finding the form factors

We would like to apply the results of the previous section to spin-1 and spin-2 conserved currents $J^\mu$ and a stress-energy tensor $T^{\mu\nu}$. These quantum operators, however, are only transverse and traceless on-shell, and in the quantum case, we need to analyse Ward identities. To proceed, we define transverse, transverse-
2.3. Decomposition of tensors

traceless and local parts of $J^\mu$ and $T^{\mu\nu}$ by

\[
j^\mu_j \equiv \pi^\mu_\alpha J^\alpha, \qquad j^\mu_{\text{loc}} \equiv J^\mu - j^\mu, \quad (2.3.14)\]

\[
t^{\mu\nu}_{\text{loc}} \equiv T^{\mu\nu} - t^{\mu\nu}, \quad (2.3.15)\]

as well as longitudinal and trace parts

\[
r = p_\mu J^\mu, \quad R^\nu = p_\mu T^{\mu\nu}, \quad R = p_\mu R^\nu, \quad T = T^\mu_\mu. \quad (2.3.16)\]

From these definitions, we then have

\[
j^\mu_{\text{loc}} = p^\mu p^2 r, \quad (2.3.17)\]

\[
t^{\mu\nu}_{\text{loc}} = p^\mu p^\nu R + \frac{p^\mu p^\nu}{p^2} R + \frac{1}{d-1} \pi^{\mu\nu} \left( T - \frac{R}{p^2} \right), \quad (2.3.18)\]

where the operator

\[
\mathcal{T}^{\mu\nu}_\alpha(p) = \frac{1}{p^2} \left[ 2p^{(\mu} \delta^{\nu)}_\alpha - \frac{p_\alpha}{d-1} \left( \delta^{\mu\nu} + (d-2) \frac{p^{\mu} p^{\nu}}{p^2} \right) \right]. \quad (2.3.19)\]

In the following, we will also use $\mathcal{T}^{\mu\nu\alpha} = \delta^{\alpha\beta} \mathcal{T}^{\mu\nu}_\beta$.

We now observe that in a CFT, all terms in (2.3.17) and (2.3.18) are computable by means of the transverse and trace Ward identities. We can therefore divide a 3-point function into two parts: the transverse-traceless part represented as in section 2.3.1, and the semi-local part (indicated by the subscript $\text{loc}$) expressible through the transverse Ward identities. For simplicity we will use the term ‘transverse-traceless part’ in all cases, even if the correlation function does not contain the stress-energy tensor.

As an example, consider

\[
\langle\langle t^{\mu_1\nu_1}(p_1) t^{\mu_2\nu_2}(p_2) O(p_3) \rangle\rangle = \Pi_{\alpha_1}^{\mu_1\nu_1}(p_1) \Pi_{\alpha_2}^{\mu_2\nu_2}(p_2) \langle\langle T^{\alpha_1\beta_1}(p_1) T^{\alpha_2\beta_2}(p_2) O(p_3) \rangle\rangle. \quad (2.3.20)\]

One can recover the full 3-point function by writing

\[
\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} O \rangle\rangle = \quad (2.3.21)\]

\[
= \langle\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} O \rangle\rangle + \langle\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} t_{\text{loc}} O \rangle\rangle + \langle\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} O \rangle\rangle + \langle\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} t_{\text{loc}} O \rangle\rangle \]

\[
= \langle\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} O \rangle\rangle - \langle\langle T^{\mu_1\nu_1} t^{\mu_2\nu_2} t_{\text{loc}} O \rangle\rangle - \langle\langle t^{\mu_1\nu_1} T^{\mu_2\nu_2} O \rangle\rangle - \langle\langle t^{\mu_1\nu_1} t^{\mu_2\nu_2} t_{\text{loc}} O \rangle\rangle. \quad (2.3.21)\]

All terms on the right-hand side apart from the first may be computed by means of Ward identities. The exact form of the Ward identities depends on the exact definition of the operators involved, but more importantly, all these terms depend on 2-point functions only.
Due to the complicated nature of contractions of the projectors (2.1.13) and (2.1.14) one might fear that it is very difficult to calculate the form factors by means of Feynman rules, given some particular QFT. Reassuringly, this is not the case, as we can see in the following example. First, we decompose the full 3-point function $\langle\langle T^{\nu_1\nu_1} T^{\mu_2\nu_2} O \rangle\rangle$ into simple tensors using the choice of momenta (2.3.4) and denote

$$\langle\langle T^{\alpha_1\beta_1} T^{\alpha_2\beta_2} O \rangle\rangle = \tilde{A}_1 p_2^\alpha_1 p_2^\beta_1 p_3^\alpha_2 p_3^\beta_2 + \tilde{A}_2 p_2^\alpha_1 p_3^\alpha_2 \delta_1^\beta_1 \beta_2 + \tilde{A}_3 \delta_1^\alpha_1 \delta_1^\beta_2 \delta_1^\beta_2 + \ldots, \tag{2.3.22}$$

where the omitted terms do not contain the tensors we have listed explicitly. Next, we apply the projectors (2.1.14). Obverse, for example, that the projector $\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1)$ is constructed from the metric and the momentum $p_1$ only, and therefore when applied to the 3-point function it cannot change the coefficient of any tensor containing $p_2^\gamma_1 p_2^\beta_1$, i.e.,

$$\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1) p_2^\alpha_1 p_2^\beta_1 = p_2^\mu_1 p_2^\nu_1 + \ldots, \tag{2.3.23}$$

where the omitted terms do not contain $p_2^\mu_1 p_2^\nu_1 = \ldots$. Using the same argument for $\Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2)$, we see that the coefficients of $p_2^\alpha_1 p_3^\beta_1 p_3^\alpha_2 p_3^\beta_2$ in (2.3.22) and $p_2^\mu_1 p_3^\nu_1 p_3^\mu_2 p_3^\nu_2$ in $\langle\langle \mu_1 \nu_1 (p_1) T^{\mu_2\nu_2} (p_2) O(p_3) \rangle\rangle$ in (2.3.20) are equal, i.e., $A_1 = \tilde{A}_1$. Similarly, we find that

$$\Pi^{\mu_1\nu_1}_{\alpha_1\beta_1}(p_1) \Pi^{\mu_2\nu_2}_{\alpha_2\beta_2}(p_2) \langle\langle T^{\alpha_1\beta_1} (p_1) T^{\alpha_2\beta_2} (p_2) O(p_3) \rangle\rangle =$$

$$= \tilde{A}_1 p_2^\mu_1 p_2^\nu_1 p_3^\mu_2 p_3^\nu_2 + \frac{1}{4} \tilde{A}_2 p_2^\mu_1 p_3^\nu_1 \delta^{\mu_2^\nu_2^1} + \frac{1}{2} \tilde{A}_3 \delta^{\mu_1^\mu_2} \delta^{\nu_1^\nu_2} + \ldots, \tag{2.3.24}$$

where the omitted terms do not contain the tensors we have listed explicitly. We therefore have

$$A_1 = \text{coefficient of } p_2^\mu_1 p_2^\nu_1 p_3^\mu_2 p_3^\nu_2 \text{ in } \langle\langle T^{\mu_1\nu_1} (p_1) T^{\mu_2\nu_2} (p_2) O(p_3) \rangle\rangle,$$

$$A_2 = 4 \cdot \text{coefficient of } p_2^\mu_1 p_3^\nu_1 \delta^{\mu_2^\nu_2^1} \text{ in } \langle\langle T^{\mu_1\nu_1} (p_1) T^{\mu_2\nu_2} (p_2) O(p_3) \rangle\rangle,$$

$$A_3 = 2 \cdot \text{coefficient of } \delta^{\mu_1^\mu_2} \delta^{\nu_1^\nu_2} \text{ in } \langle\langle T^{\mu_1\nu_1} (p_1) T^{\mu_2\nu_2} (p_2) O(p_3) \rangle\rangle. \tag{2.3.25}$$

We list the analogous formulae for all other 3-point functions in chapter 3.

### 2.3.4. Example

Let us consider a conformally coupled free scalar free massless field $\phi$ in $d$ Euclidean dimensions given by the action (1.1.64). In the presence of a non-trivial source $g^{\mu\nu}$ for the stress-energy tensor, the stress-energy tensor in given by (1.1.66). In this CFT, $O(x) = \phi^2(x)$ is a conformal primary operator of dimension $\Delta_3 = d - 2$.

For later use we quote the result for the form factors of $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} O \rangle\rangle$ in this theory. Writing down the expression for $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} O \rangle\rangle$ using the regular
Feynman rules, from (2.3.25) we may then read off expressions for the form factors. Explicitly evaluating these integrals for the case $d = 3$, we find

\[
A_1 = \frac{3(p_1^2 + p_2^2) + 12p_1p_2 + 4p_3(p_1 + p_2)}{48p_3(p_1 + p_2 + p_3)^4},
\]

\[
A_2 = \frac{2(p_1^2 + p_2^2) + p_3^2 + 6p_1p_2 + 3p_3(p_1 + p_2)}{12(p_1 + p_2 + p_3)^3},
\]

\[
A_3 = -\frac{2(p_1p_2 + p_1p_3^2 + p_3p_1^2 + p_2p_3^2 + p_3^2p_2) + 2p_1p_2p_3 + p_1^3 + p_2^3 + p_3^3}{24(p_1 + p_2 + p_3)^2},
\]

in agreement with the direct evaluation of this correlator given in [1].

2.4. Conformal Ward identities in momentum space

In section 2.2.2 we wrote down the Ward identities associated with dilatations and special conformal transformation for the case of correlators involving three scalars. In this section, we discuss the corresponding Ward identities for 3-point correlators involving insertions of the stress-energy tensor and conserved currents. First, in section 2.4.1, we obtain the dilatation and special conformal Ward identities in momentum space by Fourier transforming the well-known position space expressions; in sections 2.4.2 and 2.4.3 we then reduce these identities to a set of simple scalar equations using the tensor decomposition introduced in section 2.3.1. Finally, in section 2.4.4 we write down the transverse and trace Ward identities.

2.4.1. From position space to momentum space

Now we want to write down the Ward identities associated with dilatations and special conformal transformations in momentum space. We start from position space equations (1.3.7), (1.3.9) and (1.3.10) and we calculate their Fourier transforms in a similar manner to that discussed in [65]. Due to the translation invariance the position space correlators depend only on the differences $x_j - x_n$. Therefore, we can set $x_n = 0$ and take

\[
p_n = -\sum_{j=1}^{n-1} p_j.
\]

The Ward identities (1.3.7) and (1.3.9) then transform to

\[
0 = \left[ \sum_{j=1}^{n} \Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \right] \langle \langle O_1(p_1) \ldots O_n(p_n) \rangle \rangle,
\]

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2. Implications of conformal invariance in momentum space

\[ 0 = \left[ \sum_{j=1}^{n-1} \left( 2(\Delta_j - d) \frac{\partial}{\partial p_j^\kappa} - 2p_j^{\alpha} \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_j^\kappa} + (p_j)^{\kappa} \frac{\partial}{\partial p_j^{\alpha}} \frac{\partial}{\partial p_j^\kappa} \right) \right] \langle \langle O_1 \ldots O_n \rangle \rangle, \]  

(2.4.3)

while the additional contribution (1.3.10) transforms to

\[ 2 \sum_{j=1}^{n} \sum_{k=1}^{n_j} \left( \delta^{\mu_j k} \frac{\partial}{\partial p_j^{\alpha_j k}} - \delta^{\kappa}_{\alpha_j k} \frac{\partial}{\partial p_j^{\mu_j k}} \right) \times \]
\[ \langle \langle O_1^{\mu_1 \ldots \mu_1} (p_1) \ldots O_j^{\mu_j 1 \ldots \alpha_j k \ldots \mu_j r_j} (p_j) \ldots O_n^{\mu_n 1 \ldots \mu_n r_n} (p_n) \rangle \rangle \]  

(2.4.4)

and once again must be added to the right-hand side of (2.4.3). It will be useful to denote the differential operator obtained by summing the right-hand side of (2.4.3) and (2.4.4) as \( K^\kappa \), so that the CWIs may be compactly expressed as

\[ K^\kappa \langle \langle O_1 (p_1) \ldots O_n (p_n) \rangle \rangle = 0. \]  

(2.4.5)

In view of (2.4.4), note that \( K^\kappa \) acts non-trivially on Lorentz indices and so in fact is really of the form

\[ K^\kappa = K_{\alpha_1 1}^{\mu_1 1} \ldots K_{n r_n}^{\mu_n r_n}, \]  

(2.4.6)

however for simplicity we will omit the tensor indices on \( K^\kappa \).

In the following analysis we will focus specifically on 3-point functions. The idea will be to take the tensor decomposition the 3-point function described in section 2.3.1, then apply the operators (2.4.3) and (2.4.4) yielding differential equations for the form factors. Since by Lorentz invariance the form factors are purely functions of the momentum magnitudes, the action of momentum derivatives on form factors may be obtained using the chain rule,

\[ \frac{\partial}{\partial p_1^{\mu}} = \frac{\partial p_1}{\partial p_1^{\mu}} \frac{\partial}{\partial p_1} + \frac{\partial p_2}{\partial p_1^{\mu}} \frac{\partial}{\partial p_2} + \frac{\partial p_3}{\partial p_1^{\mu}} \frac{\partial}{\partial p_3} = \frac{p_1^{\mu}}{p_1} \frac{\partial}{\partial p_1} + \frac{p_2^{\mu}}{p_2} \frac{\partial}{\partial p_2} + \frac{p_3^{\mu}}{p_3} \frac{\partial}{\partial p_3}, \]  

(2.4.7)

noting that \( p_3 \) is fixed via (2.4.1). We may express derivatives with respect to \( p_2 \) similarly, and the final results may then be re-expressed purely in terms of the momentum magnitudes.

2.4.2. Dilatation Ward identity

Using (2.4.7), it is simple to rewrite the dilatation Ward identity (2.4.2) for a 3-point function of three conformal primary operator of any tensor structure in terms of its form factors as

\[ 0 = \left[ 2d + N_n + \sum_{j=1}^{3} \left( p_j \frac{\partial}{\partial p_j} - \Delta_j \right) \right] A_n(p_1, p_2, p_3), \]  

(2.4.8)
2.4. Conformal Ward identities in momentum space

where \(N_n\) is the tensorial dimension of \(A_n\), i.e., the number of momenta in the tensor multiplying the form factor \(A_n\) and the transverse-traceless projectors. As previously, \(\Delta_j, j = 1, 2, 3\) denote the conformal dimensions of the operators \(\mathcal{O}_j\) in the 3-point function: for a conserved current we thus have \(\Delta = d - 1\) while for a stress-energy tensor \(\Delta = d\).

The dilatation Ward identity determines the total degree of the 3-point function and hence of its form factors. In general, if a function \(A\) satisfies

\[
0 = \left[ -D + \sum_{j=1}^{3} p_j \frac{\partial}{\partial p_j} \right] A(p_1, p_2, p_3) \tag{2.4.9}
\]

for some constant \(D\) then we will refer to \(D\) as the degree of \(A\), denoted \(\text{deg}(A) = D\). (A homogeneous polynomial in momenta of degree \(D\) has dilatation degree \(D\).) Therefore (2.4.8) implies that the form factor \(A_n\) has degree

\[
\text{deg}(A_n) = \Delta_t - 2d - N_n, \tag{2.4.10}
\]

where \(\Delta_t = \Delta_1 + \Delta_2 + \Delta_3\).

2.4.3. Special conformal Ward identities

In this section, we now extract scalar equations for the form factors by inserting our tensor decomposition for the transverse-traceless part of the 3-point functions into the special conformal Ward identities. While the details are somewhat involved, the procedure is nonetheless conceptually straightforward. We will outline the method using as an example the 3-point function \(\langle\langle T_{\mu_1 \nu_1} T_{\mu_2 \nu_2} \mathcal{O} \rangle\rangle\), which captures all the important features.

Consider then the action of the CWI operator \(K^\kappa\) defined in (2.4.5) on the decomposition (2.3.21),

\[
0 = K^\kappa \langle\langle T_{\mu_1 \nu_1} T_{\mu_2 \nu_2} \mathcal{O} \rangle\rangle \tag{2.4.11}
\]

\[
= K^\kappa \langle\langle t_{\mu_1 \nu_1} t_{\mu_2 \nu_2} \mathcal{O} \rangle\rangle + K^\kappa \langle\langle t_{\mu_1 \nu_1} t_{\mu_2 \nu_2} \mathcal{O} \rangle\rangle + K^\kappa \langle\langle t_{\mu_1 \nu_1} t_{\mu_2 \nu_2} \mathcal{O} \rangle\rangle + K^\kappa \langle\langle t_{\mu_1 \nu_1} t_{\mu_2 \nu_2} \mathcal{O} \rangle\rangle;
\]

recalling that our notation for \(K^\kappa\) suppresses Lorentz indices so that in reality, e.g.,

\[
K^\kappa \langle\langle t_{\mu_1 \nu_1} t_{\mu_2 \nu_2} \mathcal{O} \rangle\rangle = K_{\alpha_1 \beta_1 \alpha_2 \beta_2} \langle\langle t_{\alpha_1 \beta_1} t_{\alpha_2 \beta_2} \mathcal{O} \rangle\rangle. \tag{2.4.12}
\]

Through a direct but lengthy calculation we find that the first term on the right-hand side of (2.4.11), \(K^\kappa \langle\langle t_{\mu_1 \nu_1} t_{\mu_2 \nu_2} \mathcal{O} \rangle\rangle\), is transverse-traceless in \(\mu_1, \nu_1\) and \(\mu_2, \nu_2\)
with respect to the corresponding momenta,

\[ 0 = \delta_{\mu_1\nu_1} [K^K \langle\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)\mathcal{O}(p_3)\rangle\rangle], \]

\[ 0 = \delta_{\mu_2\nu_2} [K^K \langle\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)\mathcal{O}(p_3)\rangle\rangle], \]

\[ 0 = p_{1\mu_1} [K^K \langle\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)\mathcal{O}(p_3)\rangle\rangle], \]

\[ 0 = p_{2\mu_2} [K^K \langle\langle t^{\mu_1\nu_1}(p_1)t^{\mu_2\nu_2}(p_2)\mathcal{O}(p_3)\rangle\rangle], \]

(2.4.13)

where we used the definitions (2.4.3) and (2.4.4) for \( K^K \) and the identities given
in appendix 2.A.8. For correlators involving conserved currents, we find that
the analogue of (2.4.13) similarly applies.

We are now free to apply transverse-traceless projectors (2.1.13) and (2.1.14)
to (2.4.11), in order to isolate equations for the form factors appearing in the
decomposition of \( \langle\langle t^{\mu_1\nu_1}t^{\mu_2\nu_2}\mathcal{O}\rangle\rangle \). Evaluating the action of \( K^K \) on the semi-local
terms in (2.4.11) via the formulae in appendix 2.A.8, we find

\[ \pi^K_{\alpha} j^K_{\alpha \mu} = \frac{2(d-2)}{p^2} \pi^K_{\mu \mu}, \]

(2.4.14)

\[ \Pi^K_{\alpha \beta} j^K_{\alpha \beta} = \frac{4d}{p^2} \Pi^K_{\alpha \mu} R^K_{\mu}, \]

(2.4.15)

\[ \pi^K_{\alpha \beta} j^K_{\alpha \beta \mu} = \Pi^K_{\alpha \beta \mu} = 0. \]

(2.4.16)

The last equation implies that any correlation function with more than one
insertion of \( t^K_{\mu \mu} \) or \( j^K_{\mu \mu} \) vanishes when the CWI operator \( K^K \) and the projectors (2.1.13)
and (2.1.14) are applied. This is because the CWI operator \( K^K \) can be written as
a sum of two terms

\[ K^K = K^K_1 \left( \frac{\partial}{\partial p^K_{\mu}} \right) + K^K_2 \left( \frac{\partial}{\partial p^K_{\nu}} \right), \]

(2.4.17)

each depending only on derivatives with respect to the appropriate momenta, hence

\[ \Pi^K_{\alpha_1 \beta_1}(p_1)\Pi^K_{\alpha_2 \beta_2}(p_2)K^K \langle\langle t^K_{\alpha_1 \beta_1}(p_1)t^K_{\alpha_2 \beta_2}(p_2)\mathcal{O}(p_3)\rangle\rangle = 0. \]

(2.4.18)

Substituting all results into (2.4.11), we find

\[ 0 = \Pi^K_{\alpha_1 \beta_1}(p_1)\Pi^K_{\alpha_2 \beta_2}(p_2)K^K \langle\langle t^K_{\alpha_1 \beta_1}(p_1)t^K_{\alpha_2 \beta_2}(p_2)\mathcal{O}(p_3)\rangle\rangle \]

\[ + \frac{4d}{p_1^K} \Pi^K_{\alpha_1 \beta_1}(p_1) \left[ p_{1\beta_1} \langle\langle T^{\alpha_1 \beta_1}(p_1)t^K_{\nu_1 \nu_2}(p_2)\mathcal{O}(p_3)\rangle\rangle \right] \]

\[ + \frac{4d}{p_2^K} \Pi^K_{\alpha_2 \beta_2}(p_2) \left[ p_{2\beta_2} \langle\langle t^K_{\mu_1 \nu_1}(p_1)T^{\alpha_2 \beta_2}(p_2)\mathcal{O}(p_3)\rangle\rangle \right]. \]

(2.4.19)

Two last terms are semi-local and may be re-expressed in terms of 2-point functions
via the transverse Ward identities. The remaining task is then to rewrite the first
2.4. Conformal Ward identities in momentum space

term of (2.4.19) in terms of form factors and extract the CWIs. Via the method of section 2.3.1, we can write the most general form of the result as

$$
\Pi_{\alpha_1,\alpha_2}^{\mu_1,\nu_1}(p_1)\Pi_{\alpha_2,\alpha_2}^{\mu_2,\nu_2}(p_2)K^{\nu_1}\langle\{t^{\alpha_1,\beta_1}(p_1)t^{\alpha_2,\beta_2}(p_2)\mathcal{O}(p_3)\}\rangle = \Pi_{\alpha_1,\alpha_1}^{\mu_1,\nu_1}(p_1)\Pi_{\alpha_2,\alpha_2}^{\mu_2,\nu_2}(p_2)\times
$$

$$
\times \left[ p_1^\nu \left( C_{11} p_2^{\alpha_1,\beta_1} p_2^{\alpha_2,\beta_2} + C_{12} p_2^{\alpha_1,\beta_1} p_3^{\alpha_2,\beta_2} + C_{13} \delta^{\alpha_1,\alpha_2} \delta^{\beta_1,\beta_2} \right) 
+ p_2^\nu \left( C_{21} p_2^{\alpha_1,\beta_1} p_2^{\alpha_2,\beta_2} + C_{22} p_2^{\alpha_1,\beta_1} p_3^{\alpha_2,\beta_2} + C_{23} \delta^{\alpha_1,\alpha_2} \delta^{\beta_1,\beta_2} \right) 
+ \left( C_{31} \delta^{\kappa,\alpha_1} p_2^{\alpha_1,\beta_2} p_3^{\alpha_2,\beta_2} + C_{32} \delta^{\kappa,\alpha_1} \delta^{\alpha_2,\beta_1} p_3^{\beta_2} 
+ C_{41} \delta^{\kappa,\alpha_2} p_2^{\alpha_1,\beta_2} p_3^{\alpha_2,\beta_2} + C_{42} \delta^{\kappa,\alpha_2} \delta^{\alpha_1,\beta_1} p_2^{\beta_2} \right) \right]. \tag{2.4.20}
$$

In this expression, the coefficients $C_{jk}$ are differential equations involving the form factors $A_1, A_2$ and $A_3$ of (2.3.7). Each CWI can then be presented in terms of the momentum magnitudes $p_j = |p_j|.$

As we can see, there are ten coefficients $C_{jk}$ in (2.4.20), so there are at most ten equations to consider. Usually not all of the CWIs, however, are independent. In this example, the $p_1 \leftrightarrow p_2$ symmetry implies that the equations following from $C_{1j}$ and $C_{2j},$ as well as $C_{3j}$ and $C_{4j},$ are pairwise equivalent.

For any 3-point function, the resulting equations can be divided into two groups: the primary and the secondary CWIs. The primary CWIs are second-order differential equations and appear as the coefficients of transverse or transverse-traceless tensors containing $p_1^\kappa$ or $p_2^\kappa,$ where $\kappa$ is the ‘special’ index in the CWI operator $K^\kappa.$ In the expression (2.4.20) above, the primary CWIs are equivalent to the vanishing of the coefficients $C_{1j}$ and $C_{2j}.$ The remaining equations, following from all other transverse or transverse-traceless terms, are then the secondary CWIs and are first-order differential equations. In the expression (2.4.20), the secondary CWIs are equivalent to the vanishing of the coefficients $C_{3j}$ and $C_{4j}.$

In the next two subsections we will examine the general form of the primary and secondary CWIs and discuss some of their properties. In section 2.5, we will return to analyse their solution for the form factors. In outline our strategy will be, first, to solve each of the primary CWIs up to an overall multiplicative constant, then second, to insert these solutions into the secondary CWIs typically allowing the number of undetermined constants to be further reduced. In the case of the correlator $\langle T^{\mu_1,\nu_1} T^{\mu_2,\nu_2} \mathcal{O} \rangle,$ for example, we will find that the final result is then uniquely determined up to one numerical constant, in agreement with the position space analysis of [22].

**Primary conformal Ward identities**

It turns out that in all cases the primary CWIs look very similar to the CWIs (2.2.10) for scalar operators. In order to write the primary CWIs in a simple way,
we define the following fundamental differential operators

\[ K_j = \frac{\partial^2}{\partial p_j^2} + \frac{d + 1 - 2\Delta_j}{p_j} \frac{\partial}{\partial p_j}, \quad j = 1, 2, 3, \]  

(2.4.21)

\[ K_{ij} = K_i - K_j, \quad j = 1, 2, 3, \]  

(2.4.22)

where \( \Delta_j \) is the conformal dimension of the \( j \)-th operator in the 3-point function under consideration. (Observe that this same operator appeared earlier in (2.2.10, 2.2.11).)

In the case of our example \( \langle\langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} O \rangle\rangle \), the primary CWIs for the form factors defined in (2.3.7) read

\[ K_{13} A_1 = 0, \quad K_{13} A_2 = 8A_1, \quad K_{13} A_3 = 2A_2, \]  

\[ K_{23} A_1 = 0, \quad K_{23} A_2 = 8A_1, \quad K_{23} A_3 = 2A_2, \]  

(2.4.23)

Note that, from the definition (2.4.22), we have

\[ K_{ii} = 0, \quad K_{ji} = - K_{ij}, \quad K_{ij} + K_{jk} = K_{ik}, \]  

(2.4.24)

for any \( i, j, k \in \{1, 2, 3\} \). One can therefore subtract corresponding pairs of equations and obtain the following system of independent partial differential equations

\[ K_{12} A_1 = 0, \quad K_{12} A_2 = 0, \quad K_{12} A_3 = 0, \]  

\[ K_{13} A_1 = 0, \quad K_{13} A_2 = 8A_1, \quad K_{13} A_3 = 2A_2. \]  

(2.4.25)

As we will prove, each equation has a unique solution up to one numerical constant. This means that at this point the 3-point function \( \langle\langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} O \rangle\rangle \) is determined by three numerical constants. After the application of the secondary CWIs this number will decrease further.

The primary CWIs for all 3-point functions are listed explicitly in chapter 3. They share the following properties:

1. All primary CWIs are second-order linear differential equations.

2. The equations for the coefficient \( A_1 \) are always homogeneous and given by (2.2.10, 2.2.11) exactly, i.e.,

\[ K_{12} A_1 = 0, \quad K_{13} A_1 = 0. \]  

(2.4.26)

3. The equations for the remaining form factors are similar to (2.2.10, 2.2.11), but they may contain a linear inhomogeneous part. For a form factor \( A_n \) multiplying a tensor of tensorial dimension \( N_n \), the only form factors \( A_j \) which can appear in the inhomogeneous part are those with \( N_j = N_n + 2 \). It is therefore always possible to solve the primary CWIs recursively, starting with \( A_1 \).

In the case of our example, the recursive structure of the equations (2.4.25) is clearly visible.
4. There is no semi-local contribution to any primary CWI. In our example, last two terms in (2.4.19) do not contribute to the primary CWIs. This conclusion is valid in general and can be checked explicitly in all cases.

5. The solution to each pair of primary CWIs is unique up to one numerical constant, as we will prove in section 2.5.

**Secondary conformal Ward identities**

The secondary CWIs are first-order partial differential equations and in principle involve the semi-local information contained in \( j_{\text{loc}}^\mu \) and \( t_{\text{loc}}^{\mu\nu} \). In order to write them compactly, we define the two differential operators

\[
L_{s,N} = p_1 (p_1^2 + p_2^2 - p_3^2) \frac{\partial}{\partial p_1} + 2p_1^2 p_2 \frac{\partial}{\partial p_2} + \left[ (2d - \Delta_1 - 2\Delta_2 + N + s)p_1^2 + (\Delta_1 - 2 + s)(p_3^2 - p_2^2) \right],
\]

\[
R_s = p_1 \frac{\partial}{\partial p_1} - (\Delta_1 - 2 + s),
\]

as well as their symmetric versions

\[
L'_{s,N} = L_{s,N} \text{ with } p_1 \leftrightarrow p_2 \text{ and } \Delta_1 \leftrightarrow \Delta_2,
\]

\[
R'_s = R_s \text{ with } p_1 \leftrightarrow p_2 \text{ and } \Delta_1 \leftrightarrow \Delta_2.
\]

These operators depend on two parameters \( N \) and \( s \) determined by the Ward identity in question. Physically, while \( N \) has no clear interpretation, the parameter \( s \) denotes the spin of the first operator insertion in the 3-point function (or the second in the case of \( L'_{s,N} \) and \( R'_s \)). For conserved currents, we therefore have \( s = 1 \) while for the stress-energy tensor \( s = 2 \).

In our example (2.3.7) one finds two independent secondary CWIs following from the coefficients \( C_{31} \) and \( C_{32} \) in (2.4.20), namely

\[
L_{2,0} A_1 + R_2 A_2 =
\]

\[
= 2d \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} p_3^{\nu_3} \text{ in } p_{1\nu_1} \langle \langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)O(p_3) \rangle \rangle,
\]

\[
L_{2,0} A_2 + 4 R_2 A_3 =
\]

\[
= 8d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_3^{\nu_3} \text{ in } p_{1\nu_1} \langle \langle T^{\mu_1\nu_1}(p_1)T^{\mu_2\nu_2}(p_2)O(p_3) \rangle \rangle,
\]

with \( \Delta_1 = \Delta_2 = d \). Note that in order to correctly extract the coefficient of a tensor, the rule (2.3.4) regarding the momenta associated with a given Lorentz index must be observed. The semi-local terms on the right-hand sides may be computed by means of transverse Ward identities, to which we now turn our attention.
2.4.4. Transverse and trace Ward identities

In this section we review briefly the transverse (diffeomorphism) and trace (Weyl) Ward identities in momentum space. These can be obtained by a direct Fourier transform of the position space expressions obtained in sections 1.3.3 and 1.3.4. In particular we will need the precise form of all semi-local terms that appear in these Ward identities since these terms are required for the explicit evaluation of the right-hand sides of the secondary CWIs such as (2.4.31, 2.4.32).

Explicit expressions for all the higher-point transverse and trace Ward identities we need are listed in chapter 3. In obtaining these expressions we have used the assumptions:

1. $O^I$ is independent of the sources, i.e.,
   \[ \frac{\delta O^I}{\delta \phi^I_0} = 0, \quad \frac{\delta O^I}{\delta A^a_\mu} = 0, \quad \frac{\delta O^I}{\delta g_{\mu\nu}} = 0. \] (2.4.33)

2. The source $\phi^I_0$ appears only as in (1.3.16), so that
   \[ \frac{\delta T_{\mu\nu}(x)}{\delta \phi^I_0(y)} = -g_{\mu\nu}(x)O(x)\delta(x - y), \quad \frac{\delta J^{\mu a}}{\delta \phi^I_0} = 0. \] (2.4.34)

3. The gauge field $A^a_\mu$ couples either through covariant derivatives or acts as an external source for the current in the form of $A^a_\mu J^{\mu a}$. This means there are no kinetic terms for $A^a_\mu$, i.e., no derivatives acting on $A^a_\mu$ in the action, hence
   \[ \frac{\delta T_{\mu\nu}(x)}{\delta A^a_\mu(y)} = F^{\rho a}_{\mu\nu}(x)\delta(x - y), \quad \frac{\delta J^{\mu a}(x)}{\delta A^a_\mu(y)} = G^{\mu\nu ab}(x)\delta(x - y) \] (2.4.35)

where $F$ and $G$ are functions of the CFT fields.

Of course, it may happen that renormalisation requires us to add counterterms violating one or more of the assumptions above, in which case the relevant Ward identities would need to be modified accordingly.

As a specific illustration of the general discussion above, let us consider the transverse Ward identity for $\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} O \rangle$ for a matter content consisting of conformal scalars, as defined in section 2.3.4. We will take the operator $O = \phi^2$. The relevant Ward identity is

\[ p^{\nu_1}_1 \langle T_{\mu_1\nu_1}(p_1)T_{\mu_2\nu_2}(p_2)O^I(p_3) \rangle = 2p^{\nu_1}_1 \langle \frac{\delta T_{\mu_1\nu_1}}{\delta g_{\mu_2\nu_2}}(p_1, p_2)O^I(p_3) \rangle, \] (2.4.36)

where $\delta T_{\mu_1\nu_1}/\delta g_{\mu_2\nu_2}$ denotes taking the functional derivative of the stress-energy tensor with respect to the metric, after which we restore the metric to its background value $g_{\mu\nu} = \delta_{\mu\nu}$. Evaluating this functional derivative explicitly using
After Fourier transforming and using the result for the 2-point function space expressions, transforming identities (1.3.43) and (1.3.44) we arrive at the following momentum the convention (2.3.4) for the Lorentz indices.

\[
\delta T_{\mu\nu}(x) \frac{\delta g^{\mu\sigma}(y)}{\delta g^{\mu\sigma}} = -\frac{1}{2} \left[ \delta_{\mu\nu} \delta_{\rho\sigma} + 2 \delta_{\mu(\rho} \delta_{\sigma)\nu} - \delta_{\mu\nu} \delta_{\rho\sigma} \delta^{\alpha\beta} \right] \delta(x - y) T_{\alpha\beta}(x) \\
+ \frac{1}{16} \left[ C^{(1)\alpha\beta}_{\mu\nu\rho\sigma} \delta(x - y) \partial_\alpha \partial_\beta + C^{(2)\alpha\beta}_{\mu\nu\rho\sigma} \partial_\alpha \delta(x - y) \partial_\beta \\
+ C^{(3)\alpha\beta}_{\mu\nu\rho\sigma} \partial_\alpha \partial_\beta \delta(x - y) \right] \mathcal{O}(x),
\]

(2.4.37)

where partial derivatives are taken with respect to \(x\) and the prefactors are

\[
C^{(1)\alpha\beta}_{\mu\nu\rho\sigma} = \delta_{\mu\nu} \delta_{\rho\sigma}^{(\alpha\beta)} + 2 \delta_{(\mu} \delta_{\nu)\rho\sigma} \delta^{\alpha\beta} - \delta_{\mu\nu} \delta_{\rho\sigma} \delta^{\alpha\beta}, \\
C^{(2)\alpha\beta}_{\mu\nu\rho\sigma} = 2 \delta_{\mu\nu} \delta_{\rho\sigma}^{(\alpha\beta)} + \delta_{(\mu(\delta_{\nu)}\rho\sigma) \delta^{\alpha\beta} - 2 \delta_{(\mu} \delta_{\nu)} \delta_{\rho\sigma} \delta^{\alpha\beta}, \\
C^{(3)\alpha\beta}_{\mu\nu\rho\sigma} = \delta_{\mu\nu} \delta_{\rho\sigma}^{(\alpha\beta)} + 2 \delta_{(\mu(\delta_{\nu)}\rho\sigma) \delta^{\alpha\beta} - 2 \delta_{(\mu} \delta_{\nu)} \delta_{\rho\sigma} \delta^{\alpha\beta} + \delta_{\rho\sigma} \delta_{\mu} \delta_{\nu}^{(\alpha\beta)}.
\]

After Fourier transforming and using the result for the 2-point function

\[
\langle \langle \mathcal{O}(p) \mathcal{O}(-p) \rangle \rangle = \frac{1}{4p}
\]

we obtain

\[
p_{1\nu_1} \langle \langle T^{\mu_1\nu_1}(p_1) T^{\mu_2\nu_2}(p_2) \mathcal{O}(p_3) \rangle \rangle = -\frac{1}{32} p_1^{\mu_1} p_2^{\mu_2} p_3^{\mu_3} + \frac{p_3^{\delta_{\mu_1\mu_2} p_2^{\nu_2} + \ldots}{32},
\]

(2.4.40)

where we have retained only the terms appearing in the right-hand sides of the secondary CWIs (2.4.31) and (2.4.32). The omitted terms do not contain the tensors listed explicitly and will play no further role in our analysis. As usual, we use the convention (2.3.4) for the Lorentz indices.

A similar considerations can be applied to the trace Ward identity. By Fourier transforming identities (1.3.43) and (1.3.44) we arrive at the following momentum space expressions,

\[
\langle \langle T(p_1) \mathcal{O}^I(p_2) \mathcal{O}^J(p_3) \rangle \rangle = -\Delta \left[ \langle \langle \mathcal{O}^I(p_2) \mathcal{O}^J(-p_2) \rangle \rangle + \langle \langle \mathcal{O}^I(p_3) \mathcal{O}^J(-p_3) \rangle \rangle \right],
\]

\[
\langle \langle T(p_1) T_{\mu\nu}(p_2) \mathcal{O}^I(p_3) \rangle \rangle = 2 \langle \langle \frac{\delta T}{\delta g^{\mu\nu}}(p_1, p_2) \mathcal{O}^I(p_3) \rangle \rangle.
\]

(2.4.41)

A complete list of all trace Ward identities is given in chapter 3.

As is well known, due to renormalisation the trace Ward identity may acquire an anomalous contribution. The exact contribution depends strongly on the specifics of the theory, but its form is universal. In this work we assume no anomalies in the transverse Ward identities (1.3.33) and (1.3.34) can appear. The anomalous contributions are therefore still transverse.

In section 2.8 we will consider in detail the divergences and anomalies in the correlation functions \(\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\) and \(\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3} \rangle\) for the case \(d = 4\).
2. Implications of conformal invariance in momentum space

Here, as an example, we consider the most general form of the trace anomalies in the correlation functions considered above, which is

\[
\langle\langle T(p_1)O^I(p_2)O^J(p_3)\rangle\rangle_{\text{anomaly}} = B_{IJ}^1,
\]

\[
\langle\langle T(p_1)T^{\mu_1\nu_2}(p_2)O^I(p_3)\rangle\rangle_{\text{anomaly}} = \Pi^{\mu_1\nu_2}_{\beta_1\beta_2}B_{IJ}^1 + \pi^{\mu_2\nu_2}(p_2)B_{IJ}^2,
\]

where the form factors \(B_{IJ}^1\), \(B_{I}^1\) and \(B_{I}^2\) are functions of the momentum magnitudes specific to the theory in question. For example, the contribution to \(\langle\langle T^{\mu_1\nu_1}O^I O^J\rangle\rangle_{\text{anomaly}}\) follows from the counterterm, [22, 66]

\[
\frac{1}{2}\int d^d x \sqrt{g}P^{IJ}\phi_0^I \Box^{-\frac{d}{2}} \phi_0^J,
\]

where \(P^{IJ}\) are numerical coefficients and we assume that \(\Delta = d\) is a non-negative integer. In this case we then find

\[
B_{IJ}^1 = (-p_1^2)^{\Delta - \frac{d}{2}} P^{IJ}.
\]

In section 2.8 we provide a short worked example of the renormalisation procedure in the case of the \(\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3}\rangle\rangle\) and \(\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} T^{\mu_3\nu_3}\rangle\rangle\) correlators and show how the anomalies arise.

2.5. Solutions to conformal Ward identities

It is a rather pleasant fact that all the primary CWIs can be solved in terms of the triple-\(K\) integrals similar to (2.2.25). We will start by analysing some properties of the triple-\(K\) integrals before proceeding to show how this class of integrals solves the primary CWIs. In particular, we will find that the solution to each primary CWI is unique up to one numerical constant. Finally, we will analyse the structure and implications of the secondary CWIs.

2.5.1. Triple-\(K\) integrals

All primary CWIs can be solved in terms of the general triple-\(K\) integral

\[
I_{\alpha\{\beta_1\beta_2\beta_3\}}(p_1, p_2, p_3) = \int_0^\infty dx x^\alpha \prod_{j=1}^3 p_j^{\beta_j} K_{\beta_j}(p_j x),
\]

where \(K_{\nu}\) is a Bessel \(K\) function. This integral depends on four parameters, namely the power \(\alpha\) of the integration variable \(x\), and the three Bessel function indices \(\beta_j\). Its arguments, \(p_1, p_2, p_3\), are magnitudes of momenta \(p_j = |p_j|\), \(j = 1, 2, 3\). In the following we will generically refer to these as \(\alpha\) and \(\beta\) parameters respectively.
It will be useful to define a reduced version $J_{N\{k_1 k_2 k_3\}}$ of the triple-$K$ integral by substituting
\[ \alpha = \frac{d}{2} - 1 + N, \quad \beta_j = \Delta_j - \frac{d}{2} + k_j, \quad j = 1, 2, 3. \] (2.5.2)
Here we assume that we concentrate on some particular 3-point function and the conformal dimensions $\Delta_j$, $j = 1, 2, 3$ are therefore fixed. In other words we define
\[ J_{N\{k_j\}} = I_{\frac{d}{2} - 1 + N\{\Delta_j - \frac{d}{2} + k_j\}}, \] (2.5.3)
where we use a shortened notation $\{k_j\} = \{k_1 k_2 k_3\}$, etc. Finally we define
\[ \Delta_t = \Delta_1 + \Delta_2 + \Delta_3, \quad \beta_t = \beta_1 + \beta_2 + \beta_3, \quad k_t = k_1 + k_2 + k_3. \] (2.5.4)
The main point of this section is to present relations showing that all primary CWIs for a given 3-point function can be solved by the triple-$K$ integrals (2.5.1). The representation (2.5.3) turns out to be extremely useful, as the parameters $N$ and $k_j$ are fixed by the primary CWIs and have no dependence on either $\Delta_j$ or $d$. If desired, these triple-$K$ integrals may also be re-expressed in terms of other familiar integrals such as Feynman or Schwinger parametrised integrals, as discussed in appendix 4.2.1.

**Region of validity, regularisation and renormalisation**

We assume all parameters and variables in the triple-$K$ integral (2.5.1) are real. From the asymptotic expansion (2.A.28) the integral converges at large $x$, however in general there may still be a divergence at $x = 0$. From the series expansion (2.A.21) and the definition (2.A.22), we see the triple-$K$ integral only converges if
\[ \alpha > \sum_{j=1}^3 |\beta_j| + 1, \quad p_1, p_2, p_3 > 0. \] (2.5.5)
If $\alpha$ does not satisfy this inequality, we can regard the triple-$K$ integral (2.5.1) as a function of $\alpha$
\[ \alpha \mapsto I_{\alpha\{\beta_1 \beta_2 \beta_3\}}(p_1, p_2, p_3). \] (2.5.6)
with the other parameters and momenta fixed and use analytic continuation. The scheme (2.2.4) is very convenient here as it does not change the indices of the Bessel functions. In terms of the parameters $\alpha$ and $\beta_j$ this corresponds to the substitutions
\[ \alpha \mapsto \alpha + \epsilon, \quad \beta_j \text{ does not change, } j = 1, 2, 3 \] (2.5.7)
in (2.5.1). Generically one finds that the limit $\epsilon \to 0$ then exists, except for cases where
\[ \alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 = -2n, \] (2.5.8)
for some non-negative \( n \). In these cases we find poles in \( \epsilon \). This can be seen as follows. The only source of the singularity is the divergence of the integral at \( x = 0 \). Therefore we can expand the integrand about \( x = 0 \) and look for possible divergences.

First, assume that the condition (2.5.8) is met but all \( \beta_j \not\in \mathbb{Z} \). We will show that in such a case we should expect single poles in \( \epsilon \). Indeed, equations (2.A.21) and (2.A.22) show that the expansion contains power terms \( x^a \) for various \( a \in \mathbb{R} \) only. Note that the function (2.5.6) is singular only if there exists a term \( x^a \) in the expansion of the integrand with \( a = -1 \). Indeed, the integral of such a term near \( x = 0 \) is

\[
\int_0^x x^a \, dx = \frac{\text{const}}{a+1},
\]

which has a single pole at \( a = -1 \) only. Note that while the triple-\( K \) integral as it stands will diverge if the expansion contains any \( x^a \) terms with \( a < -1 \), the analytic continuation will exist due to (2.5.9). Therefore, the function (2.5.6) is well-defined as long as the expansion of the integrand near \( x = 0 \) does not contain a \( 1/x \) term, which is exactly the condition (2.5.8).

Now observe that if some \( \beta_j \in \mathbb{Z} \), then the series expansion of the integrand in the triple-\( K \) integral about \( x = 0 \) contains logarithms due to (2.A.27). Since, for example

\[
\int_0^x x^a \log x \, dx = -\frac{\text{const}_1}{(a+1)^2} + \frac{\text{const}_2}{a+1},
\]

we may expect a double pole in \( \alpha \) in (2.5.6) when some of the \( \beta_j \) are integer. While the order of the pole may increase, its position remains at \( a = -1 \). Using the series expansions (2.A.21) and (2.A.27), we can see that the positions of the poles in any triple-\( K \) integral are given by (2.5.8).

**Basic properties**

Having defined triple-\( K \) integrals, we want to show that they solve the primary CWIs. We will therefore now analyse the basic properties of the triple-\( K \) integrals. The most obvious of these is the permutation symmetry

\[
I_{\alpha\{\beta_{\sigma(1)}\beta_{\sigma(2)}\beta_{\sigma(3)}\}}(p_1, p_2, p_3) = I_{\alpha\{\beta_1\beta_2\beta_3\}}(p_{\sigma^{-1}(1)}, p_{\sigma^{-1}(2)}, p_{\sigma^{-1}(3)}), \quad (2.5.11)
\]

where \( \sigma \) is any permutation of the set \( \{1, 2, 3\} \). We also have the relations

\[
\frac{\partial}{\partial p_n} I_{\alpha\{\beta_j\}} = -p_n I_{\alpha+1\{\beta_j-\delta_{jn}\}}, \quad (2.5.12)
\]

\[
I_{\alpha\{\beta_j+\delta_{jn}\}} = p_n^2 I_{\alpha\{\beta_{j-\delta_{jn}}\}} + 2\beta_n I_{\alpha-1\{\beta_j\}}, \quad (2.5.13)
\]

\[
I_{\alpha\{\beta_1\beta_2,-\beta_3\}} = p_3^{-2\beta_3} I_{\alpha\{\beta_1\beta_2\beta_3\}}, \quad (2.5.14)
\]
for any \( n = 1, 2, 3 \), as follows from the basic Bessel function relations

\[
\frac{\partial}{\partial a} [a^n K_\nu(ax)] = -xa^n K_{\nu-1}(ax),
\]

(2.5.15)

\[
K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x) = K_{\nu+1}(x),
\]

(2.5.16)

\[
K_{-\nu}(x) = K_\nu(x).
\]

(2.5.17)

Some additional properties of Bessel functions and triple-\( K \) integrals are listed in appendix 2.A.3.

**Dilatation degree of the triple-\( K \) integral**

As the triple-\( K \) integral solves CWIs, it should also solve the dilatation Ward identity (2.4.9). In other words it should have a definite dilatation dimension. Using (2.5.15) and (2.5.12) we can write

\[
\int_0^\infty dx \frac{\partial}{\partial x} \left( x^{\alpha+1} \prod_{j=1}^3 p_j^{\beta_j} K_{\beta_j}(p_j x) \right) = \\
= (\alpha + 1 - \beta_t) I_{\alpha\{\beta_k\}} - \sum_{j=1}^3 p_j^2 I_{\alpha+1\{\beta_k-\delta_jk\}} \\
= (\alpha + 1 - \beta_t) I_{\alpha\{\beta_k\}} + \sum_{j=1}^3 p_j \frac{\partial}{\partial p_j} I_{\alpha\{\beta_k\}}.
\]

(2.5.18)

The expression on the left-hand side leads to a boundary term at \( x = 0 \). In the region of convergence (2.5.5) all integrals in this expression are well-defined and the boundary term vanishes. Now we can use the analytic continuation (2.5.7) in order to argue that the analytically continued left-hand side vanishes, except in the case where (2.5.8) is satisfied. Indeed, if we regard both sides of (2.5.18) as analytic functions of \( \alpha \) with other parameters and momenta fixed, then the validity of (2.5.18) in the region (2.5.5) implies its validity in the entire domain of analyticity. Therefore, we have shown that

\[
\deg I_{\alpha\{\beta_j\}} = \beta_t - \alpha - 1, \quad \deg J_{N\{k_j\}} = \Delta_t + k_t - 2d - N
\]

(2.5.19)

provided \( \alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 \neq -2n \) for some non-negative \( n \) and independent choice of signs.

Finally we can argue that, if (2.5.8) holds, we should expect scaling anomalies in \( I_{\alpha\{\beta_j\}} \). Using the power series expansion (2.A.21) of the Bessel \( I \) functions one can see that the series expansion of the boundary term \( x^{\alpha+1} \prod_{j=1}^3 p_j^{\beta_j} K_{\beta_j}(p_j x) \) in (2.5.18) about \( x = 0 \) contains a constant piece exactly when (2.5.8) holds. This indicates that the dilatation Ward identity for the \( I_{\alpha\{\beta_j\}} \) is not satisfied in such a
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Note that this is not a strict argument since the regulator cannot be removed from the integrals appearing in (2.5.18). One should instead expand both sides in the regulator $\epsilon$ and match terms order by order.

2.5.2. Solutions to the primary conformal Ward identities

In the previous section we defined the triple-$K$ integral and analysed its basic properties. We now want to use this knowledge in order to write a solution to the CWIs. For this we need the following fundamental identity. For any $m, n = 1, 2, 3,$

$$K_{mn} J_{N\{k_j\}} = -2k_m J_{N+1\{k_j-\delta_{jm}\}} + 2k_n J_{N+1\{k_j-\delta_{jn}\}},$$

(2.5.20)

for $k_1, k_2, k_3, N \in \mathbb{R}$. The operator $K_{mn}$ is the CWI operator defined in (2.4.22). This relation is a direct consequence of the identities (2.5.12) and (2.5.13).

Let us first consider the pair of primary CWIs for the form factor $A_1$. As discussed in section 2.4.3, such CWIs are always homogeneous and take the form (2.4.26). Observe that if we set all $k_j = 0$ in (2.5.20) then $A_1 = \alpha_1 J_{N\{000\}}$ is a solution for arbitrary $N \in \mathbb{R}$ and an integration constant $\alpha_1 \in \mathbb{R}$. Furthermore, observe that, if we impose only one homogeneous equation, say $K_{12} A = 0$, then the most general solution in terms of the triple-$K$ integrals is $\alpha J_{N\{00k_3\}}$ for any $\alpha, N, k_3 \in \mathbb{R}$. In general the equation (2.5.20) is sufficient to write down solutions to all primary CWIs.

The remaining piece of information is the value of $N$. In general, if $A_n = \alpha_n J_{N\{k_j\}}$ is a form factor of tensorial dimension $N_n$, then (2.4.10) and (2.5.19) determine the value of $N = N(A_n)$ to be

$$N(A_n) = N_n + k_t.$$  

(2.5.21)

Let us see how the procedure works for our example $\langle\langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} \mathcal{O} \rangle\rangle$. The primary CWIs are given by (2.4.25) and, in particular,

\[
\begin{align*}
K_{12} A_1 &= 0, & K_{13} A_1 &= 0, \\
K_{12} A_2 &= 0, & K_{12} K_{13} A_2 &= 0, & K_{13}^2 A_2 &= 0, \\
K_{12} A_3 &= 0, & K_{12} K_{13} A_3 &= 0, & K_{12} K_{13}^2 A_3 &= 0, & K_{13}^3 A_3 &= 0,
\end{align*}
\]

(2.5.22)

Therefore, using (2.5.20) and (2.5.21), we can write the most general solution given in terms of the triple-$K$ integrals,

$$A_1 = \alpha_1 J_{4\{000\}},$$

$$A_2 = \alpha_{21} J_{3\{001\}} + \alpha_2 J_{2\{000\}},$$

$$A_3 = \alpha_{31} J_{2\{002\}} + \alpha_{32} J_{1\{001\}} + \alpha_3 J_{0\{000\}},$$

(2.5.23)
where all the $\alpha$ are numerical constants. Finally, the inhomogeneous parts of (2.4.25) fix some of these constants. When the solution above is substituted into the primary CWIs, (2.5.20) requires that

$$\alpha_{21} = 4\alpha_1, \quad \alpha_{31} = 2\alpha_1, \quad \alpha_{32} = \alpha_2.$$  (2.5.24)

The three remaining undetermined constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ multiply integrals of the form $J_{N\{000\}}$. Such integrals solve the homogeneous parts of the CWIs. Therefore the remaining constants, undetermined by the primary CWIs, will be called primary constants.

Let us summarise our results. We have analysed the primary CWIs for the $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} O \rangle\rangle$ correlation function and we found a solution

$$A_1 = \alpha_1 J_{4\{000\}},$$
$$A_2 = 4\alpha_1 J_{3\{001\}} + \alpha_2 J_{2\{000\}},$$
$$A_3 = 2\alpha_1 J_{2\{002\}} + \alpha_2 J_{1\{001\}} + \alpha_3 J_{0\{000\}},$$  (2.5.25)

with three undetermined constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. We will show shortly that this solution to the primary CWIs is indeed unique, specifying $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} O \rangle\rangle$ in momentum space up to three constants. Following application of the secondary CWIs, we will find that the number of undetermined constants is reduced to just one. The method we have described is based purely in momentum space and is applicable to all 3-point functions. Explicit solutions for all primary CWIs are listed in chapter 3.

The triple-$K$ integrals we discuss here also arise in AdS/CFT calculations of momentum space 3-point functions using a dual gravitational theory (recent papers include, e.g., [67, 68, 50]). As such, these calculations apply only to the specific CFTs dual to particular gravitational theories. In contrast, our approach here is completely general, showing that all 3-point functions of conserved currents, stress-energy tensors and scalar operators in any CFT can be expressed in terms of triple-$K$ integrals.

Finally, let us return to the issue of regularisation. In some cases, despite the finiteness of the final result, the regularisation procedure described in section 2.5.1 may still be necessary. Furthermore, it can happen that single triple-$K$ integrals may diverge on their own while the form factor they build remains finite. It is therefore essential to keep track of the expansion in the regulator $\epsilon$ carefully. In particular one must consider the primary constants as functions of the regulator and take the $\epsilon \to 0$ limit only after the substitution into the final expression for the form factors. We will show an example of this behaviour in the next section.

**More on** $\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} O \rangle$

In this section we wish to illustrate that the solution to the primary CWIs in terms of the triple-$K$ integrals can be evaluated explicitly with ease. A systematic
discussion of the evaluation of the triple-$K$ integrals will be given in section 2.6. For concreteness, consider $\langle (T_{\mu_1\nu_1}T_{\mu_2\nu_2})O \rangle$ with a scalar operator $O$ of dimension $\Delta_3 = 1$ in $d = 3$ dimensional CFT. The solution to the primary CWIs is given by (2.5.25) with constants fixed according to (2.5.24). In order to write the solution explicitly, we can use expressions (2.A.24) and (2.A.25), after which all integrals turn out to be elementary. The following integrals converge and can be easily computed

\[
J_{4(000)} = I_{\frac{3}{2}}(\frac{3}{2}, -\frac{1}{2}) = \left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{3(p_1^2 + p_2^2) + p_3^2 + 12p_1p_2 + 4p_3(p_1 + p_2)}{p_3(p_1 + p_2 + p_3)^4},
\]

\[
J_{3(001)} = I_{\frac{7}{2}}(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}) = \left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{2(p_1^2 + p_2^2) + p_3^2 + 6p_1p_2 + 3p_3(p_1 + p_2)}{(p_1 + p_2 + p_3)^3},
\]

assuming $\Delta_1 = \Delta_2 = 3$ and $\Delta_3 = 1$. The remaining integrals diverge and require a regularisation. As discussed in section 2.5.1, we consider the integrals $J_{N+\epsilon\{k_j\}}$ and we expand the result in $\epsilon$. In this manner, we find

\[
J_{2+\epsilon\{002\}} = I_{\frac{5}{2}+\epsilon\{\frac{3}{2}, \frac{1}{2}\}} = -\left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{1}{(p_1 + p_2 + p_3)^2} \left[2p_1p_2p_3 + p_1^3 + p_2^3 + p_3^3ight] + O(\epsilon),
\]

\[
J_{2+\epsilon\{000\}} = I_{\frac{7}{2}+\epsilon\{\frac{3}{2}, -\frac{1}{2}\}} = \left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{1}{p_3\epsilon} + O(\epsilon^0),
\]

\[
J_{1+\epsilon\{001\}} = I_{\frac{9}{2}+\epsilon\{\frac{3}{2}, \frac{1}{2}\}} = -\left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{p_3}{\epsilon} + O(\epsilon^0),
\]

\[
J_{0+\epsilon\{000\}} = I_{\frac{11}{2}+\epsilon\{\frac{3}{2}, -\frac{1}{2}\}} = -\left(\frac{\pi}{2}\right)^{3/2} \cdot \frac{p_2^2 + p_3^2 - p_2^2}{2p_3\epsilon} + O(\epsilon^0).
\]

As we will see the omitted terms make no contribution in our subsequent analysis. In order to further constrain the primary constants $\alpha_1, \alpha_2, \alpha_3$ we must consider the secondary CWIs. We will return to this task in section 2.5.3.

At this point we can compare the result given by (2.5.25) with the direct calculations of the 3-point function for the free scalar field carried out in section 2.3.4. We see that the form of the integrals $J_{4(000)}$, $J_{3(001)}$ and $J_{2(002)}$ match the form factors $A_1$, $A_2$ and $A_3$ in the equations (2.3.26). Therefore one finds the primary constants for this particular model to be

\[
\alpha_1 = \frac{1}{48} \left(\frac{2}{\pi}\right)^{\frac{3}{2}}, \quad \alpha_2 = 0, \quad \alpha_3 = 0.
\]

Note that the relations (2.5.24) provide a cross-check on our solution. Later, we will see that the secondary Ward identities impose two additional constraints on the primary constants that are not yet visible.
2.5. Solutions to conformal Ward identities

Uniqueness of the solution

In the previous sections we argued that all CWIs may be solved in terms of triple-K integrals (2.5.1). A case-by-case analysis confirms this and the list of complete solutions is given in chapter 3. Here we want to establish that these solutions are unique. To be more precise, we want to argue that each pair of the primary CWIs determines a form factor $A_n$ uniquely up to one numerical constant. This may be achieved by essentially the same reasoning as in section 2.2.3.

First, assume that $A_n$ satisfies a pair of homogeneous primary CWIs

$$K_{12} A_n = 0, \quad K_{13} A_n = 0,$$

(2.5.29)

together with the dilatation Ward identity (2.4.8) with tensorial dimension $N_n$.

We can then use the substitution

$$A_n(p_1, p_2, p_3) = p_3^{\Delta_1 - 2d - N_n} \left( \frac{p^2_1}{p^2_3} \right)^\mu \left( \frac{p^2_2}{p^2_3} \right)^\lambda F \left( \frac{p^2_1}{p^3_2}, \frac{p^2_2}{p^3_3} \right),$$

(2.5.30)

and proceed with the analysis analogous to that following equation (2.2.14). The substitution leads to the system of equations (2.2.16, 2.2.17) with four possible choices of parameters

$$\alpha = \frac{1}{2} \left[ N_n + \epsilon_1 \left( \Delta_1 - \frac{d}{2} \right) + \epsilon_2 \left( \Delta_2 - \frac{d}{2} \right) + \Delta_3 \right], \quad \beta = \alpha - \left( \Delta_3 - \frac{d}{2} \right),$$

$$\gamma = 1 + \epsilon_1 \left( \Delta_1 - \frac{d}{2} \right), \quad \gamma' = 1 + \epsilon_2 \left( \Delta_2 - \frac{d}{2} \right),$$

(2.5.31)

parametrised by $\epsilon_1, \epsilon_2 = \pm 1$. We can now use equation (2.2.21) and the analysis that follows. This leads to the conclusion that the only physically acceptable solution to the homogeneous part of the CWIs is given by the triple-K integral $\alpha_n J_{N_n \{000\}}(p_1, p_2, p_3)$, where $\alpha_n$ is a single undetermined constant.

In general, the primary CWIs for a form factor $A_n$ contain inhomogeneous parts. The recursive nature of the primary CWIs discussed in section 2.4.3 then allows us to solve these equations one-by-one. Since the inhomogeneous part is linear in the other form factors, every two solutions to a given pair of equations differ by a solution to the homogeneous part of the equation. The full solution to the pair of primary CWIs and the dilatation Ward identity is therefore unique up to one numerical constant.

It is important to emphasise that while the solution to each pair of primary CWIs is unique up to one primary constant, the representation in terms of triple-K integrals may not be. For example, for generic parameter values the equation (2.5.18) shows that

$$(\alpha + \beta_t I_{\alpha - 1\{\beta_1, \beta_2, \beta_3\}} = I_{\alpha \{\beta_1 + 1, \beta_2, \beta_3\}} + I_{\alpha \{\beta_1, \beta_2 + 1, \beta_3\}} + I_{\alpha \{\beta_1, \beta_2, \beta_3 + 1\}}.$$ 

(2.5.32)

One can therefore rewrite one triple-K integral as a combination of others hence the representation is not unique.
2.5.3. Solutions to the secondary conformal Ward identities

In this section we will finalise our theoretical considerations by solving the secondary CWIs. In general, the secondary CWIs lead to linear algebraic equations between the various primary constants appearing in solutions to the primary CWIs. The precise form of the secondary CWIs depends on the semi-local information provided by transverse Ward identities, which may be written in terms of the 2-point functions.

We will first return to our example from section 2.5.2 and show how the two secondary CWIs (2.4.31, 2.4.32) constrain the values of the three primary constants appearing in the solution (2.5.25) to the primary CWIs. As expected, we will find two algebraic linear equations between the three primary constants. From this we may conclude that the 3-point function \( \langle T_{\mu_1 \nu_1} T_{\mu_2 \nu_2} O \rangle \) depends on a single undetermined primary constant.

Next, we will discuss how the secondary CWIs in the general case lead into a set of algebraic equations for the primary constants. The discussion is sensitive on whether or not the regulator can be thoroughly removed from all triple-\( K \) integrals building a given 3-point function. In either cases the procedure is based on taking a zero-momentum limit. In this limit the triple-\( K \) integrals simplify and the secondary CWIs can be shown to lead to a set of equations between primary constants.

The procedure is considerably simpler in the case where the regulator can be removed from all triple-\( K \) integrals, while in the case one needs to keep the regulator special care must be taken when regulating the 2-point functions that appear in the right-hand side of the secondary CWIs.

\( \langle T_{\mu_1 \nu_1} T_{\mu_2 \nu_2} O \rangle \) for free scalars

Let us begin by discussing our example correlation function \( \langle T_{\mu_1 \nu_1} T_{\mu_2 \nu_2} O \rangle \). We derived the secondary CWIs earlier in (2.4.31) and (2.4.32), where the terms on the right-hand side of these equations are given by the transverse Ward identity (2.4.40). We now want to show that these data fix two out of three primary constants in the solution (2.5.25) of the primary CWIs. To fix the final remaining constant then requires additional physical input in the form of the specific field content.

Since the regulator \( \epsilon \) in the triple-\( K \) integrals (2.5.27) cannot be removed, we must assume that the primary constants \( \alpha_2 \) and \( \alpha_3 \) depend on the regulator \( \epsilon \) as well. As remarked earlier in section 2.5.1, while each individual component may depend on the regulator, the full expression for the form factors \( A_j \) cannot. Let us therefore define the power series expansions

\[
\alpha_j = \sum_{n=-\infty}^{\infty} \alpha_j^{(n)} \epsilon^n, \quad j = 2, 3.
\] (2.5.33)
Since the integral $J_{4\{000\}}$ is finite, we can assume that the constant $\alpha_1$ does not depend on the regulator, i.e., $\alpha_1 = \alpha_1^{(0)}$.

We start by substituting first two solutions (2.5.25) together with the series expansions (2.5.33) into the secondary CWI (2.4.31), with right-hand side given by (2.4.40). Organising the equations according to powers of $\epsilon$, upon sending $\epsilon \to 0$ all equations associated with negative powers of $\epsilon$ must vanish. In the present case, this yields $\alpha_2^{(n)} = 0$ for all $n \leq 0$. The equation coming from the $\epsilon^0$ terms then reads

$$-\frac{3}{p_3} \left( \frac{\pi}{2} \right)^{\frac{3}{2}} (\alpha_2^{(1)} + 3\alpha_1^{(0)}) = -\frac{3}{16p_3}.$$  \hspace{1cm} (2.5.34)

The same procedure may now be applied to the remaining secondary CWI (2.4.32), yielding $\alpha_3^{(n)} = 0$ for all $n \leq 0$ and

$$\left( \frac{\pi}{2} \right)^{\frac{3}{2}} \left[ 2\alpha_3^{(1)} + 3\alpha_1^{(0)} - \frac{1}{16} \left( \frac{2}{\pi} \right)^{\frac{3}{2}} \right] \frac{p_1^2 + 3p_2^2 - 3p_3^2}{p_3} + \frac{3}{4}p_3 = \frac{3}{4}p_3.  \hspace{1cm} (2.5.35)$$

Putting everything together, we have

$$\alpha_2 = \left[ -3\alpha_1 + \frac{1}{16} \left( \frac{2}{\pi} \right)^{\frac{3}{2}} \right] \epsilon + O(\epsilon^2),  \hspace{1cm} (2.5.36)$$

$$\alpha_3 = \frac{1}{2} \left[ -3\alpha_1 + \frac{1}{16} \left( \frac{2}{\pi} \right)^{\frac{3}{2}} \right] \epsilon + O(\epsilon^2),  \hspace{1cm} (2.5.37)$$

where the constant $\alpha_1$ remains undetermined by Ward identities. When we take the limit $\epsilon \to 0$, the leading terms of order $\epsilon$ in these expressions then multiply $1/\epsilon$ poles in the $J_{2+\{000\}}$, $J_{1+\{001\}}$ and $J_{0+\{000\}}$ integrals yielding the correct finite result. The omitted higher order terms in (2.5.36) and (2.5.37) make no contribution.

Finally, we can check the results of this section against the result (2.5.28) for the specific theory discussed in section 2.5.2. Inserting the value of $\alpha_1$ from (2.5.28) into (2.5.36) and (2.5.37) we indeed recover the correct result $\alpha_2 = \alpha_3 = 0$ up to insignificant $O(\epsilon^2)$ terms.

**Simplifications in the generic case**

In the previous section we substituted the full solutions to the primary CWIs into the secondary CWIs in order to extract more information about the primary constants. At first sight this procedure might appear hard to carry out in general since the triple-$K$ integrals usually cannot be expressed in terms of elementary functions. It turns out, however, that examining the zero-momentum limit leads to simple algebraic equations for the primary constants.

In this section, for reasons of simplicity, we will assume that each triple-$K$ integral in a solution to the primary CWIs can be defined by an analytic continuation
and the regulator can be completely removed. We will refer to this as the ‘generic case’ in the present and following sections. We will then analyse the remaining cases later.

In the zero-momentum limit

\[ p_3 \to 0, \quad p_1 = p_2 = p, \]  

(2.5.38)

we have

\[
p_3^{\beta_3} K_{\beta_3} (p_3 x) = \left[ \frac{2^{\beta_3 - 1} \Gamma(\beta_3)}{x^{\beta_3}} + O(p_3^2) \right] + p_3^{2\beta_3} \left[ 2^{-\beta_3 - 1} \Gamma(-\beta_3)x^{\beta_3} + O(p_3^2) \right],
\]

for \( \beta_3 \not\in \mathbb{Z} \) and

\[
K_0(p_3 x) = -\log p_3 - \log x + \log 2 - \gamma_E + O(p_3^2),
\]

(2.5.39)

\[
p_n^{\beta_3} K_n(p_3 x) = \left[ \frac{2^{n-1} \Gamma(n)}{x^n} + O(p_3^2) \right] + p_3^{2n} \left[ (-1)^{n+1} x^n \log p_3 + \text{ultralocal} + O(p_3^2) \right],
\]

(2.5.40)

for \( n = 1, 2, 3, \ldots \). From these expressions one can see that the zero momentum limit of \( p_3^{\beta_3} K_{\beta_3} (p_3 x) \) exists if \( \beta_3 > 0 \). Since for any correlation function and any form factor \( \beta_3 = \Delta_3 - \frac{d}{2} + k_3 \) with non-negative \( k_3 \), this condition is satisfied if \( \Delta_3 > \frac{d}{2} \). (For conserved currents and for the stress-energy tensor we thus have \( \beta_3 > 0 \) automatically.) We will return to discuss the case where \( \beta_3 \leq 0 \) later in the text.

Assuming \( \beta_3 > 0 \) then, we can calculate the limit of the triple-\( K \) integrals in the generic case

\[
\lim_{p_3 \to 0} I_{\alpha}(\beta_j) (p, p, p_3) = l_{\alpha}(\beta_j) \cdot p^{\beta_j - \alpha - 1},
\]

(2.5.42)

where, using the result (2.4.49), we find

\[
l_{\alpha \{\beta_k\}} = \frac{2^{\alpha - 3} \Gamma(\beta_3)}{\Gamma(\alpha - \beta_3 + 1)} \prod_{\epsilon_1, \epsilon_2 \in \{-1, 1\}} \Gamma \left( \frac{\alpha - \beta_3 + 1 + \epsilon_1 \beta_1 + \epsilon_2 \beta_2}{2} \right),
\]

(2.5.43)

which is valid away from poles of the gamma function. Since the derivatives in the L and R operators defined in (2.4.27) and (2.4.28) acting on triple-\( K \) integrals can also be expressed via (2.5.12) in terms of triple-\( K \) integrals, this procedure leads to algebraic constraints on the primary constants.
2.5. Solutions to conformal Ward identities

Let us illustrate the considerations above in the case of the correlator \( \langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\rangle \). The secondary CWIs are given by (2.4.31) and (2.4.32) with \( \Delta_1 = \Delta_2 = d \) and L and R defined by (2.4.27) and (2.4.28). The right-hand sides are semi-local and can be expressed in terms of 2-point functions by means of the transverse Ward identities. In section 2.4.4 we found the Ward identity (2.4.36), which reads

\[
p_1^{\nu_1} \langle\langle T_{\mu_1\nu_1}(p_1) T_{\mu_2\nu_2}(p_2) \mathcal{O}(p_3) \rangle\rangle = 2p_1^{\nu_1} \langle\langle \frac{\delta T_{\mu_1\nu_1}}{\delta g_{\mu_2\nu_2}} (p_1, p_2) \mathcal{O}(p_3) \rangle\rangle. \tag{2.5.44}
\]

We omit the group index on \( \mathcal{O} \) as we consider only one scalar operator. First, we want to argue that the right-hand side of (2.5.44) vanishes if \( \beta_3 > 0 \), unless some specific conditions on conformal dimensions are met. Therefore, in this section we will assume that the right-hand sides of (2.4.31) and (2.4.32) vanish, leaving a discussion of the various special cases to the following sections. Indeed, the only possibility for a non-vanishing right-hand side of (2.5.44) is if the functional derivative \( \delta T_{\mu_1\nu_1}/\delta g_{\mu_2\nu_2} \) contains the operator \( \mathcal{O} \) or its descendants. Since the dilatation degree of \( \delta T_{\mu_1\nu_1}/\delta g_{\mu_2\nu_2} \) is equal to \( d \), this requires \( d = \Delta_3 + 2n \) where \( n \) is a non-negative integer. Consider first the case \( \Delta_3 = d \). In this case, we can write the most general form of \( \delta T_{\mu_1\nu_1}/\delta g_{\mu_2\nu_2} \) which contains \( \mathcal{O} \) as

\[
\frac{\delta T_{\mu_1\nu_1}(x)}{\delta g_{\mu_2\nu_2}(y)} = [c_1 \delta_{\mu_1\nu_1} \delta_{\mu_2\nu_2} + c_2 \delta_{(\mu_1(\mu_2 \delta_{\nu_2\nu_1})]} \delta(x-y) \mathcal{O}(x) + \ldots \tag{2.5.45}
\]

where \( c_1 \) and \( c_2 \) are numerical constants. If, on the other hand, \( d = \Delta_3 + 2n \) with \( n > 0 \) then derivatives acting on both \( \mathcal{O} \) and \( \delta(x-y) \) may also appear. In all cases, the Fourier transform reads

\[
\langle\langle \frac{\delta T_{\mu_1\nu_1}(p_1, p_2) \mathcal{O}(p_3)}{\delta g_{\mu_2\nu_2}} \rangle\rangle = P_{\mu_1\nu_1\mu_2\nu_2}(p_1^2, p_2^2, p_3^2) \langle\langle \mathcal{O}(p_3) \mathcal{O}(-p_3) \rangle\rangle, \tag{2.5.46}
\]

where \( P \) is some polynomial built from momenta and the metric \( \delta_{\mu\nu} \), with kinematic dependence on squares of momenta only. This form arises from the Fourier transform of the position space expression containing derivatives acting on delta functions and on the 2-point function. Since \( \langle\langle \mathcal{O}(p_3) \mathcal{O}(-p_3) \rangle\rangle \sim p_3^{2\beta_3} \), the expression vanishes in the 3-point \( p_3 \rightarrow 0 \) limit as long as \( \beta_3 > 0 \).

We now substitute the solutions of the primary CWIs (2.5.25) into the left-hand side of (2.4.31) and take the zero-momentum limit. Assuming the regulator can be removed (see section 2.5.3 if not), the result is

\[
\frac{1}{2} \frac{\frac{d}{2} + 1 \{\frac{d}{2}, \frac{d}{2}, \Delta_3 - \frac{d}{2}\}}{p^{\Delta_3 - 2}(2+2d-\Delta_3) [\alpha_2 + \alpha_1 (\Delta_3 + 2)(\Delta_3 - d + 2)]} = 0, \tag{2.5.47}
\]

which leads to

\[
\alpha_2 = -((\Delta_3 + 2)(\Delta_3 + 2 - d)\alpha_1. \tag{2.5.48}
\]
The same reasoning as above applied to (2.4.32) leads to the equation
\[
\alpha_3 = \frac{1}{4} \Delta_3(\Delta_3 + 2)(\Delta_3 - d)(\Delta_3 + 2 - d)\alpha_1. 
\] (2.5.49)

Summarising, in this and the previous section we presented a method for extracting algebraic dependencies between the primary constants following from the secondary CWIs. The analysis was performed in the generic case, where the regulator can be removed from all triple-\(K\) integrals involved. Note that the results (2.5.48) and (2.5.49) agree with our example (2.5.36) and (2.5.37) in the leading term in \(\epsilon\) only, \(i.e.,\) they correctly predict \(\alpha_2 = \alpha_3 = 0 + O(\epsilon)\). This is due to the fact that in our example the regulator cannot be removed from each triple-\(K\) integral separately. Therefore, it does not satisfy the assumption of this section. Note, however, that the analysis of the generic case is sufficient if one is merely interested in finding the solution up to semi-local terms. This is because the possible non-generic cases arise due to the regularisation procedure, correcting the generic solution by at most semi-local terms.

**Triple-\(K\) integrals and 2-point functions**

Before we discuss the general procedure applicable to all cases, we first need to analyse the possible singularities associated with the 2-point functions. This is because the secondary CWIs connect triple-\(K\) integrals with semi-local terms expressible in terms of 2-point functions. Therefore, if the regulator is kept explicitly, the singular terms in triple-\(K\) integrals must match the singularities appearing in the 2-point functions.

An initial obstacle is that our convenient regularisation scheme (2.2.4) does not work for 2-point functions. The Fourier transform of the position space expression for a generic 2-point function is given by (2.1.3) where \(\Delta\) is a conformal dimension of a scalar operator \(O\). The singularity occurs if \(2\Delta = d + 2n\) for a non-negative integer \(n\) and is not regularised by the scheme (2.2.4).

Let us now try to find a different scheme which regularises 2-point functions and it yields results equivalent to (2.2.4, 2.5.7) when applied to the expressions entering the left-hand side of the secondary CWIs. It turns out that standard dimensional regularization has this property. We explicitly checked this in all cases discussed in the thesis. To motivate this choice consider the following integral
\[
\int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} \ldots k^{\mu_r}}{|k - p_1|^{2\delta_1} |k + p_2|^{2\delta_2}} \] (2.5.50)
regulated by shifting the space-time dimension while keeping the \(\delta_j\) parameters fixed. In terms of the \(\alpha\) and \(\beta\) parameters in (2.5.1), this corresponds to \(\alpha \mapsto \alpha - \frac{\epsilon}{2}\) and \(\beta_j \mapsto \beta_j - \frac{\epsilon}{2}\) as can be seen from (2.2.3) and (2.5.2). We can evaluate the integral (2.5.50) both by the triple-\(K\) integrals or the usual Feynman parametrised
2.5. Solutions to conformal Ward identities

integrals. This leads to equation (4.2.23), namely

\[ I_{\alpha \{ \beta_1 \beta_2 \beta_3 \}} = 2^{\alpha-3} \Gamma \left( \frac{\alpha - \beta_t + 1}{2} \right) \Gamma \left( \frac{\alpha + \beta_t + 1}{2} \right) \times \]

\[ \times \int_{[0,1]^3} dX \ D^{\frac{1}{2}(\beta_t-\alpha-1)} \prod_{j=1}^{3} x_j^{\frac{1}{2}(\alpha-1-\beta_t)+\beta_j}, \tag{2.5.51} \]

where

\[ dX = dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1), \tag{2.5.52} \]
\[ D = p_1^2 x_2 x_3 + p_2^2 x_1 x_3 + p_3^2 x_1 x_2. \tag{2.5.53} \]

The leading divergence comes from a possible zero under the first gamma function and both schemes regularise \( \alpha - \beta_t + 1 \) in the same way, shifting it by \( \epsilon/2 \).

A detailed analysis shows that all singularities and the finite part (modulo terms analytic in all momenta) in the expressions appearing in the left-hand side of the secondary CWIs match between the two schemes (i.e., dimensional regularisation, \( d \mapsto d - \epsilon \), and the scheme (2.2.4, 2.5.7)). Thus we can consistently take into account the contributions of the 2-point functions by computing them using dimensional regularization.

In the renormalised theory the divergent terms can be removed. Therefore we define constants \( c_O, c_J \) and \( c_T \) by

\[ \langle \langle O^I(p) O^J(-p) \rangle \rangle = c_O \delta^{IJ} \times \left\{ \begin{array}{ll}
p^{2\Delta-d} & \text{if } 2\Delta \neq d + 2n, \\
p^{2\Delta-d} (-\log p^2 + \text{local}) & \text{if } 2\Delta = d + 2n,
\end{array} \right. \tag{2.5.54} \]

\[ \langle \langle J^{\alpha \mu}(p) J^{\beta \nu}(-p) \rangle \rangle = c_J \delta^{ab} \pi^{\mu \nu}(p) \times \left\{ \begin{array}{ll}
p^{d-2} & \text{if } d = 3, 5, 7, \ldots \\
p^{d-2} (-\log p^2 + \text{local}) & \text{if } d = 4, 6, 8, \ldots
\end{array} \right. \tag{2.5.55} \]

\[ \langle \langle T^{\mu \nu}(p) T^{\rho \sigma}(-p) \rangle \rangle = c_T \Pi^{\mu \nu \rho \sigma}(p) \times \left\{ \begin{array}{ll}
p^d & \text{if } d = 3, 5, 7, \ldots \\
p^d (-\log p^2 + \text{local}) & \text{if } d = 4, 6, 8, \ldots
\end{array} \right. \tag{2.5.56} \]

where \( n \) is a non-negative integer and all \( O^I \) operators have the same conformal dimension \( \Delta \) and the dimensions of \( J^{\alpha \mu} \) and \( T^{\mu \nu} \) are \( d - 1 \) and \( d \) respectively. In general the normalisation constants \( c_O \) and \( c_J \) carry group indices. For simplicity we assume that the Killing form is trivial, i.e.,

\[ c_O^{IJ} = c_O \delta^{IJ}, \quad c_J^{ab} = c_J \delta^{ab}. \tag{2.5.57} \]

**Secondary conformal Ward identities in all cases**

Let us now return to the discussion of the secondary CWIs in the case where the regulator cannot be removed in certain triple-\( K \) integrals. In principle, the
procedure is simple. One must keep the explicit dependence on $\epsilon$, both in the triple-$K$ integrals as well as in the primary constants, and carry out the analysis of sections 2.5.3 and 2.5.3 order by order in the regulator. Note that if the index of a Bessel function is integral, then the expansions (2.5.40) and (2.5.41) should be used instead of (2.5.39).

The only difference with section 2.5.3 is that looking at the zero-momentum limit may not be enough. We should look at both terms following from the first and second brackets in (2.5.39), i.e., the coefficients of $p_0^3$ and $p_3^{2\beta_3}$ in the expansion in powers of $p_3$ with $p_1 = p_2 = p$. If the Bessel index is integral, then we should use (2.5.40) and (2.5.41) and look for the coefficients of $p_0^3$ and $p_3^{2\beta_3} \log p_3$. This procedure will provide a set of algebraic equations relating the primary constants.

Let us now explain why this procedure is valid for $\beta_3 < 0$ and why the remaining terms in the expansions (2.5.39) - (2.5.41) are irrelevant. First, the unitarity bound requires $-1 < \beta_3$. The unitarity bound can only be saturated by a non-composite scalar operator in a free field theory, [29] and [20]. We can therefore assume $-1 < \beta_3 < 0$. It turns out that the considerations encountered in the case $\beta_3 > 0$ remain valid here. Since the zero-momentum limit does not exist in this case, we are going to look for the coefficient of $p_0^3$ in the expansion in $p_3$. The key observation is that on the left-hand sides of the secondary CWIs such as (2.4.31, 2.4.32), the differential operators $L$ and $R$ defined by (2.4.27) and (2.4.28) do not contain derivatives with respect to $p_3$, and can only increase powers of $p_3$ by two. Therefore, the coefficient of $p_0^3$ in the series expansion in $p_3$ remains unaltered provided $-1 < \beta_3$. A similar analysis applies to the right-hand sides of the secondary CWIs.

Let us now examine why it is sufficient to look at the leading coefficients in (2.5.39) - (2.5.41) only. From (2.A.21), we know that in each successive term the power of the integration variable $x$ increases by two. After taking the zero-momentum limit, the integral (2.A.49) therefore leads to essentially the same expression as (2.5.43) with $\beta_3 \mapsto \beta_3 + 2n$ for a non-negative integer $n$ plus some finite pre-factor following from the series expansion of the Bessel $K$ function. Since the singularities manifest themselves as poles of the gamma functions, we see that the result cannot be more singular than the original $l_{\alpha\{\beta_j\}}$.

**Back to the example**

Finally, let us see how the general consideration of the previous section work in the case of the $\langle\langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} \mathcal{O} \rangle\rangle$ correlation function. First, we carry out the same analysis as in sections 2.5.3 and 2.5.3 but keep the regulator explicitly. Instead of (2.5.47), we then find

$$-\frac{1}{2} l_{\frac{d}{4} + 1 + \epsilon(\frac{d}{4} + 2, \Delta_3 - \frac{d}{4})} p_{\Delta_3 - 2 + \epsilon}(2 + 2d - \Delta_3 + \epsilon) \times \left[ a_2 + a_1((\Delta_3 - \epsilon)^2 - d(\Delta + 2 - \epsilon) + 4(1 + \Delta + \epsilon)) \right] = 0. \quad (2.5.58)$$
2.5. Solutions to conformal Ward identities

If the $\epsilon \to 0$ limit exists, we recover (2.5.47). The limit does not exist, however, if $\Delta_3$ satisfies some non-generic relations. From the definition of $l_{d+1+\epsilon(\frac{d}{2},\Delta_3-\frac{d}{2})}$ in (2.5.43), we see that this happens if at least one of the following conditions are satisfied:

- $\Delta_3 = 2 + 2n_1$,
- $\Delta_3 = d + 2 + 2n_2$,
- $\Delta_3 = d + 4 + 2n_3$,
- $\Delta_3 = 2d + 4 + 2n_4$,

where $n_1, \ldots, n_4$ are non-negative integers. The order of the singularity increases if more than one of these conditions is satisfied. Since there exists a choice of $\Delta_3$, $d$ and the $n_j$ constants such that all conditions are satisfied, we might expect a pole of order four in $\epsilon$. Note however that there is another gamma function in the denominator of (2.5.43) which becomes singular if the numerator has a pole of order four. The maximal order of the pole is therefore only three.

The discussion above leads to the conclusion that we should expand the primary constant $\alpha_2$ up to third order in $\epsilon$ as in (2.5.33), hence

$$
\alpha_2 J_{2(000)} = \frac{1}{\epsilon^3} \alpha_2^{(0)} J_{2(000)}^{(-3)} + \frac{1}{\epsilon^2} \left[ \alpha_2^{(0)} J_{2(000)}^{(-2)} + \alpha_2^{(1)} J_{2(000)}^{(-3)} \right] + \frac{1}{\epsilon} \left[ \alpha_2^{(0)} J_{2(000)}^{(-1)} + \alpha_2^{(1)} J_{2(000)}^{(-2)} + \alpha_2^{(2)} J_{2(000)}^{(-3)} \right] + O(\epsilon).
$$

With the appropriate choice of primary constants we will now see that the singular terms in the various triple-$K$ integrals building a given form factor cancel out, leaving ultralocal singular terms at most.

Let us now extract the primary constants from the secondary CWI (2.4.31). We substitute (2.5.59) into (2.5.58) and expand the result in $\epsilon$ in any possible combination of the cases itemised above. One can subsequently solve the equations starting from the most singular one. In this way, one finds

$$
\alpha_2^{(0)} = -(\Delta_3 + 2)(\Delta_3 + 2 - d)\alpha_1, \quad \alpha_2^{(2)} = -\alpha_1, \quad \alpha_2^{(1)} = (2\Delta_3 - d + 4)\alpha_1, \quad \alpha_2^{(3)} = 0.
$$

We assume here that $\alpha_1$ is a true constant, i.e., $\alpha_1 = \alpha_1^{(0)}$. These solutions are valid for the special cases listed at the beginning of this section. Notice that they still do not cover all special cases, for example they do not cover the case of the example we studied in section 2.5.3. This is because so far we have considered the equations following from the first parts of the expansions (2.5.39) - (2.5.41) only:
we must now turn our attention to the equations following from the second parts. In many cases the equations to follow will agree with (2.5.60), but in some special cases new contributions will arise.

The equation (2.5.39) and the integral (2.A.49) lead to the equation

\[
- \frac{l_d^{\frac{d}{2}+1}+\epsilon\{\frac{d}{2}+\frac{d}{2}-\Delta_3\}}{2} (2+d+\Delta_3+\epsilon) [a_2 + a_1(\Delta_3 + 2 + \epsilon)(\Delta_3 - d + 2 + \epsilon)] = 4d \cdot \text{coefficient of } p_3^{\mu_1} p_3^{\mu_2} p_3^{\nu_2} \cdot p_3^{2\Delta_3-d} p^{d-\Delta_3-2} \text{ in } p_2^{\nu_1} \left\langle \frac{\delta T_{\mu_1 \nu_1}}{\delta g^{\mu_2 \nu_2}} (p_1, p_2) \mathcal{O}^I (p_3) \right\rangle
\]

(2.5.61)

following from the coefficient of \( p_3^{2\Delta_3} \) in the series expansion (2.5.39). First note that if the \( \epsilon \to 0 \) limit exists, then the left-hand side vanishes when the solution (2.5.48) is substituted. On the other hand, we know from section 2.5.3 that the right-hand side can be non-zero only if \( \Delta_3 = d - 2 - 2n \) for some non-negative integer \( n \). Therefore, in such a case we expect the left-hand side to be more singular than the right-hand side, so that the solution (2.5.48) cancels the leading order singularity while the sub-leading terms match the right-hand side. Indeed, the left-hand side is singular if \( l_d^{\frac{d}{2}+1}\{\frac{d}{2}+\frac{d}{2}-\Delta_3\} \) is singular. Analysing the expression (2.5.43), we see that this can happen only if \( \Delta_3 = d - 2 - 2n \), where \( n \) is a non-negative integer. Note that our example analysed in section 2.5.3 is of this kind as there \( d = 3, \Delta_3 = 1 \).

Finally, we would like to extract the sub-leading equations in the case where \( \Delta_3 = d - 2 - 2n \). In order to do this, we write \( \alpha_2 = \alpha_2^{(0)} + \epsilon \alpha_2^{(1)} \) in (2.5.61) and expand the result in \( \epsilon \). At zeroth order we recover (2.5.48), after which we find

\[
C \cdot \Gamma \left( \frac{d}{2} - \Delta_3 \right) \left[ (2\Delta_3 - d + 4) \alpha_1 + \alpha_2^{(1)} \right] = 4d \cdot c_1^I c_{20},
\]

(2.5.62)

where

\[
C = \frac{(-1)^{\frac{d-\Delta_3}{2}} \Gamma(\frac{\Delta_3+2}{2}) \Gamma(\frac{\Delta_3+d+4}{2})}{\Gamma(\frac{d-\Delta_3}{2}) \Gamma(\frac{\Delta_3+4}{2})}
\]

(2.5.63)

and the constant \( c_1^K \) is defined as

\[
p_2^{\nu_1} \left\langle \frac{\delta T_{\mu_1 \nu_1}}{\delta g^{\mu_2 \nu_2}} (p_1, p_2) \mathcal{O}^I (p_3) \right\rangle \bigg|_{p_1 = p_2 = p} = c_1^K p_3^{\mu_1} p_3^{\mu_2} p_3^{\nu_2} p^{d-\Delta_3-2} \left\langle \mathcal{O}^I (p_3) \mathcal{O}^K (-p_3) \right\rangle + c_2^K p_3^{\mu_1} p_3^{\mu_2} p_3^{\nu_2} p^{d-\Delta_3} \left\langle \mathcal{O}^I (p_3) \mathcal{O}^K (-p_3) \right\rangle + \ldots
\]

(2.5.64)

By \( p_1 = p_2 = p \), we mean here the following procedure: First, the correlation function on the left-hand side is expanded in terms of simple tensors according to the convention (2.3.4), then secondly, one applies \( p_1 = p_2 = p \) to each coefficient separately.
In order to derive (2.5.62) we used (2.5.39). If \( \Delta_3 = d - 2 - 2n \) and \( 2\Delta_3 = d + 2m \) for some non-negative integers \( m \) and \( n \), i.e., if \( \beta_3 \) is an integer, then one must use instead (2.5.41). The procedure remains identical and the final result is

\[
- \frac{C}{\Gamma\left(\Delta_3 - \frac{d}{2} + 1\right)} \left(2\Delta_3 - d + 4\right)\alpha_1 + \alpha_2^{(1)} \right] = 4d \cdot c_1^I c_\Omega. \tag{2.5.65}
\]

In total, we have that

\[
\alpha_2^{(0)} = -(\Delta_3 + 2)(\Delta_3 + 2 - d)\alpha_1 \tag{2.5.66}
\]

in all cases while

\[
\alpha_2^{(1)} = (2\Delta_3 - d + 4)\alpha_1, \quad \alpha_2^{(2)} = -\alpha_1 \tag{2.5.67}
\]

if \( \Delta_3 \neq d - 2 - 2n \), where \( n \) is a non-negative integer and

\[
4d \cdot c_1^I c_\Omega = \left[\left(2\Delta_3 - d + 4\right)\alpha_1 + \alpha_2^{(1)} \right] \times C \times \left\{ \begin{array}{ll}
\frac{\Gamma\left(\frac{d}{2} - \Delta_3\right)}{\Gamma\left(\Delta_3 - \frac{d}{2} + 1\right)} & \text{if } 2\Delta_3 \neq d + 2, \\
-\frac{1}{\Gamma\left(\Delta_3 - \frac{d}{2} + 1\right)} & \text{if } 2\Delta_3 = d + 2m,
\end{array} \right. \tag{2.5.68}
\]

if \( \Delta_3 = d - 2 - 2n \), where \( m \) and \( n \) are non-negative integers.

A similar analysis may be carried out for the second CWI, (2.4.32). Putting all the ingredients together, we can now write the most general form of the correlator \( \langle \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} O^I \rangle \rangle \) for \( d = 3 \) and \( \Delta_2 = \Delta_3 = 1 \). Using the results for the triple-\( K \)-integrals from section 2.5.2 we have

\[
A_1^I = \frac{\alpha_1^I}{p_3 a_{123}^3} \left[ p_3^3 + 4p_3 a_{112} + 3(a_{112}^2 + 2b_{112}) \right] , \tag{2.5.69}
\]

\[
A_2^I = \frac{\alpha_1^I}{p_3 a_{123}^3} \left[ p_3^3 + 3p_3^2 a_{112} + 3a_{112}^2 - 3a_{112} + 8b_{112} \right] - \frac{4c_1^I c_\Omega}{p_3}, \tag{2.5.70}
\]

\[
A_3^I = \frac{\alpha_1^I (a_{112} - p_3)}{4p_3 a_{123}^3} \left[ -p_3^3 - 3p_3^2 a_{112} + p_3 (a_{112}^2 - 10b_{112}) + 3a_{112} (a_{112}^2 - 2b_{112}) \right] + \frac{c_I}{p_3} \left[ (c_1^I - 3c_2^I)(p_1^2 + p_2^2) + 3(c_1^I + c_2^I) p_3^2 \right], \tag{2.5.71}
\]

where we have defined the symmetric polynomials in momentum magnitudes

\[
a_{123} = p_1 + p_2 + p_3, \quad b_{123} = p_1 p_2 + p_1 p_3 + p_2 p_3, \quad c_{123} = p_1 p_2 p_3, \\
a_{ij} = p_i + p_j, \quad b_{ij} = p_i p_j, \tag{2.5.72}
\]

where \( i, j = 1, 2, 3 \). The solution for this correlator is thus uniquely determined up to one numerical constant \( \alpha_1^I \). The remaining constants in the solution, namely
2. Implications of conformal invariance in momentum space

$c_0$, $c'_1$ and $c'_2$, are determined by the 2-point function normalisations: $c_0$ is given in (2.5.54) while $c'_1$ and $c'_2$ are given in (2.5.64).

One can check this result against our example in section 2.5.3. From (2.4.39) and (2.4.40), the solution for the parameters is

$$c_0 = \frac{1}{4}, \quad c'_1 = -\frac{1}{16}, \quad c'_2 = 0. \quad (2.5.73)$$

2.6. Evaluation of triple-K integrals

In the preceding sections we have developed a method for calculating all 3-point functions of the stress-energy tensor, conserved currents and scalar operators in any CFT. The method operates purely in momentum space, and is based on a direct solution of the conformal Ward identities. The results we obtain are expressed in the form of triple-K integrals.

In the present section we now turn to discuss the evaluation of these triple-K integrals. The ease with which this may be accomplished depends on the dimensionality of the space $d$. In an odd number of dimensions, it turns out that all 3-point functions of conserved currents and the stress-energy tensor are expressible in terms of triple-K integrals in which the Bessel function indices are half-integer. In this case, the Bessel functions reduce to elementary functions (see appendix 2.A.3) and the triple-K integral may be straightforwardly evaluated. When the spatial dimension $d$ is even, we obtain instead triple-K integrals in which the Bessel function indices are integer. In this case, the triple-K integrals we encounter may be evaluated in terms of a single master integral through the use of a reduction scheme. In the following we will focus primarily on the case $d = 4$, but method we present extends straightforwardly to higher even dimensions.

Our reduction scheme is based on the observation that, given a triple-K integral $I_{\alpha_1,\beta_1,\beta_2,\beta_3}$, through differentiation we may easily obtain the integrals $I_{\alpha+1,\beta_1,\beta_2,\beta_3}$ and $I_{\alpha+1,\beta_1-1,\beta_2,\beta_3}$, and similarly for $\beta_2$ and $\beta_3$ by permutation. We may then start with a known integral with a sufficiently small value of $\alpha$ and obtain integrals with larger $\alpha$ through repeated differentiation. In some cases a relation for lowering $\alpha$ also exists. For 3-point functions of conserved currents and stress-energy tensors in $d = 4$ the necessary triple-K integrals are:

$$
\begin{align*}
I_{4\{111\}} & \leftrightarrow I_{5\{211\}}, I_{6\{221\}}, I_{7\{222\}}, \\
I_{2\{111\}} & \leftrightarrow I_{3\{211\}}, I_{4\{221\}}, I_{4\{311\}}, I_{5\{222\}}, I_{5\{321\}}, I_{6\{322\}}, \\
I_{0\{111\}} & \leftrightarrow I_{1\{211\}}, I_{2\{221\}}, I_{3\{222\}}, I_{3\{321\}}, I_{4\{322\}}, I_{5\{422\}}, I_{5\{332\}}, \\
I_{1\{222\}} & \leftrightarrow I_{3\{332\}}, I_{4\{333\}}.
\end{align*}
$$

(2.6.1)

The integrals are organised into four families, each with constant $\alpha - \beta_t$, where each integral within a given family may be obtained from the corresponding integral on
the leftmost end. The four leftmost integrals can then be derived from the single master integral, \( I_{0\{111\}} \), as we will show in section 2.6.4.

### 2.6.1. Reduction scheme

The elementary properties of Bessel functions imply

\[
I_{\alpha \{ \beta_1 \} \{ \beta_2 \} \{ \beta_3 \}}(p_1, p_2, p_3) = I_{\alpha \{ \beta_1 \beta_2 \beta_3 \}}(p_{\sigma^{-1}(1)}, p_{\sigma^{-1}(2)}, p_{\sigma^{-1}(3)}), \tag{2.6.2}
\]

\[
I_{\alpha \{ \beta_1 \beta_2 \beta_3 \}} = -\frac{1}{p_1} \frac{\partial}{\partial p_1} I_{\alpha - 1 \{ \beta_1 + 1 \beta_2 \beta_3 \}}, \tag{2.6.3}
\]

\[
I_{\alpha \{ \beta_1 \beta_2 - \beta_3 \}} = p_3^{-2\beta_3} I_{\alpha \{ \beta_1 \beta_2 \beta_3 \}}, \tag{2.6.4}
\]

\[
(\alpha - \beta_t) I_{\alpha - 1 \{ \beta_1 \beta_2 \beta_3 \}} = p_1^2 I_{\alpha \{ \beta_1 - 1 \beta_2 \beta_3 \}} + p_2^2 I_{\alpha \{ \beta_1 \beta_2 - 1 \beta_3 \}} + p_3^2 I_{\alpha \{ \beta_1 \beta_2 \beta_3 - 1 \}}, \quad \alpha - \beta_t \neq -2n, \tag{2.6.5}
\]

where \( n \) is a non-negative integer and the triple-\( K \) integral \( I_{\alpha \{ \beta_1 \beta_2 \beta_3 \}} \) was defined in (2.5.1). The first of these equations appeared previously as (2.5.11) and simply expresses the fact that the triple symmetry under permutation, with \( \sigma \) representing a permutation of the set \( \{1, 2, 3\} \). The second and third of these equations appeared earlier as (2.5.13) and (2.5.14), while the last follows from the second line of (2.5.18). Starting from these relations, we may now set up our reduction scheme as follows.

First, assume we are given an integral \( I_{\alpha \{ \beta_1 \beta_2 \beta_3 \}} \). Applying (2.6.3) in the form

\[
I_{\alpha + 1 \{ \beta_1 - 1 \beta_2 \beta_3 \}} = -\frac{1}{p_1} \frac{\partial}{\partial p_1} I_{\alpha \{ \beta_1 \beta_2 \beta_3 \}}, \tag{2.6.6}
\]

we increase \( \alpha \) by one while decreasing \( \beta_1 \) by one. Equivalently, this operation increases the difference \( \alpha - \beta_t \) by two but keeps the sum \( \alpha + \beta_t \) fixed.

If instead we first use equation (2.6.4), followed by (2.6.3) then (2.6.4) again, we find

\[
I_{\alpha + 1 \{ \beta_1 + 1 \beta_2 \beta_3 \}} = -p_1^{2\beta_1 + 1} \frac{\partial}{\partial p_1} \left[ p_1^{-2\beta_1} I_{\alpha \{ \beta_1 \beta_2 \beta_3 \}} \right] = \left( 2\beta_1 - p_1 \frac{\partial}{\partial p_1} \right) I_{\alpha \{ \beta_1 \beta_2 \beta_3 \}}, \tag{2.6.7}
\]

where the sum \( \alpha + \beta_t \) increases by two but the difference \( \alpha - \beta_t \) remains fixed. Similarly, applying this operation and its permutations repeatedly we obtain

\[
I_{\alpha + n \{ \beta_j + k_j \}} = (-1)^k \left[ \prod_{j=1}^{3} p_j^{2(\beta_j + k_j)} \left( \frac{1}{p_j} \frac{\partial}{\partial p_j} \right)^{k_j} \right] \left[ p_1^{-2\beta_1 - 2\beta_2 - 2\beta_3} I_{\alpha \{ \beta_j \}} \right]. \tag{2.6.8}
\]
where \( k_t = \sum_j k_j \) and the \( k_j \) are non-negative integers.

Now, both reduction relations (2.6.3) and (2.6.4) obtained by differentiation happen to increase \( \alpha \) by one. To reduce \( \alpha \) we may instead use the (2.6.5) in the form

\[
I_{\alpha-1{\beta_1\beta_2\beta_3}} = \frac{1}{\alpha - \beta_t} \left[ p_1^2 I_{\alpha{\beta_1-1,\beta_2,\beta_3}} + p_2^2 I_{\alpha{\beta_1,\beta_2-1,\beta_3}} + p_3^2 I_{\alpha{\beta_1,\beta_2,\beta_3-1}} \right],
\]

assuming \( \alpha - \beta_t \neq -2n \), where \( n \) is a non-negative integer. This equation is closely related to Davydychev’s recursion relation (3.4) introduced in [52]. Indeed, using

\[
\begin{align*}
I_{0(111)} \rightarrow (2-p_1 \frac{\partial}{\partial p_1}) \rightarrow I_{1(211)} \rightarrow -\frac{1}{p_1} \frac{\partial}{\partial p_1} \rightarrow I_{2(111)} \\
I_{1(221)} \rightarrow (2-p_2 \frac{\partial}{\partial p_2}) \rightarrow I_{3(211)} \rightarrow -\frac{1}{p_1} \frac{\partial}{\partial p_1} \rightarrow I_{4(111)} \\
I_{2(322)} \rightarrow (4-p_1 \frac{\partial}{\partial p_1}) \rightarrow I_{3(322)} \rightarrow (2-p_2 \frac{\partial}{\partial p_2}) \rightarrow I_{4(221)} \rightarrow -\frac{1}{p_1} \frac{\partial}{\partial p_1} \rightarrow I_{5(211)} \\
I_{3(332)} \rightarrow (4-p_2 \frac{\partial}{\partial p_2}) \rightarrow I_{4(322)} \rightarrow (2-p_3 \frac{\partial}{\partial p_3}) \rightarrow I_{5(222)} \rightarrow -\frac{1}{p_1} \frac{\partial}{\partial p_1} \rightarrow I_{6(221)} \\
I_{4(333)} \rightarrow (4-p_3 \frac{\partial}{\partial p_3}) \rightarrow I_{5(332)} \rightarrow (2-p_2 \frac{\partial}{\partial p_2}) \rightarrow I_{6(322)} \rightarrow -\frac{1}{p_1} \frac{\partial}{\partial p_1} \rightarrow I_{7(222)}
\end{align*}
\]

**Table 2.1:** Reduction scheme for the integrals required in the calculations of 3-point functions of conserved currents and stress-energy tensor in \( d = 4 \). All integrals can be obtained as indicated from a single master integral \( I_{0(111)} \) given by (2.4.57). The dotted line indicates the use of (2.6.9).
(4.2.17) one can rewrite Davydychev’s J integral defined in (2.1) of [52] as

\[ J(\delta_1, \delta_2, \delta_3) = \frac{4\pi^2}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(4 - \delta_t)} I_{1\{2-\delta_2-\delta_3,2-\delta_1-\delta_3,2-\delta_1-\delta_2\}}. \] (2.6.10)

Note that the rather complicated form of equation (3.4) in [52] is a consequence of the specific index structure in the triple-K integral above.

## 2.6.2. The case \(d = 4\)

Let us now apply the reduction relations above to the integrals listed in (2.6.1). Integrals within a given family have constant \(\alpha - \beta_t\) and hence are connected by (2.6.7). Similarly, integrals belonging to different families but with the same sum \(\alpha + \beta_t\) may be connected using (2.6.6). We summarise in table 2.1 the dependencies between all integrals necessary for the evaluation of 3-point functions of conserved currents and the stress-energy tensor in \(d = 4\).

Integrals within a single row (except for the first) are connected via equation (2.6.6), while integrals within a single column are related by (2.6.7). The difference \(\alpha - \beta_t\) is thus constant within each column while the sum \(\alpha + \beta_t\) is constant along each row. Similarly, the index \(\alpha\) is constant along diagonals from top-right to bottom-left, while \(\beta_t\) is constant along diagonals from the top-left to the bottom-right. In cases where there are two integrals in a given table entry, the arrows entering and leaving this entry have two labels, with the upper label referring to the upper integral and the lower label referring to the lower integral, thus, e.g.,

\[ I_{3\{222\}} = -\frac{1}{p_1} \frac{\partial}{\partial p_1} I_{2\{322\}}, \quad I_{3\{321\}} = -\frac{1}{p_3} \frac{\partial}{\partial p_3} I_{2\{322\}}. \] (2.6.11)

We assume all integrals are regularised in our standard scheme (2.5.7). In this case, the operations assigned to the arrows should be applied order by order in the regulator. Note that if one uses a different regularisation scheme which changes the values of the \(\beta\) parameters, one cannot apply the operators on the vertical lines order by order in \(\epsilon\). This is because (2.6.7) contains a \(\beta\) parameter which would become a function of the regulator.

In section 2.6.4 we will evaluate the master integral \(I_{0\{111\}}\). However, as the entries on the left side of the table are generally more singular (and more complicated) than the entries on the right, in practice it is more convenient to move in columns rather than in rows. The required expressions for the three integrals \(I_{0\{111\}}, I_{2\{111\}}\) and \(I_{1\{000\}}\), which generate all the integrals in the three rightmost columns are given in appendix 2.A.5.

Starting from \(I_{0+\epsilon\{111\}}\) we can follow the arrows in table 2.1 to obtain \(I_{2+\epsilon\{221\}}\).
Using (2.6.2) and (2.6.9) we find

\[
I_{1+\epsilon(222)} = \frac{1}{\epsilon - 4} \left[ p_1^2 I_{2+\epsilon(122)} + p_2^2 I_{2+\epsilon(212)} + p_3^2 I_{2+\epsilon(221)} \right]
\]

\[
= \frac{p_1^4 + p_2^4 + p_3^4}{2\epsilon^2}
\]

\[
+ \frac{1}{8} \left( \frac{4}{\epsilon} + 1 \right) \times \left[ \left( p_1^2 p_2^2 + \left( \frac{3}{4} - \gamma_E + \log 2 \right) p_3^4 - p_3^4 \log p_3 \right)
\]

\[
+ (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3) \right]
\]

\[
- \frac{1}{4} \left[ p_3^2 \left( 2 - p_1 \frac{\partial}{\partial p_1} \right) \left( 2 - p_2 \frac{\partial}{\partial p_2} \right) + (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3) \right] I_{0(111)}^{(0)}
\]

\[
+ O(\epsilon), \tag{2.6.12}
\]

where \( I_{0(111)}^{(0)} \) denotes the \( \epsilon^0 \) term in the series expansion in \( \epsilon \) of \( I_{0+\epsilon(111)} \). Thus all integrals in the table can be obtained from \( I_{0(111)} \).

\section*{2.6.3. Integrals in even dimensions \( d \geq 4 \)}

In the previous section we presented a comprehensive procedure for the evaluation of all triple-\( K \) integrals appearing in 3-point functions of conserved currents and the stress-energy tensor in \( d = 4 \). In this section we want to extend our analysis to all even dimensions \( d \geq 4 \). We will present a recursive procedure which yields all required integrals from the master integral \( I_{0(111)} \).

Let us start with the case \( d = 6 \). Looking at the solutions to the primary CWIs, one sees that new integrals arise only in the form factor \( A_5 \) of the \( \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle \) correlator. One of these new integrals, \( J_{0(000)} \), is equal to \( I_{2(333)} \) when \( d = \Delta_j = 6, j = 1, 2, 3 \) is used in (2.5.3). All remaining integrals in the form factor \( A_5 \) then satisfy the relation \( \alpha - \beta_t = -7 \) meaning they can be obtained from \( I_{2(333)} \) using (2.6.7). Finally, notice that by (2.6.6),

\[
- \frac{1}{p_3} \frac{\partial}{\partial p_3} I_{2(333)} = I_{3(332)}. \tag{2.6.13}
\]

Thus to obtain all required integrals in \( d = 6 \) we need to add another column to the left side of table 2.1.

This discussion generalises for general even \( d \). Assume first that we have all integrals required for the evaluation of the 3-point functions of conserved currents and the stress-energy tensor in a given even dimension. The leftmost integrals in table 2.1 have the lowest value of \( \alpha - \beta_t \) and it is these integrals which appear in the \( A_5 \) form factor in the correlator of three stress-energy tensors. For these integrals \( \alpha - \beta_t = -(d + 1) \), and the lowest value of \( \alpha + \beta_t \) is attained for the \( J_{0(000)} \) integral. For the correlator of three stress-energy tensors in \( d \) dimensions

\[
J_{0(000)} = I_{\frac{d}{2} - 1\{\frac{d}{2}\frac{d}{2}\frac{d}{2}} \tag{2.6.14}
\]
and all remaining integrals in the form factor $A_5$ can be obtained from this one by means of (2.6.7). Therefore, if one knows the integrals in a dimension $d - 2$, in order to obtain the missing integrals in dimension $d$, one needs to reduce the value of $\alpha - \beta_t$. This can be achieved by (2.6.9),

$$I_{\frac{d}{2}-1+\epsilon\{\frac{d}{2}, \frac{d}{2}\}} = -\frac{1}{d-\epsilon} \left[p_2^2 I_{\frac{d}{2}+\epsilon\{\frac{d}{2}-1, \frac{d}{2}\}} + p_3^2 I_{\frac{d}{2}+\epsilon\{\frac{d}{2}, \frac{d}{2}-1\}} + I_{\frac{d}{2}+\epsilon\{\frac{d}{2}, \frac{d}{2}, \frac{d}{2}\}}\right].$$

(2.6.15)

If one denotes $d = d' + 2$, then the integrals featuring in this expression can be obtained from the integral (2.6.14) in dimension $d'$ as follows

$$I_{\frac{d}{2}+\epsilon\{\frac{d}{2}-1, \frac{d}{2}\}} = I_{\frac{d'}{2}+1+\epsilon\{\frac{d'}{2}, \frac{d'}{2}+1, \frac{d'}{2}+1\}} = \left(d' - p_2 \frac{\partial}{\partial p_2}\right) \left(d' - p_3 \frac{\partial}{\partial p_3}\right) I_{\frac{d}{2}-1+\epsilon\{\frac{d'}{2}, \frac{d'}{2}, \frac{d'}{2}\}}.$$  

(2.6.16)

This shows that the recursive use of (2.6.16) and (2.6.7) allows analytic expressions to be found for all triple-K integrals required for the evaluation of 3-point functions of conserved currents and the stress-energy tensor in arbitrary even dimension $d \geq 4$. Visually, the procedure adds new columns to the left side of table 2.1 with lower values of $\alpha - \beta_t$. One can move down the column through the repeated use of (2.6.7), and the starting entry in each column is (2.6.14). In summary, one can find all required integrals starting from a single master integral $I_{0\{111\}}$ which we will evaluate analytically in the following section.

**2.6.4. Evaluation of the master integral**

In this section we present a method to evaluate integrals of the form $I_{\nu+1\{\nu\nu\nu\}}$, $\nu \in \mathbb{R}$. In particular, if we choose $\nu = -1$ to evaluate $I_{0\{-1-1-1\}}$, using (2.6.4) then gives

$$I_{0\{111\}} = p_1^2 p_2^2 p_3^2 I_{0\{-1-1-1\}}$$

(2.6.17)

which is the master integral we will need for the analysis of conserved currents and stress tensors.

In appendix 2.A.5 we also present expressions for $I_{2\{111\}}$ and $I_{1\{000\}}$. In particular, $I_{1\{000\}}$ is convergent and finite and is known in the literature, e.g., [52, 69].

Let us now evaluate the master integral $I_{0\{111\}}$. We will start with the more general problem of evaluating integrals of the form $I_{\nu+1\{\nu\nu\nu\}}$, $\nu \in \mathbb{R}$. To write the results in compact form, we introduce the following variables. First, we define

$$J^2 = (p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(-p_1 + p_2 + p_3)(p_1 + p_2 + p_3)$$

(2.6.18)

$$= -p_1^4 - p_2^4 - p_3^4 + 2p_1^2 p_2^2 + 2p_1 p_3^2 + 2p_2 p_3^2$$

$$= 4 \left[ p_1^2 p_2^2 - (p_1 \cdot p_2)^2 \right] = 4 \cdot \text{Gram}(p_1, p_2),$$

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where Gram is the Gram determinant. For physical momentum configurations obeying the triangle inequalities we have $J^2 \geq 0$, with $J^2 = 0$ holding if and only if the momenta are collinear. Next, we define

$$X = \frac{p_1^2 - p_2^2 - p_3^2 + \sqrt{-J^2}}{2p_2p_3}, \quad Y = \frac{p_2^2 - p_1^2 - p_3^2 + \sqrt{-J^2}}{2p_1p_3},$$

$$Z = \frac{p_3^2 - p_1^2 - p_2^2 - \sqrt{-J^2}}{2p_1p_2}. \quad (2.6.19)$$

Note that $X$ and $Y$ are symmetric under $p_1 \leftrightarrow p_2$, while the sign in front of the square root in $Z$ is different to that in $X$ and $Y$. For physical momentum configurations these variables are complex, although the entire expression for a given triple-$K$ integral remains real.

Starting with the representation (2.6.19) of the triple-$K$ integral, one can use the reduction formulae (2.6.40) - (2.6.43) in order to find

$$I_{\nu+1\{\nu\nu\nu\}} = \frac{2^{\nu-2}\Gamma(\nu)\pi}{\sin(\pi\nu)} \left[ \frac{p_3^{2\nu}}{p_1p_2}Z \cdot F_\nu(Z^2) + \frac{p_2^{2\nu}}{p_1p_3}Y \cdot F_\nu \left( \frac{Z^2}{X} \right) + \frac{p_1^{2\nu}}{p_2p_3}X \cdot F_\nu \left( \frac{Z^2}{Y} \right) \right]$$

$$+ \frac{2^{3\nu-2}\pi\Gamma(\nu + 1/2)}{\sin^2(\pi\nu)} (p_1p_2p_3)^{2\nu}(\sqrt{-J^2})^{-(2\nu+1)}. \quad (2.6.20)$$

where

$$F_\nu(x) = 2F_1(1, \nu + 1; 1 - \nu; x) \quad (2.6.21)$$

and the $X, Y, Z$ variables are defined in (2.6.19) while $J^2$ is given by (2.6.18). Note that this particular combination of parameters in the hypergeometric function appears in Legendre functions.

For generic values of $\nu$ the expression (2.6.20) is finite. In order to evaluate $I_{0\{111\}}$, however, we require $\nu = -1$ (see (2.6.17)) where (2.6.20) has singularities. In cases such as this, (2.6.20) may be series expanded in $\nu$. The relevant expansion of the hypergeometric function here is

$$F_{-1+\epsilon}(x) = 1 + F_{-1}^{(1)}(x)\epsilon + F_{-1}^{(2)}(x)\epsilon^2 + O(\epsilon^3), \quad (2.6.22)$$

where

$$F_{-1}^{(1)}(x) = 1 - \left( 1 - \frac{1}{x} \right) \log(1 - x), \quad (2.6.23)$$

$$F_{-1}^{(2)}(x) = 2 + \left( 1 - \frac{1}{x} \right) \left[ -\log(1 - x) + \log^2(1 - x) + \text{Li}_2 x \right]. \quad (2.6.24)$$

Combining everything we obtain an analytic expression for the integral

$I_{0+\epsilon\{-1+\epsilon,-1+\epsilon,-1+\epsilon\}}$. Note however that this result is given in a different regularisation scheme than our usual (2.5.7). We can easily change the regularisation
2.7. Worked example: $\langle T^{\mu_1 \nu_1} J^{\mu_2} J^{\mu_3} \rangle$

scheme through a comparison of the local terms in the triple-$K$ integrals in two regularisation schemes. This can be done without actually evaluating the integrals: we simply use (2.A.21, 2.A.22) and/or (2.A.27) to expand out two of the three Bessel $K$ functions in a given triple-$K$ integral then use the formulae (2.A.51, 2.A.53, 2.A.55). In each case, we find only a finite number of terms leading to singularities.

To illustrate this, consider for example the integral $I_{2\{111\}}$. The calculations for $I_{0\{111\}}$ are essentially identical, however, due to the fact that $I_{0+\epsilon\{111\}}$ has a double pole in $\epsilon$, the resulting expressions are much longer. The integrals in the two regularization schemes are given by

$$I_{2+\epsilon\{111\}} = \int_0^\infty dx \, x^\epsilon \left[ 1 + O(x^2) \right] \frac{p_3 K_1(p_3 x)}{p_3}$$

$$I_{2+\epsilon\{1+\epsilon,1+\epsilon,1+\epsilon\}} = \int_0^\infty dx \, x^{-\epsilon} \left[ (4p_3)^\epsilon T^2(1 + \epsilon) + O(x^2) \right] \frac{p_3^{1+\epsilon} K_{1+\epsilon}(p_3 x)}{p_3}$$

Using the expansion of the Bessel $K$ functions we find that the two integrals differ by local terms:

$$I_{2+\epsilon\{111\}} = I_{2+\epsilon\{1+\epsilon,1+\epsilon,1+\epsilon\}} + \frac{3}{2\epsilon} + \frac{3}{2}(-\gamma_E + \log 2) + O(\epsilon).$$

We can obtain $I_{0+\epsilon\{-1-1-1\}}$ from $I_{0+\epsilon\{-1+\epsilon,-1+\epsilon,-1+\epsilon\}}$ in a similar way, and then use (2.6.17). The exact analytic expression for $I_{0\{111\}}$ is given in appendix 2.A.5. The only special functions appearing in the result are dilogarithms.

2.7. Worked example: $\langle T^{\mu_1 \nu_1} J^{\mu_2} J^{\mu_3} \rangle$

Now that our general method is complete, in this section we present a full worked example, the $\langle T^{\mu_1 \nu_1} J^{\mu_2} J^{\mu_3} \rangle$ correlation function. Here we will take $J^\mu$ to be a conserved $U(1)$ current; more general results are listed in chapter 3. This correlator provides a useful test case as, while more complex than the $\langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} \rangle$ correlator we used to illustrate the method in earlier sections, it is nonetheless simpler than correlators with more stress-energy tensors.

We will also discuss the complete evaluation of all integrals in both $d = 3$ and $d = 4$ and present a concrete model, free fermions, where these correlators can be explicitly computed by standard Feynman diagrams. These results provide a nontrivial consistency check on our method.

2.7.1. Primary conformal Ward identities

We start with the analysis of primary CWIs for the $\langle T^{\mu_1 \nu_1} J^{\mu_2} J^{\mu_3} \rangle$ correlation function in general Euclidean dimension $d$. For the decomposition of the transverse-
traceless part of $\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3}\rangle\rangle$ we follow the analysis of section 2.3.1. The decomposition consists of four form factors,

$$
\langle\langle T^{\mu_1\nu_1} (p_1) J^{\mu_2} (p_2) J^{\mu_3} (p_3) \rangle\rangle = \Pi^{\mu_1\nu_1}_{\alpha_1\beta_1} (p_1) \pi^{\mu_2}_{\alpha_2} (p_2) \pi^{\mu_3}_{\alpha_3} (p_3) \left[ A_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_1^{\alpha_3} + A_2 \delta^{\alpha_2 \alpha_3} p_2^{\alpha_1} p_2^{\beta_1} \\
+ A_3 \delta^{\alpha_1 \alpha_2} p_2^{\beta_1} p_1^{\alpha_3} + A_3 (p_2 \leftrightarrow p_3) \delta^{\alpha_1 \alpha_3} p_2^{\beta_1} p_3^{\alpha_2} \\
+ A_4 \delta^{\alpha_1 \alpha_3} \delta^{\alpha_2 \beta_1} \right].
$$

(2.7.1)

Here, $p_2 \leftrightarrow p_3$ denotes exchange of the arguments $p_2$ and $p_3$, i.e., $A_3 (p_2 \leftrightarrow p_3) = A_3 (p_1, p_3, p_2)$. If on the other hand no arguments are given then the standard ordering is assumed, i.e., $A_3 = A_3 (p_1, p_2, p_3)$. Note that the form factors $A_1$, $A_2$ and $A_4$ are symmetric under $p_2 \leftrightarrow p_3$,

$$
A_j (p_1, p_3, p_2) = A_j (p_1, p_2, p_3), \quad j \in \{1, 2, 4\},
$$

(2.7.2)

while the form factor $A_3$ does not exhibit any symmetry properties.

Next, the primary CWIs can be extracted by means of the procedure described in section 2.4.3. These CWIs are

$$
\begin{align*}
K_{12} A_1 &= 0, & K_{13} A_1 &= 0, \\
K_{12} A_2 &= -2 A_1, & K_{13} A_2 &= -2 A_1, \\
K_{12} A_3 &= 0, & K_{13} A_3 &= 4 A_1, \\
K_{12} A_4 &= 2 A_3, & K_{13} A_4 &= 2 A_3 (p_2 \leftrightarrow p_3).
\end{align*}
$$

(2.7.3)

The solution follows from the analysis of section 2.5.2,

$$
\begin{align*}
A_1 &= \alpha_1 J_{4(000)}, \\
A_2 &= \alpha_1 J_{3(100)} + \alpha_2 J_{2(000)}, \\
A_3 &= 2 \alpha_1 J_{3(001)} + \alpha_3 J_{2(000)}, \\
A_4 &= 2 \alpha_1 J_{2(011)} + \alpha_3 (J_{1(010)} + J_{1(001)}) + \alpha_4 J_{0(000)}.
\end{align*}
$$

(2.7.4)

### 2.7.2. Evaluation of secondary conformal Ward identities

The independent secondary CWIs for $\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3}\rangle\rangle$ are listed in chapter 3 and read

$$
\begin{align*}
\text{L}_{2,2} A_1 + R_2 [A_3 - A_3 (p_2 \leftrightarrow p_3)] &= 2d \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle\langle T^{\mu_1\nu_1} (p_1) J^{\mu_2} (p_2) J^{\mu_3} (p_3) \rangle\rangle, \\
\text{L}_{1,2} A_1 + 2 R_1' [A_3 - A_2] &= 2d \cdot \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_1^{\mu_3} \text{ in } p_{2\mu_2} \langle\langle T^{\mu_1\nu_1} (p_1) J^{\mu_2} (p_2) J^{\mu_3} (p_3) \rangle\rangle, \\
\text{L}_{2,0} A_2 - p_1^2 [A_3 - A_3 (p_2 \leftrightarrow p_3)] &= 2d \cdot \text{coefficient of } \delta^{\mu_2 \mu_3} p_2^{\mu_1} \text{ in } p_{1\nu_1} \langle\langle T^{\mu_1\nu_1} (p_1) J^{\mu_2} (p_2) J^{\mu_3} (p_3) \rangle\rangle.
\end{align*}
$$

(2.7.5) (2.7.6) (2.7.7)
L_{2,2} A_3 - 2 R_2 A_4
\quad = 4d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle, \quad (2.7.8)

where L and R operators are defined in (2.4.27) and (2.4.28). They can be obtained by the procedure outlined in section 2.4.3. Note that there are four primary constants and four secondary CWIs. As some of the secondary CWIs are trivially satisfied, however, not all four of the primary constants are fixed, as we expect from the position space analysis [22]. Secondary CWIs that are trivially satisfied are denoted by asterisks in chapter 3 (for example (2.7.5) above is of this type).

Before solving the secondary CWIs, we must simplify the semi-local terms appearing on their right-hand sides. Differentiating (1.3.33, 1.3.34, 1.3.42) we find the following transverse and trace Ward identities,

\begin{equation}
\begin{aligned}
p_1^{\mu_1} \langle T_{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle &= \\
&= p_1^{\mu_1} \langle \frac{\delta T_{\mu_1\nu_1}}{\delta A_{\mu_3}} (p_1, p_3) J^{\mu_2}(p_2) \rangle + p_1^{\mu_1} \langle \frac{\delta T_{\mu_1\nu_1}}{\delta A_{\mu_3}} (p_1, p_2) J^{\mu_3}(p_3) \rangle \\
&\quad - p_3^{\mu_1} \langle J^{\mu_2}(p_2) J^{\mu_3}(-p_3) \rangle - p_2^{\mu_1} \langle J^{\mu_2}(p_3) J^{\mu_3}(-p_3) \rangle \\
&\quad + \delta^{\mu_3 p_{3\alpha}} \langle J^{\mu_2}(p_2) J^{\alpha}(-p_2) \rangle + \delta^{\mu_2 p_{2\alpha}} \langle J^{\mu_2}(p_3) J^{\mu_3}(-p_3) \rangle, \quad (2.7.9)
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
p_2^{\mu_2} \langle T_{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle &= \\
&= 2p_2^{\mu_2} \langle \frac{\delta J^{\mu_2}}{\delta g_{\mu_1\nu_1}} (p_2, p_1) J^{\mu_3}(p_3) \rangle + \delta_{\mu_1\nu_1} p_{1\alpha} \langle J^{\alpha}(p_3) J^{\mu_3}(-p_3) \rangle, \quad (2.7.10)
\end{aligned}
\end{equation}

\begin{equation}
\langle T(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle = \langle \frac{\delta T}{\delta A_{\mu_2}} (p_1, p_2) J^{\mu_3}(p_3) \rangle + \langle \frac{\delta T}{\delta A_{\mu_3}} (p_1, p_3) J^{\mu_2}(p_2) \rangle. \quad (2.7.11)
\end{equation}

In the next section we will extract algebraic equations between the primary constants by taking the zero-momentum limit \( p_3 \to 0 \). The details of this procedure are described in section 2.5.3. We will find that in the zero-momentum limit the right-hand sides of the secondary CWIs (2.7.5) - (2.7.7) are given by

\begin{equation}
\lim_{p_3 \to 0, p_1 = p_2 = p} \text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_1^{\mu_3} \text{ in } p_{2\mu_2} \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle = 0, \quad (2.7.12)
\end{equation}

\begin{equation}
\lim_{p_3 \to 0, p_1 = p_2 = p} \text{coefficient of } p_2^{\mu_1} p_2^{\mu_2} p_1^{\mu_3} \text{ in } p_{1\nu_1} \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle = 0, \quad (2.7.13)
\end{equation}

\begin{equation}
\lim_{p_3 \to 0, p_1 = p_2 = p} \text{coefficient of } \delta^{\mu_2\mu_3} p_1^{\mu_1} \text{ in } p_{1\nu_1} \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle = \\
\quad = \text{coefficient of } \delta^{\mu_2\mu_3} \text{ in } \langle J^{\mu_2}(p) J^{\mu_3}(-p) \rangle. \quad (2.7.14)
\end{equation}

Let us start with the first result (2.7.12). Due to conformal invariance, the only operators in \( \delta J^{\mu_2}/\delta g^{\mu_1\nu_1} \) that can give a non-vanishing result under the expectation value with the current is another current \( J^\mu \). In general, the descendants of
the current can also give a non-vanishing 2-point function with another current. In this case, however, the dilatation degree of $\delta J^\mu / \delta g^{\mu_1 \nu_1}$ is $d - 1$, and so descendants cannot appear. The most general form of the functional derivative term is therefore

$$\frac{\delta J^\mu_2}{\delta g^{\mu_1 \nu_1}} = c_1 \delta_{\mu_1 \nu_1}^2 J^\mu_2 + c_2 \delta_{(\mu_1}^2 J_{\nu_1)} + \ldots$$  \tag{2.7.15}$$

where $c_1$ and $c_2$ are numerical constants and the omitted terms may contain operators from different conformal families to that of $J^\mu$. The 2-point function then reads

$$\langle \langle \delta J^\mu_2 \delta g_{\mu_1 \nu_1} (p_2, p_1) J^{\nu_3} (p_3) \rangle \rangle = \left[ c_1 \delta_{\mu_1 \nu_1}^2 J_{\nu_3} + c_2 \delta_{(\mu_1}^2 \delta_{\nu_1)}^2 J_{\nu_3} \right] \langle \langle J^\alpha (p_3) J^{\nu_3} (-p_3) \rangle \rangle.$$  \tag{2.7.16}$$

In the limit $p_3 \to 0$, however, the 2-point function vanishes, since it behaves as $p_3^{d-2}$ and $d > 2$. The same argument works for the second term in (2.7.10) and so (2.7.12) also vanishes.

Let us now establish the remaining formulae (2.7.13) and (2.7.14). Following the same argument for the limit $p_3 \to 0$, we can restrict consideration to the following terms in (2.7.9)

$$p_{\mu_1}^\nu \langle \langle \frac{\delta T_{\mu_1 \nu_1}}{\delta A_{\mu_3}} (p_1, p_3) J^{\nu_3} (p_2) \rangle \rangle - p_{\mu_2}^\nu \langle \langle J^{\nu_3} (p_2) J^{\nu_3} (-p_2) \rangle \rangle - \delta_{\mu_2}^\nu p_{3 \nu} \langle \langle J^{\nu_3} (p_2) J^{\nu_3} (-p_2) \rangle \rangle.$$  \tag{2.7.17}$$

Using the representation (2.5.55) it is straightforward to expand the last two terms. As usual, we must use the convention (2.3.4) for the momenta associated with Lorentz indices, leading to the right-hand sides of (2.7.13) and (2.7.14). The remaining task is then to show that there are no contributions from the first term with the functional derivative.

Since the dimension of the stress-energy tensor is $d$, while that of the conserved current is $d - 1$ and that of the source $A_\mu$ is 1, the only possible contributions to the first term in (2.7.9) are

$$T_{\mu \nu} = c_3 [A_\mu J_\nu + A_\nu J_\mu] + \ldots$$  \tag{2.7.18}$$

where $c_3$ is a numerical constant and the omitted terms do not contain the current or its descendants. This definition of $c_3$ applies if the $J^\mu$ operator is the unique spin-1 conserved current in theory. If not, we can instead define the constant $c_3$ through the 2-point function

$$\langle \langle \frac{\delta T_{\mu_1 \nu_1}}{\delta A_{\mu_2}} (p_1, p_2) J^{\nu_3} (p_3) \rangle \rangle = 2c_3 \delta_{(\mu_1}^\nu_3 \langle \langle J_{\nu_1)} (p_3) J^{\nu_3} (-p_3) \rangle \rangle.$$  \tag{2.7.19}$$

After taking the functional derivative one finds that tensors $p_{\mu_1}^{\mu_2} p_{\mu_2}^{\mu_3}$ and $\delta_{\mu_2 \mu_3}$ are absent in (2.7.9).
Finally, with the definition of the $c_3$ constant as in (2.7.19), the same method can be applied to work out the zero-momentum limit of the right-hand side of the final secondary CWI (2.7.8), yielding the result

$$\lim_{p_3 \to 0} \text{coefficient of } \delta^{\mu_1 \mu_2} p_1^\mu p_3^\nu \text{ in } \langle \langle T^{\mu_1 \nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle \rangle = c_3 \cdot \text{coefficient of } \delta^{\mu_1 \mu_2} \text{ in } \langle \langle J^{\mu_1}(p) J^{\mu_2}(-p) \rangle \rangle. \quad (2.7.20)$$

### 2.7.3. Solutions to secondary conformal Ward identities

Our goal now is to analyse the additional constraints imposed by the secondary CWIs (2.7.5) - (2.7.8) on the solution (2.7.4) of the primary CWIs. We proceed as in sections 2.5.3 and 2.5.3. First, we use solutions (2.7.4) and take the zero-momentum limit $p_3 \to 0$. From (2.7.6) and (2.7.7), we then derive the following equations for the primary constants

$$\alpha_2 = \alpha_3, \quad \alpha_2 = \alpha_3, \quad (2.7.21)$$

$$d\alpha_1 + \alpha_2 = \frac{2^{3-d}}{\Gamma\left(\frac{d}{2}\right)} s_d c_J, \quad (2.7.22)$$

where

$$s_d = \begin{cases} \frac{1}{\pi} (-1)^{d-1} & \text{if } d = 3, 5, 7, \ldots, \\ (-1)^{\frac{d}{2}-1} & \text{if } d = 4, 6, 8, \ldots \end{cases} \quad (2.7.23)$$

and $c_J$ encodes the normalisation of the 2-point function as given in (2.5.55). For the right-hand sides we used (2.7.12) - (2.7.14).

The situation is more interesting for the last of the secondary CWIs (2.7.8). Assuming $d$ to be odd, we find this equation exhibits a singularity and the $\epsilon \to 0$ limit cannot be taken. Expanding in powers of $\epsilon$, the zero-momentum limit of the left-hand side is

$$\left(\frac{2}{\epsilon} - \log p^2\right) \cdot \frac{2^{\frac{d}{2}-3} d p^{d-2} \pi}{\sin\left(\frac{d \pi}{2}\right) \Gamma\left(\frac{d}{2} - 1\right)} \left[\alpha_4 + (-2 + d)\alpha_3 \right] + \ldots \quad (2.7.24)$$

where the omitted terms are of order $\epsilon^0$. Since the right-hand side is finite and does not contain logarithms, (2.7.8) forces

$$\alpha_4 = (2 - d)\alpha_3 + O(\epsilon). \quad (2.7.25)$$

With this value of $\alpha_4$, the left-hand side of (2.7.8) is finite, but does not necessarily match the right-hand side. To solve this problem we must consider a first-order correction to $\alpha_4$, i.e., we write

$$\alpha_4 = (2 - d)\alpha_3 + \epsilon\alpha_4^{(1)}. \quad (2.7.26)$$

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Using (2.7.20), the equation for $\alpha^{(1)}_4$ leads to

$$2d(d-2)\alpha_1 + d\alpha_3 + 2\alpha^{(1)}_4 = \frac{2^{5-\frac{d}{2}} s d c_3 c J}{\Gamma(\frac{d}{2} - 1)}. \quad (2.7.27)$$

This equation is valid for even $d$ as well: in this case we find that the left-hand side of (2.7.8) contains a $1/\epsilon^2$ singularity. As there is no corresponding singularity on the right-hand side we recover the same conditions (2.7.25) and (2.7.27) as above.

Summarising, we found that the primary constants in the solution to the primary CWIs (2.7.4) satisfy

$$\alpha_3 = \alpha_2, \quad \alpha_4 = -(d-2)\alpha_2$$

$$d\alpha_1 + \alpha_2 = \frac{2^{3-\frac{d}{2}} s d c J}{\Gamma(\frac{d}{2})},$$

$$2d(d-2)\alpha_1 + d\alpha_2 + 2\alpha^{(1)}_4 = \frac{2^{5-\frac{d}{2}} s d c_3 c J}{\Gamma(\frac{d}{2} - 1)}. \quad (2.7.28)$$

Through an analysis similar to that of section 2.5.3, we find there are no further constraints on the primary constants.

Our solution of the primary and secondary CWIs above depends on one undetermined primary constant as well as two different 2-point function normalisations. This result is in fact consistent with the position space result of [22] (which involves only a single 2-point function normalisation) by virtue of our different definition for the 3-point function, namely

$$\langle T_{\mu_1\nu_1}(x_1) J^{\mu_2}(x_2) J^{\mu_3}(x_3) \rangle =$$

$$= \frac{-1}{\sqrt{g(x_3)}} \frac{\delta}{\delta A_{\mu_3}(x_3)} \frac{-1}{\sqrt{g(x_2)}} \frac{\delta}{\delta A_{\mu_2}(x_2)} \frac{-2}{\sqrt{g(x_1)}} \frac{\delta}{\delta g^{\mu_1\nu_1}(x_1)} Z[g^{\mu\nu}, A_\rho]$$

$$+ \left( \frac{\delta T_{\mu_1\nu_1}(x_1)}{\delta A_{\mu_2}(x_2)} J^{\mu_3}(x_3) \right) + \left( \frac{\delta T_{\mu_1\nu_1}(x_1)}{\delta A_{\mu_3}(x_3)} J^{\mu_2}(x_2) \right). \quad (2.7.29)$$

In [22] (and similarly [47, 49]) the semi-local terms on the right-hand side of this formula are absorbed into the definition of the $\langle T_{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle$ correlator: it is these semi-local terms that are responsible, via (2.7.19), for the dependence of our solution on the additional normalisation constant $c_3$.

### 2.7.4. General form of $\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle$ in $d = 3$

Let us now focus on the special case of $d = 3$. Examining the form of the solution (2.7.4) to the primary CWIs, we find that all triple-\(K\) integrals can be evaluated in terms of elementary integrals using (2.7.20). If an integral diverges, we use the
2.7. **Worked example:** $\langle T^{\mu_1 \nu_1} J^{\mu_2} J^{\mu_3} \rangle$

regularisation (2.5.7). In this way, we find

$$J_{4\{000\}} = I_{\frac{3}{4} \{\frac{3}{4} \frac{3}{4} \frac{3}{4} \}} = 2 \left( \frac{\pi}{2} \right)^{\frac{3}{2}} \frac{4p_1 + p_2 + p_3}{(p_1 + p_2 + p_3)^4},$$

$$J_{3\{100\}} = I_{\frac{3}{4} \{\frac{3}{4} \frac{3}{4} \frac{1}{4} \}} = \left( \frac{\pi}{2} \right)^{\frac{3}{2}} \frac{9(p_1 p_2 + p_1 p_3) + 6p_2 p_3 + 8p_1^2 + 3(p_2^2 + p_3^2)}{(p_1 + p_2 + p_3)^3},$$

$$J_{2\{000\}} = I_{\frac{3}{4} \{\frac{3}{4} \frac{3}{4} \}} = \left( \frac{\pi}{2} \right)^{\frac{3}{2}} \frac{2p_1 + p_2 + p_3}{(p_1 + p_2 + p_3)^2},$$

$$J_{2\{011\}} = I_{\frac{3}{4} \{\frac{3}{4} \frac{1}{4} \frac{3}{4} \}} = -\left( \frac{\pi}{2} \right)^{\frac{3}{2}} \frac{1}{(p_1 + p_2 + p_3)^2} \left[ 2p_1 p_2 p_3 + p_1^3 + p_2^3 + p_3^3 + 2(p_1^2 p_2 + p_1 p_2^2 + p_1 p_3^2 + p_3 p_1^2 + p_2 p_3^2 + p_3 p_2^2) \right],$$

$$J_{1+\epsilon\{010\}} = I_{\frac{3}{4} + \epsilon \{\frac{3}{4} \frac{3}{4} \frac{1}{4} \}} = \left( \frac{\pi}{2} \right)^{\frac{3}{2}} \left[ -\frac{p_3}{\epsilon} + p_3 \log(p_1 + p_2 + p_3) \right. \left. + \frac{-p_1 p_2 + (\gamma E - 2)(p_1 p_3 + p_2 p_3) - p_1^2 - p_2^2 + (\gamma E - 1)p_3^2}{p_1 + p_2 + p_3} + O(\epsilon) \right],$$

$$J_{0+\epsilon\{000\}} = I_{\frac{3}{4} + \epsilon \{\frac{3}{4} \frac{3}{4} \}} = \left( \frac{\pi}{2} \right)^{\frac{3}{2}} \left[ -\frac{p_2 + p_3}{\epsilon} + (p_2 + p_3) \log(p_1 + p_2 + p_3) \right. \left. + (\gamma E - 1)(p_2 + p_3) - p_1 + O(\epsilon) \right],$$

(2.7.30)

with similar integrals following from the permutation formula (2.5.11).

Applying the secondary CWIs (2.7.28) we then obtain the final result

$$A_1 = \alpha_1 \frac{2(4p_1 + p_2 + p_3)}{(p_1 + p_2 + p_3)^4},$$

$$A_2 = \frac{2\alpha_1 p_1^2}{(p_1 + p_2 + p_3)^3} - \frac{2(2p_1 + p_2 + p_3)}{(p_1 + p_2 + p_3)^2} c_J,$$

$$A_3 = \frac{\alpha_1}{(p_1 + p_2 + p_3)^3} \left[ -2p_1^2 - p_2^2 + p_3^2 - 3p_1 p_2 + 3p_1 p_3 \right] - \frac{2(2p_1 + p_2 + p_3)}{(p_1 + p_2 + p_3)^2} c_J,$$

$$A_4 = \alpha_1 \frac{(p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(2p_1 + p_2 + p_3)}{2(p_1 + p_2 + p_3)^2} + \left( \frac{2p_1^2}{p_1 + p_2 + p_3} - p_2 - p_3 \right) c_J + 2(p_2 + p_3) c_3 c_J.$$ 

(2.7.31)

In these results we rescaled the coefficient $\alpha_1$ according to $\alpha_1 (\pi/2)^{3/2} \mapsto \alpha_1$, so as to remove the awkward factor of $(\pi/2)^{3/2}$.

The form factors build the transverse-traceless part of the correlation function according to (2.7.1). The full correlation function can then be recovered by means
of (2.3.17) and (2.3.18). Using the transverse and trace Ward identities (2.7.9, 2.7.10, 2.7.11), we find

$$
\langle \langle T^\mu_1 J^{\mu_2}(p_1) J^{\mu_3}(p_3) \rangle \rangle = \langle \langle t^\mu_1(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_3) \rangle \rangle
$$

$$
+ \frac{2}{p_3^2} \delta^{\mu_1 \mu_3} \frac{p_3^{\alpha_1}}{p_3^{\beta_1}} \langle \langle \frac{1}{\alpha_1 \beta_1} (p_3, p_1) J^{\mu_2}(p_2) \rangle \rangle
$$

$$
+ \text{everything with } (p_2, \mu_2) \leftrightarrow (p_3, \mu_3), \quad (2.7.32)
$$

where $\mathcal{T}^{\mu \nu}_\alpha$ was given in (2.3.19). Here we assume no scale anomalies are present: if anomalies occur, the additional ultralocal contribution (2.8.16) should be added to (2.7.32), as we will discuss in section 2.8.

The result (2.7.32) is the most general explicit expression for the $\langle \langle T^{\mu_1 \nu_1} J^{\mu_2} J^{\mu_3} \rangle \rangle$ correlation function in the momentum space. As we can see, it depends on one undetermined primary constant plus the normalisations of the 2-point functions.

### 2.7.5. Free fermions in $d = 3$

As a cross-check on our calculations we now consider free fermions in $d = 3$ Euclidean dimensions given by the action

$$
S = \int d^3 x \ e \left[ \bar{\psi} \gamma^\mu e a \gamma^a D_\mu \psi \right], \quad (2.7.33)
$$

where

$$
D_\mu = \nabla_\mu - i A_\mu, \quad \nabla_\mu = \partial_\mu - \frac{i}{2} \omega^{ab}_\mu \Sigma_{ab}, \quad (2.7.34)
$$

and $\omega^{ab}_\mu$ is the spin connection

$$
\omega^{ab}_\mu = e^a_\nu \partial_\mu e^{vb} + e^a_\nu e^{\sigma b} \Gamma^{\nu}_{\sigma \mu}, \quad \Sigma^{ab} = \frac{i}{4} [\gamma^a, \gamma^b]. \quad (2.7.35)
$$

Here $\Gamma^{\nu}_{\sigma \mu}$ is the Christoffel symbol associated with the metric $g_{\mu \nu}$, while $e^a_\mu$ are vielbeins satisfying $e^a_\mu e^a_{\nu a} = g_{\mu \nu}$ and the gamma matrices $\gamma^a$ satisfy $\gamma^a = e^a_\mu \gamma^a$. On flat space, we then have $\{\gamma^a, \gamma^b\} = -2 \delta^{ab}$. In $d = 3$, the spin-$\frac{1}{2}$ representation of the group $SO(3)$ is 2-dimensional and $\text{Tr}(\gamma^a \gamma^b) = -2 \delta^{ab}$.

Notice that the gauge field $A_\mu$ is treated as a source for the conserved current and is not a degree of freedom. The stress-energy tensor and the conserved current
in the presence of the sources are
\[ T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} = \bar{\psi} \gamma_\mu (\not{D}_\nu) \psi - g_{\mu\nu} \bar{\psi} \gamma_\alpha \not{D}_\alpha \psi, \] (2.7.36)
\[ J^\mu = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta A^\mu} = -\bar{\psi} \gamma^\mu \psi. \] (2.7.37)

In this case the current is associated with the \( U(1) \) symmetry, therefore we omit the group indices on \( J^\mu \). By direct calculation we find
\[ \langle \langle J^\mu (p) J^\nu (-p) \rangle \rangle = -\frac{1}{16} p_{\pi}^{\mu\nu} (p), \] (2.7.38)
\[ \langle \langle T^{\mu_1 \nu_1} (p) T^{\mu_2 \nu_2} (-p) \rangle \rangle = \frac{1}{128} p^3 \Pi^{\mu_1 \nu_1 \mu_2 \nu_2} (p). \] (2.7.39)

The transverse Ward identities can be obtained by differentiation of the equations (1.3.33, 1.3.34) and are listed in chapter 3. Some terms of the terms involve functional derivatives and may be evaluated directly from expressions (2.7.36, 2.7.37),
\[ \frac{\delta T_{\mu\nu}(x)}{\delta A_\rho(y)} = \frac{1}{2} [J_\mu \delta_\rho^\nu + J_\nu \delta_\rho^\mu - 2 J^\rho \delta_{\mu\nu}] \delta(x - y), \] (2.7.40)
\[ \frac{\delta J^\mu(x)}{\delta g^{\alpha\beta}(y)} = \frac{1}{4} [J_\beta \delta_\alpha^\mu + J_\alpha \delta_\beta^\mu] \delta(x - y), \] (2.7.41)

where the sources are turned off after the derivative is taken. All together, for this particular CFT we find
\[ c_J = -\frac{1}{16}, \quad c_T = \frac{1}{128}, \quad c_3 = \frac{1}{2}. \] (2.7.42)

where the 2-point function normalisations \( c_J \) and \( c_T \), and the constant \( c_3 \), are as defined in (2.5.55), (2.5.56) and (2.7.19) respectively.

The 3-point function can be calculated by the usual Feynman rules. Using the results of section 2.3.3, one finds
\[ A_1 = -\frac{4p_1 + p_2 + p_3}{12 (p_1 + p_2 + p_3)^4}, \]
\[ A_2 = \frac{9(p_1 p_2 + p_1 p_3) + 6p_2 p_3 + 4p_1^2 + 3(p_2^2 + p_3^2)}{24 (p_1 + p_2 + p_3)^3}, \]
\[ A_3 = \frac{6p_1 p_2 + 3p_1 p_3 + 3p_2 p_3 + 4p_1^2 + 2p_2^2 + p_3^2}{12 (p_1 + p_2 + p_3)^3}, \]
\[ A_4 = -\frac{4p_1 p_2 p_3 + 7(p_1^2 p_2 + p_1^2 p_3) - 2(p_1 p_2^2 + p_1 p_3^2) + p_2 p_3^2 + p_3 p_2^2 + 8p_1^3 - (p_2^3 + p_3^3)}{48 (p_1 + p_2 + p_3)^2}. \] (2.7.43)
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The form factors \( A_j \) are defined in the decomposition (2.7.1).

We can compare this result directly with the solution (2.7.31). Since we know the 2-point function normalisations (2.7.42) there is only one undetermined constant, \( \alpha_1 \). The solution (2.7.43) then fits perfectly with \( \alpha_1 = -\frac{1}{24} \). In fact, the secondary Ward identities provide quite a robust check on the standard QFT calculation of the 3-point function: for example, a mistake leading to the overall rescaling of all form factors in (2.7.43) by some factor would immediately lead to an inconsistency with the 2-point function normalisation constants (2.7.42).

2.7.6. General form of \( \langle T^{\mu_1 \nu_1} J^{\mu_2} J^{\mu_3} \rangle \) in \( d = 4 \)

Using the reduction scheme of section 2.6, we can write down the most general form of \( \langle T^{\mu_1 \nu_1} J^{\mu_2} J^{\mu_3} \rangle \) in \( d = 4 \). Starting from the solutions (2.7.4) and (2.7.28) for the primary and secondary CWIs, using the regularisation scheme (2.5.7) and the relations in table 2.1, page 94, after removing divergences we find

\[
A_1 = \alpha_1 I_{5(211)},
\]

\[
A_2 = -\left( 2c_J + \alpha_1 p_1 \frac{\partial}{\partial p_1} \right) I_{3(211)}^{(0)},
\]

\[
A_3 = -2 \left( c_J + \alpha_1 p_3 \frac{\partial}{\partial p_3} \right) I_{3(211)}^{(0)},
\]

\[
A_4 = 2c_J \left[ -2 + p_2 \frac{\partial}{\partial p_2} + p_3 \frac{\partial}{\partial p_3} \right] I_{1(211)}^{(0)} + 2\alpha_1 p_2 p_3 \frac{\partial^2}{\partial p_2 \partial p_3} I_{1(211)}^{(0)}
\]

\[
+ 4(c_J - c_J c_J) \left[ p_2^2 \log p_2 + p_3^2 \log p_3 - (p_2^2 + p_3^2) \left( \frac{1}{2} - \gamma_E + \log 2 \right) - \frac{1}{2} p_1^2 \right],
\]

where \( I_{\alpha+\epsilon(\beta_j)}^{(0)} \) denotes the coefficient of \( \epsilon^0 \) in the series expansion of the regulated integral \( I_{\alpha+\epsilon(\beta_j)} \) in \( \epsilon \). This is the exact, finite and fully renormalised result for the transverse-traceless part of the correlation function. The triple-K integrals appearing can be straightforwardly evaluated using the reduction scheme in table 2.1 on page 94 starting from the master integral \( I_{0(111)} \) given in appendix 2.A.5. (We have nonetheless retained the above form for its compactness.) The transverse-traceless part of the correlation function can be recovered using (2.7.1), as in the case of \( d = 3 \). The full 3-point function can then be reconstructed by means of (2.7.32). For \( d = 4 \) an anomalous contribution appears, however, due to the addition of counterterms required to render the 3-point function finite that are not of the form assumed in section 2.4.4. We will return to the treatment of anomalies shortly, in section 2.8.

2.7.7. Free fermions in \( d = 4 \)

The momentum space \( \langle T^{\mu_1 \nu_1} J^{\mu_2} J^{\mu_3} \rangle \) correlation function for free fermions in \( d = 4 \) dimensions was discussed in [47, 49]. In this section we will show how to
2.7. Worked example: $\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle$

simplify these calculations considerably using our method. We already know that the solution to the primary CWIs is given by (2.7.4). We therefore need to calculate explicitly only one primary constant, say $\alpha_1$, since the remaining constants are determined by the secondary CWIs (2.7.28).

To evaluate $\alpha_1$ we can use the standard Feynman parametrisation, which gives

$$\text{coeff. of } p_1^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_1^{\mu_3} \text{ in } \langle\langle T^{\mu_1\nu_1}(p_1) J^{\mu_2}(p_2) J^{\mu_3}(p_1) \rangle\rangle = -\frac{2}{\pi^2} \int_{[0,1]^3} \frac{x_1 x_2 x_3^2}{D},$$

(2.7.45)

where

$$dX = dx_1 dx_2 dx_3 \delta(x_1 + x_2 + x_3 - 1),$$

(2.7.46)

$$D = p_2^2 x_2 x_3 + p_3^2 x_1 x_3 + p_1^2 x_1 x_2.$$  (2.7.47)

On the other hand, using (4.2.23), we find

$$J_{4\{000\}} = I_{5\{211\}} = 96 \cdot \int_{[0,1]^3} dX \frac{x_1 x_2 x_3^2}{D}$$

(2.7.48)

and hence

$$\alpha_1 = -\frac{1}{48\pi^2}.$$

(2.7.49)

The 2-point function normalisation $c_J$ and the constant $c_3$ are

$$c_J = -\frac{1}{12\pi^2}, \quad c_3 = \frac{1}{2},$$

and so, from (2.7.28), we find

$$\alpha_2 = \alpha_3 = \frac{1}{4\pi^2}, \quad \alpha_4 = -\frac{1}{2\pi^2} - \frac{\epsilon}{6\pi^2}.$$  (2.7.51)

With these primary constants the expressions (2.7.44) represent a complete and concise solution to the transverse-traceless part of the $\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\rangle$ correlation function for free fermions in $d = 4$. The full 3-point function can be recovered as in the $d = 3$ case via (2.7.1) and (2.7.32).

The above solution can be confirmed by direct calculations. For free field theory we computed the entire $\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle\rangle$ correlator using Passarino-Veltman reduction [51]. The coefficients of the appropriate tensors were then extracted according to section 2.3.3 and the result compared with the Feynman parametrised integrals. Exact agreement was found, both for the functional form of (2.7.4) and for the constants in (2.7.51).

Our result can also be compared with those of [49]. Up to a multiplicative
factor, we find that

$$\text{coefficient of } p_2^{\mu_1} p_2^{\nu_1} p_3^{\mu_2} p_1^{\mu_3} \sim \int_{[0,1]^3} dX \frac{c_9(x_1, x_2) - 2c_8(x_1, x_2) + c_7(x_1, x_2)}{D} = \int_{[0,1]^3} dX \frac{4x_1 x_2 x_3^2}{D},$$

(2.7.52)

where the $c_j$ polynomials are defined in the table 3 of [49]. This expression agrees with (2.7.48) and indeed represents the form factor $A_1$.

### 2.8. Divergences and anomalies in $d = 4$

We discussed in previous sections the solution of the conformal Ward identities and we have seen that in certain cases the triple-$K$ integrals diverge and need to be regularised and renormalised. These infinities should be removed by means of local covariant counterterms. It turns out, however, that in some cases the counterterms break some of the symmetries and this leads to anomalies. We have already encountered this issue when discussing the 3-point function of scalar operators in section 2.2. There, we saw the triple-$K$ integral corresponding to the 3-point function of a dimension two operator in $d = 3$ diverges and the infinity can be removed by adding a local counterterm that is cubic in the source of the operator. The counterterm however is not scale invariant and this implies a trace anomaly. The anomaly then implies that the 4-point function of the stress-energy tensor with three scalar operators contains an ultra-local term that is not scheme-dependent because it is fixed unambiguously by the the anomalous Ward identity. Recall also that finite local counterterms are related to/parametrise scheme-dependence. In this case the same counterterm but with a finite coefficient is related to scheme-dependence.

In this section we would like to extend this discussion to correlators of the stress-energy tensor and of symmetry currents. For concreteness, we will focus on the case $d = 4$ but the discussion generalises to all dimensions and/or other operators. In particular, we will analyse the $\langle \langle T^{\mu_1 \nu_1} J^\mu_2 J^{\mu_3} \rangle \rangle$ and $\langle \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle \rangle$ correlators. We will show that the divergences in the solutions for the transverse-traceless parts are cancelled by counterterms. These counterterms break scale invariance and lead to trace anomalies. The anomalies that originate from infinities in 2-point functions lead to scheme-independent ultra-local terms in 3-point functions, while those originating in 3-point functions lead to ultra-local scheme-independent terms in 4-point functions. As in the discussion above, the counterterms but with finite coefficients are related to/parametrise scheme-dependence of these correlators.
2.8. Divergences and anomalies in $d = 4$

2.8.1. Counterterms and anomalies

In $d = 4$ the following counterterms can be introduced

$$S = S_{\text{CFT}}[g^{\mu\nu}, A^a_{\mu}] + \int d^4x \sqrt{g} \left[ \frac{\kappa_0}{4} F^{a\mu\nu} F_{\mu\nu} + a_0 E_4 + c_0 W^2 + b_0 R^2 + \tilde{b}_0 \Box R \right],$$

(2.8.1)

where $E_4$ is Euler density and $W^2$ is the square of Weyl tensor for the metric $g_{\mu\nu}$,

$$E_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2,$$

(2.8.2)

$$W^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2.$$

(2.8.3)

The regularisation scheme is a dimensional regularisation with $d = 4 - \epsilon$. Our conventions for the Riemann and Ricci tensors follow [70], see (5.1.1) and (5.1.2). The constants in the counterterm action are functions of $\epsilon$ and are typically divergent. These divergences are required to cancel the corresponding singularities in the regularised solutions of the primary and secondary CWIs so that a finite limit exists as we send $\epsilon \to 0$. The counterterms (2.8.1) also necessarily contribute finite pieces, however, leading to trace anomalies. By taking a functional derivative and using the results of [71] one finds the following anomalous contribution to the trace Ward identity (1.3.42)

$$\langle T \rangle = \epsilon \left[ \frac{\kappa_0}{4} F^{a}_{\mu\nu} F_{\mu\nu} + a_0 \left( E_4 + \frac{2}{3} \Box R \right) + c_0 W^2 + 12 b_0 \sqrt{g} \Box R \right].$$

(2.8.4)

With the usual representation for the anomalous trace of the stress-energy tensor

$$\langle T \rangle = \frac{\kappa}{4} F^{a}_{\mu\nu} F_{\mu\nu} + a E_4 + c W^2 + b \Box R,$$

(2.8.5)

we find the anomalies

$$\kappa = \kappa_0^{(-1)}, \quad a = a_0^{(-1)}, \quad c = c_0^{(-1)},$$

(2.8.6)

where $(\ldots)^{(-1)}$ denotes the term of order $\epsilon^{-1}$ in the series expansion of the coupling constant in the regulator.

2.8.2. $\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle$

For the $\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\mu_3} \rangle$ correlator only the second term in (2.8.1) contributes. The value of the constant $\kappa_0$ is fixed by a renormalisation of the 2-point function $\langle J^{\mu} J^{\nu} \rangle$. Using the dimensional regularisation $d = 4 - \epsilon$, we find the 2-point function

$$\langle J^{\mu}(p) J^{\nu}(-p) \rangle_{\text{reg}} = \pi^{\mu\nu}(p) p^2 \left( \frac{2 c^J}{\epsilon} - c^J \log p^2 - \kappa_0 \right).$$

(2.8.7)
We therefore need
\[ \kappa_0 = \frac{2c_J}{\epsilon} + O(\epsilon^0) \] (2.8.8)
fixing the anomaly coefficient in (2.8.5) to be
\[ \kappa = \kappa_0^{-1} = 2c_J. \] (2.8.9)

With the value of \( \kappa_0 \) fixed, we can check that all divergences in the 3-point function now cancel as well. This is a non-trivial check on our solutions and the singularities of the triple-\( K \) integrals. We find that the counterterm contribution to the solution (2.7.4) is
\[
\begin{align*}
A_{\text{anomaly}}^{\text{1}} &= O(\epsilon^0), \\
A_{\text{anomaly}}^{\text{2}} &= 2\kappa_0 = \frac{4c_J}{\epsilon} + O(\epsilon^0), \\
A_{\text{anomaly}}^{\text{3}} &= 2\kappa_0 = \frac{4c_J}{\epsilon} + O(\epsilon^0), \\
A_{\text{anomaly}}^{\text{4}} &= -\kappa_0(p_1^2 - p_2^2 - p_3^2) - 2c_3\kappa_0(p_2^2 + p_3^2) \\
&= -\frac{2c_J}{\epsilon}(p_1^2 - p_2^2 - p_3^2) - \frac{4c_3c_J}{\epsilon}(p_2^2 + p_3^2) + O(\epsilon^0),
\end{align*}
\] (2.8.10)
where the constant \( c_J \) is defined in (2.5.55) and \( c_3 \) is defined in (2.7.19). The appearance of \( c_3 \) is due to the our definition of the 3-point function (2.7.29).

The result (2.8.10) is very constraining: in particular, the coefficients of the singular terms in the regularised solution must be multiples of \( c_J \) and not the undetermined primary constant \( \alpha_1 \).

The divergences of the triple-\( K \) integrals entering the regularised solution can be evaluated using the method outlined in section 2.6.4, yielding
\[
\begin{align*}
I_{5(211)} &= \text{finite}, \\
I_{3+\epsilon(211)} &= \frac{2}{\epsilon} + O(\epsilon^0), \\
I_{1+\epsilon(211)} &= -\frac{p_2^2 + p_3^2}{\epsilon^2} + \frac{1}{\epsilon} \left[ p_2^2 \log p_2 + p_3^2 \log p_3 ight. \\
&\quad + (p_2^2 + p_3^2) \left( -\frac{1}{2} + \gamma_E - \log 2 \right) - \frac{1}{2} p_1^2 \left] \right].
\end{align*}
\] (2.8.11)
Substituting everything into the solution (2.7.4), one can check that all singularities cancel leaving a finite result. The scheme-dependent terms arise due to the \( O(\epsilon^0) \) ambiguity in the definition of \( \kappa_0 \) in (2.8.8). Looking at (2.8.10), we see that the scheme-dependence may only change the transverse-traceless part according to
\[
\begin{align*}
A_2 &\mapsto A_2 + 2\kappa_0^{(0)}, \\
A_3 &\mapsto A_3 + 2\kappa_0^{(0)}, \\
A_4 &\mapsto A_4 - \kappa_0^{(0)}(p_1^2 - p_2^2 - p_3^2) - 2c_3\kappa_0^{(0)}(p_2^2 + p_3^2),
\end{align*}
\] (2.8.12)
where \( \kappa_0^{(0)} \) is the \( O(\epsilon^0) \) part of \( \kappa_0 \).

Finally, one can look for anomalies in the trace of the stress-energy tensor. Due to the dimensional regularisation \( d = 4 - \epsilon \), the trace of the stress-energy tensor acquires a contribution from the counterterm

\[
\langle T \rangle = \frac{k}{4} F_{\mu\nu} F^{\mu\nu}, \quad \kappa = \kappa_0^{(-1)} = 2cJ. \tag{2.8.13}
\]

The anomalous term contributes to the trace Ward identity, which acquires a contribution

\[
\langle \langle T(p_1)J^\mu_2(p_2)J^{\mu_3}(p_3) \rangle \rangle_{\text{anomaly}} = \kappa \left[ p_2^{\mu_2} p_2^{\mu_3} - \frac{1}{2} (p_3^2 - p_2^2 - p_3^2) \delta^{\mu_2 \mu_3} \right] \tag{2.8.14}
\]
modifying the form of the total 3-point function. In arbitrary dimension, the form of the anomaly is

\[
\langle \langle T(p_1)J^\mu_2(p_2)J^{\mu_3}(p_3) \rangle \rangle_{\text{anomaly}} = \pi_2^{\mu_2}(p_2) \pi_3^{\mu_3}(p_3) \left[ B_1 p_3^\alpha p_1^{\alpha_3} + B_2 \delta^{\alpha_2 \alpha_3} \right] \tag{2.8.15}
\]
where the form factors \( B_1 \) and \( B_2 \) are functions of the momentum magnitudes.

The contribution to the full 3-point function can then be recovered using (2.7.32), yielding

\[
\langle \langle T^{\mu_1 \nu_1}(p_1)J^{\mu_2}(p_2)J^{\mu_3}(p_3) \rangle \rangle_{\text{anomaly}} = \frac{\pi_1^{\mu_1 \nu_1}(p_1)}{d-1} \pi_2^{\mu_2}(p_2) \pi_3^{\mu_3}(p_3) \left[ B_1 p_3^\alpha p_1^{\alpha_3} + B_2 \delta^{\alpha_2 \alpha_3} \right]. \tag{2.8.16}
\]

In our example (2.8.14), in \( d = 4 \) we find

\[
B_1 = -\kappa, \quad B_2 = -\frac{\kappa}{2} (p_2^2 - p_3^2). \tag{2.8.17}
\]

A list of all anomalies and their contributions to all correlators is given in appendix 2.A.7.

**2.8.3. \( \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle \)**

The situation for \( \langle \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle \rangle \) is very similar to that for \( \langle \langle T^{\mu_1 \nu_1} J^{\mu_2} J^{\mu_3} \rangle \rangle \). The \( \Box R \) and \( R^2 \) in the action (2.8.1) can be omitted from the analysis since the former is a total derivative while the latter does not contribute to the transverse-traceless part of the correlators, and can instead be fixed by the renormalisation of the trace part.

The 2-point function in the presence of the counterterms is

\[
\langle \langle T^{\mu_1 \nu_1}(p) T^{\mu_2 \nu_2}(-p) \rangle \rangle_{\text{reg}} = \Pi^{\mu_1 \nu_1 \mu_2 \nu_2}(p)p^4 \left[ \frac{2cT}{\epsilon} - cT \log p^2 - 4c_0 + O(\epsilon) \right], \tag{2.8.18}
\]
where we assume that the trace part was removed by adding the appropriate $R^2$ term. As we can see, the transverse-traceless part depends on $c_0$ only and requires
\[ c_0 = \frac{c_T}{2\epsilon} + O(\epsilon^0). \tag{2.8.19} \]

The anomalous contributions to the transverse-traceless part of the 3-point function are then
\[
\begin{align*}
A_1^{\text{anomaly}} &= 0, \\
A_2^{\text{anomaly}} &= -16(a_0 + c_0), \\
A_3^{\text{anomaly}} &= 8 \left[ c_0(p_1^2 + p_2^2) - a_0p_3^2 \right], \\
A_4^{\text{anomaly}} &= 8 \left[ c_0(p_1^2 + p_2^2 + 3p_3^2) - a_0(p_1^2 + p_2^2 - 2p_3^2) \right], \\
A_5^{\text{anomaly}} &= -4 \left[ a_0J^2 + c_0(p_1^2 + p_2^2 + p_3^2)^2 + 8c_g c_0(p_1^4 + p_2^4 + p_3^4) \right], \tag{2.8.20}
\end{align*}
\]
where we used the definition of the 3-point function (1.3.20). The constant $c_g$ is defined, in the case where $T_{\mu\nu}$ is the unique spin-2 conserved current, as
\[
\frac{T_{\mu_1\nu_1}(x)}{g_{\mu_2\nu_2}(y)} = 4c_g \delta_{(\mu_1(\mu_2 T_{\nu_1})\nu_2)}(x)\delta(x - y) + \ldots \tag{2.8.21}
\]
or more generally
\[
\langle\delta T_{\mu_1\nu_1}(p_1, p_2) T_{\mu_2\nu_2}(p_3)\rangle = 4c_g \delta_{\mu_1(\mu_2} \langle T_{\nu_1)\nu_2}(p_3)T_{\mu_3\nu_3}(p_3)\rangle + \ldots \tag{2.8.22}
\]
where the omitted terms do not contain tensors we have listed explicitly.

As in the case of $\langle T_{\mu_1\nu_1} J^{\mu_2} J^{\nu_3}\rangle$, we can find the divergences in the regularised form factors coming from the divergences in the triple-$K$ integrals, giving
\[
\begin{align*}
A_1^{\text{reg}} &= O(\epsilon^0), \\
A_2^{\text{reg}} &= \frac{8}{\epsilon}(16\alpha_1 + \alpha_2) + O(\epsilon^0), \\
A_3^{\text{reg}} &= -\frac{4}{\epsilon} \left[ (p_1^2 + p_2^2)c_T + p_3^2(c_T - 16\alpha_1 - \alpha_2) \right] + O(\epsilon^0), \\
A_4^{\text{reg}} &= \frac{4}{\epsilon} \left[ (p_1^2 + p_2^2 - p_3^2)(16\alpha_1 + \alpha_2 - 2c_T) - 4c_Tp_3^2 \right] + O(\epsilon^0), \\
A_5^{\text{reg}} &= \frac{2}{\epsilon} \left[ (16\alpha_1 + \alpha_2)J^2 + 2c_T(1 + 4c_g)(p_1^4 + p_2^4 + p_3^4) \right] + O(\epsilon^0), \tag{2.8.23}
\end{align*}
\]
Since the value of $c_0$ is already fixed by (2.8.19), we can use one of the form factors, say $A_2$, to find
\[
a_0 = \frac{1}{2\epsilon}(16\alpha_1 + \alpha_2 - c_T) + O(\epsilon^0). \tag{2.8.24}
\]
The immediate conclusion is that all singularities must appear with coefficients that are multiples of $16\alpha_1 + \alpha_2$ or $c_T$. As we can see, this is indeed the case. Substituting $a_0$ and $c_0$ as given by (2.8.24) and (2.8.19) into the remaining equations,
we obtain an exact cancellation of divergences. The anomaly coefficients in (2.8.5) are thus
\[ a = \frac{1}{2} (16\alpha_1 + \alpha_2 - cT), \quad c = \frac{cT}{2}. \] (2.8.25)

Finally, from (2.8.20), we find the scheme-dependent contributions take the form
\[ A_2 \mapsto A_2 - 16(a_0^{(0)} + c_0^{(0)}), \]
\[ A_3 \mapsto A_3 + 8 \left[ c_0^{(0)} (p_1^2 + p_2^2) - a_0^{(0)} p_3^2 \right], \]
\[ A_4 \mapsto A_4 + 8 \left[ c_0^{(0)} (p_1^2 + p_2^2 + 3p_3^2) - a_0^{(0)} (p_1^2 + p_2^2 - p_3^2) \right], \]
\[ A_5 \mapsto A_5 - 4 \left[ a_0^{(0)} J^2 + c_0^{(0)} (p_1^2 + p_2^2 + p_3^2)^2 + 8c_0^{(0)} (p_1^4 + p_2^4 + p_3^4) \right], \] (2.8.26)

where \( a_0^{(0)} \) and \( c_0^{(0)} \) are arbitrary terms of order \( \epsilon \) in \( a_0 \) and \( c_0 \).

### 2.9. Helicity formalism

In this section we will work entirely in \( d = 3 \) spacetime dimensions.

#### 2.9.1. Definitions

Consider conserved vector field \( j^\mu \) and the transverse-traceless, symmetric tensor \( t_{\mu\nu} \) of rank two in \( d = 3 \) dimensions. It is easy to count that both these objects contain 2 independent degrees of freedom. One can exploit this fact by rewriting the projectors (2.1.13) and (2.1.14) as
\[ \pi_{\mu\nu}(p) = \sum_{s=\pm 1} \xi^{(s)}_\mu(p) \bar{\xi}^{(s)}_\nu(p), \] (2.9.1)
\[ \Pi_{\mu\nu\rho\sigma}(p) = \frac{1}{2} \sum_{s=\pm 1} \epsilon^{(s)}_{\mu\nu}(p) \bar{\epsilon}^{(s)}_{\rho\sigma}(p), \] (2.9.2)

where \( \xi^{(s)} \) and \( \epsilon^{(s)}_{\mu\nu} \) are polarisation vectors and tensors satisfying
\[ p^\mu \epsilon^{(s)}_{\mu}(p) = p^\mu \epsilon^{(s)}_{\mu}(p) = 0, \quad \epsilon^{(s)}_{\mu\nu} = \epsilon^{(s)}_{\nu\mu}, \quad \delta^{\mu\nu} \epsilon^{(s)}_{\mu\nu} = 0, \]
\[ \bar{\epsilon}^{(s)}_{\mu\nu}(p) = \epsilon^{(s)}_{\mu\nu}(-p), \quad \bar{\xi}^{(s)}_\mu(p) = \xi^{(s)}_\mu(-p). \] (2.9.3)

The parameter of \( s \) takes two values \( s = \pm 1 \) known as *helicities*. The bar over a symbol denotes complex conjugation. By using the fact that in \( d = 3 \)
\[ \pi_{\mu\nu} \pi^{\mu\nu} = 2, \quad \Pi_{\mu\nu\rho\sigma} \Pi^{\mu\nu\rho\sigma} = 2, \] (2.9.4)
we can find
\[ \xi^{(s)}_\mu \xi^{(s')}_{\mu} = \delta^{ss'}, \quad \epsilon^{(s)}_{\mu\nu} \epsilon^{(s')}_{\mu\nu} = 2 \delta^{ss'}. \] (2.9.5)
Furthermore we define the helicity projected operators,

\[ T(p) = \delta^{\mu\nu} T_{\mu\nu}(p), \]

\[ T^{(s)}(p) = \frac{1}{2} \epsilon^{(s)}_{\mu\nu} (-p) T_{\mu\nu}(p), \]

\[ \Upsilon(p_1, p_2) = \delta^{\mu\nu} \delta^{\rho\sigma} \frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}} (p_1, p_2), \]

\[ \Upsilon^{(s)}(p_1, p_2) = \frac{1}{2} \delta^{\mu\nu} \epsilon^{(s)\rho\sigma} (-p_2) \frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}} (p_1, p_2), \]

\[ \Upsilon^{(s_1 s_2)}(p_1, p_2) = \frac{1}{4} \epsilon^{(s_1)\mu\nu} (-p_1) \epsilon^{(s_2)\rho\sigma} (-p_2) \frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}} (p_1, p_2). \]

Finally, the expressions for the polarisation tensor can be found explicitly. To do so, we first observe that the momenta \( p_i, i = 1, 2, 3 \) lie in a single plane due to momentum conservation. Taking this plane to be the \((x_1, x_3)\) plane, we may then write

\[ p_i = p_i (\sin \theta_i, 0, \cos \theta_i) \]

where without loss of generality we may choose \( 0 \leq \theta_2 \leq \pi \) and \( \pi \leq \theta_3 \leq 2\pi \) so that

\[ \cos \theta_2 = \frac{(p_2^2 - p_1^2 - p_3^2)}{2p_1 p_2}, \quad \sin \theta_2 = \frac{J}{2p_1 p_2}, \]

\[ \cos \theta_3 = \frac{(p_3^2 - p_1^2 - p_2^2)}{2p_1 p_3}, \quad \sin \theta_3 = -\frac{J}{2p_1 p_3}, \]

with \( J \) as given in (2.6.18). The required helicity tensors then follow by rotation in the \((x_1, x_3)\) plane:

\[ \epsilon^{(s_1)}(p_i) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos^2 \theta_i & i s_i \cos \theta_i & -\sin \theta_i \cos \theta_i \\ i s_i \cos \theta_i & -1 & -i s_i \sin \theta_i \\ -\sin \theta_i \cos \theta_i & -i s_i \sin \theta_i & \sin^2 \theta_i \end{pmatrix}, \]

2.9.2. Correlation functions

Correlation functions of the helicity-projected operators can easily be obtained from the transverse-traceless parts of the correlators. First observe that the semi-local parts of any correlation function vanish when contracted with polarisation tensors. Indeed, equations (2.9.1) and (2.9.2) imply that

\[ \pi_{\mu\nu} \xi^{(s)} = \xi^{(s)}_\mu, \quad \Pi^{\rho\sigma} \xi^{(s)} = \xi^{(s)}_{\mu\nu}. \]

Then, using equation (2.4.16) we can write

\[ \tilde{\xi}^{(s)} = j^{\mu}_{loc} = \tilde{\xi}^{(s)}_{\mu} j^{\mu}_{loc} = 0 \]
and similarly $\epsilon^{(s)}_{\mu\nu} t^{\mu\nu}_{\text{loc}} = 0$. To obtain correlation functions in the helicity formalism, one can therefore apply helicity projectors to the transverse-traceless parts of correlators only. Due to (2.9.15), the projectors (2.1.13) and (2.1.14) can then be removed as well. Finally, one needs to compute a small number of contractions of the helicity projectors with momenta and with the metric.

As an example, consider the $\langle\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} \mathcal{O} \rangle\rangle$ correlation function. Applying first the helicity projectors to its decomposition (2.3.7), we find
\[
\langle\langle T^{(s_1)}(p_1) T^{(s_2)}(p_2) \mathcal{O}(p_3) \rangle\rangle = \frac{1}{4} \varepsilon^{(s_1)}_{\mu_1\nu_1}(p_1) \varepsilon^{(s_2)}_{\mu_2\nu_2}(p_2) \langle\langle t^{\mu_1\nu_1}(p_1) t^{\mu_2\nu_2}(p_2) \mathcal{O}(p_3) \rangle\rangle
\]
\[
= \frac{1}{4} \left[ A_1 \varepsilon^{(s_1)}_{\mu_1\nu_1}(p_1) p_{\mu_1}^{s_1} p_{\nu_1}^{s_1} \varepsilon^{(s_2)}_{\mu_2\nu_2}(p_2) p_{\mu_2}^{s_2} p_{\nu_2}^{s_2} + A_2 \varepsilon^{(s_1)}_{\mu_1\alpha}(p_1) \varepsilon^{(s_2)}_{\mu_2\alpha}(p_2) p_{\mu_1}^{s_1} p_{\mu_2}^{s_2}
+ A_3 \varepsilon^{(s_1)}_{\alpha\beta}(p_1) \varepsilon^{(s_2)}_{\alpha\beta}(p_2) \right].
\]

(2.9.17)

The contractions with helicity tensors depend on the precise definition of the latter and also the overall dimension. For case of $d = 3$ the required contractions can be found in appendix 2.A.9 based on [1]. Using (2.5.69) - (2.5.71) for the form factors, the most general solution is
\[
\langle\langle T^{(s_1)}(p_1) T^{(s_2)}(p_2) \mathcal{O}^I(p_3) \rangle\rangle = \alpha_1^I \frac{3 p_1 p_2}{4 p_3} \left( \frac{p_1 + p_2 - p_3}{p_1 + p_2 + p_3} \right)^2 \delta^{s_1 s_2}
+ \frac{c_{\mathcal{O}} S^{(s_1 s_2)}}{32 p_1^2 p_2^2 p_3} \left[ -2 c_1^I J^2 + (c_1^I - 3 c_2^I) (p_1^2 + p_2^2) + 3(c_1^I + c_2^I) p_3 S^{(s_1 s_2)} \right].
\]

(2.9.18)

The constants $c_1^I$ and $c_2^I$ are defined in (2.5.64) and $c_{\mathcal{O}}$ is the normalisation constant of the 2-point function $\langle\langle \mathcal{O}^I \mathcal{O}^J \rangle\rangle$ defined in (2.5.54).

As a check on our results in (2.7.4), we compared our solution with that obtained in [50] for the $\langle\langle T^{\mu_1\nu_1} J^{\mu_2} J^{\nu_3} \rangle\rangle$ correlator of free scalars and fermions finding perfect agreement.

2.9.3. $\langle\langle T^{(s_1)} T^{(s_2)} T^{(s_3)} \rangle\rangle$

The application of the helicity formalism in $d = 3$ to the correlation function of three stress-energy tensors in $d = 3$ given by (3.11.21) - (3.11.25) leads to the following result
\[
\langle\langle T^{(+)}(p_1) T^{(+)}(p_2) T^{(+)}(p_3) \rangle\rangle = \frac{30 \sqrt{2} \lambda^2 p_1 p_2 p_3}{a_1^3 123} \left( \frac{1}{16 \sqrt{2} c_{123}^2} \left[ (3 a_{123}^3 - 7 a_{123} b_{123} + 5 c_{123}) + 8(p_1^3 + p_2^3 + p_3^3) c_{123} \right] \right),
\]

(2.9.19)
\[ \langle T(+)p_1 T(+)p_2 T(-)p_3 \rangle = -c_T \frac{\lambda^2 (p_1 + p_2 - p_3)^2}{16\sqrt{2}c_{123}^2} \times \]
\[ \times \left[ \frac{1}{a_{123}^2} \left( 3p_3^5 + 4p_3^4a_{12} + p_3^3(a_{12}^2 - b_{12}) + p_3a_{12}(p_3 + 4a_{12})(a_{12}^2 - 3b_{12}) 
+ a_{12}^3(3a_{12}^2 - 7b_{12}) \right) + 8(p_1^3 + p_2^3 + p_3^3)c_g \right], \quad (2.9.20) \]

where \( \lambda^2 \) is defined in (2.6.18) and all remaining variables are symmetric polynomials in magnitudes of momenta defined in (3.1.2). Notice that this solution does not depend on the primary constant \( \alpha_1 \), which features in the solution (3.11.21) - (3.11.25). The reason is that in \( d = 3 \), in position space there is one independent conformal structure less than for \( d > 3 \) [22]. The same result can be obtained directly in the momentum space, as presented in appendix 2.A.2.

Note also that the \( \langle T(+)T(+)T(-) \rangle \) part of the correlation function does not depend on \( \alpha_1 \), hence it is determined uniquely in terms of the 2-point function.

2.A. Appendix

2.A.1. Decomposition of \( \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle \) in non-conformal case

In this section we present the decomposition of the stress-energy tensor 3-point function for a general quantum field theory. As the stress-energy tensor in a general theory is no longer traceless, our arguments in the main text need some minor modifications. First, we discuss how to reconstruct the full correlation function from the purely transverse part, making use of the transverse Ward identities in a similar fashion to section 2.3.3. We then proceed to construct the general tensor decomposition of this transverse part in terms of ten independent form factors.

As in the main text, we will denote the transverse-traceless part of the stress-energy tensor by \( t^{\mu \nu} = \Pi^{\mu \nu}_{\alpha \beta} T^{\alpha \beta} \). Here, we will also make use of the purely transverse part, \( t^{\mu \nu}_T = \pi^{\mu \nu}_{\alpha \beta} T^{\alpha \beta} \), which includes a nonvanishing trace part \( (t_T)^\mu \). The difference between the stress-energy tensor and its transverse part can then be written \( \tilde{t}^{\mu \nu}_{\text{loc}} = T^{\mu \nu} - t^{\mu \nu}_T \), i.e.,

\[ \tilde{t}^{\mu \nu}_{\text{loc}} = \left( \frac{p^\mu}{p^2} \delta^\nu_\alpha + \frac{p^\nu}{p^2} \delta^\mu_\alpha - \frac{p^\mu p^\nu}{p^4} p_\alpha \right) p_\beta T^{\alpha \beta}. \quad (2.A.1) \]

To obtain the reconstruction formula, we use the Ward identity (3.11.1) to re-express \( p_\beta T^{\alpha \beta} \) in terms of 2-point functions when the expectation value of \( \tilde{t}^{\mu \nu}_{\text{loc}} \)
with other operators is taken. Defining the operator

\[ \tilde{\mathcal{L}}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(p_1, p_2, p_3) = \frac{1}{p_1^2} \left( 2p_1(\mu_1\delta^{\nu_1}_{\alpha_1}) - \frac{p_1\mu_1}{p_1^2} p_1\alpha_1 \right) \times \]

\[ \times \left[ 2\delta^{\mu_3\alpha_3}\delta^{\nu_3\alpha_3}p_1^3 \langle \delta T_{\alpha_1\beta_1}(p_1) T^{\mu_2\nu_2}(p_2) \rangle + \left( \delta^{\beta_3\alpha_3}(2p_1^{\mu_3\delta^{\nu_3}_{\alpha_3}} + p_3^{\alpha_3\delta^{\mu_3\nu_3}}) - p_3^{\alpha_1\delta^{\alpha_3\mu_3}} \delta^{\beta_3\nu_3} \right) \times \langle T_{\alpha_3\beta_3}(p_2) T^{\mu_2\nu_2}(-p_2) \rangle \right], \]

(2.4)

the reconstruction formula takes the form

\[ \langle \langle T^{\mu_1\nu_1}(p_1) T^{\mu_2\nu_2}(p_2) T^{\mu_3\nu_3}(p_3) \rangle \rangle = \langle \langle t^{\mu_1\nu_1}(p_1) t^{\mu_2\nu_2}(p_2) t^{\mu_3\nu_3}(p_3) \rangle \rangle 
+ \sum_{\sigma} \tilde{\mathcal{L}}^{\mu_{\sigma(1)}\nu_{\sigma(1)}\mu_{\sigma(2)}\nu_{\sigma(2)}\mu_{\sigma(3)}\nu_{\sigma(3)}}(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) \]

\[ - \frac{1}{p_3^2} \left( 2p_3^{\mu_3\delta^{\nu_3}_{\alpha_3}} - \frac{p_3^{\mu_3\alpha_3}}{p_3^2} \right) p_{3\beta_3} \tilde{\mathcal{L}}^{\mu_1\nu_1\mu_2\nu_2\alpha_3\beta_3}(p_1, p_2, p_3) \]

\[ - \left[ (\mu_1, \nu_1, p_1) \mapsto (\mu_2, \nu_2, p_2) \mapsto (\mu_3, \nu_3, p_3) \mapsto (\mu_1, \nu_1, p_1) \right] 
- \left[ (\mu_1, \nu_1, p_1) \mapsto (\mu_3, \nu_3, p_3) \mapsto (\mu_2, \nu_2, p_2) \mapsto (\mu_1, \nu_1, p_1) \right], \]

(2.3)

where the sum is taken over all six permutations \( \sigma \) of the set \( \{1, 2, 3\} \). Note the similarity between these expression and (3.11.3, 3.11.4).

We turn now to the tensor decomposition of the purely transverse part of the 3-point function. The most general form of this is

\[ \langle \langle t^{\mu_1\nu_1}(p_1) t^{\mu_2\nu_2}(p_2) t^{\mu_3\nu_3}(p_3) \rangle \rangle = \pi^{\mu_1}_{\alpha_1}(p_1) \pi^{\nu_1}_{\beta_1}(p_1) \pi^{\mu_2}_{\alpha_2}(p_2) \pi^{\nu_2}_{\beta_2}(p_2) \pi^{\mu_3}_{\alpha_3}(p_3) \pi^{\nu_3}_{\beta_3}(p_3) X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}, \]

(2.4)

where \( X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} \) is a general tensor built from the metric \( \delta^{\mu\nu} \) and two independent momenta, with a kinematic dependence on the momentum magnitudes \( p_1, p_2 \) and \( p_3 \). Note, however, that if \( X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} \) contains \( p_j^{\alpha_j} \) or \( p_j^{\beta_j} \) for \( j \in \{1, 2, 3\} \) then the contractions with the corresponding transverse projectors vanish. We will assume that \( X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} \) is symmetric under \( \alpha_j \leftrightarrow \beta_j \) and we use the convention (2.3.4) (explained in detail in section 2.3.1) for the momenta appearing under the various Lorentz indices:

\[ p_1, p_2 \text{ for } \mu_1, \nu_1; \ p_2, p_3 \text{ for } \mu_2, \nu_2 \text{ and } p_3, p_1 \text{ for } \mu_3, \nu_3. \]

(2.5)

The following table lists all 24 simple tensors from which \( X^{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} \) may be built.

Contracting each tensor in the table with the transverse projectors we obtain 24 transverse tensors denoted by \( P_a \), \( a = 1, 2, \ldots, 24 \). Each tensor \( P_a \) can then be
multiplied by a form factor $B_a$ to obtain the decomposition

$$
\langle\langle t^{t_{1}^{\mu_{1}^{\nu_{1}}}}(p_{1})t^{t_{2}^{\mu_{2}^{\nu_{2}}}}(p_{2})t^{t_{3}^{\mu_{3}^{\nu_{3}}}}(p_{3})\rangle\rangle = \sum_{a=1}^{24} B_{a}(p_{1},p_{2},p_{3})P_{a}^{\mu_{1}^{\nu_{1}}\mu_{2}^{\nu_{2}}\mu_{3}^{\nu_{3}}}.
$$

(2.A.6)

However, the number of independent form factors may be reduced by looking at the symmetry properties. If we denote the permutation group of the set $\{1,2,3\}$ by $S_{3}$, then the 3-point function is $S_{3}$-invariant, i.e., for any $\sigma \in S_{3}$,

$$
\langle\langle t^{t_{1}^{\mu_{1}^{\nu_{1}}}}(p_{1})t^{t_{2}^{\mu_{2}^{\nu_{2}}}}(p_{2})t^{t_{3}^{\mu_{3}^{\nu_{3}}}}(p_{3})\rangle\rangle = \langle\langle t^{t_{\sigma(1)}^{\mu_{\sigma(1)}}}(p_{\sigma(1)})t^{t_{\sigma(2)}^{\mu_{\sigma(2)}}}(p_{\sigma(2)})t^{t_{\sigma(3)}^{\mu_{\sigma(3)}}}(p_{\sigma(3)})\rangle\rangle.
$$

(2.A.7)

When contracted with the transverse projectors, the tensors at the first, fifth and the last row of the table lead to the $S_{3}$-invariant tensors. Therefore, corresponding form factors are invariant under any permutation of their arguments, for example

$$
B_{1}(p_{1},p_{2},p_{3}) = B_{1}(p_{\sigma(1)},p_{\sigma(2)},p_{\sigma(3)}),
$$

(2.A.8)

for any $\sigma \in S_{3}$. The remaining tensors transform non-trivially under the action of $S_{3}$. For concreteness, consider the second line of the table, i.e., the part of the decomposition

$$
B_{2}(p_{1},p_{2},p_{3})P_{2}^{\mu_{1}^{\nu_{1}}\mu_{2}^{\nu_{2}}\mu_{3}^{\nu_{3}}}+B_{2}(p_{1},p_{2},p_{3})P_{2}^{\mu_{1}^{\nu_{1}}\mu_{2}^{\nu_{2}}\mu_{3}^{\nu_{3}}}+B_{4}(p_{1},p_{2},p_{3})P_{4}^{\mu_{1}^{\nu_{1}}\mu_{2}^{\nu_{2}}\mu_{3}^{\nu_{3}}}.
$$

(2.A.9)

Under the action of the symmetry group the tensors $P_{2}, P_{3}, P_{4}$ shuffle among each other. For example, under the action of the transposition $(p_{1},\mu_{1},\nu_{1}) \leftrightarrow (p_{3},\mu_{3},\nu_{3})$

---

**Table 2.2:** When contracted with the transverse projectors, this table presents all 24 tensor structures in the decomposition of the transverse part of $\langle\langle T^{t_{1}^{\mu_{1}^{\nu_{1}}}}T^{t_{2}^{\mu_{2}^{\nu_{2}}}}T^{t_{3}^{\mu_{3}^{\nu_{3}}}}\rangle\rangle$. Tensors are divided into 10 orbits of the action of the symmetry group $S_{3}$, after the contractions with the transverse projectors are taken.
we obtain
\[ B_2(p_3, p_2, p_1)P_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} + B_3(p_3, p_2, p_1)P_2^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} + B_4(p_3, p_2, p_1)P_4^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}. \] (2.A.10)

Since the entire 3-point function is $S_3$-invariant, this implies that (2.A.9) and (2.A.10) are equal. Since all tensor structures $P_a$ are independent, we find
\[ B_3(p_1, p_2, p_3) = B_2(p_3, p_2, p_1), \quad B_4(p_1, p_2, p_3) = B_4(p_3, p_2, p_1). \] (2.A.11)

By analysing other symmetries we find that (2.A.9) depends on one form factor only, say $B_2$,
\[ B_2(p_1, p_2, p_3)P_2^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} + B_2(p_1 \leftrightarrow p_3)P_3^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} + B_2(p_2 \leftrightarrow p_3)P_4^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}. \] (2.A.12)

Moreover, $B_2(p_1, p_2, p_3) = B_2(p_1 \leftrightarrow p_2)$.

The described procedure reduces the number of independent form factors from 24 down to 10. The same procedure applied to the transverse-traceless part of the 3-point function reduces the number of independent tensors from 11 down to 5. In this case the decomposition is given by (2.3.10).

### 2.A.2. Degeneracy of $\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3} \rangle$ in $d = 3$

In dimension $d = 3$, a special degeneracy occurs which allows the transverse-traceless part of $\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3} \rangle$ to be decomposed in terms of only two form factors rather than five.\(^1\)

To see this, we first define the cross-product
\[ \mathbf{n} = \mathbf{p}_1 \times \mathbf{p}_2 = \mathbf{p}_2 \times \mathbf{p}_3 = \mathbf{p}_3 \times \mathbf{p}_1 \] (2.A.13)
and note that $n^2 = J^2/4$, where $J^2$ is defined in (2.6.18). Using (2.3.2) we find
\[ \delta^{\mu\nu} = \frac{4}{J^2} \left[ p_i^2 \delta_{ij}^\mu \delta_{ij}^\nu + p_i^2 \delta_{ij}^\mu \delta_{ij}^\nu - \mathbf{p}_i \cdot \mathbf{p}_j (p_i^\mu p_j^\nu + p_i^\mu p_j^\nu) + n^\mu n^\nu \right] \] (2.A.14)
for any $i, j = 1, 2, 3$ and $i \neq j$. From the fact that $\delta^{\alpha\beta} \Pi_{\alpha\beta}^{\mu\nu}(\mathbf{p}_j) = 0$, we find
\[ \Pi_{\alpha\beta}^{\mu\nu}(\mathbf{p}_j) n^\alpha n^\beta = -p^2 \Pi_{\alpha\beta}^{\mu\nu}(\mathbf{p}_j) p_{(j+1) \mod 3}^\alpha p_{(j+1) \mod 3}^\beta, \quad j = 1, 2, 3. \] (2.A.15)

We can now go back to the decomposition of the transverse-traceless part of $\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3} \rangle$, equation (3.11.5), and exchange all $\delta^{\alpha\beta}$ for (2.A.14). However, if one transverse-traceless projector is contracted with two vectors $\mathbf{n}$, then, according to (2.A.15), we can replace such a contraction with a contraction of two momenta with appropriate prefactors. Therefore, the only terms surviving in

\(^1\)From the point of view of the helicity formalism, see the end of section (4.1.9).
(3.11.5) are terms with either zero or two vectors \( n \). Hence we find only two tensor structures in the decomposition of \( \langle \langle t^{\mu_1 \nu_1} t^{\mu_2 \nu_2} t^{\mu_3 \nu_3} \rangle \rangle \),

\[
\langle \langle t^{\mu_1 \nu_1} (p_1) t^{\mu_2 \nu_2} (p_2) t^{\mu_3 \nu_3} (p_3) \rangle \rangle = \Pi_{\alpha_1 \beta_1}^{\mu_1 \nu_1} (p_1) \Pi_{\alpha_2 \beta_2}^{\mu_2 \nu_2} (p_2) \Pi_{\alpha_3 \beta_3}^{\mu_3 \nu_3} (p_3) \left[ B_1 p_2^{\alpha_1} p_2^{\beta_1} p_3^{\alpha_2} p_3^{\beta_2} p_1^{\alpha_3} p_1^{\beta_3} \right. \\
+ B_2 n^{\beta_1} n^{\beta_2} p_2^{\alpha_1} p_2^{\alpha_2} p_1^{\alpha_3} p_1^{\beta_3} + B_2(p_1 \leftrightarrow p_3) n^{\beta_1} n^{\beta_2} p_2^{\alpha_1} p_2^{\alpha_2} p_3^{\beta_2} p_3^{\beta_3} \\
+ B_2(p_2 \leftrightarrow p_3) n^{\beta_1} n^{\beta_2} p_2^{\alpha_1} p_2^{\alpha_2} p_3^{\beta_2} p_3^{\beta_3} \left. \right].
\]

The new form factors \( B_j \) are functions of the momentum magnitudes. As usual, if no arguments are specified then the standard ordering is assumed, \( B_j = B_j(p_1, p_2, p_3) \), while by \( p_i \leftrightarrow p_j \) we denote the exchange of the two momenta, e.g., \( B_1(p_1 \leftrightarrow p_3) = B_2(p_3, p_2, p_1) \).

We can now express the new form factors \( B_j \) in terms of the old ones, \( A_j \), defined in (3.11.5). Using equation (2.A.15), we write the explicit form of the contraction of two transverse-traceless projectors with a metric as

\[
\Pi_{\alpha_1 \beta_1}^{\mu_1 \nu_1} (p_1) \Pi_{\alpha_2 \beta_2}^{\mu_2 \nu_2} (p_2) \delta^{\beta_1 \beta_2} = \frac{4}{J^2} \Pi_{\alpha_1 \beta_1}^{\mu_1 \nu_1} (p_1) \Pi_{\alpha_2 \beta_2}^{\mu_2 \nu_2} (p_2) \left[ n^{\beta_1} n^{\beta_2} + \frac{1}{2} (p_3^2 - p_1^2 - p_2^2) p_3^{\beta_1} p_3^{\beta_2} \right],
\]

from which we find

\[
B_1 = A_1 + \frac{2}{J^2} \left[ (p_3^2 - p_1^2 - p_2^2) A_2(p_1, p_2, p_3) + (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3) \right] \\
+ \frac{4}{J^4} \left[ ((8p_1^2 p_2^2 - J^2) A_3 + (p_3^4 - (p_1^2 - p_2^2)^2) A_4) + (p_1 \leftrightarrow p_3) + (p_2 \leftrightarrow p_3) \right] \\
- \frac{8}{J^4} (p_1^2 + p_2^2 + p_3^2) A_5,
\]

\[
B_2 = \frac{4}{J^2} A_2 + \frac{16}{J^4} \left[ (p_3^2 - p_1^2 - p_2^2) A_3 - p_3^2 A_4 + \right. \\
\left. + \frac{1}{2} (p_2^2 - p_1^2 - p_3^2) A_4(p_1 \leftrightarrow p_3) + \frac{1}{2} (p_1^2 - p_2^2 - p_3^2) A_4(p_2 \leftrightarrow p_3) \right] \\
+ \frac{16}{J^4} A_5.
\]

Using the general expressions (3.11.21) - (3.11.25) for the form factors in \( d = 3 \), we arrive at the final result

\[
B_1 = 1920 \alpha_1 \frac{c_{123}^3}{J^4 a_{123}^2} - \frac{8 c_T}{J^4 a_{123}^2} \left[ (3 + 8c_g) a_{123}^5 (a_{123}^2 - 5b_{123}) + 24(1 + 2c_g) a_{123}^3 b_{123}^2 \\
- 8a_{123} b_{123}^3 + (3(8c_g - 1)a_{123}^4 - 48c_g a_{123}^2 b_{123} - 8b_{123}^2) c_{123} + 8a_{123} c_{123}^2 \right],
\]

(2.A.19)
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\[ B_2 = -1920 \alpha_1 \frac{c_1^2 p_3}{J^4 a_{123}^2} + \frac{64c_T}{J^4 a_{123}^2} \left[ 2(1 + c_g) p_3^4(p_3 + 2a_{12}) \right. \]
\[ + p_2^3 \left( 2(2 + c_g)a_{12}^2 - 4b_{12} \right) + p_2^2 a_{12} \left( (3 + 2c_g)a_{12}^2 - (5 + 6c_g)b_{12} \right) \]
\[ + 2p_2 \left( (1 + 2c_g)a_{12}^2(a_{12}^2 - 3b_{12}) + b_{12}^2 \right) \]
\[ - 3(1 + 2c_g)a_{12}^3b_{12} + a_{12}b_{12}^2 + (1 + 2c_g)a_{12}^5 \] \hspace{1cm} (2.A.20)

The variables used in this expression are symmetric polynomials of the momentum magnitudes as defined in (3.1.2). Note that this expression has no dependence on the primary constant \( \alpha_2 \). Therefore, in \( d = 3 \), the most general form of the correlation function \( \langle \langle T^{\mu_1 \nu_1} T^{\mu_2 \nu_2} T^{\mu_3 \nu_3} \rangle \rangle \) depends on only one undetermined primary constant and on two 2-point function normalisations \( c_T \) and \( c_g \). This is in agreement with [22], noting that the normalisation constant \( c_g \) arises through our definition of the 3-point function in (1.3.20).

Finally, while similar considerations hold for other 3-point correlators in \( d = 3 \) involving the stress-energy tensor, in these cases it turns out that the use of equation (2.A.14) does not reduce the number of independent primary constants in the final result.

2.A.3. Properties of triple-\( K \) integrals

In this appendix we list some properties of modified Bessel functions used in the main text. For further references, see e.g., [59].

The Bessel function \( I \) (modified Bessel function of the first kind) is given by the series

\[ I_\nu(x) = \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(\nu + j + 1)} \left( \frac{x}{2} \right)^{\nu + 2j}, \quad \nu \neq -1, -2, -3, \ldots \] \hspace{1cm} (2.A.21)

The Bessel function \( K \) (modified Bessel function of the second kind) is defined by

\[ K_\nu(x) = \frac{\pi}{2 \sin(\nu \pi)} \left[ I_{-\nu}(x) - I_\nu(x) \right], \quad \nu \notin \mathbb{Z}, \] \hspace{1cm} (2.A.22)
\[ K_n(x) = \lim_{\epsilon \to 0} K_{n+\epsilon}(x), \quad n \in \mathbb{Z}. \] \hspace{1cm} (2.A.23)

The finite pointwise limit for \( x > 0 \) exists for any integer \( n \). \( K_\nu \) is an even function of \( \nu \), i.e., \( K_{-\nu}(x) = K_\nu(x) \) for any \( \nu \in \mathbb{R} \). If \( \nu = \frac{1}{2} + n \), for an integer \( n \), the Bessel function reduces to elementary functions

\[ K_\nu(x) = \sqrt{\frac{\pi}{2}} e^{-x} x^{\nu - \frac{1}{2}} \sum_{j=0}^{|\nu| - \frac{1}{2}} \frac{(|\nu| - \frac{1}{2} + j)!}{j! (|\nu| - \frac{1}{2} - j)! (2x)^j}, \quad \nu + \frac{1}{2} \in \mathbb{Z}, \] \hspace{1cm} (2.A.24)
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and in particular

\[ K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi e^{-x}}{2 x^{\frac{3}{2}}}}, \quad K_{\frac{3}{2}}(x) = \sqrt{\frac{\pi e^{-x}}{2 x^{\frac{3}{2}}}(x^2 + 3x + 3)}, \quad K_{\frac{5}{2}}(x) = \sqrt{\frac{\pi e^{-x}}{2 x^{\frac{3}{2}}}(x^3 + 6x^2 + 15x + 5)}. \] 

(2.A.25)

The series expansion of the Bessel function \( K_\nu \) for \( \nu \notin \mathbb{Z} \) is given directly in terms of the expansion (2.A.21) via the definition (2.A.22). In particular

\[ K_\nu(x) = [\Gamma(-\nu)2^{-\nu-1}x^\nu + O(x^{2-\nu})] + \left[ \frac{\Gamma(\nu)2^{\nu-1}}{x^\nu} + O(x^{2+\nu}) \right], \quad \nu \notin \mathbb{Z}. \] 

(2.A.26)

For non-negative integer index \( n \), the expansion reads

\[
K_n(x) = \frac{1}{2} \left( \frac{x}{2} \right)^{-n} \sum_{j=0}^{n-1} \frac{(n-j-1)!}{j!} (-1)^j \left( \frac{x}{2} \right)^{2j} \\
+ (-1)^{n+1} \log \left( \frac{x}{2} \right) I_n(x) \\
+ (-1)^n \frac{1}{2} \left( \frac{x}{2} \right)^n \sum_{j=0}^{\infty} \frac{\psi(j+1) + \psi(n+j+1)}{j!(n+j)!} \left( \frac{x}{2} \right)^{2j}, 
\]

(2.A.27)

where \( \psi \) is the digamma function. At large \( x \), the Bessel functions have the asymptotic expansions

\[ I_\nu(x) = \frac{1}{\sqrt{2\pi x}} e^x + \ldots, \quad K_\nu(x) = \sqrt{\frac{\pi e^{-x}}{2 x}} + \ldots, \quad \nu \in \mathbb{R}. \] 

(2.A.28)

For any index \( \nu \in \mathbb{R} \), the Bessel function \( K \) satisfies the following identities

\[
\frac{\partial}{\partial a} [a^n K_\nu(ax)] = -x a^n K_{\nu-1}(ax), 
\]

(2.A.29)

\[
K_{\nu-1}(x) + \frac{2\nu}{x} K_\nu(x) = K_{\nu+1}(x), 
\]

(2.A.30)

\[
K_{-\nu}(x) = K_\nu(x). 
\]

(2.A.31)

2.A.4. Appell’s \( F_4 \) function

Appell’s \( F_4 \) function can be defined by the following double series [60, 61]

\[
F_4(\alpha, \beta; \gamma, \gamma'; \xi, \eta) = \sum_{i,j=0}^{\infty} \frac{(\alpha)_{i+j}(\beta)_{i+j}}{(\gamma)_{i}(\gamma')_{j}i!j!} \xi^i \eta^j, \quad \sqrt{\vert \xi \vert} + \sqrt{\vert \eta \vert} < 1, \] 

(2.A.32)

where \( (\alpha)_i \) is a Pochhammer symbol. Notice that

\[
F_4(\alpha, \beta; \gamma, \gamma'; \xi, \eta) = F_4(\beta, \alpha; \gamma, \gamma'; \xi, \eta) = F_4(\alpha, \beta; \gamma', \gamma; \eta, \xi). \] 

(2.A.33)
The series representation, however, is not very useful as in our case \( \xi = \frac{p_2}{p_3} \) and \( \eta = \frac{p_2}{p_3} \), so the series converges when \( p_3 > p_1 + p_2 \), which is opposite to the triangle inequality.

As in the case of ordinary hypergeometric functions, the \( F_4 \) function satisfies certain differential equations. Let \( \alpha, \beta, \gamma, \gamma' \) be fixed numbers. The following system of equations

\[
0 = \left[ \xi (1 - \xi) \frac{\partial^2}{\partial \xi^2} - \eta^2 \frac{\partial^2}{\partial \eta^2} - 2\xi\eta \frac{\partial^2}{\partial \xi \partial \eta} \right. \\
+ (\gamma - (\alpha + \beta + 1)\xi) \frac{\partial}{\partial \xi} - (\alpha + \beta + 1)\eta \frac{\partial}{\partial \eta} - \alpha \beta \left. \right] F(\xi, \eta), \tag{2.A.34}
\]

\[
0 = \left[ \eta (1 - \eta) \frac{\partial^2}{\partial \eta^2} - \xi^2 \frac{\partial^2}{\partial \xi^2} - 2\xi\eta \frac{\partial^2}{\partial \xi \partial \eta} \right. \\
+ (\gamma' - (\alpha + \beta + 1)\eta) \frac{\partial}{\partial \eta} - (\alpha + \beta + 1)\xi \frac{\partial}{\partial \xi} - \alpha \beta \left. \right] F(\xi, \eta), \tag{2.A.35}
\]

has exactly four solutions given by \([61, 62]\)

\[
F_4(\alpha, \beta; \gamma, \gamma'; \xi, \eta), 
\]

\[
\xi^{1-\gamma} F_4(\alpha + 1 - \gamma, \beta + 1 - \gamma; 2 - \gamma, \gamma'; \xi, \eta), \tag{2.A.37}
\]

\[
\eta^{1-\gamma'} F_4(\alpha + 1 - \gamma', \beta + 1 - \gamma'; \gamma, 2 - \gamma'; \xi, \eta), \tag{2.A.38}
\]

\[
\xi^{1-\gamma} \eta^{1-\gamma'} F_4(\alpha + 2 - \gamma - \gamma', \beta + 2 - \gamma - \gamma'; 2 - \gamma, 2 - \gamma'; \xi, \eta). \tag{2.A.39}
\]

The following reduction formulae can be found in \([63\) or \([61\]

\[
F_4 \left( \alpha, \beta; \alpha, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)} \right) = \frac{(1-x)^\beta (1-y)^\alpha}{1-xy}, \tag{2.A.40}
\]

\[
F_4 \left( \alpha, \beta; \beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)} \right) = (1-x)^\alpha (1-y)^\beta F_1(\alpha, 1 + \alpha - \beta; \beta; xy), \tag{2.A.41}
\]

\[
F_4 \left( \alpha, \beta; 1 + \alpha - \beta, \beta; -\frac{x}{(1-x)(1-y)}, -\frac{y}{(1-x)(1-y)} \right) = \frac{x(1-y)}{1-x} \right), \tag{2.A.42}
\]

\[
2 F_1(2\nu - 1, \nu; \nu; x) = (1-x)^{1-2\nu}. \tag{2.A.43}
\]

Integrals
Here we present the list of integrals we use in the thesis, which may be found in \([63\].
2. Implications of conformal invariance in momentum space

(i) \[
\int_0^\infty dx x^{\alpha-1} I_\lambda(x I_\mu(bx)K_\nu(cx) =
\]
\[
= \frac{2^{\alpha-2} \Gamma \left( \frac{\alpha + \lambda + \mu - \nu}{2} \right) \Gamma \left( \frac{\alpha + \lambda + \mu + \nu}{2} \right)}{\Gamma(\lambda + 1)\Gamma(\mu + 1)} \cdot \frac{a^\lambda b^\mu}{c^{\alpha + \lambda + \mu}} \times
\]
\[
\times F_4 \left( \frac{\alpha + \lambda + \mu - \nu}{2}, \frac{\alpha + \lambda + \mu + \nu}{2}; \lambda + 1, \mu + 1; \frac{a^2}{c^2}, \frac{b^2}{c^2} \right),
\]
valid for
\[
\text{Re}(\alpha + \lambda + \mu) > |\text{Re} \nu|, \quad |c| > |a| + |b|, \quad \text{Re} c > |\text{Re} a| + |\text{Re} b|.
\]

(ii) \[
\int_0^\infty dx x^{\alpha-1} K_\lambda(ax)K_\mu(bx)K_\nu(cx) =
\]
\[
= \frac{2^{\alpha-4}}{c^\alpha} [A(\lambda, \mu) + A(\lambda, -\mu) + A(-\lambda, \mu) + A(-\lambda, -\mu)],
\]
where
\[
A(\lambda, \mu) = \left( \frac{a}{c} \right)^\lambda \left( \frac{b}{c} \right)^\mu \Gamma \left( \frac{\alpha + \lambda + \mu - \nu}{2} \right) \Gamma \left( \frac{\alpha + \lambda + \mu + \nu}{2} \right) \Gamma(-\lambda)\Gamma(-\mu) \times
\]
\[
\times F_4 \left( \frac{\alpha + \lambda + \mu - \nu}{2}, \frac{\alpha + \lambda + \mu + \nu}{2}; \lambda + 1, \mu + 1; \frac{a^2}{c^2}, \frac{b^2}{c^2} \right),
\]
valid for
\[
\text{Re} \alpha > |\text{Re} \lambda| + |\text{Re} \mu| + |\text{Re} \nu|, \quad \text{Re}(a + b + c) > 0.
\]

(iii) \[
\int_0^\infty dx x^{\alpha-1} K_\mu(cx)K_\nu(cx) =
\]
\[
= \frac{2^{\alpha-3}}{\Gamma(\alpha)c^\alpha} \Gamma \left( \frac{\alpha + \mu + \nu}{2} \right) \Gamma \left( \frac{\alpha + \mu - \nu}{2} \right) \Gamma \left( \frac{\alpha - \mu + \nu}{2} \right) \Gamma \left( \frac{\alpha - \mu - \nu}{2} \right),
\]
valid for
\[
\text{Re} \alpha > |\text{Re} \mu| + |\text{Re} \nu|, \quad \text{Re} c > 0.
\]

(iv) \[
\int_0^\infty dx x^{\alpha-1} K_\nu(cx) = \frac{2^{\alpha-2}}{c^\alpha} \Gamma \left( \frac{\alpha + \nu}{2} \right) \Gamma \left( \frac{\alpha - \nu}{2} \right)
\]
valid for
\[
\text{Re} \alpha > |\text{Re} \nu|, \quad \text{Re} c > 0.
\]
$$\int_0^\infty dx \, x^{\alpha-1} \log x K_\nu(cx) = \frac{2^{\alpha-3}}{c^\alpha} \Gamma\left(\frac{\alpha + \nu}{2}\right) \Gamma\left(\frac{\alpha - \nu}{2}\right) \times$$

$$\times \left[ \psi\left(\frac{\alpha + \nu}{2}\right) + \psi\left(\frac{\alpha - \nu}{2}\right) - 2 \log \frac{c}{2} \right], \quad (2.A.53)$$

valid for

$$\text{Re } \alpha > |\text{Re } \nu|, \quad \text{Re } c > 0. \quad (2.A.54)$$

$$\int_0^\infty dx \, x^{\alpha-1} \log^2 x K_\nu(cx) = \frac{2^{\alpha-4}}{c^\alpha} \Gamma\left(\frac{\alpha + \nu}{2}\right) \Gamma\left(\frac{\alpha - \nu}{2}\right) \times$$

$$\times \left[ \left( \psi\left(\frac{\alpha + \nu}{2}\right) + \psi\left(\frac{\alpha - \nu}{2}\right) \right) \cdot \left( \psi\left(\frac{\alpha + \nu}{2}\right) + \psi\left(\frac{\alpha - \nu}{2}\right) - 4 \log \frac{c}{2} \right) \right.$$\n
$$\left. + \psi'\left(\frac{\alpha + \nu}{2}\right) + \psi'\left(\frac{\alpha - \nu}{2}\right) + 4 \log^2 \frac{c}{2} \right], \quad (2.A.55)$$

valid for

$$\text{Re } \alpha > |\text{Re } \nu|, \quad \text{Re } c > 0. \quad (2.A.56)$$

### 2.A.5. Master integral

The master integral is

$$I_{0+\epsilon\{111\}} = -\frac{p_1^2 + p_2^2 + p_3^2}{2 \epsilon^2}$$

$$+ \frac{1}{2\epsilon} \left[ -h_{1/2}(p_1^2 + p_2^2 + p_3^2) + (p_1^2 \log p_1 + p_2^2 \log p_2 + p_3^2 \log p_3) \right]$$

$$+ \frac{p_1 p_2 Z}{16} \left( \frac{1}{2} F^{(2)}(Z^2) + 2(h_1 + 2 \log p_3) F^{(1)}(Z^2) + (h_1 + 2 \log^2 p_3)^2 + 1 + \frac{1}{2} \pi^2 \right)$$

$$+ \left( p_1 \leftrightarrow p_3, Z \leftrightarrow X \text{ but } Z^2 \leftrightarrow Z \frac{X}{Y} \right) + \left( p_2 \leftrightarrow p_3, Z \leftrightarrow Y \text{ but } Z^2 \leftrightarrow Z \frac{Y}{X} \right)$$

$$- \frac{\sqrt{-J^2}}{8} \left[ \log^2 (p_1 p_2 p_3) + h_2 \log (p_1 p_2 p_3) + \frac{1}{4} h_2^2 + 1 + \frac{7}{24} \pi^2 \right.$$\n
$$\left. - (h_2 + 2 \log (p_1 p_2 p_3)) \log \sqrt{-J^2} + \log^2 \sqrt{-J^2} \right]$$

$$+ \frac{1}{8} \left[ \left( p_3^2 - \frac{1}{7} h_{2/7}^2 + \frac{7}{8} + \frac{1}{8} \pi^2 \right) + h_2 (p_1^2 + p_2^2) + 3 h_0 p_3^2 \log p_3 \right.$$\n
$$\left. + (p_1^2 + p_2^2 - p_3^2) (\log^2 p_3 - 2 \log p_1 \log p_2) \right) \right) + O(\epsilon), \quad (2.A.57)$$
where
\begin{align}
F^{(1)}(x) &= 1 - \left(1 - \frac{1}{x}\right) \log(1 - x), \tag{2.A.58} \\
F^{(2)}(x) &= 2 + \left(1 - \frac{1}{x}\right) \left[- \log(1 - x) + \log^2(1 - x) + \text{Li}_2 x\right] \tag{2.A.59}
\end{align}
are coefficients of the series expansion
\[2 F_1(1, \epsilon; 2 - \epsilon; x) = 1 + F^{(1)}(x) \epsilon + F^{(2)}(x) \epsilon^2 + O(\epsilon^3). \tag{2.A.60}\]

We have also defined
\begin{align}
J^2 &= (p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(-p_1 + p_2 + p_3)(p_1 + p_2 + p_3), \tag{2.A.61} \\
X &= \frac{p_1^2 - p_2^2 - p_3^2 + \sqrt{-J^2}}{2p_2 p_3}, \quad Y = \frac{p_2^2 - p_1^2 - p_3^2 + \sqrt{-J^2}}{2p_1 p_3}, \tag{2.A.62} \\
Z &= \frac{p_3^2 - p_1^2 - p_2^2 - \sqrt{-J^2}}{2p_1 p_2} \tag{2.A.63}
\end{align}
and the constant
\[h_\alpha = \alpha - \gamma_E + \log 2. \tag{2.A.64}\]

The master integral can be evaluated by the method presented in section 2.6.4. All remaining integrals required for the calculations of 3-point functions of conserved currents and the stress-energy tensor in any even dimension \(d \geq 4\) can be obtained from the master integral (2.A.57) by the reduction scheme. For \(d = 4\), the reduction scheme is presented in table 2.1 on page 94.

Below we present the expressions for two other integrals, \(I_{1\{222\}}\) and \(I_{1\{000\}}\), appearing in table 2.1 on page 94. Both integrals can be obtained by the method described in section 2.6.4, but the resulting expressions are simpler than the master integral \(I_{0\{111\}}\). Therefore, whenever possible, it is convenient to use them as a starting point in the reduction scheme. In particular the integral \(I_{1\{000\}}\) is convergent and known in the literature, e.g., [52, 69]
\[I_{1\{000\}} = \frac{1}{2\sqrt{-J^2}} \left[ \frac{\pi^2}{6} - 2 \log \frac{p_1}{p_3} \log \frac{p_2}{p_3} + \log \left(-X \frac{p_2}{p_3}\right) \log \left(-Y \frac{p_1}{p_3}\right) \right. \\
- \text{Li}_2 \left(-X \frac{p_2}{p_3}\right) - \text{Li}_2 \left(-Y \frac{p_1}{p_3}\right) \right]. \tag{2.A.65}\]

From this integral one can find
\[I_{4\{111\}} = -p_1 p_2 p_3 \frac{\partial^3}{\partial p_1 \partial p_2 \partial p_3} I_{1\{000\}}, \tag{2.A.66}\]
which is the top rightmost entry in table 2.1, page 94.
The second integral is
\[
I_{2+\epsilon\{111\}} = \frac{1}{\epsilon} - \left[ \frac{p_2^2Z}{2p_1p_2} \left( \tilde{F}^{(1)}(Z^2) + (h_0 + 2 \log p_3)\tilde{F}^{(0)}(Z^2) \right) + (2 \log^2 p_3 + 2h_0 \log p_3 + \frac{1}{2}h_0^2 + \frac{1}{4}\pi^2)\tilde{F}^{(-1)}(Z^2) \right] + \left( p_1 \leftrightarrow p_3, Z \leftrightarrow X \text{ but } Z^2 \leftrightarrow Z^2 \frac{X}{Y} \right) + \left( p_2 \leftrightarrow p_3, Z \leftrightarrow Y \text{ but } Z^2 \leftrightarrow Z^2 \frac{Y}{X} \right)
\]
\[
+ \frac{2p_1^2p_2^2p_3^2}{(\sqrt{-J^2})^3} \left[ \log^2(p_1p_2p_3) + h_2 \log(p_1p_2p_3) + \frac{1}{4}h_2^2 - 1 + \frac{7}{24}\pi^2 \right] - (h_2 + 2 \log(p_1p_2p_3)) \log \sqrt{-J^2} + \log^2 \sqrt{-J^2} \right] + \frac{3}{2}h_0
+ O(\epsilon),
\]  
(2.A.67)

where
\[
\tilde{F}^{(-1)}(x) = \frac{2x}{(x-1)^3},
\]  
(2.A.68)
\[
\tilde{F}^{(0)}(x) = \frac{1}{(x-1)^3} \left[ -1 + x(4 + x) - 4x \log(1 - x) \right],
\]  
(2.A.69)
\[
\tilde{F}^{(1)}(x) = \frac{4x}{(x-1)^3} \left[ -2 \log(1 - x) + \log^2(1 - x) + \text{Li}_2(x) \right]
\]  
(2.A.70)

are coefficients of the series expansion
\[
_2F_1(1,2+\epsilon; -\epsilon; x) = \frac{\tilde{F}^{(-1)}(x)}{\epsilon} + \tilde{F}^{(0)}(x) + \tilde{F}^{(1)}(x)\epsilon + O(\epsilon^2).
\]  
(2.A.71)

Note that both \(I_{1\{000\}}\) and \(I_{1\{222\}}\) can be obtained from \(I_{0\{111\}}\),
\[
I_{1\{000\}} = \frac{1}{2p_1p_2p_3} \left[ p_1 \frac{\partial^2}{\partial p_2 \partial p_3} + p_2 \frac{\partial^2}{\partial p_1 \partial p_3} + p_3 \frac{\partial^2}{\partial p_1 \partial p_2} \right] I_{0+\epsilon\{111\}},
\]  
(2.A.72)
\[
I_{2+\epsilon\{111\}} = \left[ \frac{\partial^2}{\partial p_1^2} - \frac{1}{p_1} \frac{\partial}{\partial p_1} \right] I_{0+\epsilon\{111\}}.
\]  
(2.A.73)

2.A.6. Triviality of \(\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} O \rangle\)

As our analysis shows, for any \(d \geq 3\) and \(\Delta_3\) satisfying unitarity bound the correlation functions \(\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} O \rangle\) and \(\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} J^{\mu_3} \rangle\) are trivial, i.e., they are at most semi-local. The triviality of \(\langle T^{\mu_1\nu_1} T^{\mu_2\nu_2} J^{\mu_3} \rangle\) was proved in [36] through a position space analysis. Our results independently confirm this through calculations in momentum space. In this section we discuss the triviality of \(\langle T^{\mu_1\nu_1} J^{\mu_2} O \rangle\) as an example.

The tensor decomposition of the transverse-traceless part of the \(\langle T^{\mu_1\nu_1} J^{\mu_2} O \rangle\) correlator, the primary and secondary CWIs, and the transverse Ward identities
The independent secondary CWIs are

\[ L_{2,0} A_1^I + R_2 A_2^I = \]
\[ = 2d \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} \text{ in } p_{1\nu_1} \langle \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2\alpha}(p_2) O^I(p_3) \rangle \rangle, \quad (2.A.76) \]
\[ L_{1,0} A_1^I + 2 R_1 A_2^I \]
\[ = -2(d - 2) \cdot \text{coefficient of } p_2^{\mu_1} p_3^{\mu_2} \text{ in } p_{2\mu_2} \langle \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2\alpha}(p_2) O^I(p_3) \rangle \rangle, \quad (2.A.77) \]
\[ L_{2,0} A_2^I = 4d \cdot \text{coefficient of } \delta^{\mu_1\mu_2} \text{ in } p_{1\nu_1} \langle \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2\alpha}(p_2) O^I(p_3) \rangle \rangle \quad (2.A.78) \]
and the transverse Ward identities are

\[ p_1^{\nu_1} \langle \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2\alpha}(p_2) O^I(p_3) \rangle \rangle = p_1^{\nu_1} \langle \langle \frac{\delta T^{\mu_1\nu_1}}{\delta A_2^{\mu_2}}(p_1, p_2) O^I(p_3) \rangle \rangle, \quad (2.A.79) \]
\[ p_{2\mu_2} \langle \langle T^{\mu_1\nu_1}(p_1) J^{\mu_2\alpha}(p_2) O^I(p_3) \rangle \rangle = 2 p_{2\mu_2} \langle \langle \frac{\delta J^{\mu_2\alpha}}{\delta g^{\mu_1\nu_1}}(p_2, p_1) O^I(p_3) \rangle \rangle. \quad (2.A.80) \]

If \( \beta = \Delta_3 - \frac{d}{2} > 0 \), the same reasoning as in section 2.7.2 shows that the right-hand sides of (2.A.79, 2.A.80) vanish in the zero-momentum limit. Then, by the results of section 2.5.3, in the remaining cases the coefficient of \( p_3^0 \) in the series expansion of the right-hand sides of (2.A.79, 2.A.80) is at most ultralocal. The secondary CWIs lead to the following equations

\[ 0 = \frac{(\Delta_3 - 2d - \epsilon)(\Delta_3 - d - 2 - \epsilon)}{2(\Delta_3 - d - 1 - \epsilon)} l_{\frac{d}{2} + \epsilon}^{\Delta_3 - d - 1} \times \]
\[ \times \left[ (\Delta_3^2 - \Delta_3(d - 2 + \epsilon) + \epsilon(2 + d + \epsilon)) a_I^{a_I} + \alpha_2^{a_I} \right], \quad (2.A.81) \]
\[ 0 = \frac{(\Delta_3 - 2d - \epsilon)(\Delta_3 - d - \epsilon)}{2(\Delta_3 - d - 1 - \epsilon)} l_{\frac{d}{2} + \epsilon}^{\Delta_3 - d - 1} \times \]
\[ \times \left[ ((\Delta_3 - \epsilon)^2 - d(\Delta_3 + 2 - \epsilon) + 4(\Delta_3 + 1 + \epsilon)) \alpha_1^{a_I} + 2 \alpha_2^{a_I} \right], \quad (2.A.82) \]
\[ 0 = (\Delta_3 - d - \epsilon) l_{\frac{d}{2} + \epsilon}^{\Delta_3 - d - 1} \left[ 2(2\Delta_3 - d) \alpha_1^{a_I} + \alpha_2^{a_I} \right]. \quad (2.A.83) \]

If the \( \epsilon \to 0 \) limit exists, we find three equations whose only solutions are either \( \alpha_1^{a_I} = \alpha_2^{a_I} = 0 \), or else \( \alpha_2^{a_I} = 2(d - 4) \alpha_1^{a_I} \) and \( \Delta_3 = 2 \). The second solution, however, is not valid since \( \Delta_3 = 2 \) is a special case where the regulator cannot be removed. Instead one must analyse all the special cases when the regulator cannot be removed from \( l_{\frac{d}{2} + \epsilon}^{\Delta_3 - d} \). The analysis is identical to that of section 2.5.3 and leads to the conclusion that \( \alpha_1^{a_I} = \alpha_2^{a_I} = 0 \).
Unlike in section 2.5.3, there are no additional conditions following from the coefficients of \( p_3^{2n} \) or \( p_3^{2n1} \log p_3 \) in the series expansion in \( p_3 \) of the secondary CWIs (2.A.81) - (2.A.83). Recall that such additional constraints arise when the equations following from the coefficients of \( p_3^{2n} \) or \( p_3^{2n1} \log p_3 \) are more singular than the equations following from the zero-momentum limit. In our case, it turns out that all coefficients of \( p_3^{2n} \) or \( p_3^{2n1} \log p_3 \) on the left-hand sides of (2.A.81) - (2.A.83) can be written in terms of \( \frac{1}{2} + \epsilon \left( \Delta_1 + \frac{1}{2} \right) \), accounting for all possible singularities. One can check that \( l_{\frac{1}{2} + \epsilon \left( \Delta_1 + \frac{1}{2} \right)} \) cannot be more singular than \( l_{\frac{3}{2} + \epsilon \left( \Delta_1 + \frac{1}{2} \right)} \) assuming the unitarity bound \( \Delta_3 \geq \frac{d}{2} - 1 \).

2.A.7. Form of scaling anomalies

In this appendix we list the most most general form for the anomalous part of the trace Ward identities, and evaluate the resulting contribution to the reconstruction formulæ. Following the discussion at the end of section 2.8.2, the most general anomalous contribution to the trace Ward identities takes the form

\[
\left\langle T(p_1)O^I(p_2)O^J(p_3) \right\rangle_{\text{anomaly}} = B^{IJ}_1,
\]

\[
\left\langle T(p_1)J^{\mu_2\alpha}(p_2)O^I(p_3) \right\rangle_{\text{anomaly}} = \pi^{I\alpha}(p_2)p_3^{\alpha I} \cdot B^{IJ}_1,
\]

\[
\left\langle T(p_1)J^{\mu_2\alpha_2}(p_2)J^{\mu_3\alpha_3}(p_3) \right\rangle_{\text{anomaly}} = \pi^{I\alpha_2}(p_2)\pi^{I\alpha_3}(p_3) \left[ B^{I\alpha_2_3}p_3^{\alpha I} + B^{I\alpha_2_3}\delta^{\alpha_2_3} \right],
\]

\[
\left\langle T(p_1)T^{\mu_2\nu_2}(p_2)O^I(p_3) \right\rangle_{\text{anomaly}} = \Pi^{\alpha_2\beta_2}(p_2)p_3^{\alpha I} \cdot B^{IJ}_1 + \pi^{\alpha_2\beta_2}(p_2)B^{IJ}_2,
\]

\[
\left\langle T(p_1)T^{\mu_2\nu_2}(p_2)J^{\mu_3\alpha}(p_3) \right\rangle_{\text{anomaly}} = \Pi^{\alpha_2\beta_2}(p_2)\pi^{I\alpha_3}(p_3) \left[ B^{I\alpha_2_3}p_3^{\alpha I}p_1^{\beta I} + B^{I\alpha_2_3}\delta^{\alpha_2_3}p_3^{\beta I} \right] + \pi^{\alpha_2\beta_2}(p_2)\pi^{I\alpha_3}(p_3)p_1^{\alpha I}B^{IJ}_3,
\]

\[
\left\langle T(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3) \right\rangle_{\text{anomaly}} = \Pi^{\alpha_2\beta_2}(p_2)\Pi^{\alpha_3\beta_3}(p_3) \left[ B^{I\alpha_2_3}p_3^{\alpha I}p_1^{\beta I} + B^{I\alpha_2_3}\delta^{\alpha_2_3} \right]
+ \pi^{\alpha_2\beta_2}(p_2)\Pi^{I\alpha_3\beta_3}(p_3)p_1^{\alpha I}B_1 + \Pi^{I\alpha_2\beta_2}(p_2)\Pi^{I\alpha_3\nu_3}(p_3)p_3^{\alpha I}p_1^{\beta I}B_4(p_2 \leftrightarrow p_3)
+ \pi^{\alpha_2\beta_2}(p_2)\Pi^{I\alpha_3\nu_3}(p_3)B_5,
\]

where the form factors \( B_j \) are polynomials in momenta squared \( p_j^2, j = 1, 2, 3 \). In particular \( B^{IJ}_1 \) can be non-zero only if \( 2\Delta = d + 2n \) for some non-negative integer \( n \).

The anomaly in the trace of the stress-energy tensor leads to specific anomalous contributions to the full correlation functions. These anomalous contributions can be recovered from the expressions above by the reconstruction formulæ using the results of the section 2.3.3 and equation (2.3.18). In the presence of the trace anomaly, the following expressions should be added to the right-hand sides of the
corresponding reconstruction equations, such as (3.11.4), listed in chapter 3.

\[ \langle T^{\mu_1 \nu_1}(p_1) \mathcal{O}^I(p_2) \mathcal{O}^J(p_3) \rangle_{\text{anomaly}} = \frac{\pi^{\mu_1 \nu_1}(p_1)}{d-1} B_{1}^{IJ}, \]

\[ \langle T^{\mu_1 \nu_1}(p_1) J^a(p_1) \mathcal{O}^I(p_3) \rangle_{\text{anomaly}} = \frac{\pi^{\mu_1 \nu_1}(p_1)}{d-1} \pi_{\alpha_2}^2(p_2) p_3^{\alpha_3} B_{1}^{aI}, \]

\[ \langle T^{\mu_1 \nu_1}(p_1) J^{\mu_2 a_2}(p_2) J^{\mu_3 a_3}(p_3) \rangle_{\text{anomaly}} = \]

\[ = \frac{\pi^{\mu_1 \nu_1}(p_1)}{d-1} \pi_{\alpha_2}^2(p_2) \pi_{\alpha_3}^3(p_3) \left[ B_{1}^{a_2 a_3 \alpha_2 \alpha_3} p_3^{\alpha_2} p_1^{\alpha_3} + B_{2}^{a_2 a_3 \delta a_2 a_3} \right], \]

\[ \langle T^{\mu_1 \nu_1}(p_1) T^{\mu_2 \nu_2}(p_2) \mathcal{O}^I(p_3) \rangle_{\text{anomaly}} = \frac{\pi^{\mu_1 \nu_1}(p_1)}{d-1} \left[ \Pi_{\alpha_2 \alpha_3}(p_2) p_3^{\alpha_2} p_3^{\alpha_3} B_{1}^I + \pi^{\mu_2 \nu_2}(p_2) B_{2}^I \right] \]

\[ + \left[ [(p_1, \mu_1, \nu_1) \leftrightarrow (p_2, \mu_2, \nu_2)] \right] \]

\[ \langle T^{\mu_1 \nu_1}(p_1) T^{\mu_2 \nu_2}(p_2) J^a(p_3) \rangle_{\text{anomaly}} = \frac{\pi^{\mu_1 \nu_1}(p_1)}{d-1} \langle T(p_1) T^{\mu_2 \nu_2}(p_2) J^a(p_3) \rangle_{\text{anomaly}} \]

\[ + \left[ [(p_1, \mu_1, \nu_1) \leftrightarrow (p_2, \mu_2, \nu_2)] \right] \]

\[ \frac{\pi^{\mu_1 \nu_1}(p_1) \pi^{\mu_2 \nu_2}(p_2)}{d-1} \pi_{\alpha_3}^3(p_3) B_{1}^{a \alpha_3} B_{3}, \]

\[ \langle T^{\mu_1 \nu_1}(p_1) T^{\mu_2 \nu_2}(p_2) T^{\mu_3 \nu_3}(p_3) \rangle_{\text{anomaly}} = \frac{\pi^{\mu_1 \nu_1}(p_1)}{d-1} \langle T(p_1) T^{\mu_2 \nu_2}(p_2) T^{\mu_3 \nu_3}(p_3) \rangle_{\text{anomaly}} \]

\[ \left[ [(p_1, \mu_1, \nu_1) \leftrightarrow (p_2, \mu_2, \nu_2)] \right] + \left[ [(p_1, \mu_1, \nu_1) \leftrightarrow (p_3, \mu_3, \nu_3)] \right] \]

\[ - \frac{\pi^{\mu_1 \nu_1}(p_1) \pi^{\mu_2 \nu_2}(p_2)}{d-1} \left[ \Pi_{\alpha_3 \beta_3}(p_3) B_4(p_1, p_2, p_3) + \pi^{\mu_2 \nu_3}(p_3) B_5(p_1, p_2, p_3) \right] \]

\[ \left[ [(p_1, \mu_1, \nu_1) \leftrightarrow (p_3, \mu_3, \nu_3)] \right] + \left[ [(p_2, \mu_2, \nu_2) \leftrightarrow (p_3, \mu_3, \nu_3)] \right] \]

\[ + \frac{\pi^{\mu_1 \nu_1}(p_1) \pi^{\mu_2 \nu_2}(p_2) \pi^{\mu_3 \nu_3}(p_3)}{d-1} B_5. \quad (2.85) \]

If present, such contributions should be added to the reconstruction formulae for the corresponding 3-point functions presented in chapter 3.

In particular, in $d = 4$ the anomalous trace Ward identity is

\[ \langle T \rangle = \frac{1}{2} P_{IJ} \phi_0^I \Box^{-2} \phi_0^J + \frac{1}{4} \kappa F^{\mu \nu a} F_{\mu \nu a} + a E_4 + c W^2, \quad (2.86) \]

where $W^2$ is the square of the Weyl tensor and $E_4$ is the Euler density, (2.8.3). From this we find

\[ B_{1}^{IJ} = (p_1^2)^{-1} P_{IJ}^{\mu \nu}, \quad B_{1}^{\alpha I} = 0, \]

\[ B_{2}^{\alpha \beta} = -\kappa \delta_{a_1 a_2}, \quad B_{2}^{a_2 a_3} = \frac{1}{2} (p_2^2 + p_1^2 - p_3^2) \kappa a_2 a_3, \]

\[ B_{2}^{I} = 0, \quad B_{3}^{a} = B_{2}^{a} = B_{3}^{a} = 0. \quad (2.87) \]
For $\langle T(p_1)T^{\mu_2\nu_2}(p_2)T^{\mu_3\nu_3}(p_3) \rangle$,

$$
B_1 = 8(a + c), \quad B_2 = 8(p_1^2 - p_2^2 - p_3^2)(a + c),
B_3 = -2J^2(a + c) + 4p_2^2p_3^2c, \quad B_4 = -\frac{8}{3}ap_3^2,
B_5 = \frac{4}{9}aJ^2. \quad (2.A.88)
$$

These results are in agreement with the results of sections 2.8.2 and 2.8.3.

### 2.A.8. Identities with projectors

The projectors are defined as

$$
\pi_\mu^\alpha = \delta_\mu^\alpha - \frac{p_\mu p_\alpha}{p^2}, \quad (2.A.89)
$$

$$
\Pi_{\alpha\beta}^{\mu\nu} = \frac{1}{2} \left( \pi_\mu^\alpha \pi_\nu^\beta + \pi_\mu^\beta \pi_\nu^\alpha \right) - \frac{1}{d - 1} \pi^{\mu\nu} \pi_{\alpha\beta}, \quad (2.A.90)
$$

$$
\Pi^{\mu\nu\rho\sigma} = \delta^{\rho\sigma} \delta_{\beta\alpha} \Pi^{\mu\nu}_{\alpha\beta}, \quad (2.A.91)
$$

One can find the following identities:

$$
p_\mu \pi^{\mu\nu} = p_\mu \Pi^{\mu\nu\rho\sigma} = 0,
\delta_\mu \pi^{\mu\nu} = \pi_\mu^\nu = d - 1,
\Pi^{\mu\nu\rho}_{\rho} = \delta_\rho \Pi^{\mu\nu\rho\sigma} = \pi_{\rho\sigma} \Pi^{\mu\nu\rho\sigma} = 0,
\Pi^{\mu\nu\rho}_{\rho} = \delta_\rho \Pi^{\mu\nu\rho\sigma} = \pi_{\rho\sigma} \Pi^{\mu\nu\rho\sigma} = \frac{(d + 1)(d - 2)}{2(d - 1)} \pi^{\mu\nu},
\Pi^{\mu\nu\rho\sigma} \Pi^{\mu\nu\rho\sigma} = \frac{1}{2}(d + 1)(d - 2),
\pi_{\alpha}^\mu \pi_\nu^\alpha = \pi_\nu^\mu,
\Pi_{\alpha\beta}^{\mu\nu\rho\sigma} = \Pi_{\rho\sigma}^{\mu\nu},
\Pi^{\mu\nu\rho\sigma} \Pi^{\alpha\rho\beta\sigma} = \frac{d - 3}{2(d - 1)} \Pi^{\mu\nu\rho\sigma}. \quad (2.A.92)
$$

Basic derivatives can be calculated directly. Denoting $\partial_\mu = \frac{\partial}{\partial p_\mu}$ we find

$$
\partial_\kappa \pi_{\mu\nu} = -\frac{p_\mu}{p^2} \pi_{\nu\kappa} - \frac{p_\nu}{p^2} \pi_{\mu\kappa},
\partial_\kappa \Pi^{\mu\nu\rho\sigma} = -\frac{p_\mu}{p^2} \Pi^{\nu\rho\kappa} - \frac{p_\nu}{p^2} \Pi^{\mu\rho\kappa} - \frac{p_\rho}{p^2} \Pi^{\kappa\nu\rho} - \frac{p_\sigma}{p^2} \Pi^{\mu\kappa\rho},
\pi_{\alpha}^\mu \partial_\kappa \pi_\nu^\alpha = -\frac{p_\nu}{p^2} \pi_{\mu\kappa}^\mu,
\pi_{\mu\kappa} \partial_\alpha \pi_\nu^\alpha - \pi_{\mu\alpha} \partial_\alpha \pi_\nu^\kappa = -(d - 2) \frac{p_\nu}{p^2} \pi_{\mu\kappa} + \frac{p_\kappa}{p^2} \pi_{\mu\nu}^\nu. \quad (2.A.93)
$$
\[
\Pi_{\alpha\beta} \partial_\kappa \Pi_{\rho\sigma} = - \frac{p_\rho}{p^2} \Pi_{\mu\nu} - \frac{p_\sigma}{p^2} \Pi_{\mu\nu},
\]
\[
\Pi_{\kappa\beta} \partial_\alpha \Pi^\gamma_{\rho\sigma} - \Pi_{\mu\alpha} \beta \partial_\alpha \Pi^\gamma_{\rho\sigma} = - \frac{1}{2} \frac{d - 1}{p^2} [p_\beta \Pi_{\mu\nu} + p_\sigma \Pi_{\mu\nu}^\gamma] + \frac{p_\kappa}{p^2} \Pi_{\mu\nu}^\gamma. \tag{2.A.94}
\]

Analogous expressions with two derivatives are
\[
\pi_{\alpha}^\mu \partial^2 \pi_{\alpha}^\nu = - \frac{2}{p^2} \pi_{\alpha}^\mu,
\]
\[
p_\alpha \pi_{\mu}^\nu \partial_\alpha \pi_{\nu}^\beta = \frac{p_\nu}{p^2} \pi_{\mu}^\kappa,
\]
\[
\Pi_{\alpha\beta} \partial^2 \Pi_{\rho\sigma} = - \frac{4}{p^2} \Pi_{\mu\nu},
\]
\[
p_\gamma \Pi_{\alpha\beta} \partial_\gamma \partial_\kappa \Pi_{\rho\sigma} = \frac{p_\rho}{p^2} \Pi_{\mu\nu} + \frac{p_\sigma}{p^2} \Pi_{\mu\nu}. \tag{2.A.95}
\]

For the semi-local operators defined in (2.3.14) and (2.3.15) we find
\[
\pi_{\alpha}^\mu \partial_\kappa j_{\text{loc}}^\alpha = \frac{1}{p^2} \pi_{\alpha}^\mu r,
\]
\[
\pi_{\alpha}^\mu \partial_\alpha j_{\text{loc}}^\kappa - \pi_{\mu}^\alpha \partial_\kappa j_{\text{loc}}^\alpha = \frac{d - 3}{p^2} \pi_{\mu}^\kappa r + \frac{1}{p^2} \pi_{\mu}^\kappa \pi_{\alpha}^\beta \partial_\alpha r - \frac{p_\kappa}{p^2} \pi_{\mu}^\alpha \partial_\alpha r,
\]
\[
\pi_{\alpha}^\mu \partial^2 j_{\text{loc}}^\alpha = \frac{2}{p^2} \pi_{\mu}^\alpha \partial_\alpha r,
\]
\[
p_\alpha \pi_{\mu}^\nu \partial_\alpha \partial_\kappa j_{\text{loc}}^\beta = - \frac{2}{p^2} \pi_{\mu}^\kappa r + \frac{1}{p^2} \pi_{\mu}^\kappa p_\sigma \partial_\alpha r,
\]
\[
\Pi_{\alpha\beta} \partial_\kappa t_{\text{loc}}^{\alpha\beta} = \frac{2}{p^2} \Pi_{\alpha\kappa} R^\alpha,
\]
\[
\Pi_{\alpha\beta} \partial_\kappa t_{\text{loc}}^{\alpha\beta} - \Pi_{\mu\alpha} \beta \partial_\kappa t_{\text{loc}}^{\alpha\beta} = \frac{d - 2}{p^2} \Pi_{\mu\kappa}^\alpha R^\alpha + \frac{p_\beta}{p^2} \Pi_{\mu\kappa}^\alpha \partial_\beta R^\alpha - \frac{p_\kappa}{p^2} \Pi_{\mu\alpha}^\beta \partial_\alpha R^\beta,
\]
\[
\Pi_{\alpha\beta} \partial^2 t_{\text{loc}}^{\alpha\beta} = \frac{4}{p^2} \Pi_{\alpha\kappa}^\beta \partial_\alpha R^\beta,
\]
\[
p_\gamma \Pi_{\alpha\beta} \partial_\gamma \partial_\kappa t_{\text{loc}}^{\alpha\beta} = - \frac{4}{p^2} \Pi_{\alpha\kappa}^\beta R^\beta + \frac{2}{p^2} \Pi_{\alpha\kappa}^\beta p_\beta \partial_\beta R^\alpha. \tag{2.A.96}
\]

2.A.9. Constructions with helicity tensors

This appendix summarises our notation for the various contractions of helicity tensors. In \(d = 3\) the helicity tensors satisfy
\[
\Pi_{\mu\nu\rho\sigma}(p) = \frac{1}{2} \sum_{s = \pm 1} \epsilon_{\mu\nu}^{(s)}(p) \epsilon_{\rho\sigma}^{(s)}(p), \tag{2.A.97}
\]
\[
\epsilon_{\mu\nu}^{(s)}(p) \epsilon_{\mu\nu}(p) = 2 \delta^{ss'}, \tag{2.A.98}
\]
\[
\bar{\epsilon}_{\mu\nu}^{(s)}(p) = \epsilon_{\mu\nu}^{(s)}(-p). \tag{2.A.99}
\]
where overbar denotes the complex conjugation.

We use the following symbols,

\[ \theta^{(s_3)}(p_i) = \epsilon_{\alpha \beta}^{(s_3)}(-p_3)p_1^\alpha p_1^\beta = \epsilon_{\alpha \beta}^{(s_3)}(-p_3)p_2^\alpha p_2^\beta, \]  
\[ (2.A.100) \]

\[ \theta^{(s_2 s_3)}(p_i) = \epsilon_{\alpha_1 \beta_1}^{(s_2)}(-p_2)\epsilon_{\alpha_2 \beta_2}^{(s_3)}(-p_3)\delta_{\alpha_1 \alpha_2} \delta_{\beta_1 \beta_2} = \epsilon_{\alpha \beta}^{(s_2)}(-p_2)\epsilon_{(s_3)\alpha \beta}(-p_3), \]  
\[ (2.A.101) \]

\[ \theta^{(s_1 s_2 s_3)}(p_i) = \epsilon_{\alpha_1 \beta_1}^{(s_1)}(-p_1)\epsilon_{\alpha_2 \beta_2}^{(s_2)}(-p_2)\epsilon_{\alpha_3 \beta_3}^{(s_3)}(-p_3)t^{\alpha_1 \alpha_2 \alpha_3} \delta_{\beta_1 \beta_2 \beta_3}, \]  
\[ (2.A.102) \]

where \( t_{\alpha_1 \alpha_2 \alpha_3} = \delta_{\alpha_1 \alpha_2} p_{1 \alpha_3} + \delta_{\alpha_2 \alpha_3} p_{2 \alpha_1} + \delta_{\alpha_3 \alpha_1} q_{3 \beta_2}. \) In addition, the following contractions arise in the holographic analysis

\[ \Theta^{(s_3)}(p_i) = \pi_{\alpha \beta}^{(s_3)}(p_n)\epsilon_{\alpha \beta}^{(s_3)}(-p_3) = -\theta_1^{(s_3)}(p_i), \quad n = 1, 2, 3, \]  
\[ (2.A.103) \]

\[ \Theta^{(s_2 s_3)}(p_i) = \pi_{\alpha \beta}^{(s_2)}(p_1)\epsilon_{\alpha \gamma}^{(s_2)}(-p_2)\epsilon_{\beta \gamma}^{(s_3)}(-p_3) = \theta^{(s_2 s_3)}(p_i) - \frac{p_1^\alpha p_1^\beta}{p_1^2} \epsilon_{\alpha \gamma}^{(s_2)}(-p_2)\epsilon_{\beta \gamma}^{(s_3)}(-p_3), \]  
\[ (2.A.104) \]

\[ \Theta^{(s_1 s_2 s_3)}(p_i) = \delta^{\alpha_1 \beta_2} \delta^{\alpha_2 \beta_3} \delta^{\alpha_3 \beta_1} \epsilon_{\alpha_1 \beta_1}^{(s_1)}(-p_1)\epsilon_{\alpha_2 \beta_2}^{(s_2)}(-p_2)\epsilon_{\alpha_3 \beta_3}^{(s_3)}(-p_3). \]  
\[ (2.A.105) \]

All the symbols depend on magnitudes of momenta only.

Define

\[ S_1 = -p_1^2 + (s_2 p_2 + s_3 p_3)^2, \]
\[ S_2 = -p_2^2 + (s_3 p_3 + s_1 p_1)^2, \]
\[ S_3 = -p_3^2 + (s_1 p_1 + s_2 p_2)^2. \]  
\[ (2.A.106) \]

Using the exact presentation (2.9.14) one can find the following expressions for the defined symbols,

\[ \theta^{(s_3)}(p_i) = \frac{J^2}{4\sqrt{2} p_3^2}, \]  
\[ (2.A.107) \]

\[ \theta^{(s_2 s_3)}(p_i) = \frac{1}{8 p_2^2 p_3^2} S_1^2, \]  
\[ (2.A.108) \]

\[ \theta^{(s_1 s_2 s_3)}(p_i) = \frac{J^2}{32 \sqrt{2} p_1^2 p_2^2 p_3^2} (S_1 + S_2 + S_3)^2 \]
\[ = \frac{J^2}{32 \sqrt{2} p_1^2 p_2^2 p_3^2} (s_1 p_1 + s_2 p_2 + s_3 p_3)^4, \]  
\[ (2.A.109) \]

and similarly,

\[ \Theta^{(s_3)}(p_i) = -\frac{J^2}{4\sqrt{2} p_n^2 p_3^2}, \quad n = 1, 2, \quad \Theta^{(s_3)}(p_i) = 0, \]  
\[ (2.A.110) \]
2. Implications of conformal invariance in momentum space

\[ \Theta^{(s_2s_3)}(p_i) = \frac{1}{8p_1^2p_2^3}S_1^2 - \frac{J^2}{16p_1^2p_2^2p_3^2}S_1, \quad (2.A.111) \]

\[ \frac{p_1^2}{p_1^2}p_1^2\epsilon^{(s_2)}(-p_2)\epsilon^{(s_3)}(-p_3) = \frac{J^2}{16p_1^2p_2^2p_3^2}S_1, \quad (2.A.112) \]

\[ \Theta^{(s_1s_2s_3)}(p_i) = -\frac{1}{16\sqrt{2}p_1^2p_2^2p_3^2}S_1S_2S_3, \quad (2.A.113) \]

where

\[ J^2 = (p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(-p_1 + p_2 + p_3)(p_1 + p_2 + p_3) \quad (2.A.114) \]

is defined in (2.6.18).

We define the helicity projected operators as

\[ T(p) = \delta^{\mu\nu}T_{\mu\nu}(p), \quad (2.A.115) \]

\[ T^{(s)}(p) = \frac{1}{2}\epsilon^{(s)\mu\nu}(-p)T_{\mu\nu}(p), \quad (2.A.116) \]

\[ \Upsilon(p_1, p_2) = \delta^{\mu\nu}\delta^{\rho\sigma}\frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}}(p_1, p_2), \quad (2.A.117) \]

\[ \Upsilon^{(s)}(p_1, p_2) = \frac{1}{2}\delta^{\mu\nu}\epsilon^{(s)\rho\sigma}(-p_2)\frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}}(p_1, p_2), \quad (2.A.118) \]

\[ \Upsilon^{(s_1s_2)}(p_1, p_2) = \frac{1}{4}\epsilon^{(s_1)\mu\nu}(-p_1)\epsilon^{(s_2)\rho\sigma}(-p_2)\frac{\delta T_{\mu\nu}}{\delta g^{\rho\sigma}}(p_1, p_2). \quad (2.A.119) \]