Conformal symmetry and holographic cosmology

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Chapter 5

General relativity and cosmology

5.1. Basic tools

In this section we would like to discuss some basic tools which we will use in our analysis. These tools cover well-known, standard results in general relativity, see [70, 76, 77]. Our conventions follow [70], we will work both in Euclidean and mostly plus (− + + . . . +) Lorentzian signatures and

\[
R_{\mu\nu} = R_{\alpha\mu\alpha\nu}.
\]

5.1.1. ADM decomposition

The ADM formalism (Arnowitt, Deser, Misner) is a Hamiltonian approach to gravity. Consider a \( D \) dimensional manifold with Riemannian or Lorentzian structure \( g \) and a foliation \( \Sigma_z \) parametrised by \( z \). In the Lorentzian case the foliation is assumed to be spacelike, i.e., the metric induced on \( \Sigma \) is Riemannian. In such case \( z \) is interpreted as time. For that reason, both in the Lorentzian and Euclidean cases, we will denote the derivative with respect to the \( z \) coordinate by a dot.

The normal unit vector to \( \Sigma_z \) and the induced metric \( \gamma_{\mu\nu} \) on \( \Sigma_z \) can be written as

\[
\gamma^\mu_\nu = \delta^\mu_\nu - n^\mu n_\nu.
\]

The tensor \( \gamma^\mu_\nu (z) \) is a projector onto \( \Sigma_z \) and therefore \( \gamma^\mu_\nu |_{\Sigma_z} = \delta^\mu_\nu \). Finally, we can introduce a vector field \( z^\mu \) by \( z^\mu \partial_\mu z = 1 \). The lapse and shift functions \( N \) and \( N^\mu \)
are defined as the normal and tangent components of $z^\mu$ to $\Sigma_z$,

$$Nn^\mu = (g_{\alpha\beta}z^\alpha n^\beta) n^\mu, \quad N^\mu = \gamma^\mu_\alpha z^\alpha. \quad (5.1.4)$$

With all these definitions one can check that the following 1-forms

$$d\hat{x}^\mu = dx^\mu - (N^\mu + Nn^\mu) dz \quad (5.1.5)$$

constitute a basis for the cotangent space $T^*\Sigma_z$. The full metric $g_{\mu\nu}$ can be then decomposed as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \sigma N^2 dz^2 + \gamma_{\mu\nu}(d\hat{x}^\mu + N^\mu dz)(d\hat{x}^\nu + N^\nu dz). \quad (5.1.6)$$

The parameter $\sigma$ is a sign, $\sigma = \pm 1$, depending on the signature:

- $\sigma = +1$ for the Euclidean signature,
- $\sigma = -1$ for the mostly plus Lorentzian signature ($- + + \ldots +$).

Each slice of the foliation $\Sigma_z$ can be characterised by its intrinsic and extrinsic curvatures, respectively,

$$\hat{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}[\gamma_{\alpha\beta}], \quad (5.1.7)$$
$$K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu} = \gamma^\alpha_{\mu} \nabla^\alpha n_{\nu} = \frac{1}{2N} \left( \hat{\nabla}_{\mu} N_{\nu} - \hat{\nabla}_{\nu} N_{\mu} \right), \quad (5.1.8)$$

where $\mathcal{L}_n$ is the Lie derivative in the direction of the vector $n$ and $\hat{\nabla}_{\mu}$ is a covariant derivative of the metric $\gamma$. One can show that $\hat{\nabla}_{\mu}$ is the projection of the full covariant derivative $\nabla_{\mu}$ onto $\Sigma_z$. Now various projections of the full Riemann tensor $R_{\mu\nu\rho\sigma}$ on directions parallel and transverse to the foliation can be expressed in terms of the following Gauss-Codazzi equations,

$$\gamma^\alpha_{\mu} \gamma^\beta_{\nu} \gamma^\gamma_{\rho} \gamma^\delta_{\sigma} R_{\alpha\beta\gamma\delta} = \hat{R}_{\mu\nu\rho\sigma} + K_{\mu\sigma} K_{\nu\rho} - K_{\mu\rho} K_{\nu\sigma}, \quad (5.1.9)$$
$$\gamma^\alpha_{\mu} n^\beta R_{\alpha\beta} = \hat{\nabla}^\alpha K^\beta_{\mu} - \hat{\nabla}_{\mu} K^\beta, \quad (5.1.10)$$
$$n^\alpha n^\beta R_{\mu\alpha\nu\beta} = -n^\alpha \nabla^\alpha K_{\mu\nu} - K_{\mu\alpha} K^\alpha_{\nu}, \quad (5.1.11)$$

where $K = K^\alpha_{\alpha}$. Finally, one can manipulate the Gauss-Codazzi equations in order to obtain the following form

$$K^2 - K_{\mu\nu} K^{\mu\nu} - \hat{R} = 2G_{\mu\nu} n^\mu n^\nu, \quad (5.1.12)$$
$$\hat{\nabla}^\alpha K^\mu_{\mu} - \hat{\nabla}_{\mu} K = G_{\alpha\beta} \gamma^\alpha_{\mu} n^\beta, \quad (5.1.13)$$
$$\mathcal{L}_n K_{\mu\nu} + K K_{\mu\nu} - 2K_{\mu}^\alpha K_{\alpha\nu} - \hat{R}_{\mu\nu} = -\gamma^\alpha_{\mu} \gamma^\beta_{\nu} R_{\alpha\beta}, \quad (5.1.14)$$

where $G_{\mu\nu}$ denotes Einstein tensor for the full metric $g_{\mu\nu}$. The right hand sides of these equations can be connected to the stress-energy tensor of matter via the Einstein equations
5.1. Basic tools

5.1.2. Action principle

In this thesis we will consider Einstein equations both in Euclidean and Lorentzian signatures. We will consider almost exclusively a single scalar coupled to matter, for which the action is

\[ S = \frac{\sigma}{2\kappa^2} \int d^D x \sqrt{|g|} \left[ -R + \partial_\mu \Phi \partial^\mu \Phi + 2V(\Phi) \right] \quad (5.1.15) \]

where \( \kappa^2 = 8\pi G_D \), \( G_D \) is the \( D \)-dimensional Newton constant and \( V(\Phi) \) is a potential for \( \Phi \). This action can be supplemented by the Gibbons-Hawking term if the boundary of the manifold is non-empty.

In this normalisation there is an overall \( \kappa^{-2} \) factor in front of the entire action. This is a natural normalisation from the string theory point of view as we will discuss in section 6.1.1. Another convention commonly used is a cosmological convention, where \( 1/(2\kappa^2) \) multiplies the Ricci scalar only. It can be obtained from (5.1.15) by a redefinition of the scalar field as \( \Phi \mapsto \kappa \Phi \) together with the redefinition of the coupling constants in the potential. Finally, in section 6.6 we will use yet another convention, where we redefine the potential in (5.1.15) according to \( V(\Phi) \mapsto \kappa^2 V(\Phi) \).

The equations of motion following from (5.1.15) are

\[ G_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad \Box \Phi - V'(\Phi) = 0 \quad (5.1.16) \]

where

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad \kappa^2 T_{\mu\nu} = \frac{2\sigma}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} S_{\text{matter}}. \quad (5.1.17) \]

The sign is chosen such that in both Euclidean and Lorentzian signatures we obtain the same stress-energy tensor. In case of (5.1.15),

\[ T_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \left[ g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi + 2V(\phi) \right]. \quad (5.1.18) \]

Let us now return to the ADM formalism. Using Gauss-Codazzi equations one finds that the action (5.1.15) takes form

\[ S = \frac{1}{2\kappa^2} \int d^D x \sqrt{\gamma} N \left[ K_{\mu\nu} K^{\mu\nu} - K^2 + \frac{1}{N^2} \left( \dot{\Phi} - N^\mu \partial_\mu \Phi \right)^2 \right. \]
\[ + \sigma \left( -\dot{R} + \gamma^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + 2V(\Phi) \right) \]. \quad (5.1.19) \]

Using (5.1.8) one can rewrite the action purely in terms of \( N \), \( N^\mu \) and \( \gamma_{\mu\nu} \). Since \( N \) and \( N^\mu \) have no dynamics, the Hamilton equations are constraint equations

\[ \frac{\delta S}{\delta N} = 0, \quad \frac{\delta S}{\delta N^\mu} = 0 \quad (5.1.20) \]
known as Hamiltonian and momentum constraints respectively. The canonical momenta for the metric and the scalar field are

\[
\Pi_{\mu\nu} = -\frac{2}{\sqrt{\gamma}} \delta \gamma_{\mu\nu} \frac{\delta L}{\delta \gamma_{\mu\nu}} (\kappa^2 L) = K_{\mu\nu} - \gamma_{\mu\nu} K, \tag{5.1.21}
\]

\[
\Pi_{\Phi} = \frac{1}{\sqrt{\gamma}} \delta L \frac{\delta L}{\delta \Phi} (\kappa^2 L) = \frac{1}{N^2} \dot{\Phi}, \tag{5.1.22}
\]

and the Hamilton equations are

\[
\dot{\Pi}_{\mu\nu} = -\frac{2}{\sqrt{\gamma}} \delta \gamma_{\mu\nu} (\kappa^2 L), \quad \dot{\Pi}_{\Phi} = -\frac{1}{\sqrt{\gamma}} \delta \Phi (\kappa^2 L). \tag{5.1.23}
\]

All these equations result in (5.1.12) - (5.1.14) with right hand sides expressed via the Einstein equations.

In section 6.6 we will use the gauge choice where \( N = 1 \) and \( N^\mu = 0 \), so the metric (5.1.6) takes form

\[
ds^2 = \sigma dz^2 + \gamma_{ij} dx^i dx^j. \tag{5.1.24}
\]

In this case direction \( z \) is perpendicular to the foliation at each point, therefore we may assume that the Latin indices take values \( i = 1, 2, \ldots, D - 1 \) while Greek indices take values \( \mu = z, 1, 2, \ldots, D - 1 \). In this gauge the extrinsic curvature (5.1.8) is

\[
K_{ij} = \frac{1}{2} \dot{\gamma}_{ij} \tag{5.1.25}
\]

and the Gauss-Codazzi equations read

\[
K^2 - K_{ij} K^{ij} - R = T_{zz}, \tag{5.1.26}
\]

\[
\nabla_i K^i_j - \nabla_j K = \frac{1}{2} T_{jz}, \tag{5.1.27}
\]

\[
\dot{K}^i_j + K K^i_j - R^i_j - \frac{1}{2} \left( T^i_j - \frac{1}{d-1} \delta^i_j T \right), \tag{5.1.28}
\]

where \( T = T^\mu_\mu \) is the trace of the stress-energy tensor and \( \dot{K}^i_j = \partial_z (\gamma^{jk} K_{kj}) \).

### 5.1.3. Vacuum solutions

An important class of spacetimes are solutions with the cosmological constant \( \Lambda \). The action (5.1.15) takes form

\[
S = \frac{\sigma}{2\kappa^2} \int d^D x \sqrt{|g|} \left[ -R + 2\Lambda \right], \tag{5.1.29}
\]
The homogeneous and isotropic solutions to the Einstein equations can be found as follows. Let $\mathbb{R} \times S^D$ be a compactification of the usual Minkowski space $\mathbb{R}^{D+1}$ with coordinates $X_0, X_1, \ldots X_D$ and the standard metric

$$d s^2 = -dX_0^2 + \sum_{j=1}^{D} dX_j^2.$$  \hspace{1cm} (5.1.30)

Euclidean anti-de Sitter space (EAdS, AdS), known also as the hyperbolic space, is defined as a Riemannian submanifold of $\mathbb{R}^{D+1}$ satisfying

$$-X_0^2 + \sum_{j=1}^{D} X_j^2 = -L_{\text{AdS}}^2,$$  \hspace{1cm} (5.1.31)

with the induced metric. The parameter $L_{\text{AdS}} > 0$ is the radius of the AdS space. This version of AdS is called Euclidean, since the induced metric is positive definite. To see it notice that by definition $|X_0| > L_{\text{AdS}}$ so one can introduce the coordinates

$$X_0 = L_{\text{AdS}} \cosh u, \quad X_j = L_{\text{AdS}} \sinh u e_j, \quad j = 1, \ldots, D,$$  \hspace{1cm} (5.1.32)

where $e_j$ are the standard coordinates for the unit sphere $S^{D-1} \subseteq \mathbb{R}^D$. These coordinates cover the entire AdS space and the induced metric is

$$d s^2 = L_{\text{AdS}}^2 \left[ du^2 + \sinh^2 u \, d\Omega_{D-1}^2 \right],$$  \hspace{1cm} (5.1.33)

which is Euclidean indeed.

Due to the fact that $|X_0| > L_{\text{AdS}}$, the manifold has two identical components as in figure 5.1. By Euclidean AdS we mean a single component with $X_0 > 0$. In a similar fashion one can define a Lorentzian version of AdS by starting from space with two time-like directions. Such a Lorentzian AdS is a connected space and its relation to the Euclidean AdS is similar to the relation between Schwarzschild and Euclidean Schwarzschild geometry. In this thesis we will concentrate on the Euclidean version of AdS, but interesting phenomena can arise when the Lorentzian dynamics is considered.

Let us define two more coordinate systems that will be useful in the upcoming analysis. Consider coordinates on AdS denoted by $(z, \bm{x})$, where $\bm{x}$ is a $(D-1)$ dimensional vector and $z \in (0, \infty)$. The coordinates are defined as

$$X_0 = \frac{L_{\text{AdS}}}{2z} (1 + x^2 + z^2),$$  \hspace{1cm} (5.1.34)

$$X_j = \frac{L_{\text{AdS}}x_j}{z}, \quad j = 1, \ldots, D-1,$$  \hspace{1cm} (5.1.35)

$$X_D = \frac{L_{\text{AdS}}}{2z} (1 - x^2 - z^2).$$  \hspace{1cm} (5.1.36)
5. General relativity and cosmology

Figure 5.1: On the left: two dimensional hyperbolic (AdS) space (blue) and de Sitter space (yellow). Note, however, that the metric on the dS space should be Lorentzian, not Euclidean as suggested by the plot. On the right: a model of the hyperbolic space as a Poincaré disc. The boundary of the disc is infinitely far away from its center. (M.C. Escher, ‘Circle Limit III’)

These coordinates do not cover the entire AdS space, since $X_0 + X_D = L_{AdS}/z > 0$. However, the induced metric is particularly simple,

$$ds^2 = \frac{L_{AdS}^2}{z^2} \left[ dz^2 + dx^2 \right]. \quad (5.1.37)$$

Yet another useful coordinate system we are going to use in cosmology can be obtained by the substitution

$$z = e^{H r}, \quad y = L_{AdS} x, \quad H = L_{AdS}^{-1}, \quad (5.1.38)$$

which leads to the metric

$$ds^2 = dr^2 + e^{-2H r} dy^2. \quad (5.1.39)$$

Lorentzian de Sitter (dS) space can be defined as a Lorentzian submanifold of the Minkowski space $\mathbb{R}^{D+1}$ satisfying

$$- X_0^2 + \sum_{j=1}^{D} X_j^2 = L_{dS}^2, \quad (5.1.40)$$

with the induced metric. In this case the substitution

$$X_0 = L_{dS} \sinh u, \quad X_j = L_{dS} \cosh u \ e_j, \quad j = 1, \ldots, D, \quad (5.1.41)$$

leads to the Lorentzian metric

$$ds^2 = L_{dS}^2 \left[ -du^2 + \cosh^2 u \ d\Omega_{D-1}^2 \right]. \quad (5.1.42)$$

222
Essentially the same substitutions as for the AdS metric with some signs changed lead to the following metrics on the dS spacetime,

\[ ds^2 = \frac{L_{\text{dS}}^2}{z^2} \left[ -dz^2 + dx^2 \right], \quad (5.1.43) \]

\[ ds^2 = -dt^2 + e^{2Ht} dy^2, \quad H = L_{\text{dS}}^{-1}. \quad (5.1.44) \]

As we can see the Euclidean AdS space and Lorentzian dS space can be related by some analytic continuations of coordinates or radii. In some sense we can think about the dS space as a space with the positive square of the radius \( L^2 > 0 \) and the AdS space as the same space with \( L^2 < 0 \). Indeed, both de Sitter and anti-de Sitter space are homogeneous spaces,

\[ \text{AdS}_D \equiv \frac{SO(D,1)}{SO(D)}, \quad \text{dS}_D \equiv \frac{SO(D,1)}{SO(D-1,1)} \quad (5.1.45) \]

and hence they admit metrics with a constant curvature. By direct calculations one shows that all metrics discussed have the constant Ricci scalar,

\[ R = \frac{D(D-1)}{L^2}, \quad (5.1.46) \]

where \( L^2 = L_{\text{dS}}^2 \) for de Sitter space and \( L^2 = -L_{\text{AdS}}^2 \) for anti-de Sitter space. This means that the AdS space solves vacuum Einstein equations with a negative cosmological constant while the dS space solves them with a positive one,

\[ \Lambda = \frac{(D-1)(D-2)}{2L^2}. \quad (5.1.47) \]

In particular the full Riemann and Ricci tensors can be rewritten in terms of the metrics as

\[ R_{\mu\nu\rho\sigma} = \frac{1}{L^2} \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right), \quad (5.1.48) \]

\[ R_{\mu\nu} = R_{\mu\alpha\nu}^\alpha = \frac{D-1}{L^2} g_{\mu\nu}. \quad (5.1.49) \]

Both de Sitter and anti-de Sitter spaces have a boundary. In the original coordinates the boundary occurs for \( |X_0| \to \infty \), which corresponds to \( z \to 0 \) in (5.1.43, 5.1.37). The boundary of dS space is spacelike, but the boundary of (Lorentzian) AdS is timelike. Although the boundary is infinitely far away from the interior of AdS, it has a peculiar property that – in the Lorentzian version – light reaches the boundary in a finite global time. Therefore the Lorentzian AdS space is not hyperbolic. This means that, in order to have an unambiguous evolution, initial conditions must be supplied by a specification of boundary conditions.

Since both de Sitter and anti-de Sitter space are quotient spaces of the Lie group \( SO(D,1) \), their symmetry group is precisely \( SO(D,1) \). For the AdS space
in coordinates (5.1.37), the subgroup isomorphic to the Poincaré group $P_D = SO(D) \times \mathbb{R}^D \subseteq SO(1, D)$ acts as rotations and translations

$$x^\mu \mapsto \Lambda^\mu_\alpha x^\alpha + a^\mu, \quad (5.1.50)$$

for a matrix of rotation $\Lambda$ in $\mathbb{R}^D$ and a vector $a \in \mathbb{R}^D$, leaving the $z = 0$ plane invariant. The scalings by $\lambda$ act as

$$z \mapsto \lambda z, \quad x^\mu \mapsto \lambda x^\mu \quad (5.1.51)$$

and the inversions,

$$z \mapsto \frac{z}{z^2 + \vec{x}^2}, \quad x^\mu \mapsto \frac{x^\mu}{z^2 + \vec{x}^2}. \quad (5.1.52)$$

These transformations generate the conformal group $SO(D, 1)$ and they map the boundary at $z = 0$ to itself. Furthermore at the boundary these transformations reduce to conformal transformations (1.1.12).

For more details on the geometry of dS and AdS spaces see [78, 79]. As a side comment notice that if one started with the de Sitter space construction (5.1.40) in Euclidean rather than Lorentzian space with the positive sign in front of $X_0^2$, one would obtain the usual sphere $S^D$ of radius $L_{dS}$.

5.2. From cosmology to inflation

5.2.1. The shape of the Universe

Einstein equations of relativity provide a unique opportunity to investigate the history and the dynamics of our entire Universe. In the year 1927 Georges Lemaître theorised the possibility that our Universe can expand or contract with time [80]. The prediction of the expanding Universe was confirmed by Edwin Hubble in 1929 [81], based on observations of distant galaxies. Hubble found that there exists a correlation between the distance $R$ to a galaxy and its velocity $v$,

$$v = H_0 R, \quad (5.2.1)$$

where $H_0$ is a constant known as the Hubble constant. The Hubble constant changes with time and its value today is still quite uncertain. The direct observations of distant galaxies yield values,

$$H_0^{\text{HST}} = 73.8 \pm 2.4 \text{ (km/s)/Mpc}, \quad (5.2.2)$$
$$H_0^{\text{6dF}} = 67.0 \pm 3.2 \text{ (km/s)/Mpc}, \quad (5.2.3)$$

where pc stands for a parsec, $1\text{pc} \approx 3.086 \cdot 10^{15}\text{m}$. The $H_0^{\text{HST}}$ value was obtained by observation via the Hubble Space Telescope [82], while $H_0^{\text{6dF}}$ follows from the 6dF
Galaxy Survey [83]. On the other hand, the values of Hubble constant obtained by the satellites WMAP [84] and Planck [85] are respectively,

\[ H_0^\text{WMAP9} = 69.32 \pm 0.80 \text{ (km/s)/Mpc}, \]
\[ H_0^\text{Planck} = 67.80 \pm 0.77 \text{ (km/s)/Mpc}. \]

These measurements have much smaller uncertainty, but they are not direct. The value of the Hubble constant is obtained from the measurements of the Cosmic Microwave Background by the fit to the ΛCDM model, which we will discuss in section 5.6. The direct measurements do not depend on the underlying model, but have much bigger uncertainty. For a comparison of various measurements of the Hubble constant, see figure 5.2.

![Figure 5.2: The values of the Hubble constant today measured by various experiments, from top to bottom: [85, 84, 82, 86, 87, 88, 89, 90]. The first two measurements are indirect results based on the measurements of the Cosmic Microwave Background. The next three values are obtained by direct observations of velocities of distant galaxies. The remaining three measurements are based on geometrical methods such as gravitational lensing and redshift. Source: [91].](image)

One could be concerned that the value of the Hubble constant depends both on position and direction of the observation in spacetime. However, all observations mentioned above show that this is not the case and our Universe is homogeneous and isotropic on cosmological scales. This means that on average the density of matter is constant at any point and at any direction. Obviously, there exist regions
with large matter densities due to the gravitational interactions, such as galaxies, planets or black holes, but on scales much larger than the size of a galaxy the Universe looks extremely homogeneous and isotropic.

With the assumption of homogeneity and isotropy we can build a basic but – as it turns out – very realistic model of our Universe. As references, consult standard textbooks, e.g., [70, 77, 92, 93]. The isotropy of the Universe requires that the metric locally takes form

\[ ds^2 = -dt^2 + a^2(t) \left[ e^{2B(r)} dr^2 + r^2 d\Omega^2 \right], \]  

where \(d\Omega^2\) is a volume element of the unit sphere and \(a(t)\) and \(B(r)\) are arbitrary functions. The homogeneity requires that the Ricci scalar is constant in space, which leads to the Friedman-Robertson-Walker (FRW) metric

\[ ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]. \]  

The constant \(k\) is related to the spatial curvature of a single time slice at \(t = 0\). It can be assumed that \(k \in \{-1, 0, +1\}\), as one can always rescale \(r\) appropriately.

The two parameters in the FRW metric: \(a(t)\) and \(k\) depend on the content of the Universe. The scale factor \(a(t)\) determines the physical distances in space and its dependence on time describes the velocity of the expansion or contraction. If we choose two points with coordinates \((t, x_j), j = 1, 2\), then their physical distance on the slice of constant time at \(t\) is \(R = a(t)d(x_1, x_2)\), where \(d\) is a geodesic distance in the slice. Therefore the rate of change of the distance \(R\) between the two points is

\[ v = \frac{dR}{dt} = \frac{R}{a} \frac{da}{dt}. \]  

By comparing with empirical law (5.2.1) we have

\[ H = \frac{\dot{a}}{a}, \]  

where \(\dot{a}\) denotes the derivative of \(a(t)\) with respect to \(t\). Note that \(H\) changes with time. The values of various parameters in our Universe today will be denoted by superscript 0, e.g., \(H_0\).

The constant \(k\) in the metric (5.2.7) determines spatial geometry of the Universe,

1. If \(k = 0\) then the FRW metric reduces to

\[ ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2 d\Omega^2 \right]. \]  

The term in brackets is a flat space metric, so spatial slices are flat. Such a case is known as the flat universe and all observations point to the conclusion, that our Universe is flat.
2. If $k = 1$ then by the substitution $r = \sin \phi$ we obtain

$$ds^2 = -dt^2 + a^2(t) \left[ d\phi^2 + \sin^2 \phi \, d\Omega_2^2 \right]. \tag{5.2.11}$$

The term in brackets is a metric on the unit sphere $S^3$. Such a universe, called *closed*, has spheres as spatial slices with radius changing in time according to $a(t)$.

3. If $k = -1$ then by the substitution $r = \sinh u$ we obtain

$$ds^2 = -dt^2 + a^2(t) \left[ du^2 + \sinh^2 u \, d\Omega_2^2 \right]. \tag{5.2.12}$$

The term in brackets is the metric on the 3-dimensional hyperbolic (EAdS) space we found in (5.1.33). Topologically, the constant time sections are diffeomorphic with $\mathbb{R}^3$ and such a model of a universe is called *open*.

### 5.2.2. Dynamics

With the most general homogeneous and isotropic Universe described by the FRW metric (5.2.7), we can now analyse the dynamics of the scale factor $a(t)$ described by the Einstein equations,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \tag{5.2.13}$$

where $G_{\mu\nu}$ is the Einstein tensor for the FRW metric (5.2.7). We use supergravity conventions where the Newton constant multiplies the entire action (5.1.15).

In this section we will consider a simple model where the stress-energy tensor is given *a priori*. In particular we give no action principle for matter fields and we simply assume that the stress-energy tensor is that of the perfect isotropic fluid,

$$\kappa^2 T_{\mu\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{5.2.14}$$

where $\rho$ and $p$ are density and pressure of matter. We may assume in this section that the contribution from the cosmological constant $\Lambda$ is incorporated into the stress-energy tensor.

Each form of matter or energy satisfies some equation of state, $F(\rho, p) = 0$. We will assume a simple form of this equation,

$$p = w\rho \tag{5.2.15}$$

where $w$ is a number. The most typical values of $w$ are listed in the following table,
5. General relativity and cosmology

<table>
<thead>
<tr>
<th>Type of matter</th>
<th>$w$ parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relativistic particles: radiation</td>
<td>$w = \frac{1}{3}$</td>
</tr>
<tr>
<td>Non-relativistic particles: matter</td>
<td>$w = 0$</td>
</tr>
<tr>
<td>Cosmological constant: dark energy</td>
<td>$w = -1$</td>
</tr>
</tbody>
</table>

Furthermore, the stress-energy tensor must satisfy the conservation equation $\nabla_\mu T^\mu_\nu = 0$. By evaluating this on the FRW background the equation following from $\nu = 0$ component reads

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0,$$

(5.2.16)

where dot denotes a derivative with respect to $t$. By solving this equation we find the dependence between the energy density and the scale factor for all types of matter described by the equation of state (5.2.15). We find

$$\rho(t) = \rho_0 a(t)^{-3(1+w)}$$

(5.2.17)

for some integration constant $\rho_0$.

Finally, we can insert the FRW metric (5.2.7) to the Einstein equations (5.2.13) and derive the equations for the evolution of the scale factor, known as Friedman equations,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}\rho - \frac{k}{a^2},$$

(5.2.18)

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3p).$$

(5.2.19)

In fact these two equations are not independent, as they are related by Bianchi identities. Before we substitute the information on the matter content delivered by the equations (5.2.15) and (5.2.17) let us describe some general features of the Friedman equations. Since the left hand side of (5.2.18) is positive, if the universe is dominated by the negative cosmological constant giving $\rho < 0$ as follows from (5.1.18), one necessarily needs $k = -1$. On the other hand if $\rho > 0$ and $p > 0$ equation (5.2.19) leads to the conclusion that $\ddot{a} < 0$, which means the expansion of the Universe slows down. Thus not earlier than time

$$\tau_H \leq \frac{a}{\dot{a}} = H^{-1}$$

(5.2.20)

ago we would have $a = 0$. This means that the isotropic and homogeneous universe with positive energy density has an initial singularity or the Big Bang. A finite time ago all distances between all points were zero. We can assume that this initial singularity happens at $t = 0$. A distance from the initial singularity that a null particle can travel in time $t$ is then

$$\tau = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da'}{a'^2} \frac{1}{a'H(a')}.$$  (5.2.21)
This is a maximal distance that can be seen at time $t$ after the initial singularity and is called the **Hubble horizon**.

The Friedman equations can be solved exactly for the matter types (5.2.15). Exact solutions can be found in [70], and for the universe filled with matter or radiation the following conclusions apply:

1. If $k = 1$ then at some finite time $\dot{a} = 0$ and after that $\dot{a} < 0$. This means the universe collapses to $a = 0$ at some finite time in the future.

2. If $k = 0$ then the universe expands forever, but $\lim_{t \to \infty} \dot{a}(t) = 0$.

3. If $k = -1$ then the universe expands forever and $\lim_{t \to \infty} \dot{a}(t) > 0$.

The solutions for $k = 0$ are particularly simple and read

$$a(t) = a_0 t^{\frac{2}{3(1+w)}}, \quad (5.2.22)$$

where $a_0$ is some constant. This turns out to be the physical case, since all observations as well as the theory of inflation suggest that our Universe is extremely close to be flat and we can set $k = 0$ henceforth.

A determination of the value of $k$ in our Universe can be achieved as follows. Define

$$\Omega = \frac{\rho}{3H^2}, \quad \rho_{\text{crit}} = 3H_0^2 \quad (5.2.23)$$

and rewrite (5.2.18) as

$$\Omega - 1 = \frac{k}{a^2 H^2}, \quad \Omega = \frac{\rho}{\rho_{\text{crit}}}. \quad (5.2.24)$$

Therefore $k = 0$ ($k < 0, k > 0$) only if $\Omega = 1$ ($\Omega < 1, \Omega > 1$), which corresponds to $\rho = \rho_{\text{crit}}$ ($\rho < \rho_{\text{crit}}, \rho > \rho_{\text{crit}}$). The measurements of the density of the Universe [84, 91, 94, 95, 96] lead to the conclusion that today $\rho \cong \rho_{\text{crit}}$ and therefore we will consider models with $k = 0$ only. Note, however, that the value of $\Omega$ depends on time and if it is not exactly equal to one, the geometry of the Universe will be more and more curved in time, see figure 5.3

### 5.2.3. Need for inflation

The standard Big Bang cosmology is a very successful model of the cosmological evolution, confirmed by hundreds of measurements and observations. However, it presents a few problems, which require an explanation:

1. **Horizon problem.** Due to the expansion of the Universe, two distant regions separated by an angle of about $1^\circ$ in the sky should have never been in a causal contact with one another. This means that one should expect to see
5. General relativity and cosmology

Figure 5.3: The graph shows the evolution of the scale factor $a(t)$ depending on the content of the universe. Since our Universe is dominated by dark energy and matter, one can write $\Omega = \Omega_m + \Omega_\Lambda$ where $\Omega_m$ is a density of matter with $w = 0$ in the equation of state (5.2.15) and $\Omega_\Lambda$ is a density of dark energy with $w = -1$. In our Universe $\Omega_m \sim 0.3$, $\Omega_\Lambda \sim 0.7$, as we will discuss in section 5.6.

about one million causally disconnected regions in the sky. If so, it is a very disturbing fact that all the Universe looks so homogeneous. Without interactions, the causally disconnected domains should not equilibrate, leading to variable formations in each region. While it is theoretically possible that the initial conditions were extremely homogeneous, it does not seem very probable.

2. Flatness problem. The radius of the constant time slices at time $t$ of the FRW metric is

$$R = \frac{H^{-1}}{\sqrt{|\Omega - 1|}}.$$  \hspace{1cm} (5.2.25)

For $k = 0$ one finds $R = \infty$, while for $k = \pm 1$ we find the appropriate radius of the sphere or the hyperbolic space $R = a^{-1}$. By looking at (5.2.24), one finds that the radius of curvature always grows in time. Since the Universe we observe today is very close to be flat, this implies that the initial conditions were extremely fine-tuned. Again, since this seems very improbable, either we should find a reason why $R = \infty$ exactly or explain why the curvature was extremely small at the very early Universe.

3. Entropy problem. The flatness problem can be reiterated in terms of the entropy problem. Since for relativistic particles the entropy $S \sim a^3 T^3$, one
can show that the small initial curvature implies small entropy of the Universe. Since the evolution of the Universe is approximately adiabatic, and the observed entropy now is huge, one needs a mechanism that generates a large amounts of entropy at the very early Universe.

4. **Magnetic monopoles.** Many fundamental theories such as string theory requires an existence of magnetic monopoles. Since these were not observed [97, 98], there should exist a mechanism that dilutes them to the densities small enough for them not to be detected.

5. **Spectrum of the Cosmic Microwave Background.** The whole Universe is filled with an electromagnetic radiation known as the *Cosmic Microwave Background* (CMB). Interestingly, the CMB was detected by accident [99], and since then it became the most important prediction of the Big Bang theory, since it delivers quantitative data. The CMB radiation is almost thermal with temperature today $T = 2.725$ K, but the small fluctuations of order $\Delta T/T \approx 10^{-5}$ carry an important imprint of the inflationary era. We will discuss the properties of the CMB in section 5.6.

All the problems listed above would be solved, if there was a phase in the very early Universe when

$$\dot{a} > 0, \quad \ddot{a} > 0. \quad (5.2.26)$$

Such a phase is called *inflation*, since the Universe is ‘inflated’: all distances in spacetime grow very rapidly. For the inflation to take place, equation (5.2.19) implies that

$$p < -\frac{\rho}{3}. \quad (5.2.27)$$

This condition then can be satisfied by an addition of a positive cosmological constant that has $p = -\rho$. In next section we will find a particular example of matter that satisfies this peculiar condition.

It turns out that the inflation in principle solves all the five problems listed above. Particular quantitative analysis requires a specific model, but the generic features are as follows

1. **Horizon problem.** The horizon problem would be solved, if the Hubble horizon had shrunk during the inflation, so that two points that were in a causal contact before the inflation would become causally disconnected afterwards, see figure 5.4. For this to happen it is enough that in equation (5.2.21),

$$\frac{d}{dt} \frac{1}{aH} < 0 \iff \ddot{a} > 0. \quad (5.2.28)$$
2. Flatness problem. Looking at the equation (5.2.24) we can write

\[
\frac{|\Omega - 1|_{\text{end}}}{|\Omega - 1|_{\text{start}}} = \left( \frac{\dot{a}_{\text{start}}}{\dot{a}_{\text{end}}} \right)^2.
\] (5.2.29)

During the inflationary stage $\dot{a}$ grows very rapidly, usually exponentially fast. Therefore the size of the spatial curvature at the end of the inflation is many orders of magnitudes smaller than at the beginning. Therefore, even if $k \neq 0$ in our Universe, after the inflation $|\Omega - 1|$ is extremely small.

3. Entropy problem. The inflation itself is an adiabatic process, but at some point the Universe should exhibit a phase transition to the radiation dominated phase. During such a transition the energy stored previously in highly energetic degrees of freedom that driven the inflation must be transferred into low energy degrees of freedom and the production of a huge number of particles takes place. This period of the history of our Universe is known as \textit{reheating}. We will not discuss the entropy problem in detail, but one can estimate that the entropy produced in this way would be sufficient to explain the entropy of the Universe today.
4. **Magnetic monopoles.** As we are not focusing on this problem, let us only mention that the magnetic monopoles problem would be solved if the energy required for their creation was sufficiently large. Then, the magnetic monopoles would have not been created during or after the inflation, which would have necessarily diluted their number present at the pre-inflationary universe.

5. **Spectrum of the Cosmic Microwave Background.** Today, this is the most robust argument for the inflation, since the Cosmic Microwave Background (CMB) delivers a quantitative checks on inflationary models. We will discuss the features of the CMB in the following sections.

### 5.3. Inflation

In the previous section we discussed why it is strongly believed that the very early Universe underwent a phase known as the inflation. In this section we will build some basic models and analyse their phenomenology. The simplest model is based on the observation that a universe dominated by the positive dark energy expands exponentially. This simple fact will resolve the horizon problem, the flatness problem and the magnetic monopoles problem. Since it is the Cosmic Microwave Background (CMB) that delivers the most interesting information about inflation, various models will differ by the characteristics of the predicted spectrum of the CMB. Therefore most of the remaining part of this chapter will be devoted to quite non-trivial calculations of the features of the CMB. The fundamentals of the inflationary theory is covered in standard textbooks, *e.g.*, [93, 92, 100, 101, 102, 103].

#### 5.3.1. Pure dark energy

The most primitive model of inflation is an expansion of the universe due to a positive cosmological constant. In such case the Friedman equations (5.2.18) and (5.2.19) with \( k = 0 \) lead to the de Sitter solution discussed in section 5.1.3 with the scale parameter

\[
a(t) = a_e e^{H(t-t_e)},
\]

where \( H = \dot{a}/a \) is a constant Hubble parameter and \( a_e \) and \( t_e \) are arbitrary constants. \( t_e \) is interpreted as the time when the inflation ends and \( a_e \) is the scale factor at \( t_e \). In such a model the scale factor grows exponentially fast,

\[
\frac{a_e}{a_b} = e^{H(t_e-t_b)},
\]

where \( a_b \) denotes the scale factor at the beginning of the inflation. One can show that this model solves the horizon and flatness problems if the inflation took place
sometime between $10^{-36}$ and $10^{-32}$ seconds after the Big Bang. During the inflation the Universe grew by the factor of at least $10^{78}$.

The main drawback of this simple model is the absence of any other degrees of freedom. Therefore the inflation never ends and the Standard Model particles are not produced. In particular there is no Cosmic Microwave Background. On the other hand, one can expect that due to the extreme dilution, the magnetic monopoles problem could be solved, if the monopoles were heavy enough.

### 5.3.2. Introducing inflaton

A more realistic model than the dark energy model requires a single scalar field $\Phi$ called *inflaton* with the potential energy dominating over the kinetic energy. We will consider the Lorentzian action in the supergravity normalisation,

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{|g|} \left[ R - \partial_\mu \Phi \partial^\mu \Phi - 2V(\Phi) \right].$$

(5.3.3)

Let us now find a background solution, i.e., the solution consistent with the flat FRW cosmology,

$$ds^2 = -dt^2 + a^2(t)dx^2, \quad \Phi(t, x) = \phi(t),$$

(5.3.4)

depending on time $t$ only. The derivatives with respect to $t$ will be denoted by a dot. The *Friedman equations* following from (5.1.16) are

$$H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{2}{(D-1)(D-2)} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right],$$

(5.3.5)

$$\frac{\ddot{a}}{a} = -\frac{2}{D-1} \left[ \frac{1}{2} \ddot{\phi}^2 - \frac{1}{D-2} V(\phi) \right],$$

(5.3.6)

$$0 = \ddot{\phi} + (D-1) \frac{\dot{a}}{a} \dot{\phi} + V'(\phi).$$

(5.3.7)

As previously, the second equation follows from the first one when the Bianchi identities $\nabla_\mu G^{\mu\nu} = 0$ are used. Moreover, by comparing both equations we find

$$\dot{H} = \frac{d}{dt} \frac{\dot{a}}{a} = -\frac{1}{D-2} \ddot{\phi}^2.$$ 

(5.3.8)

The right hand side of this equality is non-positive, which leads to the conclusion that $H$ is a non-increasing function of time. This means that $\dot{\phi}$ can change sign only at times $t_0$ such that $\dot{H}(t_0) = 0$, therefore $\phi$ is a piecewise monotonic function on intervals where $\dot{H} < 0$. Thus we can invert $\phi(t)$ and express $H$ in terms of the field through the function $W$ defined as

$$H(t) = -\frac{1}{D-2} W(\phi(t)).$$

(5.3.9)
5.3. Inflation

Equation (5.3.8) leads to

\[ W'(\phi) = \dot{\phi}. \]  

(5.3.10)

Finally, one can substitute these results to (5.3.7) to find that the potential \( V \) can be expressed in terms of \( W \) as

\[ V(\phi) = -\frac{1}{2} W''(\phi) + \frac{D - 1}{2(D - 2)} W^2(\phi). \]  

(5.3.11)

All together, we have shown that the equations of motion (5.3.5) - (5.3.7) for the scalar coupled to the gravity (5.3.3) can be reduced to the set of first order equations,

\[ H = \frac{\dot{a}}{a} = -\frac{1}{D - 2} W(\phi), \]  

(5.3.12)

\[ \dot{\phi} = W'(\phi), \]  

(5.3.13)

\[ V(\phi) = -\frac{1}{2} W'^2(\phi) + \frac{D - 1}{2(D - 2)} W^2(\phi), \]  

(5.3.14)

on the domain of monotonicity of \( \phi \). For its relations to supergravity, the function \( W \) is called a fake superpotential. The applications of the fake superpotential to cosmology date back to [104], where it was called as Hubble function.

5.3.3. Slow-roll inflation

Let us finally see how the model (5.3.3) leads to inflation. The stress-energy tensor (5.1.18) has the form (5.2.14) with

\[ \rho = \frac{1}{2} \dot{\phi}^2 + V(\phi), \]  

(5.3.15)

\[ p = \frac{1}{2} \dot{\phi}^2 - V(\phi). \]  

(5.3.16)

Therefore, if the potential energy dominates the kinetic energy, i.e., \( \dot{\phi}^2 \ll V(\phi) \) we have \( \rho \approx -p \) approximately. We recognize the equation of state for dark energy and effectively we obtain the model discussed in section 5.3.1 with approximately exponential expansion (5.3.1) with

\[ H \approx \sqrt{\frac{2V(\phi)}{(D - 1)(D - 2)}}. \]  

(5.3.17)

The conditions we imposed means that the field rolls slowly down the potential and hence the name: slow-roll inflation. This model of inflation was developed by Andrei Linde in [105]. It is also reasonable to assume that \( |\dot{\phi}| \ll 1 \), so that the slow-rolling phase lasts long enough.
In the further analysis it is convenient to define the following parameters,

\[ \epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = -\frac{\ddot{\phi}}{H\dot{\phi}}. \]  

(5.3.18)

Note that the conventions for the definition of \( \eta \) vary among the textbooks. These variables are well-defined in any model based on the action (5.3.3). If, however, one sticks to the slow-roll inflation, then the above assumptions lead to the conclusion that the background solution is essentially given by the exponential model with the cosmological constant only, and \( \epsilon \) and \( \eta \) measure small departures from such a geometry. In the discussion here we assumed \( \ddot{\phi}^2 \ll V(\phi) \) and \( |\ddot{\phi}| \ll 1 \) that can be expressed as \( |\epsilon_{SR}| < 1 \) and \( |\eta_{SR}| < 1 \). We added a subscript \( SR \) to indicate that these assumptions hold only in the slow-roll inflationary models. In such cases, equations (5.3.5) - (5.3.7) lead to the following set of slow-roll conditions,

\[ \begin{align*}
\dot{\phi}^2 & \ll V(\phi), & \frac{V''}{V} & \ll H^2, \\
\ddot{\phi} & \ll (D - 1)H\dot{\phi}, & V'' & \ll H^2, \\
\epsilon_{SR} & \approx \frac{1}{D - 2} \left( \frac{V'}{V} \right)^2 & \eta_{SR} & \approx \frac{V''}{V} - \epsilon_{SR}.
\end{align*} \]  

(5.3.19)

In the slow-roll inflation the universe expands due to the large positive cosmological constant, as in the model discussed in section 5.3.1. The power of the inflaton is that now one can consider small fluctuations around the background solution, both for the inflaton and for the metric. These fluctuations, quantum in nature at the beginning of the inflation, are stretched to gigantic sizes in the process. Therefore, small inhomogeneities in the inflaton field lead to the emergence of the structure in the Universe such as galaxies and stars. Since the initial quantum fluctuations were essentially random, the visible structure today reflects this randomness and their various statistical characteristics can be measured in the CMB. The exact features of the CMB depend on a particular model under considerations.

Also the entropy problem can be solved by coupling the inflaton and metric to the Standard Model particles. However, we will not pursue this direction and the discussion of the problem can be found in the standard textbooks.

The slow-roll inflation is only one of a vast number of inflationary models. The most popular models are:

1. **Old inflation**: the original Guth’s proposal [106]. The model is based on the quantum tunnelling of the inflaton from an unstable vacuum to a stable one.

2. **Eternal inflation** [107, 108]: taking the old inflation to a new level. The Universe is filled up with a slowly-changing dilaton field that has a complicated
landscape of vacua. Therefore, the inflation lasts forever, but its speed differs in time and space.

3. **Hiltop inflation** [109, 110]. This model is closely related to the slow-roll inflation. The difference is that the inflation starts as the inflaton starts rolling down the hill from an unstable critical point of the potential. We will find this kind of the inflationary model in section 7.2. The inflaton profile resembles an instantonic solution.

4. **Large-field models** such as chaotic inflation [111], or natural inflation [112]. In these models one obtains larger tensor amplitudes than in other inflationary models.

5. **Multi-field inflation**, e.g., [113, 114, 115]: there is more than one inflating field.

6. **String gas cosmology** [116, 117]: the inflation is embedded into string theory.

7. **Holographic inflation**: the main point of this thesis and many more.

   All the presented models share two additional features. Firstly, the inflating fields are spin-0 scalars. If this assumption is not met, some particular direction in the Universe would be preferred, but this is not observed. Secondly, all inflationary models predict at least one more unknown particle: the inflaton. One can wonder whether the only scalar particle in the Standard Model, the Higgs boson, could be the inflaton [118]. However, due to the small mass of the Higgs and unitarity issues it is very unlikely that this is the case [119, 120].

### 5.4. Inflaton on de Sitter background

#### 5.4.1. Perturbation of inflaton

As a starter we will calculate the spectrum of perturbations of the inflaton, assuming a fixed gravitational background given by the de Sitter metric

\[
ds^2 = -dt^2 + a^2(t)dx^2, \quad a(t) = e^{Ht}. \tag{5.4.1}
\]

This is the usual setup of the quantum field theory on the curved background [71, 121]. We will keep \( a(t) \) parameter explicitly for now, since we will use these results later on. In general we will denote the Hubble parameter

\[
H = \frac{\dot{a}}{a}. \tag{5.4.2}
\]
On the de Sitter solution $H$ is a true constant and equal to the parameter in (5.4.1) also denoted by $H$. We want to solve the equation of motion $\Box \Phi - V'(\Phi) = 0$ on this background, which reads

$$\ddot{\Phi} + (D - 1) \frac{\dot{a}}{a} \dot{\Phi} - \frac{1}{a^2} \Box_0 \Phi + V'(\Phi) = 0,$$

(5.4.3)

where $\Box_0 = \delta^{ij} \partial_i \partial_j$ is the d’Alambertian in the $x$ directions. We will solve (5.4.3) perturbatively by assuming that the potential has the following form

$$V(\Phi) = V_0 + \frac{1}{2} m^2 \Phi^2 + \frac{a_3}{3} \Phi^3 + \frac{a_4}{4} \Phi^4 + \ldots$$

(5.4.4)

and we will work with the perturbative expansion in all couplings $a_j, j \geq 3$. The field $\Phi$ then has a perturbative expansion in all couplings as well. We will consider only first order perturbation by writing

$$\Phi(t, x) = \phi(t) + \delta \phi(t, x).$$

(5.4.5)

The equation for the background solution $\phi(t)$ following from (5.4.3) is

$$\ddot{\phi} + dH \dot{\phi} + m^2 \phi = 0,$$

(5.4.6)

where $d = D - 1$. For the de Sitter background the equation becomes linear with two independent solutions

$$\phi(t) = C_+ t^{\Delta_+} + C_- t^{\Delta_-}$$

(5.4.7)

with two undetermined constants $C_\pm$ and

$$\Delta_\pm = \frac{dH}{2} \left[-1 \pm \sqrt{1 - \left(\frac{2m}{dH}\right)^2}\right].$$

(5.4.8)

Let us now turn to the perturbation $\delta \phi$. First assume $a_n = 0$ for all $n$. Since we consider the flat FRW universe, we can Fourier transform $\delta \phi$ in $x$ directions by writing

$$\delta \phi(t, x) = \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \delta \phi(t, k)$$

$$= \int \frac{d^d k}{(2\pi)^d} \left[a_k u_k(t)e^{ik \cdot x} + a_k^* u_k^*(t)e^{-ik \cdot x}\right],$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{ik \cdot x} \left[a_k u_k(t) + a_k^* u_k^*(t)\right],$$

(5.4.9)
where $d = D - 1$, $u_k(t)$ and $u_k^*(t)$ are the conjugate Fourier modes satisfying (5.4.3) and $a_k$ and $a_{-k}^*$ are conjugate coefficients. The $u_k(t)$ modes depend on the magnitude $k = |k|$, since (5.4.3) reads

$$\ddot{u}_k + dH \dot{u}_k + \left( \frac{k^2}{a^2} + m^2 \right) u_k = 0. \quad (5.4.10)$$

Notice the following:

- For wavelengths $\lambda \ll H^{-1}$ or equivalently $k \gg aH$ we can neglect the $dH \dot{u}_k$ term in (5.4.10) and the equation becomes the equation for the harmonic oscillator with the frequency dependent on time. Such a regime is called subhorizon.

- For superhorizon modes with $k \ll aH$ or equivalently $\lambda \gg H^{-1}$ we can neglect the $k^2/a^2$ term in (5.4.10) and the resulting equation has constant coefficients. If $m = 0$, then the mode is constant outside the horizon and if $m \neq 0$ but small, the solution depends slightly on $k$.

The equation of motion (5.4.10) can be simplified by means of the substitutions

$$u_k(t) = \frac{v_k(t)}{a(t)}, \quad \text{d}\tau = \frac{dt}{a(t)}. \quad (5.4.11)$$

The new time variable $\tau$ is called the conformal time. The equation (5.4.10) reads now

$$v''_k + v_k \left[ -\frac{1}{2}(d - 1) \frac{a''}{a} - \frac{1}{4}(d - 1)(d - 3) \frac{a'^2}{a^2} + \left( \frac{k^2}{a^2} + m^2 a^2 \right) \right] = 0, \quad (5.4.12)$$

where prime denotes the derivative with respect to $\tau$.

From now on let us work on the de Sitter background, where $a(t) = e^{Ht}$ and by choosing the integration constants we have

$$\tau = -\frac{1}{H} e^{-Ht}, \quad a(\tau) = -\frac{1}{H \tau}. \quad (5.4.14)$$

With this choice the far past in the original time variable $t$ corresponds to $\tau \to -\infty$, but the far future corresponds to $\tau \to 0$. In general $\tau < 0$. The equation (5.4.12) simplifies to

$$v''_k + v_k \left[ k^2 - \frac{1}{\tau^2} \left( \frac{1}{4}(d^2 - 1) - \frac{m^2}{H^2} \right) \right] = 0. \quad (5.4.15)$$

The general solution is

$$v_k(\tau) = \sqrt{-\tau} \left[ c_1 H^{(1)}(k\tau) + c_2 H^{(2)}(k\tau) \right], \quad (5.4.16)$$
where
\[ \nu = \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}} \]  
(5.4.17)
and \( H_{(j)}^\nu \), \( j = 1, 2 \) denote the Hankel functions of the first and second kind, see [59] and \( c_j \) are two undetermined integration constants. We should fix the integration constants by requiring that in the far past, i.e., for \( \tau \to -\infty \) we have only incoming planar waves,
\[ v_{\mathtt{k}}(-\infty) = \frac{e^{-ik\tau}}{\sqrt{2k}}, \]
(5.4.18)
where the normalisation is fixed by the Wronskian condition \( v^*v' - vv'^* = -i \). Such a choice of vacuum is called the Bunch-Davies vacuum. This leads to the solution
\[ v_{\mathtt{k}}(\tau) = \frac{\sqrt{\pi}}{2} e^{i(\nu + \frac{1}{2})\pi} \sqrt{-\tau} H_{(1)}^\nu (-\mathtt{k}\tau) \]
(5.4.19)
where \( K_{\nu} \) is the Bessel function \( K \). This is the first appearance of the Bessel \( K \) functions in cosmology. Note that in the massless limit \( m = 0 \) the solution simplifies to elementary functions,
\[ v_{\mathtt{k}}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left( 1 + \frac{i}{k\tau} \right). \]
(5.4.20)

5.4.2. 2-point function

At this point we can quantise the system by the imposition of the canonical commutation relations,
\[ [a_{\mathtt{k}}, a_{\mathtt{k'}}^\dagger] = (2\pi)^d \delta(\mathtt{k} + \mathtt{k'}), \]
(5.4.21)
and the 2-point function in momentum space is then
\[ \langle \delta\phi(\tau, \mathtt{k})\delta\phi(\tau, \mathtt{k'}) \rangle = (2\pi)^d \delta(\mathtt{k} + \mathtt{k'}) \frac{|\tau|}{\pi} |K_{\nu}(ik\tau)|^2. \]
(5.4.22)
If no state is specified, the expectation value is always taken in the Bunch-Davies vacuum. We can return to the original time variable and use (5.4.14). Then we can expand the result around \( t = \infty \) to find its late time behaviour to be
\[ \langle \delta\phi(\mathtt{k})\delta\phi(\mathtt{k'}) \rangle \approx (2\pi)^d \delta(\mathtt{k} + \mathtt{k'}) \Gamma(\nu) \frac{4^{\nu-1}}{a^2 k\pi} \left( \frac{k}{aH} \right)^{1-2\nu}. \]
(5.4.23)
When the late time behaviour is considered, we will omit the indication of the time dependence in the cosmological correlation functions. For a massless field in \( D = 4 \) we have \( \nu = 3/2 \) and the 2-point function simplifies at late times to
\[ \langle \delta\phi(\mathtt{k})\delta\phi(\mathtt{k'}) \rangle \approx (2\pi)^3 \delta(\mathtt{k} + \mathtt{k'}) \frac{H^2}{2k^3}. \]
(5.4.24)
5.4. Inflaton on de Sitter background

We call a scalar 2-point function *scale invariant* if its dependence on momentum is \( k^{-d} \), where \( d = D - 1 \). The reason is that such a 2-point function does not depend on any dimensionful parameters other than \( H \).

The *spectral index* or *tilt* \( n_\phi \) is defined as

\[
  n_\phi - d = \frac{d}{d \log k} \log \langle \delta\phi(k)\delta\phi(-k) \rangle.
\]

(5.4.25)

For a scale-invariant 2-point function we find \( n_\phi = 0 \). As in the first part of the thesis by a double bracket we denote the 2-point function without the Dirac delta,

\[
\langle \delta\phi(k)\delta\phi(k') \rangle = (2\pi)^d \delta(k + k') \langle \delta\phi(k)\delta\phi(-k) \rangle
\]

(5.4.26)

with a similar notation for the higher-point correlation functions, see (2.3.1).

5.4.3. 3-point function

Now let us turn to higher-point functions. We will discuss an example of calculations of the 3-point function of the inflaton perturbation in case of \( m = 0 \). The 3-point function for a free field vanishes and therefore we should expect that the interactions play the most important role here. Let us consider a simple potential of the form

\[
V(\Phi) = g^3 \Phi^3.
\]

(5.4.27)

The 3-point function is by definition

\[
\langle \Omega(\tau) | :\delta\phi(\tau, k_1)\delta\phi(\tau, k_2)\delta\phi(\tau, k_3) : | \Omega(\tau) \rangle = \langle 0 | U_\tau^{-1} :\delta\phi(\tau, k_1)\delta\phi(\tau, k_2)\delta\phi(\tau, k_3) : U_\tau | 0 \rangle,
\]

(5.4.28)

where \( | \Omega(\tau) \rangle = U_\tau | 0 \rangle \) denotes the state obtained by the evolution of the Minkowski vacuum \( | 0 \rangle \) in the far past to the time \( \tau \). This is by definition the Bunch-Davies vacuum. Here we denote the evolution operator

\[
U_\tau = \exp \left( -i \int_{-\infty}^{\tau} H_{\text{int}}(\tau')d\tau' \right)
\]

(5.4.29)

and in our case

\[
H_{\text{int}}(\tau) = \frac{g}{3} \int d^d x a^d(\tau)\Phi^3(\tau, x),
\]

(5.4.30)

where \( d = D - 1 \). By expanding in the coupling \( g \) and keeping the leading term only we have

\[
\langle :\delta\phi(\tau, k_1)\delta\phi(\tau, k_2)\delta\phi(\tau, k_3) : \rangle \approx -i \int_{-\infty}^{\tau} d\tau' a^d(\tau')(0) \langle :\delta\phi(\tau, k_1)\delta\phi(\tau, k_2)\delta\phi(\tau, k_3) : H_{\text{int}}(\tau') | 0 \rangle \]

\[
\approx \int_{-\infty}^{\tau} \langle :\delta\phi(\tau, k_1)\delta\phi(\tau, k_2)\delta\phi(\tau, k_3) : H_{\text{int}}(\tau') | 0 \rangle
\]

241
\[ \int_{-\infty}^{\tau} d\tau' a^d(\tau') \int \frac{d^D q_1}{(2\pi)^D} \int \frac{d^D q_2}{(2\pi)^D} \times \]

\[ \times \langle 0|:\delta \phi(\tau, k_1)\delta \phi(\tau, k_2)\delta \phi(\tau, k)\rangle :\delta \phi(\tau', q_1)\delta \phi(\tau', q_2 - q_1)\delta \phi(\tau', -q_2)|0 \rangle \]

\[ = \frac{-ig}{3} \int_{-\infty}^{\tau} d\tau' a^d(\tau') \int \frac{d^D q_1}{(2\pi)^D} \int \frac{d^D q_2}{(2\pi)^D} \times \]

\[ \times \left[ v_{k_1}(\tau)v_{k_2}(\tau)v_{k_3}(\tau)v_{q_2 - q_1}(\tau')v_{q_2}^*(\tau')v_{q_2}^*(\tau')v_{k_2}^*(\tau')v_{k_3}^*(\tau')\langle 0|a_{k_1}a_{k_2}a_{k_3}a_{q_1}^\dagger a_{q_2}^\dagger a_{q_2}^\dagger|0 \rangle \right. \]

\[ - v_{q_1}(\tau)v_{q_2 - q_1|}(\tau)v_{q_2}(\tau)v_{k_2}^*(\tau')v_{k_2}^*(\tau')v_{k_3}^*(\tau')\langle 0|a_{q_1}a_{q_2 - q_1}a_{q_2}^\dagger a_{q_2}^\dagger a_{q_2}^\dagger|0 \rangle \]

\[ = 4g \text{Im} \left[ v_{k_1}(\tau)v_{k_2}(\tau)v_{k_3}(\tau) \int_{-\infty}^{\tau} d\tau' a^d(\tau')v_{k_1}^*(\tau')v_{k_2}^*(\tau')v_{k_3}^*(\tau') \right]. \quad (5.4.31) \]

We should comment on the choice of the vacuum. In case of a free theory the vacuum \( |0 \rangle \) is defined as the usual Minkowski vacuum at far past with the canonical normalisation coming from the Wronskian condition. In case of interactions one should take the Minkowski vacuum at far past of the full interacting theory. One can obtain such a vacuum state by the standard procedure of the projection of the free theory vacuum state on the subspace of minimal energy by deforming the integration contour into a slightly imaginary direction, see [21, 72]. In the case of the integral in (5.4.31) the deformation is \( \tau' \rightarrow \tau' + i\epsilon|\tau'| \), where \( \epsilon \) is a small parameter. This corresponds to the choice of the Bunch-Davies vacuum. Moreover, it regularises the integral, which is usually severely divergent.

Let us finalise the calculations by considering de Sitter background (5.4.14). Using (5.4.19) the integral to be done is

\[ (-1)^d \int_{-\infty}^{\tau} d\tau' \left( \frac{i}{\pi H^2 \tau'} \right)^{\frac{d}{2}} K_\nu(ik_1 \tau')K_\nu(ik_2 \tau')K_\nu(ik_3 \tau'). \quad (5.4.32) \]

This is the first instance of the triple-\( K \) integral in our cosmological analysis. We will not pursue its evaluation, since the example of the inflaton on the fixed background is not physical. In the next section we will consider its backreaction on the gravity and then we will arrive at the triple-\( K \) integrals with \( \tau \rightarrow 0 \). In the late time limits we will usually omit the implicit time dependence in the correlation function.

### 5.5. Cosmological perturbations

In the previous sections we considered perturbations of inflaton field on the fixed gravitational background. However, we should also consider the perturbations of gravity, since the inflaton is coupled to gravity in the action (5.3.3). In total we expect three propagating degrees of freedom: one scalar and two transverse-traceless spin-2. To see it, we must either fix the gauge freedom in the choice of the metric
or to find such a combination of the variables that is gauge independent. It will be convenient to work in the ADM formalism.

### 5.5.1. ADM formulation

Consider the ADM decomposition (5.1.6) of the metric,

\[ \text{d}s^2 = -N^2\text{d}t^2 + \gamma_{ij}(\text{d}x^i + N^i\text{d}t)(\text{d}x^j + N^j\text{d}t) \]  

(5.5.1)

where

\[ N = 1 + \delta N(t,\mathbf{x}), \quad N_i = g_{ij}N^j = \delta N_i(t,\mathbf{x}), \]

\[ \gamma_{ij} = a^2(t)(\delta_{ij} + h_{ij}(t,\mathbf{x})), \]  

(5.5.2)

and \( i, j = 1, 2, 3 \). For example, the perturbation of \( g_{tt} \) in (5.5.1) is

\[ \delta g_{tt} = -2(\delta \delta N + \delta N^2) + \frac{\delta N_i \delta N_i}{a^2}, \]  

(5.5.3)

where the repeated Latin indices are summed with the Kronecker delta as a metric. The perturbations of shift vector and \( h_{ij} \) can be decomposed as

\[ \delta N_i = a^2(\partial_i \nu + \nu_i), \quad h_{ij} = -2\psi\delta_{ij} + 2\partial_i\partial_j\chi + 2\partial_i\omega_j + \gamma_{ij}, \]  

(5.5.4)

where \( \nu, \chi \) and \( \psi \) are scalars, \( \nu_i \) and \( \omega_i \) are transverse vectors satisfying \( \partial_i \nu_i = \partial_i \omega_i = 0 \) and \( \gamma_{ij} \) is symmetric transverse and traceless, \( \sum_i \gamma_{ii} = 0 \).

We will work in the comoving gauge where the perturbations of inflaton vanish, \( i.e., \)

\[ \Phi(t,\mathbf{x}) = \phi(t), \quad \delta \phi(t,\mathbf{x}) = 0. \]  

(5.5.5)

This means that all degrees of freedom are encoded in two functions that we will be denoted as \( \zeta(t,\mathbf{x}) \) and \( \hat{\gamma}_{ij}(t,\mathbf{x}) \) and call primordial perturbations. The first function is a scalar perturbation of the curvature and the second one is a symmetric transverse-traceless tensor perturbation, \( i.e., \) gravity waves. They are defined as perturbations of the spatial part of the metric,

\[ \gamma_{ij} = a^2e^{2\zeta}[e^{\hat{\gamma}}]_{ij} = a^2e^{2\zeta}\left(\delta_{ij} + \hat{\gamma}_{ij} + \frac{1}{2}\hat{\gamma}_{ik}\hat{\gamma}_{kj} + \ldots\right). \]  

(5.5.6)

Now one can use the formulae listed above to find perturbations of the entire metric (5.5.1) in the ADM formalism. It can be shown [75] that the \( \zeta \) and \( \hat{\gamma} \) variables can be defined – up to second order in perturbations – in an gauge independent way as follows,

\[ \zeta = -\psi - \frac{H}{\phi}\delta \phi \]  

(5.5.7)

\[ -\psi^2 + \left(\dot{H} - \frac{H\ddot{\phi}}{\phi}\right)\frac{\delta \phi^2}{2\phi^2} + \frac{H}{\phi^2}\delta \phi \ddot{\phi} + \frac{H}{\phi}(\partial_k \chi + \omega_k)\partial_k \delta \phi + \frac{1}{4} \hat{\pi}_{ij} X_{ij}, \]  

(5.5.8)
5. General relativity and cosmology

\[ \dot{\gamma} = \gamma_{ij} + \dot{\Pi}_{ijkl} X_{kl}, \]  
\hspace{1cm} \text{(5.5.9)}

where
\[ X_{ij} = - \frac{\partial_i \delta \phi \partial_j \delta \phi}{a^2 \dot{\phi}^2} - \frac{2 \delta N_i (\partial_j \delta \phi)}{a^2 \dot{\phi}} - \frac{\delta \dot{h}_{ij}}{\dot{\phi}} - 2 \partial_i (\partial_k \chi + \omega_k) h_{jk} \]
\hspace{1cm} \text{(5.5.10)}

\[ - (\partial_k \chi + \omega_k) \partial_k h_{ij} + \partial_i (\partial_k \chi + \omega_k) \partial_j (\partial_k \chi + \omega_k) + 2 \dot{\phi} \gamma_{ij} - \frac{1}{2} \gamma_{ik} \gamma_{kj} \]

and \( \dot{\pi}_{ij} \) and \( \dot{\Pi}_{ijkl} \) are defined in (2.1.20) and (2.1.21). If one is interested in 2-point functions only, then only the leading, first order terms in (5.5.7) and (5.5.9) can be considered. These can be found in the standard textbooks.

5.5.2. Equations of motion

Now the action (5.3.3) can be rewritten in the ADM formalism as in (5.1.19),
\[ S = \frac{1}{2 \kappa^2} \int d^D x \sqrt{|g|} N \left[ K_{\mu \nu} K^{\mu \nu} - K^2 + \frac{1}{N^2} \left( \dot{\Phi} - N^\mu \partial_\mu \Phi \right)^2 \right. \]
\[ + \left. \left( \dot{R} - \gamma^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi - 2 V(\Phi) \right) \right], \]
\hspace{1cm} \text{(5.5.11)}

where
\[ K_{\mu \nu} = \frac{1}{2} \mathcal{L}_n \gamma_{\mu \nu} = \frac{1}{2N} \left( \dot{\gamma}_{\mu \nu} - \hat{\nabla}_\mu N_\nu - \hat{\nabla}_\nu N_\mu \right). \]
\hspace{1cm} \text{(5.5.12)}

Equations of motion for the first order perturbations can be worked out from this action, see e.g., [122] and they read
\[ 0 = \ddot{\zeta}_k + \left( 3H + \frac{\dot{\epsilon}}{\epsilon} \right) \dot{\zeta}_k + \frac{k^2}{a^2} \zeta_k, \]
\hspace{1cm} \text{(5.5.13)}

\[ 0 = \ddot{\gamma}_k + 3H \dot{\gamma}_k + \frac{k^2}{a^2} \gamma_k, \]
\hspace{1cm} \text{(5.5.14)}

where the Fourier modes are defined as
\[ \zeta(t, x) = \int \frac{d^3 k}{(2\pi)^3} \left[ a_k \zeta_k(t) e^{i k \cdot x} + a_k^* \zeta_k^*(t) e^{-i k \cdot x} \right], \]
\hspace{1cm} \text{(5.5.15)}

\[ \dot{\gamma}^{(s)}(t, x) = \int \frac{d^3 k}{(2\pi)^3} \left[ b_k^{(s)} \dot{\gamma}_k(t) e^{i k \cdot x} + b_k^{(s)*} \dot{\gamma}_k^*(t) e^{-i k \cdot x} \right], \]
\hspace{1cm} \text{(5.5.16)}

Here \( H \) and \( \epsilon \) stand for the Hubble parameter and the slow-roll parameter,
\[ H = \frac{\dot{a}}{a}, \quad \epsilon = - \frac{\dot{H}}{H^2}, \]
\hspace{1cm} \text{(5.5.17)}

which are fixed by the background solution, but are generally time-dependent. In slow-roll inflation one assumes \(|\epsilon| < 1\), but the equations (5.5.13) and (5.5.14) are exact as long as one considers first order perturbations on top of the FRW solution.
5.5. Cosmological perturbations

As we can see, equation (5.5.14) is exactly the same as equation (5.4.10) for a free massless scalar on the de Sitter background. Its solution in conformal time \( \tau \) is therefore given by (5.4.19). The first equation (5.5.13) for the scalar perturbation is similar to (5.4.10) up to the \( \dot{\epsilon}/\epsilon \) term. In the leading order in slow-roll we can account for this correction at late times by solving the free massless equation and evaluating the cosmological parameters at the horizon crossing time, i.e., at time \( t_*(k) \) such that \( k = a(t_*)H(t_*) \). We will also denote \( a_* = a(t_*) \) and \( H_* = H(t_*) \). The reason is that at late times \( \zeta \) is approximately constant as we have seen in (5.4.19), while at early times the field is in the vacuum and its wavefunction is accurately given by the WKB approximation. For more details, see [72].

5.5.3. 2-point functions

Using canonical quantisation we promote the coefficients \( a_k \) and \( b_k^{(s)} \) to operators satisfying

\[
[a_k, a_{-k'}] = (2\pi)^3 \delta(k + k'), \quad [b_k^{(s)}, b_{-k'}^{(s)\dagger}] = (2\pi)^3 \delta(k + k')\delta^{ss'}.
\] (5.5.18)

The 2-point functions of the primordial perturbations are

\[
\langle \zeta(t, k)\zeta(t, k') \rangle = (2\pi)^3 \delta(k + k')|\zeta_k(t)|^2,
\] (5.5.19)

\[
\langle \hat{\gamma}^{(s)}(t, k)\hat{\gamma}^{(s')}(t, k') \rangle = (2\pi)^3 \delta(k + k')\delta^{ss'}|\hat{\gamma}_k(t)|^2.
\] (5.5.20)

Notice that these expressions depend on the amplitude of the momentum \( k \) only. If one can solve equations (5.5.13) and (5.5.14) exactly, this will give an exact 2-point function in the absence of interactions.

Similarly as in case of the inflaton, we can define the spectral indices as

\[
n_S(k) - 4 = \frac{d}{d \log k} \log \langle \zeta(k)\zeta(-k) \rangle,
\] (5.5.21)

\[
n_T(k) - 3 = \frac{d}{d \log k} \log \langle \hat{\gamma}^{(s)}(k)\hat{\gamma}^{(s)}(-k) \rangle,
\] (5.5.22)

which measure a departure of the 2-point functions from scale invariance. For historical reasons, \( n_S = 1 \) and \( n_T = 0 \) correspond to the scale-invariant scalar and tensor 2-point functions respectively.

In case of the slow-roll inflation, one can solve equations (5.5.13) and (5.5.14) in the leading order in \( \epsilon \), which leads to the late time correlation functions,

\[
\langle \zeta(k)\zeta(-k) \rangle_{SR} = \frac{\kappa^2 H_*^2}{4\epsilon_* k^3} \left[ 1 + O(\epsilon_*, \eta_*) \right],
\] (5.5.23)

\[
\langle \hat{\gamma}^{(s)}(k)\hat{\gamma}^{(s')}(k) \rangle_{SR} = \frac{2\kappa^2 H_*^2 \delta^{ss'}}{k^3} \left[ 1 + O(\epsilon_*, \eta_*) \right],
\] (5.5.24)
where the asterisk denotes the evaluation at the horizon crossing time $t_*$ satisfying 
\[ k = a_* H_* = a(t_*) H(t_*) \]. Their dependence on the momentum is hidden in the fact 
that the Hubble parameter and $\epsilon_{SR}$ are evaluated at the horizon crossing time. 
Using the conditions (5.3.19) one finds

\[
\begin{align*}
    n_S - 1 &= \frac{d}{d \log k} \log \frac{H_*^2}{\epsilon_*} \approx \frac{1}{H_*} \frac{d}{dt_*} \log \frac{H_*^2}{\epsilon_*} \approx 2\eta_* - 4\epsilon_*, \\
    n_T &= \frac{d}{d \log k} \log H_*^2 = \frac{1}{H_*} \frac{d}{dt_*} \log H_*^2 \approx -2\epsilon_*. 
\end{align*}
\]

(5.5.25)

(5.5.26)

Note that these results are of order one in slow-roll parameters. Therefore, the 
slow-roll inflation predicts only minor deviations from the scale-invariance of the 
2-point functions.

It is customary to define the power spectra $\Delta^2_S$ and $\Delta^2_T$ as

\[
\begin{align*}
    \Delta^2_S(k) &= \frac{k^3}{2\pi^2} \langle \zeta(k) \zeta(-k) \rangle_{SR} = \frac{\kappa^2 H_*^2}{8\pi^2 \epsilon_*} [1 + O(\epsilon_*, \eta_*)], \\
    \Delta^2_T(k) &= \frac{2k^3}{\pi^2} \langle \hat{\gamma}^{(s)}(k) \hat{\gamma}^{(s)}(-k) \rangle_{SR} = \frac{4\kappa^2 H_*^2}{\pi^2} [1 + O(\epsilon_*, \eta_*)].
\end{align*}
\]

(5.5.27)

(5.5.28)

The expressions for the 2-point functions expanded to the higher order in slow-
roll are known [123, 124]. For the comparison with our holographic cosmology, we 
will need the first order results for the scalar perturbations,

\[
\langle \zeta(k) \zeta(-k) \rangle_{SR} = \frac{\kappa^2 H_*^2}{4\epsilon_* k^3} [1 + (2 - \log 2 - \gamma_E)(2\epsilon_* + \eta_*) - \epsilon_*].
\]

(5.5.29)

In general one can show [125] that for any non-negative integer $n$, 

\[
\frac{d^n}{d \log k^n} (n_S - 1) \sim \text{slow-roll}^{n+1},
\]

(5.5.30)

where by ‘slow-roll’ we denote small parameters appearing in the higher terms of 
the expansion (5.5.29). These comprise $\epsilon$ and $\eta$ but also other slow-roll parameters 
following from the estimates of higher derivatives of $V$ and $\phi$ in the background 
solution.

### 5.5.4. Hamiltonian formalism

For the analysis of 3-point functions of the primordial perturbations it is convenient 
to work in the Hamiltonian formalism. The reason is that, in order to calculate the 
3-point functions as in section 5.4.3, we must evaluate the interaction Hamiltonians 
for the primordial perturbations $\zeta$ and $\hat{\gamma}$. To do it define the canonical momenta

\[
\Pi = \frac{\delta}{\delta \zeta}(\kappa^2 L), \quad \Pi_{ij} = \frac{\delta}{\delta \hat{\gamma}}(\kappa^2 L),
\]

(5.5.31)
where \( L \) is the Lagrangian in (5.5.11). Next, since the perturbation \( \tilde{\gamma}_{ij} \) is transverse-traceless, we can project it onto the helicity basis as explained in section 2.9. We define

\[
\tilde{\gamma}^{(s)}(k) = \frac{1}{2} \tilde{\gamma}_{ij}(k) \epsilon^{(s)}_{ij}(-k), \quad \Pi^{(s)}(k) = \frac{1}{2} \Pi_{ij}(k) \epsilon^{(s)}_{ij}(-k),
\]

(5.5.32)

where the projectors \( \epsilon^{(s)}_{ij} \) are defined in (2.9.2).

The dynamics is given by the Hamilton equations which in momentum space read,

\[
\dot{\zeta}(k) = (2\pi)^3 \frac{\delta}{\delta \Pi(-k)} (\kappa^2 \mathcal{H}), \quad \dot{\tilde{\gamma}}^{(s)}(k) = \frac{1}{2} (2\pi)^3 \frac{\delta}{\delta \Pi^{(s)}(-k)} (\kappa^2 \mathcal{H}),
\]

\[
\dot{\Pi}(k) = -(2\pi)^3 \frac{\delta}{\delta \zeta(-k)} (\kappa^2 \mathcal{H}), \quad \dot{\Pi}^{(s)}(k) = -\frac{1}{2} (2\pi)^3 \frac{\delta}{\delta \tilde{\gamma}^{(s)}(-k)} (\kappa^2 \mathcal{H}),
\]

(5.5.33)

where \( \mathcal{H} \) is the Hamiltonian following from (5.5.11). In order to extract it, one needs to find the interactions between the primordial perturbations in (5.5.11).

We can divide the Hamiltonian into a free and interacting part,

\[
\mathcal{H} = \mathcal{H}_{\text{free}} + \mathcal{H}_{\text{int}}.
\]

(5.5.34)

The free part can be written as \( \mathcal{H}_{\text{free}} = H_{\zeta\zeta} + H_{\tilde{\gamma}\tilde{\gamma}} \), where the subscript denotes types of interactions. Starting from (5.5.11) one can show [122, 72, 103] that

\[
\kappa^2 H_{\zeta\zeta} = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{4a^3\epsilon} \Pi(k)\Pi(-k) + a\epsilon k^2\zeta(k)\zeta(-k) \right],
\]

(5.5.35)

\[
\kappa^2 H_{\tilde{\gamma}\tilde{\gamma}} = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{4}{a^3} \Pi^{(s)}(k)\Pi^{(s)}(-k) + \frac{a}{4} k^2\tilde{\gamma}^{(s)}(k)\tilde{\gamma}^{(s)}(-k) \right].
\]

(5.5.36)

One can also check that these Hamiltonians lead to equations (5.5.13) and (5.5.14). These Hamiltonians are valid in general, as the equations (5.5.13) and (5.5.14) hold in general. Note that the lack of interactions between \( \zeta \) and \( \tilde{\gamma} \) explains why

\[
\langle\langle \zeta(k)\tilde{\gamma}(-k) \rangle\rangle = 0.
\]

(5.5.37)

For the interaction part of the Hamiltonian we use the perturbative expansion. We look for the triple interactions in (5.5.11) since we are interested in 3-point functions only and without loop corrections. On general grounds we can write

\[
H_{\text{int}} = H_{\zeta\zeta\zeta} + H_{\zeta\zeta\tilde{\gamma}} + H_{\zeta\tilde{\gamma}\tilde{\gamma}} + H_{\tilde{\gamma}\tilde{\gamma}\tilde{\gamma}},
\]

(5.5.38)

where the particular components are complicated, theory dependent expressions.

Before we analyse their form closer, let us discuss the case of the slow-roll inflation, where the Hamiltonian (5.5.38) is known.
5.5.5. 3-point functions in slow-roll inflation

Expressions for the four terms in the Hamiltonian (5.5.38) have been found in the slow-roll approximation, see [75]. In this case, however, various tricks introduced in [72] allow to simplify the calculations considerably, see also [100, 103, 40]. For example, it turns out that by using the following field redefinition,

\[ \zeta = \zeta_c + \left( \frac{\ddot{\phi}}{2 \dot{\phi} H} + \frac{\epsilon}{4} \right) \zeta_c^2 + \frac{\epsilon}{2} \partial^{-2} (\zeta_c \partial^2 \zeta_c) + \ldots, \tag{5.5.39} \]

one can rewrite the essential part of \( H_{\zeta\zeta\zeta} \) as

\[ H_{\zeta\zeta\zeta} = -\frac{1}{\kappa^2} \int d^4 x \, 4 \epsilon^2 a^5 H \zeta_c^2 \partial^{-2} \dot{\zeta}_c + \ldots \tag{5.5.40} \]

where the omitted terms are higher order in slow-roll parameters or vanish outside the horizon. Using the interaction Hamiltonian following from this interaction one finds in the same way as described in section 5.4.3

\[ \langle \langle \zeta_c(k_1) \zeta_c(k_2) \zeta_c(k_3) \rangle \rangle_{SR} = \frac{\kappa^4 H^4_*}{\epsilon_* c_{123}^2} \sum_{i<j} k_i^2 k_j^2 \frac{1}{4a_{123}} + \ldots \tag{5.5.41} \]

in the leading order in slow-roll parameters. The full 3-point function in the original variable in the leading order in slow-roll parameters \( \epsilon_* \) and \( \eta_* \) defined in (5.3.18) reads

\[ \langle \langle \zeta(k_1) \zeta(k_2) \zeta(k_3) \rangle \rangle_{SR} = \frac{\kappa^4 H^4_*}{32 \epsilon_* c_{123}^2} \left[ 2 \eta_* \sum_j k_j^3 + \epsilon_* \left( a_{123}^3 - 2a_{123} b_{123} - 16c_{123} + 8 \frac{b_{123}^2}{a_{123}} \right) \right], \tag{5.5.42} \]

where \( a_{123}, b_{123} \) and \( c_{123} \) are symmetric polynomials in amplitudes of momenta,

\[ a_{123} = k_1 + k_2 + k_3, \quad b_{123} = k_1 k_2 + k_1 k_3 + k_2 k_3, \quad c_{123} = k_1 k_2 k_3. \tag{5.5.43} \]

The index \( SR \) reminds us that this form of the 3-point function holds for the slow-roll inflation in the first order in \( \epsilon_* \) and \( \eta_* \) in late time approximation.

In a similar fashion one can obtain the remaining correlation functions for the primordial perturbations,

\[ \langle \langle \zeta(k_1) \zeta(k_2) \dot{\gamma}^{(+)}(k_3) \rangle \rangle_{SR} = \frac{\kappa^4 H^4_*}{16 \sqrt{2} \epsilon_* a_{123}^2 c_{123}^2 k_3^2} \left[ a_{123}^3 - a_{123} b_{123} - c_{123} \right], \tag{5.5.44} \]

\[ \langle \langle \zeta(k_1) \dot{\gamma}^{(+)}(k_2) \dot{\gamma}^{(+)}(k_3) \rangle \rangle_{SR} = -\frac{\kappa^4 H^4_*}{128 b_{23}^2 k_1^2} \left[ (k_1 - a_{23}^2) (k_2 - a_{23}^2) (k_1 + 2b_{23}) - \frac{8b_{23}^2}{k_1 a_{123}} \right], \tag{5.5.45} \]

248
\[ \langle \zeta k_1 \hat{\gamma}(+)(k_2) \hat{\gamma}(-)(k_3) \rangle_{SR} = -\frac{\kappa^4 H_*^4}{128 b_{23}^6 k_1^2} (k_1^2 - a_{23}^2 + 4b_{23})^2 \times \]
\[ \times \left[ (k_1^2 - a_{23}^2 + 2b_{23}) - \frac{8b_{23}^2}{k_1 a_{123}} \right]. \] (5.5.46)

\[ \langle \hat{\gamma}(+)(k_1) \hat{\gamma}(+)(k_2) \hat{\gamma}(+)(k_3) \rangle_{SR} = \frac{\kappa^4 H_*^4}{64\sqrt{2}} \frac{J^2 a_{123}^2}{c_{123}^6} (a_{123}^3 - a_{123}b_{123} - c_{123}), \] (5.5.47)

\[ \langle \hat{\gamma}(+)(k_1) \hat{\gamma}(+)(k_2) \hat{\gamma}(-)(k_3) \rangle_{SR} = \frac{\kappa^4 H_*^4}{64\sqrt{2}} \frac{J^2}{a_{123}^2 c_{123}^6} (k_3 - a_{12})^4 \times \]
\[ \times (a_{123}^3 - a_{123}b_{123} - c_{123}), \] (5.5.48)

where

\[ a_{ij} = k_i + k_j, \quad b_{ij} = k_i k_j \] (5.5.49)

for \( i, j = 1, 2, 3 \) and \( J^2 \) is defined in (2.6.18) as

\[ J^2 = (p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(-p_1 + p_2 + p_3)(p_1 + p_2 + p_3). \] (5.5.50)

As long as the Hamiltonians in do not contain parity violating terms, the correlation functions are symmetric in the sense that

\[ \langle \zeta \zeta \hat{\gamma}(+) \rangle = \langle \zeta \zeta \hat{\gamma}(+) \rangle, \quad \langle \zeta \hat{\gamma}(+) \hat{\gamma}(+) \rangle = \langle \zeta \hat{\gamma}(+) \hat{\gamma}(+) \rangle, \]
\[ \langle \hat{\gamma}(+) \hat{\gamma}(+) \hat{\gamma}(+) \rangle = \langle \hat{\gamma}(+) \hat{\gamma}(+) \hat{\gamma}(+) \rangle, \] (5.5.51)

so only the relative number of helicities matters. All the above results can be found in [72, 75]. More on the structure of 3-point correlation functions in the slow-roll inflation, including parity violating terms can be found in [126, 127, 29, 128].

### 5.5.6. Response functions

In the previous sections we calculated 2- and 3-point functions of scalar and tensor perturbations produced during the slow-roll inflation and propagated to late times. We were starting from the inflationary action (5.5.11) and we worked mostly in the leading order in the slow-roll parameters \( \epsilon \) and \( \eta \), (5.3.18).

In this section we want to return to the Hamiltonian formalism of section 5.5.4 and introduce a more general analysis valid for any inflationary scenario with the scalar and spin-2 propagating degrees of freedom. For the details, consult [39, 40, 75].

First notice that the free Hamiltonian (5.5.35) and (5.5.36) is valid in general, since the equations (5.5.13) and (5.5.14) are derived without any approximations.
where

\[ H_{\zeta \zeta \zeta} = \frac{1}{k^2} \int \left[ \sum \Omega_{ij} \right] \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} \frac{d^3 k_3}{(2\pi)^3}. \]  

(5.5.53)

Exact expressions for the functions \( A, B, C, D \) depend on a particular model. For slow-roll inflation these were calculated in [40].

Now we can define the response functions \( \Omega_{[2]} \) and \( \Omega_{[3]} \) as in the usual response theory,

\[ \Pi(k_1) = \Omega_{[2]}(-k_1)\zeta(k_1) + \int \left[ \sum \Omega_{ij}(k_2, k_3) \zeta(-k_2)\zeta(-k_3) + \ldots \right] \]  

(5.5.54)

where

\[ \left[ \sum \Omega_{ij} \right] = (2\pi)^3 \delta(\sum k_j) \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} \frac{d^3 k_3}{(2\pi)^3}. \]  

(5.5.55)

One can show that the linear response function \( \Omega_{[2]}(k) \) depends on the magnitude of the momentum \( k = |k| \) only. Similarly, the response function \( \Omega_{[3]}(k_1, k_2) \) can be written entirely in terms of the momenta amplitudes

\[ k_1 = |k_1|, \quad k_2 = |k_2|, \quad k_3 = |k_3| = | - k_1 - k_2 | \]  

(5.5.56)

as discussed in section 2.2.2. We will usually omit the arguments of the response functions, assuming \( \Omega_{[3]} = \Omega_{[3]}(k_1, k_2, k_3) \).

The Hamilton equations (5.5.33) lead to the equations satisfied by the response functions,

\[ 0 = \dot{\Omega}_{[2]}(k) + \frac{1}{2a^3 \epsilon} \Omega_{[2]}^2(k) + 2a \epsilon k^2, \]  

(5.5.57)

\[ 0 = \dot{\Omega}_{[3]}(k_1, k_2, k_3) + \frac{1}{2a^3 \epsilon} \left( \Omega_{[2]}(k_1) + \Omega_{[2]}(k_2) + \Omega_{[2]}(k_3) \right) \Omega_{[3]}(k_1, k_2, k_3) \]  

\[ + \mathcal{X}(k_1, k_2, k_3), \]  

(5.5.58)

where

\[ \mathcal{X}(k_1, k_2, k_3) = 3A_{123} + B_{123} \Omega_{[2]}(k_1) + B_{213} \Omega_{[2]}(k_2) + B_{312} \Omega_{[2]}(k_3) \]  

\[ + C_{123} \Omega_{[2]}(k_2) \Omega_{[2]}(k_3) + C_{213} \Omega_{[2]}(k_1) \Omega_{[2]}(k_3) + C_{312} \Omega_{[2]}(k_1) \Omega_{[2]}(k_2) \]  

\[ + 3D_{123} \Omega_{[2]}(k_1) \Omega_{[2]}(k_2) \Omega_{[2]}(k_3). \]  

(5.5.59)
5.6. Experimental evidence for inflation

As we mentioned at the beginning of this chapter, the observation of the inhomogeneities in the Cosmic Microwave Background (CMB) is a strong evidence for the inflation. In this section we will shortly discuss how the calculations carried out in the previous sections fit into the actual measurements.

The most recent measurements of the Cosmic Microwave Background (CMB) were carried out by the Planck satellite [85], see figure 5.5, and are in the good
Figure 5.5: The maps of the Cosmic Microwave background obtained by the Planck satellite [85] in all its channels. The bright area in the middle is the Milky Way galaxy. Before the analysis of the Cosmic Microwave Background can be carried out, such bright objects must be cut off, so that only the residual CMB is analysed.

agreement with the previous measurements, e.g., [84, 129, 99]. The CMB is an electromagnetic, almost thermal radiation filling up the Universe. Its temperature is \( T = 2.7260 \pm 0.0013 \) K [130] and the small fluctuations on top of the thermal distribution are of the order \( \Delta T/T \approx 10^{-5} \).

Due to the evolution of the Universe, the shape of the spectrum of the CMB in the figure 5.6 does not resemble a simple shape given by (5.5.23). There is a few well-understood reasons for that. Firstly, as we live at a fixed position within the Universe, we cannot measure the 3-dimensional distribution of the radiation, but rather a 2-dimensional one extending on the sphere of the sky. Therefore, in order to compare with the experiment, all theoretical computations must be recalculated in terms of spherical observables. For example, by using the spherical harmonics we can write

\[
\zeta(x) = \int_0^\infty dk \sum_{l,m} \zeta_{lm}(k) Z_{klm}(x, \theta, \phi),
\]

where

\[
Z_{klm}(z, \theta, \phi) = \sqrt{\frac{2}{\pi k^j l k_x}} Y_{lm}(\theta, \phi),
\]
5.6. Experimental evidence for inflation

where $j_l$ denotes the spherical Bessel function. By orthonormality of $Z_{klm}$ we have

$$
\langle \zeta_{lm}(k)\zeta_{lm'}(k') \rangle = \langle \zeta(k)\zeta(k') \rangle \delta_{ll'} \delta_{mm'}.
$$

(5.6.3)

Similar, but more involved calculations are required for 3-point functions, see [131, 132].

Another problem is that while the inflationary models predict an average spectrum, we observe its particular realisation in our part of the Universe. For example, the monopole moment $l = 1$ is unknown, since it is impossible to measure the move of the Earth with respect to the background geometry. This means that we must account for this cosmic variance by adding additional error bars on all our observations. It turns out that on average the cosmic variance behaves like $l^{-1/2}$, therefore it is negligible for large multiple numbers $l$.

The final problem is the evolution of the perturbations $\zeta$ and $\hat{\gamma}$ from the inflationary period to our time. This is a non-trivial problem that requires solving equations of motion for the evolution of matter and electromagnetic field coupled to gravity, see [133, 134, 92, 100]. These equations are usually solved numerically and the resulting spectrum is fitted to the data. In this way one can predict the spectrum of the CMB based on the spectrum of $\zeta$ and $\hat{\gamma}$. The actual measurements and the fit to the $\Lambda$CDM model is presented in figure 5.6.

Any inflationary model delivers predictions for correlation functions of the primordial perturbations. Up to date only the scalar power spectrum $\langle \zeta\zeta \rangle$ was measured precisely. It is convenient to parametrise the spectrum as

$$
\Delta^2_S(k) = \frac{k^3}{2\pi^2} \langle \zeta(k)\zeta(-k) \rangle = \Delta^2(k_0) \left( \frac{k}{k_0} \right)^{n_S(k) - 1},
$$

(5.6.4)

where $k_0$ is some reference scale and $n_S$ is a spectral index defined in (5.5.21). For the Planck satellite the reference scale is $k_0 = 0.05$ Mpc$^{-1}$. The spectral index was determined by the measurements of the CMB at various wavelengths. The Planck satellite measured nine different frequencies between 30 and 857 GHz. While it is enough to determine the average value of the spectral index, its dependence on the momentum could not be determined precisely. By expanding in log $k$ one can parametrise

$$
\langle \zeta(k)\zeta(-k) \rangle = \frac{2\pi^2}{k^3} \Delta^2(k_0) \left( \frac{k}{k_0} \right)^{n_S - 1 + \frac{1}{2} \frac{dn_S}{d\log k} \log \frac{k}{k_0} + \frac{1}{8} \frac{d^2n_S}{d\log k^2} \log^2 \frac{k}{k_0} + ...},
$$

(5.6.5)

where $n_S$, $\frac{dn_S}{d\log k}$ and $\frac{d^2n_S}{d\log k^2}$ are evaluated at $k_0$. The first derivative of $n_S$ is known as the running of the spectral index, the second derivative is the running of the running and so on. It turns out that the spectrum is almost scale-invariant with a very small amplitude,

$$
\Delta^2(k_0) = (2.23 \pm 0.16) \cdot 10^{-9}
$$

(5.6.6)
so that indeed $\Delta(k_0) \sim 10^{-5}$.\textsuperscript{1} The values of the remaining parameters depend slightly on whether one assumes that $\frac{d^2n_S}{d\log k^2} = 0$ or not, see Table 5.1. When

<table>
<thead>
<tr>
<th>$n_S(k_0) - 1$</th>
<th>$\frac{d^2n_S}{d\log k^2} = 0$</th>
<th>$\frac{d^2n_S}{d\log k^2}$ fitted</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dn_S}{d\log k}(k_0)$</td>
<td>$-0.0404 \pm 0.0063$</td>
<td>$-0.0432 \pm 0.068$</td>
</tr>
<tr>
<td>$\frac{d^2n_S}{d\log k^2}(k_0)$</td>
<td>$-0.013 \pm 0.009$</td>
<td>$0.000 \pm 0.016$</td>
</tr>
<tr>
<td>$\frac{d^2n_S}{d\log k^2}(k_0)$</td>
<td>$0$</td>
<td>$0.017 \pm 0.016$</td>
</tr>
</tbody>
</table>

Table 5.1: Values of the spectral tilt and its running measured by the Planck satellite, [138]. The fitted values depend on the assumption whether the second running $\frac{d^2n_S}{d\log k^2}$ is assumed to vanish or not.

combined with previous measurements, the value of the spectral index is $n_S(k_0) - 1 = -0.0392 \pm 0.0054$, which excludes $n_S = 1$ at 6$\sigma$ level. On the other hand

\textsuperscript{1}All results in this section are given with the uncertainty at 68% level.
5.6. Experimental evidence for inflation

The inflationary models predict that the subsequent runnings of the spectral index should be suppressed by higher and higher powers of slow-roll parameters, (5.5.30).

The amplitude of the tensor modes in our Universe is very small. So far, no detection of the tensor modes has been reported [91], but hopefully new data from the Planck satellite to be released in 2014 may shed some new light on the perturbations of $\hat{\gamma}$.

The 3-point function of the scalar mode $\zeta$ has not been detected yet. Since all 3-point functions are functions of three amplitudes of momenta, one usually considers some typical configurations parametrised by a single constant $f_{NL}$. The equalateral and local non-Gaussianities are [139, 132],

$$\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle_{\text{local}} = 2\Delta^2(k_0)f_{NL}^{\text{local}} \left[ \frac{1}{(k_1k_2)^{4-n_S}} + \frac{1}{(k_2k_3)^{4-n_S}} + \frac{1}{(k_3k_1)^{4-n_S}} \right],$$

(5.6.7)

$$\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle_{\text{equil}} = 6\Delta^2(k_0)f_{NL}^{\text{equiv}} \left[ \frac{2}{(k_1k_2k_3)^2(4-n_S)/3} - \frac{1}{(k_1k_2)^{4-n_S}} - \frac{1}{(k_2k_3)^{4-n_S}} - \frac{1}{(k_3k_1)^{4-n_S}} + \frac{1}{k_1^{4-n_S}/3}k_2^{2(4-n_S)/3}k_3^{4-n_S} + 5 \text{ permutations} \right].$$

(5.6.8)

The Planck satellite results are [140],

$$f_{NL}^{\text{local}} = 2.7 \pm 5.8, \quad f_{NL}^{\text{equiv}} = -42 \pm 75.$$  

(5.6.9)

We can compare the results listed so far with the general form of the slow-roll correlation functions (5.5.23) and (5.5.24). As we can see, the tensor-to-scalar ratio,

$$r_{SR}^2(k) = \frac{\Delta_T^2}{\Delta_S^2} = 32\epsilon_*.$$  

(5.6.10)

With the sensitivity of the Planck satellite, one obtains an estimate $\epsilon_* \lesssim 0.01$. Similarly, comparing the forms of non-Gaussianities (5.6.7) and (5.6.8) with the 3-point function (5.5.42) we find $f_{NL} = O(\epsilon_*)$ in slow-roll models, which is consistent with the measurements.

Let finish this section with a comment on how the remaining cosmological parameters such as the Hubble constant or the curvature $\Omega$ can be extracted. The basic idea is that the evolution of the CMB depends heavily on the content of the Universe and its expansion rate. For example, the position of the first peak in the spectrum in figure 5.6 is determined mostly by the abundance of the dark energy in the Universe.

The accepted model that fits all the features of the spectrum in figure 5.6 is called $\Lambda CDM$ model (Cosmological Constant & Cold Dark Matter). The basic
version contains six parameters: the amplitude $\Delta^2(k_0)$ and the spectral index $n_S(k_0)$ of the primordial scalar fluctuations, the Hubble constant $H_0$, densities of matter and dark energy respectively,

$$\Omega_m = \frac{\rho_m}{\rho_{\text{crit}}}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_{\text{crit}}},$$

(5.6.11)

where $\rho_{\text{crit}}$ is the critical density defined in (5.2.23) and the optical depth $\tau$. This last parameter specifies a probability $P = e^{-\tau}$ that the photon emitted after the decoupling of the CMB from other particles was scattered. In principle this parameter is computable, but known astrophysical models deliver only some crude estimates. The measured value of the optical depth is $\tau = 0.089 \pm 0.014$. The values of the remaining parameters determined by the Planck satellite are

$$\Omega_m = 0.314 \pm 0.020, \quad \Omega_\Lambda = 0.686 \pm 0.020,$$

(5.6.12)

see figure 5.7. Note that $\Omega_m + \Omega_\Lambda = 1.00 \pm 0.03$. This is consistent with the flat Universe and the inflation as discussed in section 5.2.2.

With the current resolution in the measurements of the CMB, the $\Lambda$CDM model can be extended to contain more parameters. In particular, one can measure the abundance of our usual baryonic matter to be about 4.5%. These measurements are in great agreement with other observations, e.g., [94, 95]. Also, the age of our Universe can be calculated to be $t_0 = 13.81 \pm 0.06$ Gyr.

Figure 5.7: Approximate content of our Universe today, based on the Planck and WMAP results.