Chapter 1

Introduction

This thesis deals with connections between graph parameters and invariants of the orthogonal group and some of its subgroups. We consider basically two types of graph parameters: partition functions of vertex-coloring models and partition functions of edge-coloring models.

An edge-coloring model over a field $\mathbb{F}$ is a map $h$ from the set of all multisets on $k$ elements (for some $k \in \mathbb{N}$) to $\mathbb{F}$. Given a coloring of the edges of a graph $H$ with $k$ colors, at every vertex $v$ of $H$ we obtain a multiset of colors by looking at the colors of the edges incident with $v$; applying $h$ to it we get a number in $\mathbb{F}$; taking the product of these numbers over all vertices of $H$ one obtains the weight associated to the coloring of the edges. The partition function of $h$, $p_h$, is the map from the set of all graphs to $\mathbb{F}$ given by the sum over all $k$-colorings of the edges of a graph $H$ of the weights associated to the coloring:

$$p_h(H) = \sum_{\phi : E(H) \to \{1, \ldots, k\}} \prod_{v \in V(H)} h(\phi(\delta(v))),$$

(1.1)

where $\phi(\delta(v))$ denotes the multiset of colors of the edges incident with $v$. The partition function of an edge-coloring model can be seen as a generalization of the number of homomorphism of the linegraph of $H$ into the linegraph of a graph $G$. See section 3.3 for more details.

A vertex-coloring model over $\mathbb{F}$ is a symmetric $n \times n$ matrix. Given a coloring of the vertices of a graph $H$ with $n$ colors, it associates to the colors of the endpoints of each edge a number (the entry of the matrix corresponding to the pair of colors at the endpoints of the edge). The partition functions of a vertex-coloring model is a similar expression as (1.1), except that the role of vertices and edges is interchanged. For a graph $H$, the partition function
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of a vertex-coloring model can be seen as a generalization of the number of homomorphisms of $H$ into a graph $G$. See section 3.2 for more details.

Edge- and vertex-coloring models can be viewed as generalizations of statistical models. Their partition functions were introduced as graph parameters by de la Harpe and Jones [28], where they are called vertex models and spin models respectively. We choose to call them edge- and vertex coloring models to emphasize their connection with graph and linegraph coloring. We refer to Section 3.1 for more about the statistical mechanics origin.

One of the things we are concerned with in this thesis is the question: which graph parameters are partition functions of edge-coloring models? It turns out that there is an action of the orthogonal group on the set of edge-coloring models which leaves the partition function invariant. To give an answer to this question and other related questions, we need to use tools from (classical) invariant theory (the First and Second Fundamental Theorem for the orthogonal group), algebraic geometry (Hilbert’s Nullstellensatz) and the theory of (affine) algebraic groups. This gives an interesting connection between partition functions of edge-coloring models and invariants of the orthogonal group. Before we say more about this we will first sketch some background and motivation for our work.

1.1 Background and motivation

The study of partition functions of edge-coloring models is part of the recently emerged field of graph limits and graph partition functions. This is an exciting new area of research which has connections to extremal graph theory, probability theory, topology, basic commutative algebra and invariant theory. This, and many more things, have been described by Lovász in his beautiful recent book entitled: ‘Large Networks and Graph Limits’ [40].

Characterization of partition functions of vertex-coloring models

As Lovász writes in the introduction of his book [40], about ten years ago Freedman, who was interested in applications to quantum computing, asked the question: which graph parameters are partition functions of (real) vertex-coloring models? This question was solved a few months later by Freedman, Lovász and Schrijver [24]. To prove their characterization, Freedman, Lovász and Schrijver equipped the set of labeled graphs with the structure of a semigroup and introduced the concept of graph algebras and vertex-connection matrices (infinite matrices indexed by the set of all labeled graphs). The results were later generalized by Lovász and Schrijver to semidefinite functions on
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semigroups [42] and to semidefinite functions on certain categories [43]. The tools they developed turned out to be very useful to tackle other related questions. In particular, they were used by Lovász and Sós to characterize so-called generalized quasi random graphs [44]. Moreover, the structure of these graph algebras has connections to algebras of tensors invariant under certain subgroups of the symmetric group [41].

At around the same time Razborov [51] introduced the concept of a flag algebra, which he used to solve a longstanding problem in extremal graph theory [52]. These flag algebras turned out to be closely related to the graph algebras introduced by Freedman, Lovász and Schrijver (cf. [40] and [43]).

**Characterization of partition functions of edge-coloring models**

Motivated by a question of Freedman, Lovász and Schrijver, Szegedy [66] proved a characterization of partition functions of real edge-coloring models quite similar to the characterization of partition functions of real vertex-coloring models. Szegedy’s characterization is based on a different type of infinite matrices; edge-connection matrices. However, whereas the proof of Freedman, Lovász and Schrijver is based on basic properties of finite dimensional commutative algebras, Szegedy used the First Fundamental Theorem for the orthogonal group and the Positivstellensatz. This proof method inspired Schrijver [58] to give a characterization of algebras of tensors that are invariant under a subgroup of the orthogonal group. Szegedy [66, 67] also studied connections between partition functions of vertex-coloring models and complex edge-coloring models. In particular, he showed that the first are a special case of the latter, which led him to ask for which vertex-coloring models the edge-coloring model can be taken to be real-valued.

**Large networks and graph limits**

Parallel to the characterization of partition functions, in [45], which as awarded the Fulkerson prize in 2012, Lovász and Szegedy initiated a theory of limits of dense graphs. In particular, they defined a notion of convergence for a sequence of simple graphs based on homomorphism densities and exhibited a natural limit object for a convergent sequence of graphs. This was further developed by Borgs, Chayes, Lovász, Sós and Vesztergombi in [7], where they study various variants of the cut-metric related to the topology on the space of graphs as defined in [45].

The theory of graph limits has deep connections to Szemerédi’s regularity Lemma [46], to graph property testing [5] and to exchangeable random graphs [18]. We refer to the book by Lovász [40] for many more details.
1.2 Contributions

Motivated by questions that arose in the field of graph limits and graph partition functions, we present in this thesis various results about partition functions of edge-coloring models. In particular, we characterize which graph parameters are such partition functions. Furthermore, we determine the rank of the edge-connection matrices of partition functions of edge-coloring models (this is done by giving a combinatorial parametrization of certain algebras of tensors), and we characterize which partition functions of vertex-coloring models are partition functions of real edge-coloring models. Furthermore, we develop, analogues to the theory of graph limits, the first step for limits of edge-coloring models. For a more detailed overview of this thesis see below.

One can view our contributions as deepening the connection between the field of graph limits and partition functions and that of the invariant theory of the orthogonal group. Szegedy [66] was the first to observe that partition functions of (real) edge-coloring models and the invariant theory of the (real) orthogonal are intimately connected. In this thesis we will make use of this important observation and use various techniques from classical and geometric invariant theory. One could say that some of the results in this thesis are merely an application of these invariant-theoretical techniques. For example, from the point of view of invariant theory, one could consider the characterization of partition functions of edge-coloring models as a purely invariant-theoretical statement about the action of the orthogonal group acting on some polynomial ring with infinitely many variables. However, we rather speak of an interesting interaction between combinatorics and invariant theory. For example, the result by Schrijver [58] about characterizing algebras of tensors invariant under subgroups of the orthogonal group was motivated and inspired by its connection to combinatorics. Moreover, using combinatorial objects such as graphs or fragments to parametrize certain polynomial or tensor invariants might be the most natural way of looking at them.

1.3 Outline of this thesis

This thesis is roughly organized as follows: in Chapters 2–4 we state definitions, preliminaries and some more background; Chapters 5–8 contain the heart of this thesis. We will now give a more detailed outline.

Chapter 2 Preliminaries
Here we introduce important concepts such as labeled graphs, fragments and
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connection matrices. These concepts are probably not so well known and will be used in several parts of this thesis. We will also state some basic definitions and set up some notation about graphs.

Chapter 3 Partition functions of edge- and vertex-coloring models
In this chapter we define partition functions of edge- and vertex-coloring models and we say something about their statistical mechanics background. Furthermore, we show how the orthogonal group acts on edge-coloring models and we explain why this action leaves the partition function invariant. We also give some further background on partition functions of edge- and vertex-coloring models.

Chapter 4 Invariant theory
In this chapter we give a very brief introduction to invariant theory. In particular, we state the First and Second Fundamental Theorem for the orthogonal group, and we state a theorem about the uniqueness and existence of so-called closed orbits.

Chapter 5 Characterizing partition functions of edge-coloring models
In this chapter we give a characterization of partition functions of edge-coloring models with values in an algebraically closed field of characteristic zero. Furthermore, we characterize when the edge-coloring model can be taken to be of finite rank. To do so we use Hilbert’s Nullstellensatz and the First and Second Fundamental Theorem for the orthogonal group. Our proof is much inspired by Szegedy’s proof [66] for real edge-coloring models. This is based on joint work with Jan Draisma, Dion Gijswijt, Laci Lovász and Lex Schrijver, which appeared in the Journal of Algebra [19].

Chapter 6 Connection matrices and algebras of invariant tensors
Here we consider the rank of edge-connection matrices and relate this to the dimension of algebras of tensors that are invariant under certain subgroups of the orthogonal group. In particular, we give a combinatorial parametrization of such algebras. For real edge-coloring models this is based on [53], which appeared in the European Journal of Combinatorics. For edge-coloring models with values in an algebraically closed field of characteristic zero this is based on joint work with Jan Draisma, which appeared in the Journal of Algebraic Combinatorics [20]. The proofs are based on, and inspired by, the aforementioned result of Schrijver, characterizing algebras of invariant tensors [58].
Chapter 7. Edge-reflection positive partition functions of vertex-coloring models
In this chapter, we give an answer to the question by Szegedy asking for which vertex-coloring model its partition function is the partition function of a real edge-coloring model. We use ideas from Kempf and Ness [33] and a generalization of the Hilbert-Mumford theorem, a deep result from geometric invariant theory. Except for Section 7.2, this is based on [54] of which an extend abstract appeared in The Seventh European Conference on Combinatorics, Graph Theory and Applications, Eurocomb 2013. Section 7.2 is based on joint work with Jan Draisma [20, Section 6].

Chapter 8. Compact orbit spaces in Hilbert spaces and limits of edge-coloring models
Motivated by the theory of graph limits, we construct limits of edge-coloring models in this chapter. To do so we prove a general result about compact orbit spaces in Hilbert spaces. This result can be applied to construct limit objects for certain sequences of edge-coloring models as well as limit objects for convergent sequences of graphs. This is based on joint work with Lex Schrijver [55].