Graph parameters and invariants of the orthogonal group
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Chapter 5

Characterizing partition functions of edge-coloring models

In this chapter we characterize which graph invariants are partition functions of edge-coloring models over an algebraically closed field of characteristic zero.

This chapter is based on joint work with Jan Draisma, Dion Gijswijt, Laci Lovász and Lex Schrijver \[19\] except for Section 5.2 which is based on unpublished joint work with Lex Schrijver and Dion Gijswijt.

5.1 Introduction

Motivated by a question of Freedman (see the preface of the book by Lovász \[40\]), Freedman, Lovász and Schrijver characterized partitions functions of real vertex-coloring models in terms of rank and positive semidefiniteness conditions for the vertex-connection matrices.

Theorem 5.1 (Freedman, Lovász and Schrijver \[24\]). Let $f : \mathcal{G} \to \mathbb{R}$ be a graph invariant. Then there exists $a \in \mathbb{R}^n_{>0}$ and a symmetric matrix $B \in \mathbb{R}^{n \times n}$ such that $f(H) = p_{a,B}(H)$ for all $H \in \mathcal{G}$ if and only if $f$ is multiplicative, reflection positive and $\text{rk}(N_{f,l}) \leq n^l$ for all $l \in \mathbb{N}$.

In an earlier version of their paper Freedman, Lovász and Schrijver conjectured that a similar characterization holds for partition functions of real edge-
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coloring models. This was proved by Szegedy [66]. The characterization is as follows.

**Theorem 5.2** (Szegedy [66]). Let \( f : G' \to \mathbb{R} \) be a graph invariant. Then there exists a real edge-coloring model \( h \) such that \( f = p_h \) if and only if \( f \) is multiplicative and edge-reflection positive.

Whereas the proof of Theorem 5.1 in [24] makes use of some basic properties of finite dimensional commutative algebras, Szegedy [66] proved Theorem 5.2 using the First Fundamental Theorem of invariant theory for the orthogonal group and the Positivestellensatz (real Nullstellensatz). This connection with invariant theory and algebra has been further developed by Schrijver [59], giving an alternative (and shorter) proof of Theorem 5.2. He also used this idea to characterize partition functions of vertex-coloring models with \( a = 1 \), the all-ones vector, [60, 61].

In this chapter we give a characterization of partition functions of edge-coloring models with values in an algebraically closed field of characteristic zero. So throughout this chapter \( \mathbb{F} = \mathbb{F} \). Moreover, we characterize when the edge-coloring model can be taken to be of finite rank (see definition below). To state our results we need to introduce some definitions.

For a graph \( H = (V, E) \), \( U \subseteq V \) and any \( s : U \to V \), define
\[
E_s := \{ us(u) \mid u \in U \} \quad \text{and} \quad H_s := (V, E \cup E_s)
\]
(adding multiple edges if \( E \) intersects \( E_s \)). Let \( S_U \) denote the group of permutations of \( U \).

**Theorem 5.3.** Let \( \mathbb{F} = \mathbb{F} \) and let \( f : G \to \mathbb{F} \) be a graph invariant. Then \( f = p_h \) for some \( k \)-color edge-coloring model over \( \mathbb{F} \) if and only if \( f \) is multiplicative and for each graph \( H = (V, E) \) and each \( U \subseteq V \) of size \( k + 1 \) and each \( s : U \to V \),
\[
\sum_{\pi \in S_U} \text{sgn} (\pi) f(H_{s \circ \pi}) = 0. \quad (5.2)
\]

We will prove Theorem 5.3 in Section 5.4. Recently, based on Theorem 5.3, Schrijver [62] found a characterization of partition functions of complex edge-coloring models in terms of rank growth of the edge-connection matrices.

For a \( k \)-color edge-coloring model \( h \), its moment matrix \( M_h \) is defined by
\[
M_h(\alpha, \beta) = h(\alpha^{a} + \beta), \quad \text{for} \quad \alpha, \beta \in \mathbb{N}^k.
\]

Abusing language we say that \( h \) has rank \( r \) if \( M_h \) has rank \( r \). For any graph \( H = (V, E), U \subseteq V \) and \( s : U \to V \), let \( H/s \) be the graph obtained by contracting all edges in \( E_s \).
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**Theorem 5.4.** Let $\mathbb{F} = \mathbb{F}$ and let $f : G \to \mathbb{F}$ be the partition function of a $k$-color edge-coloring model over $\mathbb{F}$. Then $f = p_h$ for some $k$-color edge-coloring model over $\mathbb{F}$ of rank at most $r$ if and only if for each graph $H = (V, E)$ and each $U \subseteq V$ of size $r + 1$ and each $s : U \to V \setminus U$,

$$
\sum_{\pi \in S_U} \text{sgn}(\pi) f(H / s \circ \pi) = 0.
$$

We will prove Theorem 5.4 in Section 5.5. The conditions in Theorem 5.4 imply those in Theorem 5.3 for $k := r$. Indeed for each $u \in U$ we can add a new vertex $u'$ and a new edge $uu'$ to $H$, thus obtaining a graph $H'$. Then (5.4) for $H', U'$ and $s'(u') = s(u)$ gives (5.2) for $H, U, s$. This implies that if a graph parameter $f : G \to \mathbb{F}$ is multiplicative and satisfies (5.4), for all $H, U$ and $s$, then $f$ is the partition function of an $r$-color edge-coloring model over $\mathbb{F}$.

Let us illustrate Theorem 5.4 by showing that it implies that the partition function of a vertex-coloring model is also the partition of an edge-coloring model. This was already shown by Szegedy in [66], where he even constructs the edge-coloring model from the vertex-coloring model (cf. Lemma 7.1). Let $(a, B)$ be an $n$-color vertex-coloring model over $\mathbb{F}$. Let $H = (V, E)$ be a graph, take $U \subset V$ of size $n + 1$ and let $s : U \to V \setminus U$. Then

$$
\sum_{\pi \in S_U} \text{sgn}(\pi) p_{a, B}(H / s \circ \pi) = \sum_{\phi : V \setminus U \to [n]} \sum_{\pi \in S_U} \text{sgn}(\pi) \prod_{v \in V \setminus U} a_{\phi(v)} \prod_{uv \in E(H / s \circ \pi)} B_{\phi(u), \phi(v)}.
$$

For fixed $\phi : V \setminus U \to [n]$ there exists $u_1, u_2 \in U$ such that $\phi(s(u_1)) = \phi(s(u_2))$. Let $\rho \in S_U$ be the transposition interchanging $u_1$ and $u_2$. Then the contribution of $\pi$ and $\pi \circ \rho$ will cancel each other. Hence (5.5) is zero.

Our proofs of both Theorem 5.3 and 5.4 are based on the First and Second Fundamental Theorem of invariant theory for the orthogonal group and Hilbert’s Nulstellensatz. They are much inspired by Szegedy’s proof of Theorem 5.2.

The rest of this chapter is organized as follows. In Section 5.2 we discuss a question of Szegedy concerning finite rank edge-coloring models, which has motivated the results in this chapter. In Section 5.3 we develop the invariant-theoretical framework necessary to prove both Theorems 5.3 and 5.4. The proofs of these theorems are given in the subsequent sections. Finally, in Section 5.6 we state analogues results for directed graphs.
5.2 Finite rank edge-coloring models

Using an explicit description of finite rank edge-coloring models, Szegedy [67] showed that partition functions of finite rank edge-coloring models over $C$ can be seen as limits of partition functions of vertex-coloring models over $C$. In particular, the vertex-connection matrices of these partition functions have exponentially bounded rank growth.

**Proposition 5.5** (Szegedy [67]). Let $h$ be a $k$-color edge-coloring model over $C$ of rank $r$. Then $\text{rk} \left( N_{p_h,l} \right) \leq r^l$ for all $l$.

Let us give a short proof.

**Proof.** Define the $(N^k)^l \times G_l$ matrix $A$ by

$$A(\alpha_1, \ldots, \alpha_l, H) := \sum_{\psi: E \rightarrow [k]} \prod_{v \in V \setminus [l]} h(\psi(\delta(v)))$$

for $H = (V, E) \in G_l$ and $(\alpha_1, \ldots, \alpha_l) \in (N^k)^l$. Then $N_{p_h,l} = A^T M_h^{\otimes l} A$. Hence $\text{rk} \left( N_{p_h,l} \right) \leq \text{rk} \left( M_h^{\otimes l} \right) = r^l$. \qed

This result made Szegedy ask the question whether there exists a graph parameter $f: G \rightarrow C$ whose vertex-connection matrices have exponentially bounded rank growth and which is not the partition function of an edge-coloring model. The answer to this question turns out to be positive as we will describe below.

Recall the graph parameter $f_x: G \rightarrow C$ for $x \in C$ from Example 2.1.

$$f_x(H) = \begin{cases} x^{c(H)} & \text{if } H \text{ is 2-regular}, \\ 0 & \text{otherwise}, \end{cases} \quad (5.7)$$

where $c(H)$ denotes the number of connected components of $H$. We will show below that $\text{rk} \left( N_{f,2,l} \right) \leq 4^l$, but first we will show that $f_{-2}$ it is not the partition function of an edge-coloring model.

**Proposition 5.6.** The graph parameter $f_x$ is the partition function of an edge-coloring model over $C$ if and only if $x \in \mathbb{N}$.

**Proof.** Suppose first that $x = k \in \mathbb{N}$. Define $h: C[x_1, \ldots, x_k] \rightarrow C$ by

$$h(x^a) = \begin{cases} 1 & \text{if } x^a = x_i^2 \text{ for some } i \in [k], \\ 0 & \text{otherwise}. \end{cases} \quad (5.8)$$
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Then it is easy to see that \( f_k = p_h \).

We will now show that \( f_x \) is not the partition function of a \( k \)-color edge-coloring model for any \( k \in \mathbb{N} \), if \( x \in \mathbb{C} \setminus \mathbb{N} \). Fix any \( k \in \mathbb{N} \). Consider the graph \( H = ([k + 1], \emptyset) \) and define \( s : [k + 1] \to [k + 1] \) by \( s(i) = i \) for all \( i \). Then for \( \pi \in S_{k+1} \), \( H \circ s \pi \) consists of exactly \( o(\pi) \) cycles, where \( o(\pi) \) denotes the number of orbits of the permutation \( \pi \). So

\[
\sum_{\pi \in S_{k+1}} \text{sgn}(\pi) f_x(H \circ s \pi) = \sum_{\pi \in S_{k+1}} \text{sgn}(\pi) x^{o(\pi)},
\]

which is a polynomial \( p \) in \( x \) of degree \( k + 1 \) with leading coefficient 1. As by the above and by Theorem 5.3, \( p(x) = 0 \) for \( x = 0, \ldots, k \), it follows that \( p(x) = x(x - 1) \ldots (x - k) \). Hence \( p(x) \neq 0 \) for \( x \notin \mathbb{N} \) and so Theorem 5.3 implies that \( f_x \) is not the partition function of a \( k \)-color edge-coloring model over \( \mathbb{C} \).

Note that the proof of Proposition 5.6 actually shows that if \( x \in \mathbb{F} \setminus \mathbb{N} \), then \( f_x \) is not the partition function of any edge-coloring model over \( \mathbb{F} \) for any algebraically closed field \( \mathbb{F} \) of characteristic zero.

Proposition 5.7. The rank of \( N_{f_{-2,l}} \) is bounded by \( 4^l \) for all \( l \).

Proof. Write \( Q_l := Q_l(f_{-2}) \). The first thing to note is that \( Q_l \) is spanned by labeled graphs that are disjoint unions of \( K^*_1 \)'s, \( C^*_1 \)'s and \( K^**_2 \)'s. Indeed, since \( f_{-2} \) is only nonzero on 2-regular graphs, this already implies that we can restrict ourselves to disjoint unions of \( K^*_1 \)'s, \( C^*_1 \)'s and paths with both endpoints labeled. Since any path with two endpoints labeled is equivalent modulo \( f_{-2} \) to a multiple of \( K^**_2 \), the claim follows.

For \( i \in \mathbb{N} \), let \( A_i \) be the submatrix of \( N_{f_{-2,2i}} \) indexed by \( 2i \)-labeled graphs on \( 2i \) vertices that are disjoint unions of labeled edges (these are exactly the fully labeled perfect matchings on \( 2i \) vertices). Using that the submatrix of \( N_{f_{-2,l}} \) indexed by disjoint unions of \( K^*_1 \)'s, \( C^*_1 \)'s and \( K^**_2 \)'s, has a special block structure, it follows that

\[
\text{rk}(N_{f_{-2,l}}) = \sum_{i=0}^{\lfloor l/2 \rfloor} \binom{l}{2i} 2^{l-2i} \text{rk}(A_i).
\]
Next, note that $A_2$ is given by

$$A_2 = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix}. \tag{5.11}$$

So $\text{rk}(A_2) = 2$ and we we see that

$$\text{+} + \text{+} = 0 \quad \text{in } Q_l. \tag{5.12}$$

We will refer to \table as a \textit{crossing pair}. By (5.12) we can replace a crossing pair by a linear combination of pairs of edges that are crossing. We will refer to this as \textit{uncrossing}. Note that after uncrossing a crossing pair in a perfect matching, the two new matchings obtained both contain fewer crossing pairs than the original one. This implies that the row space of $A_i$ is spanned by the perfect matchings that do not contain crossing pairs. We will call these matchings \textit{noncrossing}. The number of such matchings is bounded by $\binom{2i}{i} \leq 4^i$, as each noncrossing perfect matching uniquely determines a subset of $[2i]$ of size $i$ by looking at the left points of each edge. So, as $\text{rk}(A_i) \leq 4^i$, (5.10) implies that $\text{rk}(N_{f-2,l}) \leq 4^l$.

\subsection{5.2.1 Catalan numbers and the rank of $N_{f-2,l}$}

Using representation theory of the symmetric group, we determine the rank of $N_{f-2,l}$ exactly. We will see that it is exactly the Catalan number $C_l$. In fact, an explicit computation of the rank of the vertex-connection matrices of $f_x$ can be determined in this way for any $x \in \mathbb{Z}$. It can be derived from [26, Theorem 3.1]. We refer to [57] for an introduction to the representation theory of the symmetric group.

The Catalan numbers form a sequence of natural numbers that occur in various counting problems. In his book, Stanley [64] gave a list of exercises with 66 possible interpretations of the Catalan numbers. The list of interpretations keeps on growing. Currently, there are 207; see [65]. For $n \geq 0$, the $n$-th Catalan
number is defined as
\[ C_n := \frac{1}{n+1} \binom{2n}{n}. \] (5.13)

Let us give an interpretation.

**Lemma 5.8** (Exercise 19(o) in [64]). \( C_n \) is equal to the number of noncrossing perfect matchings on \([2n]\).

To compute \( C_{n+1} \) from \( C_0, \ldots, C_n \), we can by Lemma 5.8 do the following. We start by putting an edge from 1 to any \( i \in [2n+2] \). Then, since edges are not allowed to cross, we are left with finding the number of noncrossing perfect matchings under the first edge times the number of noncrossing perfect matchings right from the endpoint of the first edge. This in particular implies that \( i \) should be even. Hence
\[ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}. \] (5.14)

The symmetric group \( S_{2n} \) acts on the set of perfect matchings on \([2n]\), \( M_n \), by permuting the endpoints of the edges. For example for \( n = 2 \) and \( \tau = (23) \in S_4 \),
\[ \tau(\circ\circ\circ\circ) = \circ\circ\circ\circ \] (5.15)
Now note that the matrix \( A_n \) is \( S_{2n} \)-equivariant, i.e., for each \( N, M \in M_n \) and \( \tau \in S_{2n} \), we have \( A_n(\tau N, \tau M) = A_n(N, M) \). Let \( M_0 \) be the matching on \([2n]\) with edges 12, 34, \ldots, \((2n-1)n\), let \( S_n \subset S_{2n} \) be the subgroup permuting the odd positions and let \( v \in \mathbb{F} M_n \) be defined by
\[ v := \sum_{\tau \in S_n} \text{sgn}(\tau) \tau M_0. \] (5.16)

We claim that \( A_n v \neq 0 \). Indeed,
\[ (A_n v)(M_0) = \sum_{\tau \in S_n} \text{sgn}(\tau)(-2)^{c(M_0 \cup \tau M_0)} = \sum_{\tau \in S_n} \text{sgn}(\tau)(-2)^{o(\tau)}. \] (5.17)
So by the proof of Proposition 5.6, \( A_n v \neq 0 \).

As \( v \) is a generator of the Specht module \( S^\lambda \), where \( \lambda \) is the partition of \( 2n \) given by \((2,2,\ldots,2)\), and as \( A_n \) is \( S_{2n} \)-equivariant, Schur’s lemma implies that \( v \in \text{Im} A_n \). Hence the rank of \( A_n \) is at least the dimension of \( S^\lambda \).

The dimension of \( S^\lambda \) is equal to the number of ways to place the numbers 1, 2, \ldots, 2n in a \( n \times 2 \) array such that both the columns and the rows are increasing. (The number of standard Young tableaux of shape \( \lambda \)).
number is known to be equal to $C_n$ (cf. [64, Exercise 19(ww)]). It follows that $\text{rk}(A_n) \geq \dim(S^1) = C_n$. As by the proof of Proposition 5.7, $\text{rk}(A_n)$ is bounded by the number of noncrossing perfect matchings, Lemma 5.8 implies that $\text{rk}(A_n)$ is equal to $C_n$.

Viewing $C^*_1$ as a matching of a vertex to itself we may say that the dimension of $Q_l$ is equal to the number of noncrossing matchings on $[2l]$. So to compute the dimension of $Q_l$ for $l \geq 1$, we can choose to put on the first position an isolated vertex, a loop or the left vertex of an edge and then continue recursively. Setting $\dim(Q_{-1}) = \dim(Q_0) = 1$, this gives rise to the following recurrence relation for $\dim(Q_l)$:

$$
\dim(Q_l) = 2 \dim(Q_{l-1}) + \sum_{i=0}^{l-2} \dim(Q_i) \dim(Q_{l-2-i})
\quad = \sum_{i=0}^{l} \dim(Q_{l-1}) \dim(Q_{l-1-i}). \quad (5.18)
$$

Now note that $\dim(Q_l)$ satisfies the same recurrence relation as $C_{l+1}$ in (5.14). As $\dim(Q_{-1}) = \dim(Q_0) = C_0 = C_1$, it follows that $\dim(Q_l) = C_{l+1}$ for all $l$. We will summarize it as a theorem.

**Theorem 5.9.** The rank of $N_{f_{-2,l}}$ is equal to $C_{l+1}$.

As a corollary to the proof of Theorem 5.9 and (5.10), we obtain the following recurrence relation for the Catalan numbers, previously obtained by Xin and Xu [68].

**Corollary 5.10.** The Catalan numbers satisfy the following recurrence equation:

$$
C_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} 2^{n-2i} C_i. \quad (5.19)
$$

### 5.3 Framework

Here we develop the framework used for the proof of both Theorem 5.3 and 5.4.

Let $k \in \mathbb{N}$. Introduce a variable $y_\alpha$ for each $\alpha \in \mathbb{N}^k$ and define the ring $T$ of polynomials in these (infinitely) many variables:

$$
T := \mathbb{F}[y_\alpha \mid \alpha \in \mathbb{N}^k]. \quad (5.20)
$$
Note that there is a bijection between the variables $y_\alpha$ and the monomials $x^{\alpha} = \prod_{i=1}^{k} x_i^{\alpha_i}$ in $R = \mathbb{F}[x_1, \ldots, x_k]$. In this way functions $h : \mathbb{N}^k \rightarrow \mathbb{F}$ correspond to elements of $R^*$. The action of $O_k$ on $R$ induces an action of $O_k$ on $T$ via the bijection between the variables of $T$ and the monomials of $R$. Equivalently, using the action of $O_k$ on $R^*$, define $gq(h) = q(g^{-1}h)$ for $g \in O_k, q \in T$ and $h : \mathbb{N}^k \rightarrow \mathbb{F}$.

Define $p : \mathcal{G} \rightarrow T$ by

$$p(H) := \sum_{\phi : E(H) \rightarrow [k]} \prod_{v \in V(H)} y_\phi(\delta(v)),$$  \hspace{1cm} (5.21)

where we view $\phi(\delta(v))$ as a multisubset of $[k]$, which we identify with its characteristic vector in $\mathbb{N}^k$. Note that $p(H) = p(H')$ for isomorphic graphs $H$ and $H'$. Now extend $p$ linearly to $\mathbb{F}\mathcal{G}$ to obtain an algebra homomorphism $p : \mathbb{F}\mathcal{G} \rightarrow T$. (Recall that $\mathbb{F}\mathcal{G}$ is the semigroup algebra of $(\mathcal{G}, \cdot)$, where the product of two graphs is just their disjoint union.) Using the First and Second Fundamental Theorem for the orthogonal group we characterize the kernel $\text{Ker} \ p$ and the image $\text{Im} \ p$ of $p$. The characterization of $\text{Im} \ p$ is similar to the one given by Szegedy [66].

To characterize $\text{Ker} \ p$, let $I$ be the subspace of $\mathbb{F}\mathcal{G}$ spanned by the quantum graphs

$$\frac{\prod_{ij \in E} x_{ij}}{\sum_{\pi \in S_U} \text{sgn}(\pi) H_{s \circ \pi}},$$  \hspace{1cm} (5.22)

where $H = (V, E)$ is a graph, $U \subseteq V$ with $|U| = k + 1$, and $s : U \rightarrow V$.

**Proposition 5.11.** We have $\text{Im} \ p = T^{O_k}$ and $\text{Ker} \ p = I$.

**Proof.** For $n \in \mathbb{N}$, let $\mathcal{G}_n$ be the collection of graphs with vertex set $[n]$. Let $S\mathbb{F}^{n \times n}$ be the set of symmetric matrices in $\mathbb{F}^{n \times n}$. For any linear space $X$ let $\mathcal{O}(X)$ denote the set of regular functions on $X$ (the algebra generated by the linear functions on $X$). Then $\mathcal{O}(S\mathbb{F}^{n \times n})$ is spanned by the monomials $\prod_{ij \in E} x_{ij}$ in the variables $x_{ij}$, where $([n], E)$ is a graph. Here $x_{ij} = x_{ji}$ are the standard coordinate functions on $S\mathbb{F}^{n \times n}$, while taking $ij$ as unordered pair.

Let $\mathbb{F}\mathcal{G}_n$ be the space of formal $\mathbb{F}$-linear combinations of elements of $\mathcal{G}_n$. Let $T_n$ be the set of homogenous polynomials in $T$ of degree $n$. We set $p_n := p|\mathbb{F}\mathcal{G}_n$. So $p_n : \mathbb{F}\mathcal{G}_n \rightarrow T_n$. Hence it suffices to prove, for each $n$,

$$\text{Im} \ p = T^{O_k}_n$$  \hspace{1cm} (5.23)

To show (5.23), we define linear functions $\mu, \sigma$ and $\tau$ so that the following
diagram commutes:

\[
\begin{array}{ccc}
FG_n & \xrightarrow{p_n} & T_n \\
\mu & \downarrow & \sigma \\
O(S\mathbb{F}^{n\times n}) & \xrightarrow{\tau} & O(\mathbb{F}^{k\times n})
\end{array}
\]  

(5.24)

Define \(\mu\) by

\[
\mu(\prod_{ij\in E} x_{i,j}) := H
\]  

(5.25)

for any graph \(H := ([n], E)\). Define \(\sigma\) by

\[
\sigma(\prod_{j=1}^{n} \prod_{i=1}^{k} z_{i,j}^{\alpha_{i,j}}) := \prod_{j=1}^{n} y_{\alpha_{j}}
\]  

(5.26)

for \(\alpha \in \mathbb{N}^{k\times n}\), where \(z_{i,j}\) are the standard coordinate functions on \(\mathbb{F}^{k\times n}\) and where \(\alpha_j := (\alpha_{1,j}, \ldots, \alpha_{k,j}) \in \mathbb{N}^k\). Then \(\sigma\) is \(O_k\)-equivariant for the natural action of \(O_k\) on \(O(\mathbb{F}^{k\times n})\). Finally, define \(\tau\) by

\[
\tau(q)(z) := q(z^T z)
\]  

(5.27)

for \(q \in O(S\mathbb{F}^{n\times n})\) and \(z \in \mathbb{F}^{k\times n}\). Now (5.24) commutes; in other words,

\[
p_n \circ \mu = \sigma \circ \tau.
\]  

(5.28)

To prove it, consider any monomial \(q := \prod_{ij\in E} x_{i,j}\) in \(O(S\mathbb{F}^{n\times n})\), where \(H := ([n], E)\) is a graph. Then for any \(z \in \mathbb{F}^{k\times n}\),

\[
\tau(q)(z) = q(z^T z) = \prod_{ij\in E} \sum_{h=1}^{k} z_{h,i} z_{h,j} = \sum_{\phi:E\rightarrow[k]} \prod_{i\in[n]} \prod_{e\in\delta(i)} z_{\phi(e),i'}
\]  

(5.29)

So, by definition (5.26) of \(\sigma\) and (5.25) of \(\mu\),

\[
\sigma(\tau(q)) = \sum_{\phi:E\rightarrow[k]} \prod_{i\in[n]} y_{\phi(\delta(i))} = p_n(H) = p_n(\mu(q))
\]  

(5.30)

This proves (5.28).

Note that \(\tau\) is an algebra homomorphism, but \(\sigma\) and \(\mu\) generally are not. (\(FG_n\) and \(T_n\) are not algebras.) The latter two functions are surjective. Moreover, \(\mu\) is bijective and restricted to the \(S_n\)-invariant part of its domain, \(\sigma\) is bijective.
By the FFT for $O_k$ (cf. Theorem 4.4), $\text{Im } \tau = (O(\mathbb{F}^{k \times n}))^{O_k}$. Hence, as $\mu$ and $\sigma$ are surjective, and as $\sigma$ is $O_k$-equivariant,

$$\text{Im } p_n = p_n(FG_n) = p_n(\mu(O(S\mathbb{F}^{n \times n}))) = \sigma(\tau(O(S\mathbb{F}^{n \times n}))) = \sigma((O(\mathbb{F}^{k \times n}))^{O_k}) = T_{\mu}^O_k.$$  \hfill (5.31)

The last equality follows from the fact that $\sigma$ is $O_k$-equivariant, so that we have $\subseteq$. To see $\supseteq$, take any $q \in T_{\mu}^O_k$, as $\mu$ is surjective, $q = \sigma(r)$ for some $r \in O(\mathbb{F}^{k \times n})$. Then, by Lemma 4.2, $q = \sigma(\rho_{O_k}(r))$, where $\rho_{O_k}$ is the Reynolds operator of $O_k$. This proves the first statement in (5.23).

To see that $\mathcal{I} \cap F\mathcal{G}_n \subseteq \text{Ker } p_n$, let $H = ([n], E)$ be a graph, $U \subset [n]$ with $|U| = k + 1$, and $s : U \to [n]$. Then $\sum_{\pi \in S_U} \text{sgn}(\pi) H_{s,\pi}$ belongs to $\text{Ker } p_n$, as

$$p(\sum_{\pi \in S_U} \text{sgn}(\pi) H_{s,\pi}) = \sum_{\phi : E \cup E_s \to [k]} \sum_{\pi \in S_U} \text{sgn}(\pi) \prod_{i \in [n]} y_{\phi(\delta_{H_{s,\pi}}(i))}. \hfill (5.32)$$

For fixed $\phi$, there exist distinct $u_1, u_2 \in U$ such that $\phi(u_1, s(u_1)) = \phi(u_2, s(u_2))$. So if $\rho$ is the permutation of $U$ interchanging $u_1$ and $u_2$, we have that the terms in (5.32) corresponding to $\pi$ and $\pi \circ \rho$ cancel. Hence (5.32) is zero.

We finally show $\text{Ker } p_n \subseteq \mathcal{I}$. By the SFT for $O_k$ (cf. Theorem 4.5), (as $\mathbb{F}$ is algebraically closed) $\text{Ker } \tau$ is the ideal in $O(S\mathbb{F}^{n \times n})$ generated by the $(k + 1) \times (k + 1)$ minors of $S\mathbb{F}^{n \times n}$. Then

$$\mu(\text{Ker } \tau) \subseteq \mathcal{I}. \hfill (5.33)$$

To prove (5.33), it suffices to show that for any $(k + 1) \times (k + 1)$ submatrix $N$ of $S\mathbb{F}^{n \times n}$ and any graph $H = ([n], E)$ one has

$$\mu(\text{det}(N) \prod_{ij \in E} x_{i,j}) \in \mathcal{I}. \hfill (5.34)$$

There is a subset $U$ of $[n]$ with $|U| = k + 1$ and an injective function $s : U \to [n]$ such that $\{(u, s(u)) \mid u \in U\}$ forms the diagonal of $N$. So

$$\text{det}(N) = \sum_{\pi \in S_U} \text{sgn}(\pi) \prod_{u \in U} x_{u, s, \pi(u)}. \hfill (5.35)$$

Then

$$\mu(\text{det}(N) \prod_{ij \in E} x_{i,j}) = \sum_{\pi \in S_U} \text{sgn}(\pi) \mu(\prod_{u \in U} x_{u, s, \pi(u)} \cdot \prod_{ij \in E} x_{i,j}) = \sum_{\pi \in S_U} \text{sgn}(\pi) H_{s,\pi} \in \mathcal{I}, \hfill (5.36)$$
by definition of $I$. This proves (5.33).

To prove $\text{Ker } p_n \subseteq I$, let $\gamma \in \text{Ker } p_n$. Then $\gamma = \mu(q)$ for some $q \in \mathcal{O}(S\mathbb{F}^{n \times n})$. Hence $\sigma(\tau(q)) = p(\mu(q)) = p(\gamma) = 0$. We may assume that $q$ is $S_n$-invariant since $p$ is isomorphism-invariant (cf. Lemma 4.2). As $\sigma$ is bijective on $(\mathcal{O}(\mathbb{F}^{k \times n}))^{S_n}$, this implies that $\tau(q) = 0$. Hence $\gamma = \mu(q) \in \mu(\text{Ker } \tau) \subseteq I$. This finishes the proof of the second statement in (5.23).

5.4 Proof of Theorem 5.3

**Theorem 5.3.** Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $f : G \to \mathbb{F}$ be a graph invariant. Then $f = p_n$ for some $k$-color edge-coloring model over $\mathbb{F}$ if and only if $f$ is multiplicative and for each graph $H = (V, E)$ and each $U \subseteq V$ of size $k + 1$ and each $s : U \to V$,

$$\sum_{\pi \in S_U} \text{sgn}(\pi) f(H_{s \circ \pi}) = 0.$$  \hspace{1cm} (5.2)

**Proof.** We fix $k$. Necessity of the conditions (5.2) follows from the fact $\text{Ker } p = I$ by Proposition 5.11.

To prove sufficiency, we must show that the polynomials $p(H) - f(H)$ have a common zero. Here $f(H)$ denotes the constant polynomial with value $f(H)$. A common zero means an element $y : \text{N}^k \to \mathbb{F}$, with for all $H \in G (p(H) - f(H))(y) = 0$, equivalently, $p_y(H) = f(H)$, as required.

As $f$ is multiplicative, $f$ extends linearly to an algebra homomorphism $f : \mathbb{F}G \to \mathbb{F}$. By the condition in Theorem 5.3, $f(I) = 0$. So by Proposition 5.11 $\text{Ker } p \subseteq \text{Ker } f$. Hence there exists an algebra homomorphism $\hat{f} : p(\mathbb{F}G) \to \mathbb{F}$ such that $\hat{f} \circ p = f$; that is such that the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{F}G & \xrightarrow{f} & \mathbb{F} \\
p & \searrow & \downarrow \hat{f} \\
T\mathbb{O}_k. & & \\
\end{array}$$ \hspace{1cm} (5.37)

Let $I$ be the ideal in $T$ generated by the polynomials $p(H) - f(H)$ for $H \in \mathcal{G}$. Let $\rho_{\mathbb{O}_k}$ denote the Reynolds operator of $\mathbb{O}_k$ on $T$. (This exists by reductiveness of $\mathbb{O}_k$ and the fact that $T$ has a canonical direct sum decomposition into finite dimensional $\mathbb{O}_k$-modules.) By Proposition 5.11 and the fact that $\rho_{\mathbb{O}_k}(qr) = \rho_{\mathbb{O}_k}(q)r$ for $q \in T$ and $r \in T\mathbb{O}_k$ (cf. (4.2)), $\rho_{\mathbb{O}_k}(I)$ is the ideal in $p(\mathbb{F}G) = T\mathbb{O}_k$. 

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generated by the polynomials \( p(H) - f(H) \). This implies, as \( \hat{f}(p(H)) - f(H) = 0 \), that
\[
\hat{f}(\rho_{O_k}(I)) = 0,
\]
(5.38)
hence \( 1 \notin I \).

If \(|F|\) is uncountable (e.g. if \( F = \mathbb{C} \)), the Nullstellensatz for countably many variables (Lang [37]) yields the existence of a common zero \( y \).

To prove the existence of a common zero \( y \) for general algebraically closed fields \( F \) of characteristic 0 let, for any \( d \in \mathbb{N} \), \( N^k_{\leq d} := \{ \alpha \in \mathbb{N}^k \mid |\alpha| \leq d \} \), where \( |\alpha| := \sum_{i=1}^k \alpha_i \) and let
\[
Y_d := \{ z : N^k_{\leq d} \to F \mid q(z) = \hat{f}(q) \text{ for each } q \in F[y_{\alpha} \mid \alpha \in N^k_{\leq d}]} \). (5.39)
So \( Y_d \) consists of the common zeros of the polynomials \( p(H) - f(H) \), where \( H \) ranges over the graphs of maximum degree \( d \).

By the Nullstellensatz, as \( N^k_{\leq d} \) is finite, \( Y_d \neq \emptyset \). Note that \( Y_d \) is a fiber of the quotient map
\[
\pi : F^{N^k_{\leq d}} \to F^{N^k_{\leq d}} / O_k.
\]
(5.40)
So by Theorem 4.7, \( Y_d \) contains a unique Zariski-closed \( O_k \)-orbit \( C_d \).

Let \( pr_d \) be the projection \( z \mapsto z_{\leq d} := z|_{N^k_{\leq d}} \) for \( z : N^k_{\leq d'} \to F \) with \( d' \geq d \). (It is convenient to allow \( d' = \infty \) here.) Note that if \( \infty > d' \geq d \), then \( pr_d(C_{d'}) \) is an \( O_k \)-orbit contained in \( Y_d \). Hence
\[
\dim(C_d) \leq \dim(pr_d(C_{d'})) \leq \dim(C_{d'}),
\]
(5.41)
where \( \dim \) denotes the Krull-dimension. As \( \dim(C_d) \leq \dim(O_k) \) for all \( d \in \mathbb{N} \), there is \( d_0 \) such that for each \( d \geq d_0 \), \( \dim(C_d) = \dim(C_{d_0}) \). Hence we have equality throughout in (5.41).

By uniqueness of the orbit of minimal Krull-dimension, this implies that for each \( d' \geq d \geq d_0 \), \( C_d = pr_d(C_{d'}) \). Hence there exists \( y : N^k \to F \) such that \( y_{\leq d} \in C_d \) for each \( d \geq d_0 \). This \( y \) is as required. \( \square \)

### 5.5 Proof of Theorem 5.4

**Theorem 5.4.** Let \( F = F \) and let \( f : G \to F \) be the partition function of a \( k \)-color edge-coloring model over \( F \). Then \( f = p_h \) for some \( k \)-color edge-coloring model over \( F \) of rank at most \( r \) if and only if for each graph \( H = (V, E) \) and each \( U \subseteq V \) of size \( r + 1 \) and each \( s : U \to V \setminus U \),
\[
\sum_{\pi \in S_U} sgn(\pi)f(H/s \circ \pi) = 0.
\]
(5.4)
Characterizing partition functions of edge-coloring models

Proof. Necessity can be seen as follows. Choose \( y : \mathbb{N}^k \to \mathbb{F} \) with \( \text{rk}(M_y) \leq r \) and let \( H = (V, E) \) be a graph. Choose \( U \subseteq V \) with \(|U| = r + 1\) and \( s : U \to V \setminus U \). Then

\[
\sum_{\pi \in S_U} \text{sgn}(\pi) p_y(H/s \circ \pi)
= \sum_{\varphi : E \to [k]} \sum_{\pi \in S_U} \text{sgn}(\pi) \prod_{u \in U} y_{\varphi(\delta(u) \cup \delta(s(\pi(u))))} \cdot \prod_{v \in V \setminus (U \cup s(U))} y_{\varphi(\delta(v))}
= \sum_{\varphi : E \to [k]} \det((y_{\varphi(\delta(u) \cup \delta(s(\pi(v))))})_{u,v \in U}) \prod_{v \in V \setminus (U \cup s(U))} y_{\varphi(\delta(v))} = 0.
\]

To see sufficiency, let \( \mathcal{J} \) be the ideal in \( \mathbb{F}G \) be the ideal spanned by the quantum graphs

\[
\sum_{\pi \in S_U} \text{sgn}(\pi) H/s \circ \pi,
\]

where \( H = (V, E) \) is a graph, \( U \subseteq V \) with \(|U| = r + 1\) and \( s : U \to V \setminus U \). Let \( J \) be the ideal in \( R \) generated by the polynomials \( \det(N) \) where \( N \) is an \((r + 1) \times (r + 1)\) submatrix of \( M_y \).

Proposition 5.12. \( \rho_{O_k}(J) \subseteq p(\mathcal{J}) \).

Proof. It suffices to prove that for any \((r + 1) \times (r + 1)\) submatrix \( N \) of \( M_y \) and any monomial \( a \in T \), \( \rho_{O_k}(a \det(N)) \in p(\mathcal{J}) \). Let \( a \) have degree \( d \), and let \( n := 2(r + 1) + d \). Let \( U := [r + 1] \) and let \( s : U \to [n] \setminus U \) be defined by \( s(i) = r + 1 + i \) for \( i \in U \).

We use the framework of Proposition 5.11 with \( \tau \) as in (5.27). For each \( \pi \in S_{r+1} \) we define linear functions \( \mu_{\pi} \) and \( \sigma_{\pi} \) so that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{F}G_m & \xrightarrow{p_m} & T_m \\
\downarrow{\mu_{\pi}} & & \downarrow{\sigma_{\pi}} \\
\mathcal{O}(S\mathbb{F}^{n \times n}) & \xrightarrow{\tau} & \mathcal{O}(\mathbb{F}^{k \times n})
\end{array}
\]

where \( m := r + 1 + d = n - (r + 1) \).

The function \( \mu_{\pi} \) is defined by

\[
\mu_{\pi}(\prod_{ij \in E} x_{i,j}) := H/s \circ \pi
\]
for any graph $H = ([n], E)$. It implies that for each $q \in \mathcal{O}(S^m \times n)$,
\begin{equation}
\sum_{\pi \in S_{r+1}} \text{sgn}(\pi) \mu_\pi(q) \in \mathcal{J},
\end{equation}
by definition of $\mathcal{J}$.

Next, $\sigma_\pi$ is defined by
\begin{equation}
\sigma_\pi \left( \prod_{j=1}^{n} \prod_{i=1}^{k} \alpha_{ij} \right) := \prod_{j=1}^{r+1} y_{\alpha_j + \alpha_{r+1} + \pi(j)} \cdot \prod_{j=2r+3}^{n} y_{\alpha_j}
\end{equation}
for any $\alpha \in \mathbb{N}^{k \times n}$. So
\begin{equation}
a \det(N) = \sum_{\pi \in S_{r+1}} \text{sgn}(\pi) \sigma_\pi(u)
\end{equation}
for some monomial $u \in \mathcal{O}(\mathbb{F}^k \times n)$. Note that $\sigma_\pi$ is $O_k$-equivariant.

Now one directly checks that the diagram (5.44) commutes, that is,
\begin{equation}
p \circ \mu_\pi = \sigma_\pi \circ \tau.
\end{equation}
By the FFT, $\rho_{O_k}(u) = \tau(q)$ for some $q \in \mathcal{O}(S^m \times n)$. Hence $\sigma_\pi(\rho_{O_k}(u)) = \sigma_\pi(\tau(q)) = p(\mu_\pi(q))$. Therefore, using (5.48) and (5.46),
\begin{equation}
\rho_{O_k}(a \det(N)) \in p(\mathcal{J}),
\end{equation}
as required. \hfill \square

Since $f$ is the partition function of a $k$-color edge-coloring model, there exists an algebra homomorphism $\hat{f} : T \to \mathbb{F}$, such that $\hat{f} \circ p = f$ (cf. (5.37)). If the conditions in Theorem 5.4 are satisfied, then $f(\mathcal{J}) = 0$, and hence with Proposition 5.12
\begin{equation}
\hat{f}(\rho_{O_k}(J)) \subseteq \hat{f}(p(\mathcal{J})) = f(\mathcal{J}) = 0.
\end{equation}
With (5.38) this implies that $1 \notin I + J$, where $I$ is again the ideal generated by the polynomials $p(H) - f(H)$ for graphs $H$. The proof of Theorem 5.3 now shows that $I + J$ has a common zero, as required. Indeed, we just have to replace $Y_d$ by
\begin{equation}
Y'_d := \{ z \in Y_d \mid \text{rk}(M_z) \leq r \},
\end{equation}
where for $z : \mathbb{N}_{\leq d}^k \to \mathbb{F}$, we set $M_z(\alpha, \beta) = 0$ if $|\alpha + \beta| > d$. Then $Y'_d \neq \emptyset$, by the Nullstellensatz, since $1 \notin I + J$. As $\text{rk}(M_{g z}) = \text{rk}(M_z)$ for all $g \in O_k$, it follows that $Y'_d$ is closed and $O_k$-stable. So the unique Zariski-closed orbit $C_d \subseteq Y_d$ is by Theorem 4.7 contained in $Y'_d$. The rest of the proof can now be copied from the proof of Theorem 5.3. \hfill \square
5.6 Analogues for directed graphs

Similar results hold for directed graphs, with similar proofs, now by applying the FFT and SFT for GL($F^k$) (cf. [25, Section 5.2] and [25, Section 11.2] respectively). The corresponding models were also considered by de la Harpe and Jones [28]. We only state the results.

Let $D$ denote the collection of all directed graphs. Directed graphs are finite and may have loops and multiple edges. A map $f : D \to F$ is called a directed graph parameter if it assigns the same value to isomorphic directed graphs. The directed partition function of a $2k$-color edge-coloring model $y$ is the directed graph parameter $p_y : D \to F$ defined for any directed graph $D = (V, A)$ by

$$p_y(D) := \sum_{\kappa : A \to [k]} \prod_{v \in V} y(\kappa(\delta^-(v)), \kappa(\delta^+(v))).$$

Here $\delta^-(v)$ and $\delta^+(v)$ denote the sets of arcs entering $v$ and leaving $v$, respectively. Moreover, $(\kappa(\delta^-(v)), \kappa(\delta^+(v)))$ stands for the concatenation of the vectors $\kappa(\delta^-(v))$ and $\kappa(\delta^+(v)) \in N^k$, so as to obtain a vector in $N^{2k}$.

Call a function $f : D \to F$ multiplicative if $f(\emptyset) = 1$ and $f(D_1D_2) = f(D_1)f(D_2)$ for all $D_1, D_2 \in D$. Again, $D_1D_2$ denotes the disjoint union of $D_1$ and $D_2$. Moreover, for any directed graph $D = (V, A)$, any $U \subseteq V$, and any $s : U \to V$, define

$$A_s := \{(u, s(u)) \mid u \in U\} \quad \text{and} \quad D_s := (V, A \cup A_s).$$

**Theorem 5.13.** Let $F = F$. A directed graph parameter $f : D \to F$ is the directed partition function of some $2k$-color edge-coloring model over $F$ if and only if $f$ is multiplicative and for each directed graph $D = (V, A)$, each $U \subseteq V$ with $|U| = k + 1$, and each $s : U \to V$:

$$\sum_{\pi \in S_U} \text{sgn}(\pi)f(D_{s\circ \pi}) = 0.$$  

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For any directed graph $D = (V, A)$, $U \subseteq V$, and $s : U \to V$, let $D/s$ be the directed graph obtained from $D_s$ by contracting all arcs in $A_s$.

**Theorem 5.14.** Let $F = F$ and let $f$ be the directed partition function of a $2k$-color edge-coloring model over $F$. Then $f$ is the directed partition function of a $2k$-color edge-coloring model over $F$ of rank at most $r$ if and only if for each directed graph $D = (V, A)$, each $U \subseteq V$ with $|U| = r + 1$, and each $s : U \to V \setminus U$:

$$\sum_{\pi \in S_U} \text{sgn}(\pi)f(D/s \circ \pi) = 0.$$