Chapter 6

Connection matrices and algebras of invariant tensors

This chapter deals with a connection between connection matrices of partition functions of edge- and vertex-coloring models and algebras of tensors that are invariant under certain subgroups of the orthogonal group. Based on characterizations of these invariant algebras we characterize the rank of edge-connection matrices of partition functions of edge-coloring models as the dimension of the algebras of tensors invariant under the subgroup of the orthogonal group stabilizing the edge-coloring model. The corresponding result for the rank of vertex-connection matrices of partition functions of vertex-coloring models was proved by Lovász [41] using different ideas.

This chapter is based on joint work with Jan Draisma [20] and on [53].

6.1 Introduction

Let \((a, B)\) be a real twin-free \(n\)-color vertex-coloring model (i.e. \(B\) has no two equal rows and \(a_i > 0\) for all \(i \in [n]\)). In [41] Lovász determined the rank of the vertex-connection matrices of \(p_{a, B}\). To describe his result we need some definitions.

Let \(\text{Aut}(a, B) \subseteq S_n\) be the automorphism group of the weighted graph \(G(a, B)\), i.e., the subgroup of the group of all permutations of \([n]\) preserving both the vertex- and edge weights of \(G(a, B)\). The group \(S_n\) has a natural action on \([n]^l := \{\phi : [l] \to [n]\}\), for any \(l\), via \((\pi \cdot \phi)(i) = \pi(\phi(i))\), for \(\pi \in S_n\) and \(\phi \in [n]^l\).
Theorem 6.1 (Lovász [41]). Let \((a, B)\) be a real twin-free \(n\)-color vertex-coloring model. Then

\[
\text{rk} \left( N_{p,a,B,t} \right) = \text{the number of orbits of the action of Aut}(a, B) \text{ on } \left[ n \right]^I.
\]  \tag{6.1}

Theorem 6.1 has applications in the study of generalized quasi-random graphs (see [44, 40]). It is natural to ask whether a similar result holds for the rank of edge-connection matrices of partition functions of (real) edge-coloring models. This question was posed by Szegedy [66] and by Borgs, Chayes, Lovász, Sós and Vesztergombi [6].

In this chapter we will show that a similar result indeed holds for the rank of edge-connection matrices of partition functions of both real and complex edge-coloring models. To state our results we need to introduce some definitions.

Let \(V := F^k\). (Recall that \(F\) denotes any field of characteristic zero.) Let \(e_1, \ldots, e_k\) denote the standard basis for \(V\) and let \((\cdot, \cdot)\) denote the standard symmetric bilinear form on \(V\); i.e., \((e_i, e_j) = \delta_{i,j}\). The orthogonal group \(O_k = O_k(F)\) is the group of \(k \times k\) matrices over \(F\) that leave this bilinear form invariant, i.e., \(g \in O_k\) if and only if \(g^Tg = I\). For an edge-coloring model \(h \in R^*\) (recall that \(R = F[x_1, \ldots, x_k]\)), define

\[
\text{Stab}(h) := \{ g \in O_k(F) \mid gh = h \}. \tag{6.2}
\]

The action of \(O_k\) on \(V\) extends naturally to \(V \otimes l\) for any \(l \in \mathbb{N}\). Let \(G \subseteq O_k\) be a subgroup. Recall that

\[
(V \otimes l)^G = \{ v \in V \otimes l \mid gv = v \text{ for all } g \in G \}. \tag{6.3}
\]

Now we can state our characterization. For real valued edge-coloring models the following result holds.

**Theorem 6.2.** Let \(h\) be a \(k\)-color edge-coloring model over \(\mathbb{R}\). Then, for any \(t \in \mathbb{N}\),

\[
\text{rk} \left( M_{p,h,t} \right) = \dim \left( (V \otimes t)^{\text{Stab}(h)} \right). \tag{6.4}
\]

Theorem 6.2 will be proved in Section 6.2.

To see the similarity between Theorem 6.2 and Theorem 6.1, let \(e_1, \ldots, e_n\) be the standard basis of \(W := \mathbb{R}^n\). Then the set \([n]^I\) corresponds to the standard basis of \(W \otimes l\) via \([n]^I \ni \phi \leftrightarrow e_\phi := e_{\phi(1)} \otimes \cdots \otimes e_{\phi(l)}\) and the action of \(S_n\) on \([n]^I\) induces an action on \(W \otimes l\). With these definitions, (6.1) now reduces to

\[
\text{rk} \left( N_{p,a,B,t} \right) = \dim(W \otimes l)^{\text{Aut}(a,B)}, \tag{6.5}
\]

showing the similarity between Theorem 6.2 and Theorem 6.1.

For edge-coloring models with values in an algebraically closed field \(F\) of characteristic zero a similar result as Theorem 6.2 holds.

52
6.1. Introduction

Theorem 6.3. \(\text{Let } G = F \text{ and let } h \text{ be a } k\text{-color edge-coloring model over } F. \text{ Then there exists a } k\text{-color edge-coloring model } h' \text{ over } F \text{ such that } p_h = p_{h'}, \text{ and such that for any } t \in \mathbb{N},\)
\[
\text{rk}(M_{p_h,t}) = \dim((V^\otimes t)^{\text{Stab}(h')}).
\]  
(6.6)

We will prove Theorem 6.3 in Section 6.2.

We cannot simply take \(h' = h\) in Theorem 6.3, as the following example shows.

Example 6.1. \(\text{Let } G = F \text{ and let } i \in F \text{ be a square root of } -1 \text{ and set } k := 2. \text{ Consider the edge-coloring model } h : F[x_1, x_2] \to F \text{ given by}\)
\[
h(x_1^a x_2^b) = \begin{cases} 
1 & \text{if } a = 1 \text{ and } b = 0, \\
i & \text{if } a = 0 \text{ and } b = 1, \\
0 & \text{otherwise}. 
\end{cases}
\]  
(6.7)

Note that for any graph \(G\) with at least one vertex we have \(p_h(G) = 0. \text{ Indeed, if } G \text{ contains an isolated vertex or a vertex of degree at least 2, then } p_h(G) = 0. \text{ Otherwise, } G \text{ is a perfect matching. Since } p_h(K_2) = h(x_1)^2 + h(x_2)^2 = 0, \text{ the claim follows. So the rank of } M_{p_h,1} \text{ is equal to zero. It is not difficult to see that } \text{that } \text{Stab}(h) = \{I\}. \text{ Hence } \text{rk}(M_{p_h,1}) \neq \dim((V^{\text{Stab}(h)})) = 2. \text{ More generally, the following holds: } \text{rk}(M_{p_h,t}) = \dim((V^\otimes t)^{O_2}). \text{ The edge-coloring model } h' \equiv 0 \in F[x_1, x_2]^* \text{ does the job.}\)

There is however a class of edge-coloring models for which we can take \(h = h'.\)

Theorem 6.4. \(\text{Let } G = F, \text{ let } u_1, \ldots, u_n \in V \text{ be distinct vectors that span a non-degenerate subspace of } V \text{ and let } a_1, \ldots, a_n \in F^*. \text{ Let } h \text{ be the edge-coloring model defined by } h(p) := \sum_{i=1}^n a_i p(u_i), \text{ for } p \in R. \text{ Then, for any } t \in \mathbb{N},\)
\[
\text{rk}(M_{p_h,t}) = \dim((V^\otimes t)^{\text{Stab}(h)}).
\]  
(6.8)

The proof of Theorem 6.4 depends on a result from Section 7.2 but we will prove it in Section 6.2.

The outline for the rest of this chapter is as follows. In the next section we develop the necessary framework to prove Theorem 6.2 and Theorem 6.3. Based on this framework, we use a theorem of Schrijver \[58\], characterizing algebras of the form \(T(V)^G\) for subgroups of the (real) orthogonal group, to prove Theorem 6.2. In the algebraically closed case we cannot use Schrijver’s result as it uses the compactness of the real orthogonal group. Instead, we prove an algebraic version of this result (cf. Theorem 6.11) and use the framework of Section 6.2.
5.3 and the existence and uniqueness of closed orbits to prove Theorem 6.3. In Section 6.3 we will use this approach, based on a characterization of tensors invariant under subgroups of $S_n$ (cf. Theorem 6.16), to give different (but not necessarily simpler) proof of Theorem 6.1. Finally, in Section 6.4 we provide proofs of Theorem 6.11 and Theorem 6.16.

6.2 The rank of edge-connection matrices

As follows from Example 6.1, the real and the algebraically closed case are different. However, the proofs of Theorem 6.2 and Theorem 6.3 have the same structure. We first develop the common framework for both cases and then we will specialize to $\mathbb{F} = \mathbb{R}$ and algebraically closed fields separately. Throughout this section we let $V := \mathbb{F}^k$ and we let $h$ denote any $k$-color edge-coloring model over $\mathbb{F}$ unless indicated otherwise.

6.2.1 Algebra of fragments

Recall from Section 2.2 that $F_l$ is the set of all $l$-fragments. Let $\mathbb{F}F_l$ denote the linear space consisting of (finite) formal $\mathbb{F}$-linear combinations of $l$-fragments; they are called quantum fragments. Extend the gluing operation, $\ast$, bilinearly to $\mathbb{F}F_l \times \mathbb{F}F_l$. Let

$$A := \bigoplus_{l=0}^{\infty} \mathbb{F}F_l. \quad (6.9)$$

Make $A$ into a graded associative algebra by defining, for $F \in F_l$ and $H \in F_t$, the tensor product $F_1 \otimes F_2$ to be the disjoint union of $F_1$ and $F_2$, where the open end of $F_2$ labeled $i$ is relabeled to $l + i$.

Set

$$I_l(h) := \{ x \in \mathbb{F}F_l \mid p_h(x \ast F) = 0 \text{ for all } l\text{-fragments } F \} \quad (6.10)$$

and let $I(h) := \bigoplus_{l=0}^{\infty} I_l(h)$. Note that $I_l(h)$ is the kernel of the $l$-th edge-connection matrix of $p_h$. Observe that

$$\text{rk}(M_{p_h,l}) = \dim(\mathbb{F}F_l/I_l(h)). \quad (6.11)$$

Let $T(V) := \bigoplus_{n=0}^{\infty} V^\otimes n$ be the tensor algebra of $V$ (with product the tensor product). For $\phi : [n] \to [k]$ define $e_{\phi} := e_{\phi(1)} \otimes \cdots \otimes e_{\phi(k)}$. The $e_{\phi}$ form a basis for $V^\otimes n$. We will write $(\cdot, \cdot)$ to denote the nondegenerate symmetric bilinear form on $V^\otimes n$ induced by $(\cdot, \cdot)$ for any $n$. We will now exhibit a natural homomorphism from $A$ to $T(V)$. 54
6.2. The rank of edge-connection matrices

For an \( l \)-fragment \( F \) we denote its edges (including half edges) by \( E(F) \) and its vertices (not including open ends) by \( V(F) \). Moreover, we will identify the half edges of \( F \) with the set \([l]\). Let \( F \in \mathcal{F}_l \) and let \( \phi : [l] \to [k] \). Define

\[
\h_{\phi}(F) := \sum_{\psi : E(F) \to [k]} \prod_{v \in V(F)} h(\prod_{e \in \delta(v)} x_{\psi(e)}).
\]  

(6.12)

We can now extend the map \( \rho_h : G \to \mathcal{F} \) to a linear map \( \rho_h : \mathcal{A} \to T(V) \) by defining

\[
\rho_h(F) = \sum_{\phi : [l] \to [k]} \h_{\phi}(F)e_{\phi},
\]  

(6.13)

for \( F \in \mathcal{F}_l \), for \( l \geq 0 \).

Note that for \( F_1, F_2 \in \mathcal{F}_l \),

\[
\rho_h(F_1 \ast F_2) = \sum_{\phi : [l] \to [k]} \h_{\phi}(F_1)\h_{\phi}(F_2) = (\rho_h(F_1), \rho_h(F_2)).
\]  

(6.14)

For \( \mathcal{F} = \mathbb{R} \), (6.14) implies that for \( \gamma = \sum_{i=1}^{n} \lambda_i F_i \in \mathbb{R}\mathcal{F}_l \),

\[
\rho_h(\gamma \ast \gamma) = \sum_{\phi : [l] \to [k]} \sum_{i,j=1}^{n} \lambda_i \lambda_j \h_{\phi}(F_i)\h_{\phi}(F_j) \geq 0,
\]  

(6.15)

showing the easy part of Theorem 5.2.

It is not difficult to see that \( \rho_h \) is a homomorphism of algebras. By (6.14) it follows that \( \text{Ker} \rho_h \subseteq \mathcal{I}(h) \). This gives rise to the following definition: we call an edge-coloring model \( h \) nondegenerate if \( \text{Ker} \rho_h = \mathcal{I}(h) \). Equivalently, \( h \) is nondegenerate if the algebra \( \rho_h(\mathcal{A}) \) is nondegenerate with respect to the bilinear form on \( T(V) \) (induced by that on \( V \)). So for nondegenerate \( h \) we have \( \mathcal{A}/\mathcal{I}(h) \cong \rho_h(\mathcal{A}) \). In particular, by (6.11), we have the following lemma.

Lemma 6.5. Let \( h \) be a nondegenerate \( k \)-color edge-coloring model. Then, for any \( t \in \mathbb{N} \),

\[
\text{rk}(M_{\rho_h,t}) = \dim(\rho_h(\mathcal{A}) \cap V^\otimes t).
\]  

(6.16)

6.2.2 Contractions

In this subsection we introduce contractions for tensors and fragments, and we show that \( \rho_h \) preserves these.

For \( 1 \leq i < j \leq l \in \mathbb{N} \) the contraction \( C^l_{ij} \) is the unique linear map

\[
C^l_{ij} : V^\otimes l \to V^\otimes l-2
\]  

(6.17)

\[
v_1 \otimes \ldots \otimes v_l \mapsto (v_i, v_j)v_1 \otimes \ldots \otimes v_{i-1} \otimes v_{i+1} \ldots \otimes v_{j-1} \otimes v_{j+1} \otimes \ldots \otimes v_l.
\]
A subspace $A$ of $T(V)$ is called graded if $A = \bigoplus_{l=0}^{\infty} (V^\otimes l \cap A)$. A graded subspace $A$ of $T(V)$ is called contraction closed if $C_{ij}^l(a) \in A$ for all $a \in A \cap V^\otimes l$ and $i < j \leq l \in \mathbb{N}$. Note that for any subgroup $G \subseteq O_k$, $T(V)^G = \bigoplus_{l=0}^{\infty} (V^\otimes l)^G$ is a graded and contraction closed subalgebra of $T(V)$ as, by definition, contractions are $O_k$-invariant.

We now define a contraction operation for fragments. For $1 \leq i < j \leq l \in \mathbb{N}$, the contraction $C_{ij}^l : \mathcal{F}_l \to \mathcal{F}_{l-2}$ is defined as follows: for $F \in \mathcal{F}_l$, $C_{ij}^l(F)$ is the $(l-2)$-fragment obtained from $F$ by connecting the half edges incident with the open ends labeled $i$ and $j$ into one single edge (deleting these open ends), and then relabeling the remaining open ends $1, \ldots, l-2$ such that the order is preserved. See Figure 6.1 for an example.

\[ C_{ij}^l(F) = \sum_{\phi: [l] \to [n]} h_\phi(F)C_{ij}^l(e_\phi) = \sum_{\phi: [l] \to [n]} h_\phi(F)C_{ij}^l(e_\phi) = \sum_{\psi: [l-2] \to [n]} h_\phi(C_{ij}^l(F))e_\psi = p_h(C_{ij}^l(F)). \] (6.18)

The basic $l$-fragment $F_l$ is the $l$-fragment that contains one vertex and $l$ open ends connected to this vertex, labeled $1$ up to $l$. Recall that $K_{2\bullet}^\bullet$ denotes the edge which has exactly two open ends and note that $p_h(K_{2\bullet}^\bullet) = \sum_{i=1}^{k} e_i \otimes e_i$. By relabeling $(K_{2\bullet}^\bullet)^{\otimes m}$ for $m \in \mathbb{N}$, we see that by the Tensor FFT for $O_k$ (cf. Theorem 4.3), the image of $p_h$ contains all $O_k$-invariant tensors.

Let $F$ be an $l$-fragment without circles with $V(F) = [n]$ and $|E(F)| = m$, such that its underlying graph is connected. Then either $F = K_{2\bullet}^\bullet$ or $F$ can be obtained from the fragment $\bigotimes_{i=1}^{n} F_{d(i)}$ by applying $m-l$ contractions to it; see Figure 6.2.
6.2. The rank of edge-connection matrices

Figure 6.2: Obtaining a 3-fragment by contracting the product of two basic fragments.

Let us summarize the properties of the map \( p_h \).

**Proposition 6.6.** The image of \( p_h \) is a graded contraction-closed algebra that contains \( T(V)^{O_k} \). Moreover, \( p_h(A) \) is generated by the images of the basic fragments and \( K_2^{••} \) as a contraction-closed algebra.

### 6.2.3 Stabilizer subgroups of the orthogonal group

For \( l \in \mathbb{N} \), we write \( h_l \) for the restriction of \( h \) to the space of homogenous polynomials of degree \( l \). We think of \( h_l \) as a symmetric tensor as follows:

\[
(h_l, e_{\phi}) = h_l(x_{\phi}),
\]

where for a map \( \phi : [l] \to [k] \), we define the monomial \( x_{\phi} \in \mathbb{F}[x_1, \ldots, x_k] \) by \( x_{\phi} := \prod_{i=1}^{k} x_{\phi(i)} \). This gives a natural \( O_k \)-equivariant embedding of \( \mathbb{F}^{\mathbb{N}_l} \) into \( V^\otimes l \). Indeed, as \( V \) and \( V^* \) are isomorphic \( O_k \)-modules (cf. (3.17)), we have for any \( \phi : [l] \to [k] \):

\[
(gh_l)(x_{\phi}) = h_l(g^{-1}x_{\phi}) = (h_l, g^{-1}e_{\phi}) = (gh_l, e_{\phi}).
\]

For a subset \( A \subseteq T(V) \), define the **pointwise stabilizer** of \( A \) by

\[
\text{Stab}(A) := \{ g \in O_k \mid ga = a \text{ for all } a \in A \}.
\]

The next proposition shows that \( \text{Stab}(h) \) is equal to \( \text{Stab}(p_h(A)) \).

**Proposition 6.7.** Let \( h \) be an edge-coloring model. Then \( \text{Stab}(h) = \text{Stab}(p_h(A)) \).

**Proof.** Let \( l \in \mathbb{N} \). Then \( p_h(F_l) = h_l \) (viewing \( h_l \) as a symmetric tensor). So in particular, \( gp_h(F_l) = p_{gh}(F_l) \) for each \( g \in O_k \). Since \( p_h(A) \) is generated, as a contraction-closed algebra, by \( K_2^{••} \) and the basic fragments and since contractions are by definition \( O_k \)-invariant, it follows that for any \( l \)-fragment \( F \), \( p_{gh}(F) = gp_h(F) \) for each \( g \in O_k \). This implies that \( g \in \text{Stab}(h) \) if and only if \( g \in \text{Stab}(p_h(A)) \). \( \square \)
6.2.4 The real case

Here we will give a proof of Theorem 6.2. So $\mathbb{F} = \mathbb{R}$ (and $h$ denotes a $k$-color edge-coloring model over $\mathbb{R}$).

First note that, by (6.15), $h$ is clearly nondegenerate. So by Lemma 6.5, it suffices to prove the following combinatorial parametrization of the tensors invariant under $\text{Stab}(h)$.

**Theorem 6.8.** Let $h$ be a $k$-color edge-coloring model over $\mathbb{R}$. Then

$$p_h(A) = T(V)^{\text{Stab}(h)}. \tag{6.22}$$

A crucial ingredient in the proof of Theorem 6.8 is the characterization by Schrijver [58] of subalgebras of the tensor algebra that are of the form $T(V)^G$ for subgroups $G$ of the real orthogonal group.

**Theorem 6.9** (Schrijver [58]). Let $A \subseteq T(V)$. Then $A = T(V)^G$ for some subgroup $G \subseteq O_k$ if and only if $A$ is a graded contraction-closed subalgebra of $T(V)$ that contains $T(V)^{O_k}$.

We can now give a proof of Theorem 6.8.

**Proof of Theorem 6.8.** By Proposition 6.6, $p_h(A)$ is a graded contraction-closed subalgebra of $T(V)$ that contains $T(V)^{O_k}$. So we can apply Theorem 6.9 to see that $p_h(A) = T(V)^G$, for some subgroup $G$ of $O_k$. Now note that $G \subseteq \text{Stab}(p_h(A))$, implying that $T(V)^{\text{Stab}(p_h(A))} \subseteq T(V)^G$. Moreover, $T(V)^G = p_h(A) \subseteq T(V)^{\text{Stab}(p_h(A))}$. Hence $T(V)^{\text{Stab}(p_h(A))} = T(V)^G$. As $\text{Stab}(h) = \text{Stab}(p_h(A))$ by Proposition 6.7, this proves the theorem. \[\square\]

6.2.5 The algebraically closed case

Here we will give a proof of Theorem 6.3. So $\mathbb{F}$ denotes an algebraically closed field from now on.

Just as in the real case, we will state a combinatorial parametrization of the tensors invariant under $\text{Stab}(h')$, for certain nondegenerate edge-coloring models $h'$ over $\mathbb{F}$, which implies Theorem 6.3 by Lemma 6.5.

**Theorem 6.10.** Let $\mathbb{F} = \overline{\mathbb{F}}$ and let $h$ be a $k$-color edge-coloring model over $\mathbb{F}$. Then there exists a nondegenerate $k$-color edge-coloring model $h'$ over $\mathbb{F}$ such that $p_h(H) = p_{h'}(H)$ for all $H \in \mathcal{G}$ and such that

$$p_{h'}(A) = T(V)^{\text{Stab}(h')} \tag{6.23}$$
We cannot proceed in the same way as in Section 6.2.4 for two reasons. The first reason being that any edge-coloring model over \( F \) is not automatically nondegenerate (cf. Example 6.1). To circumvent this issue, we will find and edge-coloring model \( h' \) such that \( h' \leq d \) is contained in the unique closed orbit in \( O_k h' \leq d \) for \( d \) large enough and show that \( h' \) is nondegenerate. The second reason is that the proof of Theorem 6.9 in [58] uses the compactness of the real orthogonal group and hence it does not apply to \( O_k(F) \), as it is not compact.

Derksen (private communication, 2006) completely characterized which subalgebras of \( T(V) \) are the algebras of \( G \)-invariant tensors for some reductive group \( G \subseteq O_k \), but we do not need the full strength of his result to prove Theorem 6.10. Instead, we state a sufficient condition for a subalgebra of \( T(V) \) to be the algebra of \( G \)-invariants for some reductive group \( G \subseteq O_k \).

**Theorem 6.11.** Let \( F = \overline{F} \) and let \( A \subseteq T(V) \) be a graded contraction closed subalgebra containing \( T(V)^{O_k} \). If \( \text{Stab}(A) = \text{Stab}(w) \) for some \( w \in A \) whose \( O_k \)-orbit is closed in the Zariski topology, then \( A = T(V)^{\text{Stab}(A)} \) and moreover \( \text{Stab}(A) \) is a reductive group.

We will prove this theorem in Section 6.4. Now we will use it to prove Theorem 6.10.

**Proof of Theorem 6.10.** The proof consists basically of checking the conditions in Theorem 6.11. It is based upon the framework developed in Section 5.3 and the proof of Theorem 5.3.

Let

\[
Y_d := \{ z : \mathbb{N}^{\leq d} \to \overline{F} \mid p_z(H) = p_h(H) \text{ for each graph } H \text{ of maximum degree at most } d \}. \tag{6.24}
\]

Then \( Y_d \) is the fiber of \( h_{\leq d} \) under the quotient map \( \pi : \overline{F}^{N_k \leq d} \to \overline{F}^{N_k \leq d} // O_k \) (cf. (5.40)). In the same way as in the proof of Theorem 5.3 we choose \( h' \) such that \( h_{\leq d} \) is in the unique closed \( O_k \)-orbit \( C_d \) in \( Y_d \) for each \( d \geq d_0 \) for \( d_0 \) large enough.

We will now show that this \( h' \) is as required. First note that \( \text{Stab}(h') = \cap_{e \geq 0} \text{Stab}(h'_{\leq e}) \). Since the ring of regular functions of \( O_k \) is Noetherian it follows that there exists \( e \) such that \( \text{Stab}(h') = \text{Stab}(h'_{\leq e}) \). We may assume that \( e \geq d_0 \).

Let \( F = \sum_{0 \leq k \leq e} F_k \), the sum in \( \mathcal{A} \) of the first \( e+1 \) basic fragments. Write \( w := p_{h'}(F) \) and note that \( w \) is the image of \( h'_{\leq e} \) under the natural \( O_k \)-equivariant embedding of \( \overline{F}^{N_k \leq e} \) into \( \bigoplus_{k=0}^e V^\otimes k \). Then

\[
\text{Stab}(w) = \text{Stab}(h'). \tag{6.25}
\]
Moreover, as we can view $C_e$ and $Y_e$ as subvarieties of $\bigoplus_{k=0}^c V^\otimes k$, it follows that the $O_k$-orbit of $w$ is Zariski closed. By Proposition 6.7, $\text{Stab}(p_{h'}(A)) = \text{Stab}(w)$. By Proposition 6.6, $p_h(A)$ is a graded contraction-closed subalgebra that contains $T(V)^{O_k}$. So we can apply Theorem 6.11 to find that $p_{h'}(A) = T(V)^{\text{Stab}(h')}$. Moreover, we find that $\text{Stab}(h')$ is reductive. From this we conclude that $h'$ is nondegenerate.

Indeed, suppose that $p_{h'}(x) \neq 0$ for some $x \in A$. Then there exists $y \in T(V)$ such that $(p_{h'}(x), y) \neq 0$. Since $\text{Stab}(h')$ is reductive we can write $T(V) = T(V)^{\text{Stab}(h')} \oplus W$ with $W$ stable under $\text{Stab}(h')$. Write $y = v + w$ with $v \in T(V)^{\text{Stab}(h')}$ and $w \in W$. As $p_{h'}(x) \in T(V)^{\text{Stab}(h')}$, we have for each $g \in \text{Stab}(h')$ and $u \in T(V)$,

$$
(p_{h'}(x), gu) = (g^{-1} p_{h'}(x), u) = (p_{h'}(x), u). \tag{6.26}
$$

So Lemma 4.2 implies that $(p_{h'}(x), w) = 0$. It follows that $h'$ is nondegenerate.

Using a result from Section 7.2, the proof of Theorem 6.4 is now basically done.

**Theorem 6.4.** Let $F = \mathbb{F}$, let $u_1, \ldots, u_n \in V$ be distinct vectors that span a nondegenerate subspace of $V$ and let $a_1, \ldots, a_n \in \mathbb{F}^*$. Let $h$ be the edge-coloring model defined by $h(p) := \sum_{i=1}^n a_i p(u_i)$, for $p \in R$. Then, for any $t \in \mathbb{N}$,

$$
\text{rk} \ (M_{p_h,t}) = \dim \left( (V^\otimes t)^{\text{Stab}(h)} \right). \tag{6.27}
$$

**Proof.** By Theorem 7.7 for $d \geq 3n$, the orbit of $h_{\leq d}$ is closed. It follows by the proof of Theorem 6.10 and by Lemma 6.5 that $\text{rk} \ (M_{p_h,t}) = \dim (V^\otimes t)^{\text{Stab}(h)}$. \hfill $\square$

### 6.3 The rank of vertex-connection matrices

In this section we will give a proof of Theorem 6.1 using the ideas from the previous section. Since the groups we are dealing with are finite, we do not have to differentiate between fields that are algebraically closed or not. Throughout this section, $(a, B)$ will denote any $n$-color vertex-coloring model over $\mathbb{F}$ unless indicated otherwise. Moreover, we set $W := \mathbb{F}^n$.

#### 6.3.1 Another algebra of labeled graphs

Recall from Section 2.2 that $G_l$ denotes the set of $l$-labeled graphs. Let $\mathbb{F} G_l$ be the vectorspace consisting of finite formal linear combinations of $l$-labeled
6.3. The rank of vertex-connection matrices

graphs. Let
\[ Q := \bigoplus_{l=0}^{\infty} \mathbb{F} G_l, \tag{6.28} \]
and make it into an associative algebra by defining for \( H \in G_l \) and \( F \in G_k \), \( H_1 \otimes H_2 \) to be the disjoint union of \( H_1 \) and \( H_2 \) where we add \( l \) to all the labels of \( H_2 \) so that \( F \otimes H \in G_{l+k} \) and extend this bilinearly to \( Q \times Q \). Note that \( FH \) and \( F \otimes H \) are different if the number of labels is positive.

Let \( e_1, \ldots, e_n \) be the standard basis for \( W = \mathbb{F}^n \). Let for any \( w \in W \), \( (\cdot, \cdot)_w \) be the symmetric bilinear form on \( W \times W \) defined by
\[ (e_i, e_j)_w := w_i \delta_{ij}. \tag{6.29} \]
Note that taking \( w \) the all ones vector, we obtain the standard bilinear form.

Write \( G := G(a, B) \) and extend \( p_{a, B} \) to a linear map \( p_{a, B} : Q \to T(V) \) by defining, for \( H \in G_l \),
\[ p_{a, B}(H) = \sum_{\phi: [l] \to [n]} \text{hom}_\phi(H, G) e_{\phi}, \tag{6.30} \]
where for \( \phi: [l] \to [n] \) and \( H \in G_l \) we define
\[ \text{hom}_\phi(H, G) := \sum_{\psi: V(H) \to [n]} \prod_{v \in V(H) \setminus [l]} a_{\psi(v)} \cdot \prod_{uv \in E(H)} B_{\psi(u), \psi(v)}. \tag{6.31} \]

Recall from Section 2.2.1 that we extended graph parameters to labeled graphs by setting \( f(H) := f([H]) \) for \( H \in G_l \) and a graph parameter \( f \). So to avoid confusion, we will write \( \text{hom}(H, G) \) if we mean \( p_{a, B}([H]) \); by \( p_{a, B}(H) \) we mean an \( l \)-tensor as defined by (6.30). Now note that for any \( H_1, H_2 \in G_l \),
\[ \text{hom}(H_1 \cdot H_2, G) = \sum_{\phi: [l] \to [n]} \prod_{i \in [l]} a_{\phi(i)} \text{hom}_\phi(H_1, G) \text{hom}_\phi(H_2, G). \tag{6.32} \]

Note that when \( \mathbb{F} = \mathbb{R} \) and \( a_i > 0 \) for each \( i \in [n] \), (6.32) implies, similarly to (6.15), that \( \text{hom}(\cdot, G) \) is reflection positive.

Clearly, \( p_{a, B} \) is a homomorphism of algebras. We call the pair \( (a, B) \) nondegenerate if the image of \( p_{a, B} \) is nondegenerate with respect to \( (\cdot, \cdot)_a \). As in the edge-coloring model case we have the following result.

**Lemma 6.12.** Let \( (a, B) \) be a nondegenerate twin-free \( n \)-color vertex-coloring model. Then, for any \( l \in \mathbb{N} \),
\[ \text{rk}(N_{p_{a, B}, l}) = \text{dim}(p_{a, B}(\mathbb{F} G_l)). \tag{6.33} \]
6.3.2 Some operations on labeled graphs and tensors

We define some operations on labeled graphs and tensors and show how they are related via the map $p_{a,B}$.

Let $\circ : W^{\otimes 2} \times W^{\otimes 2} \to W^{\otimes 2}$ be the linear map defined by $(C \circ D)_{ij} = C_{ij}D_{ij}$, for $C, D \in W^{\otimes 2}$. This operation is called the Schur product. Note that for two 2-labeled graphs $H_1$ and $H_2$ we have

$$p_{a,B}(H_1 \cdot H_2) = p_{a,B}(H_1 \circ p_{a,B}(H_2)).$$

We next define contraction-like operations for labeled graphs and tensors. For $i < j \leq l \in \mathbb{N}$ define the labeled contraction $K^l_{i,j} : G_l \to G_{l-1}$ by identifying for $H \in G_l$, the labeled vertices $i$ and $j$ as one vertex, giving the vertex label $i$ and relabeling the remaining labeled vertices $1, \ldots, i-1, i+1, \ldots, l-1$ in the same order. Note that if $i$ and $j$ are connected by an edge one creates a loop at vertex $i$. We now define the corresponding operation for tensors. For $l \in \mathbb{N}$ and $i < j \leq l$

$$K^l_{i,j} : W^{\otimes l} \to W^{\otimes l-1}$$

is the unique linear map defined by

$$e_{t_1} \otimes \cdots \otimes e_{t_l} \mapsto \delta_{t_i,t_j} e_{t_1} \otimes \cdots \otimes e_{t_{i-1}} \otimes e_{t_{i+1}} \otimes \cdots \otimes e_{t_l}. \quad (6.35)$$

Then it is easy to see that

$$K^l_{i,j}(p_{a,B}(H)) = p_{a,B}(K^l_{i,j}(H)) \quad (6.36)$$

for each $l \in \mathbb{N}$, $i < j \leq l$ and $H \in G_l$.

We now define an unlabeling operation for labeled graphs and for tensors. For any $l$ and $i \in [l]$ define $U^l_i : G_l \to G_{l-1}$ by unlabeling the $i$-th vertex and then relabeling the remaining vertices in the same order. Moreover, define

$$U^l_i : W^{\otimes l} \to W^{\otimes l-1}$$

to be the unique linear map satisfying

$$v_1 \otimes \cdots \otimes v_l \mapsto (v_i, 1)_a v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes \cdots \otimes v_l. \quad (6.37)$$

Then it is easy to see that $p_{a,B}$ preserves unlabeling, that is for all $H \in G_l$ and any $i \in [l]$ we have

$$U^l_i(p_{a,B}(F)) = U^l_i(p_{a,B}(F)). \quad (6.38)$$

We define one more operation on two-tensors (i.e. matrices). Let $A$ be the diagonal matrix defined by $A_{i,i} = a_i$ for $i \in [n]$. For $C, D \in W^{\otimes 2}$ we define

$$C * D := CAD, \quad (6.39)$$

62
the (ordinary matrix) product of the matrices $C, A$ and $D$. Note that $C \ast D$ is equal to $C_{2,3}^4 (C, D)$, the contraction of $C \otimes D$ with respect to $(\cdot, \cdot)_q$. Let $J \in W^{\otimes 2}$ denote the all-ones matrix and let $I$ denote the identity matrix.

**Lemma 6.13.** Let $B \subset W^{\otimes 2}$ be an algebra with $\ast$-product, generated by $B$ and $J$ and which is closed under taking the Schur product. If $(a, B)$ is twin free and if $\sum_{i \in S} a_i \neq 0$ for all $S \subseteq [n]$, then $I$ and $A^{-1}$ are contained in $B$.

**Proof.** Put an equivalence relation on $[n] \times [n]$ by $(i, j) \sim (i', j')$ if and only if $C_{i,j} = C_{i', j'}$ for all $C \in B$. Let $M_1, \ldots, M_t$ be the incidence matrices of the equivalence classes of $\sim$. Then

$$M_i \in B \text{ for } i = 1, \ldots, t. \quad (6.40)$$

To see (6.40), let $C = \sum_{i=1}^t c_i M_i \in B$ be a matrix for which the number of distinct coefficients is maximal. Then all $c_i$ are distinct. For suppose this is not true. We may assume that $c_1 = c_2$. By definition of the equivalence relation, there exists $D = \sum_{i=1}^t d_i M_i \in B$ such that $d_1 \neq d_2$. Pick a nonzero number $x$ such that if $c_i \neq c_j$, then $xc_i + d_i \neq xc_j + d_j$. Then $xC + D \in B$ contains more distinct coefficients than $C$. A contradiction.

Now pick interpolating polynomials $p_1, \ldots, p_t$ such that $p_i(c_j) = \delta_{i,j}$ (cf. [17, Lemma 2.9]). Then, since $B$ is closed under the Schur product, $p_i(C) = M_i \in B$. This proves (6.40).

Observe that for each $i$, $M_i = M_i^T$ for some $j$, since $B$ and $J$ are symmetric. Moreover, as $J \in B$ we have $\sum_{i=1}^t M_i = J$. Now suppose that $I \notin B$. Then there exists $i \neq j$ and $k$ such that $C_{i,j} = C_{k,k}$ for all $C \in B$. As no two rows of $B$ are equal, there exist $s, t$ such that $(M_s)_{i,t} = 0$ and $(M_s)_{i,t} = 1$. Since the $M_i$ sum up to $J$, there exists $l \neq s$ such that $(M_l)_{i,t} = 1$. So $(M_l \ast M_i^T)_{i,j} \neq 0$. (Here we use that $\sum_{i \in S} a_i \neq 0$ for all $S \subseteq [n]$.) But since $C_{i,j} = C_{k,k}$ for all $C \in B$, we have that

$$(M_l \ast M_i^T)_{i,j} = (M_l \ast M_i^T)_{k,k} = 0, \quad (6.41)$$

since the rows of $M_s$ and $M_l$ have disjoint support. A contradiction. So we conclude that $I \in B$.

Now observe that $A = I \ast I \in B$. Hence, as $B$ contains the $M_i$, we find that $A^{-1} \in B$. \hfill \Box

We now summarize the properties of the image of $p_{a,B}$.

**Proposition 6.14.** If $(a, B)$ is twin free and if $\sum_{i \in S} a_i \neq 0$ for each $S \subseteq [n]$, then the image of $p_{a,B}$ is a graded contraction-closed subalgebra of $T(W)$ that contains $T(W)^{S_n}$.  

63
Theorem 6.15. Let $(a, B)$ be a twin-free $n$-color vertex-coloring model over $\mathbb{F}$, such that $\sum_{i \in I} a_i \neq 0$ for all $I \subseteq [n]$. Then

$$p_{a, B}(Q) = T(W)^{\text{Aut}(a, B)}. \tag{6.43}$$

Proof of Theorem 6.15 By Lemma 6.12 and Theorem 6.15 we only need to show that $(a, B)$ is nondegenerate. Suppose that $0 \neq v \in T(W)^{\text{Aut}(a, B)}$. Then there
6.4 Proofs of Theorem 6.11 and Theorem 6.16

exists $w \in T(W)$ such that $(v, w)_a \neq 0$. Then, as $v$ is $\text{Aut}(a, B)$-invariant, we have

$$(v, w)_a = \frac{1}{|\text{Aut}(a, B)|} \sum_{\pi \in \text{Aut}(a, B)} (v, \pi w)_a. \quad (6.44)$$

As $\sum_{\pi \in \text{Aut}(a, B)} \pi w \in T(W)^{\text{Aut}(a, B)}$, Theorem 6.15 implies that $p_{a, B}(Q)$ is nondegenerate.

To prove Theorem 6.15 we use a characterization of subalgebras of $T(W)$ that are algebras of $G$-invariants for subgroups $G$ of $S_n$.

**Theorem 6.16.** Let $A \subseteq T(W)$. Then $A = T(W)^G$ for some subgroup $G \subseteq S_n$ if and only if $A$ is a graded contraction-closed subalgebra of $T(W)$ that contains $T(W)^{S_n}$.

We will prove this theorem in Section 6.4. Now we will use it to prove Theorem 6.15.

**Proof of Theorem 6.15** By Proposition 6.14 we can apply Theorem 6.16 to find that we have $p_{a, B}(Q) = T(W)^G$, for some subgroup $G \subseteq S_n$.

We finish the proof by showing that $G = \text{Aut}(a, B)$. First note that $a = U_1^2(h_2)$ and that $B \in p_{a, B}(Q)$ hence $G \subseteq \text{Aut}(a, B)$. To see the converse, just observe that $T(W)^{\text{Aut}(a, B)} \subseteq p_{a, B}(Q) = T(W)^G$, as for each $l$-labeled graph $H$, each $\phi : [l] \rightarrow [n]$ and each $\pi \in \text{Aut}(a, B)$, we have that $\text{hom}_{\pi, \phi}(H, G(a, B)) = \text{hom}_{\phi}(H, G(a, B))$ implying that $p_{a, B}(H)$ is invariant under $\text{Aut}(a, B)$.

**Remark.** Our proof of Theorem 6.1 is probably more involved than the proof of Lovász [41], but it has the advantage that it also works for $(a, B)$ where not all $a$ are positive, as long as the condition $\sum_{i \in S} a_i \neq 0$ for each $S \subseteq [n]$ is satisfied. In fact, the method by Lovász only requires that $(a, B)$ is nondegenerate, which is immediate if all $a_i$ are positive. It follows from our results that, if $\sum_{i \in S} a_i \neq 0$ for each $S \subseteq [n]$, then $(a, B)$ is nondegenerate. We do not know whether this can be shown directly, neither do we know whether we can remove this condition.

### 6.4 Proofs of Theorem 6.11 and Theorem 6.16

Both proofs are based on Schrijver’s proof of Theorem 6.9 and have a similar structure. We will first prove Theorem 6.16 since proving Theorem 6.11 requires more advanced machinery.

**Theorem 6.16.** Let $A \subseteq T(W)$. Then $A = T(W)^G$ for some subgroup $G \subseteq S_n$ if and only if $A$ is a graded contraction-closed subalgebra of $T(W)$ that contains $T(W)^{S_n}$. 


Proof. The 'only if' part is clear. To see the 'if' part, let $A \subseteq T(W)$ be a graded contraction-closed algebra containing $T(W)^{S_n}$.

Let $G := \{ \pi \in S_n \mid \pi a = a \text{ for all } a \in A \}$. We will show that $A = T(W)^G$, where the inclusion $A \subseteq T(W)^G$ is direct. To see the opposite inclusion, let $X := S_n/G$ be the set of left $G$-cosets and define functions $f_{v,w} : X \to \mathbb{F}$ by $f_{v,w}(\pi G) := (\pi v, w)$, for $\pi \in S_n$, $v \in A \cap W \otimes^k$ and $w \in W \otimes^k$, for any $k$. This is well defined since if $\pi \in G$, then $\pi v = v$. Note that $f_{v,w}f_{v',w'} = f_{v \otimes v', w \otimes w'}$. By nondegeneracy, $f_{h_2,w}$ is the constant one function for some $w \in W \otimes^2$.

Let $F$ be the algebra spanned by the functions $f_{v,w}$, for $v \in A \cap W \otimes^k$ and $w \in W \otimes^k$ and $k \in \mathbb{N}$. If $\pi G \neq \pi' G$, then by definition of $G$ there exists $v \in A \cap W \otimes^k$ for some $k$ such that $\pi^{-1}\pi' v \neq v$. So there exists $w \in W \otimes^k$ such that $f_{v,w}(\pi G) \neq f_{v,w}(\pi' G)$. So for each $\pi G \neq \pi' G \in X$, $F$ contains a function $f$ such that $f(\pi G) = 1$ and $f(\pi' G) = 0$, as $F$ contains the all-ones function. Since $F$ is an algebra it follows that $F = \mathbb{F}^X$. (This is actually the Stone-Weierstrass theorem for continuous functions on finite sets.)

Now let $x \in (W \otimes^k)^G$ for some $k$. Then for any $\pi \in S_n$ we can write

$$\pi x = \sum_{\phi : [k] \to [n]} f_{\phi}(\pi)e_{\phi^1}$$

(6.45)

for certain functions $f_{\phi} : S_n \to \mathbb{F}$. Since $x$ is $G$-invariant, the $f_{\phi}$ are actually functions on $X$. So we can write (6.45) as

$$\pi x = \sum_{\phi,j} f_{\phi,i,\phi,j}(\pi G)e_{\phi^1}$$

(6.46)

for certain $\phi,j \in A \cap W \otimes^k$ and $w_{\phi,i} \in W \otimes^k$. Multiplying (6.46) by $\pi^{-1}$ we obtain, (as $(\cdot,\cdot)$ is $S_n$-invariant),

$$x = \sum_{\phi,i,j} (\pi\phi,i,\pi^{-1}w_{\phi,i}) \pi^{-1}e_{\phi} = \sum_{\phi,i,j} (\phi,i,\pi^{-1}w_{\phi,i}) \pi^{-1}e_{\phi}. \quad (6.47)$$

Now note that $(\phi,i,1) \pi^{-1}w_{\phi,i}) \pi^{-1}e_{\phi}$ is equal to a series of contractions $K_{\phi,i}$ applied to $\phi,i \otimes \pi^{-1}(w_{\phi,i} \otimes e_{\phi})$. Hence

$$x = \sum_{\phi,i} K_{\phi,i}(\phi,i \otimes \left(\frac{1}{n!} \sum_{\pi \in S_n} \pi^{-1}(w_{\phi,i} \otimes e_{\phi})\right)),$$

(6.48)

implying that $x \in A$, as $A$ contains $T(W)^{S_n}$ and is a graded subalgebra of $T(W)$ that is closed under contractions. This finishes the proof of the theorem. \qed

66
The proof of Theorem 6.11 has the same structure as the proof of Theorem 6.16 but since the orthogonal group is a non-compact group, certain details require more advanced algebraic techniques.

**Theorem 6.11.** Let $\mathcal{F} = \mathcal{F}$ and let $A \subset T(V)$ be a graded contraction closed sub-algebra containing $T(V)^{O_k}$. If $\text{Stab}(A) = \text{Stab}(w)$ for some $w \in A$ whose $O_k$-orbit is closed in the Zariski topology, then $A = T(V)^{\text{Stab}(A)}$ and moreover $\text{Stab}(A)$ is a reductive group.

**Proof.** Let $w \in A$ be such that $G := \text{Stab}(w)$ equals $\text{Stab}(A)$. Write $w = w_1 + \ldots + w_t$ with $w_j \in W_j := V^{\otimes n_j}$ the homogeneous components of $w$, and assume that $O_k w \subset W := \bigoplus_{j=1}^t W_j$ is closed. The map $O_k \to W$ given by $g \mapsto gw$ induces an isomorphism $O_k/G \to O_k w$ of quasi affine varieties (cf. [30, Section 12] or [9, Theorem 1.16]). As $O_k w$ is closed, both varieties are affine and moreover regular functions on $O_k w$ extend to regular functions (polynomials) on $W$. So they are generated by $W_j^*$ for $j = 1, \ldots, t$. This means that any regular function on $O_k/G$ is a linear combination of functions of the form

$$\begin{align*}
g G & \mapsto (gw_1, u_1)^{d_1} \cdots (gw_t, u_t)^{d_t} \\
& = (w_1^{\otimes d_1} \otimes \cdots \otimes w_t^{\otimes d_t}, g^{-1}(u_1^{\otimes d_1} \otimes \cdots \otimes u_t^{\otimes d_t})), \tag{6.49}
\end{align*}$$

where $d_1, \ldots, d_t$ are natural numbers and $u_j \in W_j$ for all $j$. Since $A$ is a graded algebra, the tensor products of the $w_j$ are contained in $A$. So we find that every regular function on $O_k/G$ is a linear combination of functions of the form $g G \mapsto (gq, u) = (q, g^{-1} u)$ with $u \in T(V)$ and $q \in A$ in the same graded component of $T(V)$.

Clearly, $A$ is contained in $T(V)^G$. To prove the converse, let $a \in (V^{\otimes k})^G$. Let $z_1, \ldots, z_s$ be a basis of $V^{\otimes k}$. Then we can write,

$$ga = \sum_{i=1}^s f_i(g) z_i, \tag{6.50}$$

for all $g \in O_k$, where the $f_i$ are regular functions on $O_k$. Since $gha = ga$ for all $h \in G$ it follows that the $f_i$ induce regular functions on $O_k/G$. By the above, for each $i = 1, \ldots, s$, we can write

$$f_i(g) = \sum_j (q_{i,j}, g^{-1} u_{i,j}), \tag{6.51}$$

for certain $q_{i,j} \in A$ and $u_{i,j} \in T(V)$. Multiplying both sides of (6.50) by $g^{-1}$ we obtain

$$\begin{align*}
\text{for all } g \in O_k: a &= \sum_{ij} (q_{i,j}, g^{-1} u_{i,j}) g^{-1} z_i = \sum_{ij} K_{i,j} (q_{i,j} \otimes g^{-1} (u_{i,j} \otimes z_i)), \tag{6.52}
\end{align*}$$
where $K_{ij}$ denotes a certain series of contractions. Let $\rho_{O_k}$ be the Reynolds operator of $O_k$. Then we have
\[ a = \sum_{i,j} K_{ij} (q_{ij} \otimes \rho_{O_k}(u_{ij} \otimes z_i)). \] (6.53)

In the case where $\mathbb{F} = \mathbb{C}$, this follows immediately by integrating (6.52) over $g$ in the compact real orthogonal group (with respect to the Haar measure). In the general case this follows, by reductiveness of $O_k$, from Lemma 4.2.

To complete the proof note that $q_{ij} \in A$ and $\rho_{O_k}(u_{ij} \otimes z_i) \in T(V)^{O_k} \subseteq A$. As $A$ is a contraction closed subalgebra of $T(V)$ it follows that $a \in A$.

Finally, since $O_k / G$ is affine, Matsushima’s Criterion (see [1] for an elementary proof) implies that $\text{Stab}(A) = G$ is a reductive group. \qed