Chapter 8

Compact orbit spaces in Hilbert spaces and limits of edge-coloring models

We prove an abstract theorem about compact orbit spaces in Hilbert spaces. As a consequence we derive the existence of limits of certain sequences of edge-coloring models.

This chapter is based on joint work with Lex Schrijver [55].

8.1 Introduction

In [45] (which was awarded the Fulkerson prize in 2012) Lovász and Szegedy develop a theory of limits of dense graphs (here dense means that the number of edges is proportional to the number of vertices squared). The theory of graph limits has many connections to other areas of discrete mathematics, computer science and statistical mechanics. We refer to the book by Lovász [40] for details and references.

We shall now describe one of the main results from [45], but first we need to introduce a few definitions. For two simple graphs $H$ and $G$, we define the homomorphism density of $H$ in $G$ by

$$t(H, G) := p_{1/n, B}(H) = \frac{1}{n^{|V(H)|}} \text{hom}(H, G),$$  \hspace{1cm} (8.1)

where $n$ is the number of nodes of $G$, $B$ is the adjacency matrix of $G$ and $1/n$
denotes the vector with all entries equal to $1/n$. Then $t(H, G)$ is the probability that a random map from $V(H)$ to $V(G)$ is homomorphism. Central in the theory of graph limits is the following definition. A sequence $(G_n)$ of simple graphs is called convergent if for each simple graph $H$, $(t(H, G_n))$ is a convergent sequence of real numbers.

The main result in [45] is the discovery of a natural limit object for a convergent sequence of graphs, which we will now describe. A graphon is a symmetric Lebesgue measurable function $W : [0,1]^2 \to [0,1]$. For a graphon $W$ and a graph $H = ([k], E)$ define $t(H, W)$ by

$$t(H, W) := \int_{[0,1]^k} \prod_{ij \in E} W(x_i, x_j) dx_1 \cdots dx_k.$$ \hspace{1cm} (8.2)

In the context of de la Harpe and Jones [28], we may view $t(H, W)$ as the partition function of $W$.

We can view a simple graph $G = ([n], E)$ as a $\{0,1\}$-valued graphon $W_G$ by scaling its adjacency matrix, i.e.,

$$W_H(x,y) := \begin{cases} 
1 & \text{if } ([nx], [ny]) \in E, \\
0 & \text{otherwise.}
\end{cases} \hspace{1cm} (8.3)$$

Then $t(H, G) = t(H, W_G)$ for each simple graph $H$. So (8.2) generalizes (8.1). Lovász and Szegedy [45] showed that graphons are natural limit objects of convergent graph sequences in the following sense.

**Theorem 8.1** (Lovász and Szegedy [45]). Let $(G_n)$ be a convergent sequence of simple graphs. Then there exists a graphon $W$ such that $\lim_{n \to \infty} t(H, G_n) = t(H, W)$ for each simple graph $H$.

We can view Theorem 8.1 as describing limit objects for certain convergent sequences of vertex-coloring models. From that perspective, the following definition is natural. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A sequence $(h^n)$ of edge-coloring models over $\mathbb{F}$ is called convergent if for each simple graph $H$, $(p_{h^n}(H))$ is a convergent sequence in $\mathbb{F}$.

If we would allow all graphs in this definition, and if the number of colors of each $h^n$ is bounded, by $k$ say, then its is easy to see by Theorem 5.2 in the real case, and by Theorem 5.3 in the complex case, that there exists a $k$-color edge coloring model $h$ such that $\lim_{n \to \infty} p_{h^n}(H) = p_h(H)$ for all graphs $H$. However, if the number of colors grows we can not represent the limit parameter as the partition function of an ordinary edge-coloring model, as the following example shows.

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Example 8.1. Consider for $n \in \mathbb{N}$ the edge-coloring model: $h^n \in \mathbb{R}[x_1, \ldots, x_n]^*$ defined by $x^a \mapsto 1$ if $x^a = x_i^a$ for $i \in [n]$ and $x^a \mapsto 0$ otherwise. Then $(h^n)$ is convergent. Indeed, $p_{h^n}(H) = 1$ if $H$ is the disjoint union of regular graphs of degree at most $n$ and 0 otherwise, implying that $\lim_{n \to \infty} p_{h^n}(H) = 1$ if $H$ is the disjoint union of regular graphs and 0 otherwise. Let $f$ denote the limit graph parameter. Then $f$ is not the partition function of any $k$-color edge-coloring model, for any $k \in \mathbb{N}$.

Indeed, let $k \in \mathbb{N}$ and let for $i = 1, \ldots, k+1$, $H_i$ be an $i$-regular graph. Fix for each $i$ an edge $u_i v_i$ from $H_i$ and let $H'_i$ be the graph where this edge is removed. Let $H$ be the disjoint union of the $H'_i$. Define $s : \{u_1, \ldots, u_{k+1}\} \to V(H)$ by $s(u_i) = v_i$ for $i = 1, \ldots, k+1$. Now note that

$$\sum_{\pi \in S_{k+1}} \text{sgn}(\pi) f(H_{s\circ \pi}) = f(H_s) = 1. \quad (8.4)$$

So by Theorem 5.3 it follows that $f$ is not the partition function of any $k$-color edge-coloring model over $\mathbb{C}$ (neither over any algebraically closed field of characteristic zero).

The limit graph parameter $f$ can be described as the partition function of $h \in \mathbb{R}[x_1, x_2, \ldots]^* \to \mathbb{R}$ defined by $h(x^a) = 1$ if $x^a = x_i^a$ for $i \in \mathbb{N}$ and $h(x^a) = 0$ otherwise.

We shall show that under some boundedness conditions there exists a natural limit object for each convergent sequence of edge-coloring models $(h^n)$, which, as in the example above, is an infinite color edge-coloring model, just as a graphon can be considered as a vertex-coloring model with an (uncountably) infinite number of states. This answers a question posed by Lovász [39] and also, in a slightly different form, by Kannan [31].

To do so, we state in the next section an abstract theorem about compact orbit spaces in Hilbert spaces (cf. Theorem 8.2), which generalizes a result from Lovász and Szegedy [46] and as such it allows to show Theorem 8.1. Moreover, it allows to construct limit objects for certain convergent sequences of edge-coloring models.

### 8.2 Compact orbit spaces in Hilbert spaces and applications

We state a theorem on compact orbit space in Hilbert spaces. In [55] this is done for real Hilbert spaces only. It is straightforward to extend the results to complex Hilbert spaces, which we will do here. We moreover show how the
theorem applies to limits of both graphs and edge-coloring models. Throughout this section \( F \) denotes either \( \mathbb{R} \) or \( \mathbb{C} \).

### 8.2.1 Compact orbit spaces in Hilbert spaces

We start with a few definitions. Let \( \mathcal{H} \) be a (complex or real) Hilbert space, i.e., \( \mathcal{H} \) is a linear space equipped with an inner product \( \langle \cdot, \cdot \rangle \), (which is linear in the first argument and conjugate linear in the second argument) such that \( \mathcal{H} \) is complete with respect to the norm topology induced by the inner product. We denote the 2-norm of \( x \in \mathcal{H} \) by \( \| x \| \), where \( \| x \| = \sqrt{\langle x, x \rangle} \), and the Hilbert metric is denoted by \( d_2 \), where \( d_2(x, y) = \| x - y \| \) for \( x, y \in \mathcal{H} \). By \( B(\mathcal{H}) \) we denote the closed unit ball in \( \mathcal{H} \).

For a bounded subset \( R \subset \mathcal{H} \) we define a seminorm \( \| \cdot \|_R \) and a pseudometric \( d_R \) on \( \mathcal{H} \) by for \( x, y \in \mathcal{H} \):

\[
\| x \|_R := \sup_{r \in R} |\langle x, r \rangle| \quad \text{and} \quad d_R(x, y) := \| x - y \|_R.
\] (8.5)

We use the topology induced by this pseudometric only if we explicitly mention it, otherwise we use the topology induced by the ordinary Hilbert norm. Note that if \( R \subseteq B(\mathcal{H}) \), then, by Cauchy-Schwarz, \( d_R(x, y) \leq d_2(x, y) \) for any \( x, y \in \mathcal{H} \).

A subset \( \mathcal{W} \) of \( \mathcal{H} \) is called weakly compact if it is compact in the weak topology on \( \mathcal{H} \). (A set \( U \) is open in the weak topology if for each \( u \in U \), there exist \( n \in \mathbb{N} \), \( y_i \in \mathcal{H} \) and \( \epsilon_i > 0 \) for \( i = 1, \ldots, n \) such that \( U \) contains \( \bigcap_{i=1}^n \{ x \in \mathcal{H} \mid |\langle u - x, y_i \rangle| < \epsilon_i \} \).) By the Banach-Alaoglu Theorem (cf. [15, Theorem V.3.1] and the Principle of Uniform Boundedness (cf. [15, Theorem III.14.1]), for any \( \mathcal{W} \subseteq \mathcal{H} \):

\[
\mathcal{W} \text{ closed, bounded and convex} \Rightarrow \mathcal{W} \text{ weakly compact}
\]

\[
\mathcal{W} \text{ weakly compact} \Rightarrow \mathcal{W} \text{ bounded.}
\] (8.6)

Let \( G \) be a group acting on a topological space \( X \). The orbit space \( X/G \) is the quotient space of \( X \) taking the orbits of \( G \) as classes. We can now state our result on compact orbit space in Hilbert spaces.

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A *seminorm* is a norm except that nonzero elements may have norm 0. A pseudometric is a metric except that distinct points may have distance 0. One can turn a pseudometric space into a metric space by identifying points at distance 0, but for our purposes it is notationally easier and sufficient to maintain the original space. Notions like convergence pass easily over to pseudometric spaces, but limits need not be unique.
Theorem 8.2. Let $\mathcal{H}$ be a Hilbert space and let $G$ be a group of unitary transformations of $\mathcal{H}$. Let $W$ and $R$ be $G$-stable subsets of $\mathcal{H}$, with $W$ weakly compact and $R^k / G$ compact for each $k \in \mathbb{N}$. Then $(W, d_R) / G$ is compact.

We postpone the proof of Theorem 8.2 to Section 8.3. Schrijver [65] found a nice application of it to low-rank approximation of polynomials. We will not describe this here. We will now show some applications of it to limits of both graphs and edge-coloring models.

8.2.2 Application of Theorem 8.2 to graph limits

Here we will show how Theorem 8.2 can be used to prove Theorem 8.1. In this subsection, measures are Lebesgue measure.

Let $\mathcal{H} := L^2([0,1]^2)$, the Hilbert space of all square integrable functions $[0,1]^2 \to \mathbb{R}$. Let $R$ be the collection of functions $\chi^A \times \chi^B$, where $A, B$ are measurable subsets of $[0,1]$ and where $\chi^A$ and $\chi^B$ denote the incidence functions of $A$ and $B$ respectively. Let $S_{[0,1]}$ be the group of measure space automorphisms of $[0,1]$. The group $S_{[0,1]}$ act naturally on $\mathcal{H}$ by $\pi W(x,y) = W(\pi^{-1}x, \pi^{-1}y)$ for $W \in \mathcal{H}$ and $\pi \in S_{[0,1]}$. Moreover, $R^k / S_{[0,1]}$ is compact for each $k$. (This can be derived from the fact that for each measurable $A \subseteq [0,1]$ there exists $\pi \in S_{[0,1]}$ such that $\pi(A)$ is an interval up to a set of measure 0 (cf. [19]).)

Let $W_0 \subseteq \mathcal{H}$ be the set defined by all $[0,1]$-valued functions $W$ such that $W(x,y) = W(y,x)$ for all $x, y \in [0,1]$, that is, $W_0$ is the set of all graphons. Then $W_0$ is a closed bounded and convex $S_{[0,1]}$-stable subset of $\mathcal{H}$. So by (8.6) and by Theorem 8.2 we recover Theorem 5.1 fromLovász and Szegedy [46]:

$$(W_0, d_R) / S_{[0,1]}$$

is compact. (8.7)

Note that $t(H, W) = t(H, \pi W)$ for each $\pi \in S_{[0,1]}$, simple graph $H$ and graphon $W$. Two graphons $W, W' \in W_0$ are considered to be the same if there exists $\pi \in S_{[0,1]}$ such that $\pi W = W'$. So one might say that the graphon space is compact with respect to $d_R$.

By $G_{\text{sim}}$ we denote the set of all simple graphs. In [45], Lovász and Szegedy showed that the map $\tau : (W_0, d_R) \to \mathbb{R}^{G_{\text{sim}}}$ defined by $\tau(W)(H) := t(H, W)$ is continuous (here the restriction to simple graphs is really necessary). Since $(W_0, d_R) / S_{[0,1]}$ is compact, and since $\tau$ is $S_{[0,1]}$-invariant, the image of $\tau$ in $\mathbb{R}^{G_{\text{sim}}}$ is compact. Hence each sequence $\tau(W_1), \tau(W_2) \ldots \in \mathbb{R}^{G_{\text{sim}}}$ of partition functions of graphons such that $t(H, W_i)$ converges for each simple graph $H$ converges to the partition function $\tau(W) \in \mathbb{R}^{G_{\text{sim}}}$ of some graphon $W$. So, as simple graphs can be viewed as graphons, this gives a limit behavior of simple graphs, that is, it implies Theorem 8.1.
8.2.3 Application of Theorem 8.2 to edge-coloring models

We will now show how Theorem 8.2 can be applied to (limits of) edge coloring models. We will again extend the results of [55] to the complex setting. First we need to extend our definition of an edge-coloring model to a Hilbert space setting. After that we will state our main results about limits of edge-coloring models, postponing the proofs to Section 8.4.

We will use a different, but universal, model of Hilbert space. Let $C$ be a finite or infinite set, and consider for $F = C$ or $F = \mathbb{R}$, the Hilbert space $l^2(C)$, the set of all functions $f : C \to F$ with $\sum_{c \in C} |f(c)|^2 < \infty$, having norm $\|f\| := (\sum_{c \in C} |f(c)|^2)^{1/2}$. The inner product on $l^2(C)$ is defined by $\langle f, h \rangle := \sum_{c \in C} f(c) \overline{h(c)}$ for $f, h \in l^2(C)$.

Define for each $k = 0, 1, \ldots$:

$$H_k := l^2(C^k). \quad (8.8)$$

As usual, $H_k^{S_k}$ denotes the set of elements of $H_k$ that are invariant under the natural action $S_k$ on $H_k$. We call an element $h = (h_k)_{k \in \mathbb{N}}$ of $\prod_{k=0}^{\infty} H_k^{S_k}$ a $C$-color edge-coloring model. Note that for finite $C$ this agrees with our original definition of a $|C|$-color edge-coloring model, because we can view $h \in \prod_{k=0}^{\infty} H_k^{S_k}$ as a linear map on $C[x_1, \ldots, x_{|C|}]$ via the identification of symmetric tensors in $H_k$ with homogeneous polynomials of degree $k$. Let $G_0 \subset G$ be the set of all graphs without loops. The partition function of $h$ is the graph parameter $p_h : G_0 := G_0 \to F$ defined by,

$$p_h(H) := \sum_{\phi : E \to C} \prod_{v \in V} h_{d(v)}(\phi(\delta(v))) \quad (8.9)$$

for a loopless graph $H = (V, E)$. Recall that $d(v)$ denotes the degree of the vertex $v$. Moreover, if $\delta(v)$ consists of the edges $e_1, \ldots, e_k$ (in some arbitrary order), then $\phi(\delta(v)) = (\phi(e_1), \ldots, \phi(e_k)) \in C^k$. As $h_k$ is $S_k$-invariant the order is irrelevant. We will show below (cf. (8.22)) that the sum (8.9) is absolutely convergent. Hence $p_h$ is well-defined. The next example shows that it is necessary for $H$ to not have loops.

**Example 8.2.** Define $h \in H_2^{S_2}$ by

$$h(i, j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases} \quad (8.10)$$

and let $H = C_1$. Then $\|h\|^2 < \infty$, but $p_h(H) = \sum_{k=1}^{\infty} 1/k$ and this series does not converge.
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Define
\[ \pi : \prod_{k=0}^{\infty} \mathcal{H}_k^S \rightarrow \mathcal{G}_{\text{sim}} \text{ by } \pi(h)(H) = p_h(H) \] (8.11)

for \( H \in \mathcal{G}_{\text{sim}} \). It is not difficult to show that \( \pi \) is continuous on \( \prod_{k=0}^{\infty} \mathcal{H}_k^S \), even if we replace \( \mathcal{G}_{\text{sim}} \) by \( \mathcal{G}_0 \).

Let \( O(H) \) denote the group of invertible linear transformations of the real Hilbert space \( l^2(C, \mathbb{R}) \) that preserve the inner product. We call \( O(H) \) the orthogonal group. Note that \( O(H) \) is a subgroup of the group of unitary transformations of \( l^2(C, \mathbb{C}) \).

The tensor power \( l^2(C)^{\otimes k} \) embeds naturally in \( l^2(C^k) \). In fact, \( l^2(C^k) \) is the completion of \( l^2(C)^{\otimes k} \). Hence the group \( O(H) \) acts naturally on \( H_k \) for each \( k \). Just as in the finite case, the partition functions of edge-coloring models are invariant under the orthogonal group. This follows directly from the case where \( |C| \) is finite (cf. the proof of Proposition 6.7), as soon as we show that we can extend the definition of \( p_h \) to fragments, which we will do in Section 8.4.

The standard orthonormal basis for \( H_k \) is given by the set \( \{ e_\phi \mid \phi : [k] \rightarrow C \} \), where for \( \phi : [k] \rightarrow C \), \( e_\phi := e_{\phi(1)} \otimes \ldots \otimes e_{\phi(k)} \), and where \( e_c \) for \( c \in C \) is the orthonormal basis for \( l^2(C) \) given by \( e_c(c') = \delta_{c,c'} \). Define
\[ R_k := \{ r_1 \otimes \ldots \otimes r_k \mid r_1, \ldots, r_k \in B(H_1) \} \subset H_k. \] (8.12)

We will show that \( \pi \) is continuous on \( \prod_{k=0}^{\infty} B_k \), when \( B_k := B(H_k)^S \) is equipped with the metric \( d_{R_k} \).

**Theorem 8.3.** The map \( \pi \) is continuous on \( \prod_{k=0}^{\infty} (B_k, d_{R_k}) \).

From Theorem 8.2 we will derive:

**Theorem 8.4.** The space \( (\prod_{k=0}^{\infty} (B_k, d_{R_k}))/O(H) \) is compact.

The proofs of Theorem 8.3 and 8.4 will be given in Section 8.4. Note that since \( \pi \) is \( O(H) \)-invariant, Theorems 8.3 and 8.4 imply:

**Corollary 8.5.** The image \( \pi(\prod_{k=0}^{\infty} B_k) \) of \( \pi \) is compact.

This implies:

**Corollary 8.6.** Let \( h^1, h^2 \ldots \in \prod_{k=0}^{\infty} B_k \) be a convergent sequence of edge-coloring models. Then there exists \( h \in \prod_{k=0}^{\infty} B_k \) such that for each simple graph \( H \),
\[ \lim_{n \rightarrow \infty} p_{h^n}(H) = p_h(H). \] (8.13)
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The corollary holds more generally for sequences in $\prod_{k=0}^{\infty} \lambda_k B_k$ for any fixed sequence $\lambda_0, \lambda_1, \ldots \in \mathbb{F}$.

Since $l^2(C)$ embeds naturally in $l^2(C')$ for $C \subseteq C'$, all edge-coloring models with a any finite number of states embed into $\prod_{k=0}^{\infty} (l^2(\mathbb{N}^k))^S_k$. So just as Theorem 8.1 describes a limit behavior of finite graphs, Corollary 8.6 describes a limit behavior of finite-state edge-coloring models, answering a question of Lovász [39]. Since we can think symmetric $k$-tensors as edge-coloring models, Corollary 8.6 also describes a limit behavior of symmetric tensors, providing an answer to a question of Kannan [31].

We end this section with two questions. In [7], Borg, Chayes, Lovász, Sós and Vesztergombi show that the map $\tau : W_0/S_{[0,1]} \rightarrow [0,1]^{\mathcal{G}_{\text{sim}}}$ satisfies that if $\tau(W) = \tau(W')$, then $W'$ is contained in the closure of the $S_{[0,1]}$-orbit of $W$.

**Question 1.** Is it true that if $\pi(h) = \pi(h')$ for any $h, h' \in \prod_{k=0}^{\infty} B_k$, then $h'$ is contained in the closure of the $O(\mathcal{H})$-orbit of $h$?

The image of the map $\tau$ was characterized by Lovász and Szegedy [45] in terms of some form of reflection positivity.

**Question 2.** Can one give a characterization of the image of $\pi$ for $\mathbb{F} = \mathbb{R}$ in terms of some form of edge-reflection positivity?

### 8.3 Proof of Theorem 8.2

We start by proving that a weakly compact space equipped with the $d_R$ metric (with $R$ bounded) is complete.

**Proposition 8.7.** Let $\mathcal{H}$ be a Hilbert space and let $R, W \subseteq \mathcal{H}$ with $R$ bounded and $W$ weakly compact. Then $(\mathcal{W}, d_R)$ is complete.

**Proof.** Let $a_1, a_2, \ldots \in \mathcal{W}$ be a Cauchy sequence with respect to $d_R$. We must show that it has a limit in $\mathcal{W}$ with respect to $d_R$. We may assume that $\mathcal{H}$ is separable, otherwise we can replace $\mathcal{H}$ by the closure of the linear span of the $a_i$.

Then, as $\mathcal{W}$ is weakly compact, the sequence has a weakly convergent subsequence (cf. [15] Theorem V.5.1], say with limit $a \in \mathcal{W}$. Then $a$ is the required limit, that is, $\lim_{n \to \infty} d_R(a_n, a) = 0$. For choose $\epsilon > 0$. As $a_1, a_2, \ldots$ is Cauchy with respect to $d_R$, there is a $p$ such that $d_R(a_n, a_m) < 1/2\epsilon$ for $n, m \geq p$. Since $a$ is the weak limit of a subsequence of the $a_i$, there is for each $r \in R$ an $m \geq p$ such that $|\langle a_m - a, r \rangle| < 1/2\epsilon$. This implies, by the triangle inequality, that for each $n \geq p$,

$$|\langle a_n - a, r \rangle| \leq |\langle a_n - a_m, r \rangle| + |\langle a_m - a, r \rangle| < \epsilon.$$  \hfill (8.14)
So \(d_R(a_n, a) \leq \varepsilon\) if \(n \geq p\). \(\square\)

Let \(G\) be a group acting on a pseudometric space \((X, d)\) that leaves \(d\) invariant. Define a pseudometric \(d/G\) on \(X\) by, for \(x, y \in X\):

\[
d/G(x, y) := \inf_{g \in G} d(x, gy). \tag{8.15}\]

Since \(d\) is \(G\)-invariant, \((d/G)(x, y)\) is equal to the distance of the \(G\)-orbits \(Gx\) and \(Gy\). Any two points \(x, y\) on the same \(G\)-orbit have \((d/G)(x, y) = 0\). If we identify points of \((X, d/G)\) that are on the same orbit, the topological space obtained is homeomorphic to the orbit space \((X, d)/G\) of the topological space \((X, d)\).

**Proposition 8.8.** Let \((X, d)\) be a complete metric space and let \(G\) be a group that acts on \((X, d)\), leaving \(d\) invariant. Then \((X, d/G)\) is complete.

**Proof.** Let \(a_1, a_2, \ldots \in X\) be a Cauchy sequence with respect to \(d/G\). Then it has a subsequence \(b_1, b_2, \ldots\) such that \((d/G)(b_k, b_{k+1}) < 2^{-k}\) for all \(k\).

Let \(g_1 = 1 \in G\). If \(g_k \in G\) has been chosen, let \(g_{k+1} \in G\) such that \(d(g_kb_k, g_{k+1}b_{k+1}) < 2^{-k}\). Then \(g_1b_1, g_2b_2, \ldots\) is a Cauchy sequence with respect to \(d\). Hence it has a limit \(b\) say. Then \(\lim_{k \to \infty} (d/G)(b_k, b) = 0\), implying \(\lim_{n \to \infty} (d/G)(a_n, b) = 0\). \(\square\)

Let \(\mathcal{H}\) be a Hilbert space and let \(R \subseteq \mathcal{H}\). For any \(k \geq 0\), define

\[
Q_k := \{ \lambda_1r_1 + \ldots + \lambda_k r_k \mid r_i \in R, |\lambda_i| \leq 1 \text{ for } i = 1, \ldots, k\}. \tag{8.16}\]

For any pseudometric \(d\), let \(B_d(Z, \varepsilon)\) denote the set of points at most distance \(\varepsilon\) from \(Z\). The following is a form of ‘weak Szemerédi regularity’. (cf. Lemma 4.1 of Lovász and Szegedy [46], extending a result of Fernandez de la Vega, Kannan, Karpinski and Vempala [23].)

**Proposition 8.9.** If \(R \subseteq B(\mathcal{H})\), then for each \(k \geq 1\):

\[
B(\mathcal{H}) \subseteq B_{d_R}(Q_k, 1/\sqrt{k}). \tag{8.17}\]

**Proof.** Choose \(a \in B(\mathcal{H})\) and set \(a_0 := a\). If, for some \(i \geq 0\), \(a_i\) has been found, and \(\|a_i\|_R > 1/\sqrt{k}\), choose \(r \in R\) with \(|\langle a_i, r \rangle| > 1/\sqrt{k}\). Define \(a_{i+1} := a_i - \langle a_i, r \rangle r\). Then, with induction,

\[
\|a_{i+1}\|^2 = \|a_i\|^2 - 2|\langle a_i, r \rangle|^2 + |\langle a_i, r \rangle|^2 \|r\|^2 = \|a_i\|^2 - |\langle a_i, r \rangle|^2 (2 - \|r\|^2) \\
\leq \|a_i\|^2 - |\langle a_i, r \rangle|^2 \leq \|a_i\|^2 - 1/k \leq 1 - (i+1)/k. \tag{8.18}\]

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Moreover, since $|\langle a_i, r \rangle| \leq 1$, we know by induction that $a - a_i \in Q_i$. By (8.18), as $\|a_{i+1}\|^2 \geq 0$, the process terminates for some $i \leq k$. For this $i$ one has $\|a_i\|_R \leq 1/\sqrt{k}$. Hence, since $Q_i \subseteq Q_k$,

$$d_R(a, Q_k) \leq d_R(a, Q_i) \leq d_R(a, a - a_i) = \|a_i\|_R \leq 1/\sqrt{k}. \tag{8.19}$$

We can now give a proof of Theorem 8.2.

**Theorem 8.2.** Let $\mathcal{H}$ be a Hilbert space and let $G$ be a group of unitary transformations of $\mathcal{H}$. Let $\mathcal{W}$ and $R$ be $G$-stable subsets of $\mathcal{H}$, with $\mathcal{W}$ weakly compact and $R^k/G$ compact for each $k \in \mathbb{N}$. Then $(\mathcal{W}, d_R)/G$ is compact.

**Proof.** As $R/G$ is compact, $R$ is bounded. So by (8.6), we may assume that both $R$ and $\mathcal{W}$ are contained in $B(\mathcal{H})$.

By Propositions 8.7 and 8.8, $(\mathcal{W}, d_R/G)$ is complete. So it suffices to show that $(\mathcal{W}, d_R/G)$ is totally bounded; that is for each $\varepsilon > 0$, $\mathcal{W}$ can be covered by finitely many $d_R/G$-balls of radius $\varepsilon$. For suppose $a_1, a_2, \ldots$ is some sequence in $\mathcal{W}$. Then there exists a ball $B_1$ of $d_R/G$-radius $2^{-1}$ containing infinitely many of the $a_i$. Let $N_1 := \{ n \in \mathbb{N} \mid a_n \in B_1 \}$. If $B_k$ and $N_k$ have been chosen, choose a ball $B_{k+1}$ of $d_R/G$-radius $2^{-k-1}$, such that $N_{k+1} := \{ n \in N_k \mid a_n \in B_{k+1} \}$ is infinite. Now choose for $k \geq 1$, $n_k \in N_k$ with $n_k > n_{k-1}$ and set $b_k := a_{n_k}$. Then $(d_R/G)(b_k, b_{k+1}) \leq 2^{-k-1}$. Hence $b_1, b_2, \ldots$ forms a Cauchy sequence in $(\mathcal{W}, d_R/G)$ and thus has a limit $b \in \mathcal{W}$, proving compactness of $(\mathcal{W}, d_R/G)$.

Now we will show that $(\mathcal{W}, d_R/G)$ is totally bounded. Let $\varepsilon > 0$ and set $k := \lfloor 4/\varepsilon^2 \rfloor$. As $R^k/G$ is compact, $Q_k/G$ is compact (since the function $R^k \times \{ \lambda \mid |\lambda| \leq 1 \}^k \to Q_k$ mapping $(r_1, \ldots, r_k, \lambda_1, \ldots, \lambda_k)$ to $\lambda_1 r_1 + \ldots + \lambda_k r_k$ is continuous, surjective and $G$-equivariant.) Hence (as $d_R \leq d_2$) $(Q_k, d_R)/G$ is compact, equivalently, $(Q_k, d_R/G)$ is compact. Therefore, there exists some finite set $F$ such that $Q_k \subseteq B_{d_R/G}(F, 1/\sqrt{k})$. Then by Proposition 8.9 and the triangle inequality,

$$\mathcal{W} \subseteq B(\mathcal{H}) \subseteq B_{d_R}(Q_k, 1/\sqrt{k}) \subseteq B_{d_R/G}(Q_k, 1/\sqrt{k}) \subseteq B_{d_R/G}(F, 2/\sqrt{k}) \subseteq B_{d_R/G}(F, \varepsilon). \tag{8.20}$$

**8.4 Proofs of Theorem 8.3 and 8.4.**

Throughout this section, $F$ denotes either $\mathbb{R}$ or $\mathbb{C}$. 92
We start by showing some properties of the map \( \pi \), after which we will prove Theorem 8.3.

For an \( l \)-fragment \( F = ([n], E) \) without loops nor open edges and \( h = (h_v)_{v \in [n]} \in \prod_{v \in [n]} B_{d(v)} \) define \( p_h(F) \in \mathcal{H}_l \) by

\[
p_h(F)(c_1, \ldots, c_l) = \sum_{\phi: E \to C} \prod_{v \in V} h_v(\phi(\delta(v))).
\]

Then

\[
\|p_h(F)\| \leq \prod_{v \in [n]} \|h_v\|.
\]

This in particular shows that the sum (8.9) is absolutely convergent and that (8.21) is well-defined. We prove (8.22) by induction on \(|E \setminus [l]|\). The case \( E = [l] \) being trivial. Let \( |E \setminus [l]| \geq 1 \) and choose an edge \( ab \in E \setminus [l] \). Set \( E' = E \setminus \{ab\} \), \( \delta'(v) := \delta(v) \setminus \{ab\} \) and \( d'(v) = \|\delta'(v)\| \) for each \( v \in [n] \). Let \( F' \) be the fragment obtained from \( F \) by deleting the edge \( ab \). For \( c_1, \ldots, c_m \in C \) and \( h \in \mathcal{H}_{k+1} \)

\[
h(c_1, \ldots, c_m) \text{ is the element of } \mathcal{H}_{k+1} \text{ defined by } h(c_1, \ldots, c_m)(c_{m+1}, \ldots, c_k) = h(c_1, \ldots, c_k).
\]

Since

\[
|p_h(F)(c_1, \ldots, c_l)| \leq \sum_{\phi: E \to C} \prod_{v \in [n]} |h_v(\phi(\delta(v)))|,
\]

we may assume that \( h \) takes values in \( \mathbb{R}_{\geq 0} \). Then

\[
p_h(F)(c_1, \ldots, c_l) = \sum_{\phi: E' \to C} \sum_{c \in C} h_a(\phi(\delta'(a)), c) h_b(\phi(\delta'(b)), c) \cdot \prod_{v \in [n]\setminus\{a,b\}} h_v(\phi(\delta(v)))
\]

by Cauchy-Schwarz. Now define \( h'_v = h_v \) for \( v \notin \{a, b\} \) and for \( v \in \{a, b\} \), \( h'_v \in \mathcal{H}_{d'(v)} \) is defined by

\[
h'_v(c_1, \ldots, c_{d'(v)}) := \|h_v(c_1, \ldots, c_{d'(v)})\|.
\]

Then the last line of (8.24) is equal to \( p_{h'}(F')(c_1, \ldots, c_l) \). Since \( \|h'_v\| = \|h_v\| \) for all \( v \in V \), (8.24) implies with induction that

\[
\|p_h(F)\| \leq \|p_{h'}(F')\| \leq \prod_{v \in [n]} \|h_v\|.
\]
This proves (8.22).

Next, for a graph without loops \( H = ([n], E) \) define a function
\[
\pi_F : \prod_{v \in [n]} \mathcal{H}^S_{d(v)} \to F \quad \text{by} \quad \pi_F(h) := \sum_{\phi : E \to C} \prod_{v \in [n]} h_v(\phi(\delta(v))) \tag{8.27}
\]
for \( h = (h_v)_{v \in [n]} \in \prod_{v \in [n]} \mathcal{H}^S_{d(v)} \).

**Proposition 8.10.** For a simple graph \( H = (V, E) \), the map \( \pi_H \) is continuous on \( \prod_{v \in V} (B_{d(v)}, d^S_{\delta(v)}) \).

**Proof.** We start by showing that for each \( u \in V \),
\[
|\pi_H(h)| \leq \|h_u\|_{R_{d(u)}} \prod_{v \in V \setminus \{u\}} \|h_v\|. \tag{8.28}
\]

To see this, let \( N(u) \) be the set of neighbors of \( u \), \( H' = H - u \), \( \delta'(v) := \delta(v) \setminus \delta(u) \) for \( v \in V \setminus \{u\} \) and \( d'(v) = |\delta'(v)| \). As above, define for \( v \neq u \), \( h'_v \in \mathcal{H}^S_{d'(v)} \) by \( h'_v = h_v \) if \( v \notin N(u) \) and \( h'_v(c_1, \ldots, c_{d'_v(v)}) = \|h_v(c_1, \ldots, c_{d'_v(v)})\| \) if \( v \in N(u) \). Again, \( \|h'_v\| = \|h_v\| \) for all \( v \). Then
\[
|\pi_H(h)| = \left| \sum_{\phi : E \to C} \prod_{v \in V} h_v(\phi(\delta'(v))) \right| \leq \sum_{\phi : E \to C} \prod_{v \in N(u)} \|h_v(\phi(\delta'(v)))\| \prod_{v \in (V(H') \setminus N(u))} \|h_v(\phi(\delta'(v)))\| \leq \sum_{\phi : E(H') \to C} \|h_u\|_{R_{d(u)}} \prod_{v \in V(H')} |h_v'(\phi(\delta'(v)))| \leq \|h_u\|_{R_{d(u)}} \prod_{v \in V(H')} \|h_v\|,
\]
where the inequalities follow from the definition of \( \| \cdot \|_{R_{d(u)}} \) and from \( (8.22) \) (applied to \( H' \)). This proves (8.28).

Next, identify \( V \) with \([n]\) and let \( g, h \in \prod_{v \in [n]} B_{d(v)} \). For \( u = 1, \ldots, n \) define \( p^u \in \prod_{i \in [n]} B_{d(i)} \) by \( p^u_I := g_I \) if \( i < u \), \( p^u_I := g_u - h_u \) and \( p^u_I := h_i \) if \( i > u \). Moreover, for \( u = 0, \ldots, n \) define \( q^u \in \prod_{i \in [n]} B_{d(i)} \) by \( q^u_I := g_i \) if \( i \leq u \) and \( q^u_I := h_i \) if \( i > u \). So \( q^n = g \) and \( q_0 = h \). By the multilinearity of \( \pi_H \) we have \( \pi_H(q^n) - \pi_H(q^{n-1}) = \pi_H(p^u) \). Hence by (8.28) we have the following, proving the proposition,
\[
|\pi_H(g) - \pi_H(h)| = \left| \sum_{u=1}^n (\pi_H(q^u) - \pi_H(q^{u-1})) \right| = \left| \sum_{u=1}^n \pi_H(p^u) \right| \leq \sum_{u=1}^n \|p^u\|_{R_{d(u)}} = \sum_{u=1}^n \|g_u - h_u\|_{R_{d(u)}}. \tag{8.30}
\]
8.4. Proofs of Theorem 8.3 and 8.4

We can use Proposition 8.10 to prove Theorem 8.3.

**Theorem 8.3.** The map \( \pi \) is continuous on \( \prod_{k=0}^{\infty} (B_k, d_{R_k}) \).

*Proof.* For each simple graph \( H \), the function \( \psi : \prod_{k=0}^{\infty} B_k \rightarrow \prod_{v \in V(H)} B_{d(v)} \) mapping \( (h_k)_{k=0}^{\infty} \) to \( (h_{d(v)})_{v \in V(H)} \) is continuous. As \( \pi(\cdot)(H) = \pi_H(\psi(\cdot)) \), the theorem follows from Proposition 8.10. \( \square \)

Note that we really need simple graphs in Theorem 8.3, as the following example shows.

**Example 8.3.** Let \( H = C_2 := \emptyset \) and let \( h^n \in B_2 \) be defined by

\[
h^n(i, j) := \begin{cases} n^{-1/2} & \text{if } i = j \leq n, \\ 0 & \text{otherwise}. \end{cases}
\]  

Then \( \|h^n\|^2 = p_H(h^n) = 1 \) for all \( n \), but \( \lim_{n \to \infty} \|h^n\|_{d_{R_2}} = 0 \). So \( \pi_H \) is not continuous with respect to \( d_{R_2} \).

It is easy to see that Theorem 8.3 remains true if we replace \( B_{d(i)} \) by \( \lambda_i B_{d(i)} \) for any \( \lambda_0, \lambda_1, \ldots \in \mathbb{F} \). (As it only affects the bound in \( H \) by a factor of \((\max_{v \in V} |\lambda_{d(v)}|)^{-1/3}\). But \( \pi_H \) is not continuous on \( \prod_{i \in \{n\}} (H^S_{d(i)}(\cdot), d_{R_d(i)}) \), as the following example shows.

**Example 8.4.** Define \( h^n \in \mathcal{H}^S_{22} \) by

\[
h^n(i, j) := \begin{cases} n^{-1/3} & \text{if } i = j \leq n, \\ 0 & \text{otherwise}. \end{cases}
\] 

Then for \( H = C_3 \), we have \( p_H(h^n) = 1 \) for all \( n \), but \( \lim_{n \to \infty} \|h^n\|_{d_{R_2}} = 0 \).

However, with respect to the Hilbert metric we have continuity (and even differentiability) on \( \prod_{k=0}^{\infty} H^S_k \). Indeed, let \( H = ([n], E) \) be a graph without loops, and let \( h, x \in \prod_{i \in \{n\}} H^S_{d(i)} \). Let for \( i = 1, \ldots, n \), \( y^i \in \prod_{i \in \{n\}} H^S_{d(i)} \) be defined by \( y^i_j := x_i \) and \( y^i_j := h_j \) if \( i \neq j \). Then by (8.22) and by the multilinearity of \( \pi_H \),

\[
\pi_H(h + x) = \pi_H(h) + \pi_H(y^1) + \ldots + \pi_H(y^n) + o(x).
\] 

This implies that the derivative of \( \pi_H(\cdot) \) at \( h \) is the linear map \( D(\pi_H, h) : \prod_{i \in \{n\}} H^S_{d(i)} \rightarrow \mathbb{F} \) given by \( x = (x_i)_{i \in \{n\}} \mapsto \pi_H(y^1) + \ldots + \pi_H(y^n) \). One can similarly find that \( \pi_H(\cdot) \) is \( k \) times differentiable for any \( k \).
We can realize the derivative $D(\pi_H, h)$ as the image of a quantum fragment (assuming for simplicity that $\mathbb{F} = \mathbb{R}$). For a graph $H = (V, E)$, let for $v \in V$, $F_v$ be the quantum $d(v)$-fragment obtained from $H$ by deleting vertex $v$, but keeping all the edges adjacent to $v$ as open ends, and taking the sum over all possible labelings of the open ends. Then

$$D(\pi_H, h) = \left( \frac{1}{d(v)} p_h(F_v) \right)_{v \in V} \in \prod_{v \in V} \mathcal{H}^d_{d(v)},$$

where we identify a Hilbert space with its dual space.

Remark. In [59] Schrijver characterizes partition functions of (finite color) edge-coloring models over $\mathbb{R}$ using these derivatives. Perhaps they can also be used to characterize the image of the map $\pi$.

### 8.4.2 Proof of Theorem 8.4

Here we give a proof of Theorem 8.4. But first we show:

**Proposition 8.11.** Let $(X_1, \delta_1), (X_2, \delta_2), \ldots$ be complete metric spaces and let $G$ be a group acting on each $X_k$, leaving $\delta_k$ invariant ($k = 1, 2, \ldots$). Then $(\prod_{k=1}^\infty X_k) / G$ is compact if and only $(\prod_{k=1}^t X_k) / G$ is compact for each $t$.

**Proof.** Necessity being direct, we show sufficiency. We may assume that space $X_k$ has diameter at most $1/k$. Let $A := \prod_{k=1}^\infty X_k$, and let $d$ be the supremum metric on $A$ (i.e. $d(a, b) := \sup_k \delta_k(a_k, b_k)$ for $a = (a_k)$ and $b = (b_k)$). Then $d$ is $G$-invariant and $\prod_{k=1}^\infty (X_k, \delta_k)$ is $G$-homeomorphic with $(A, d)$. Indeed, a set $B_d(x, \epsilon)$ is open in $\prod_{k=1}^\infty (X_k, \delta_k)$, as it only gives open conditions for $k < 1/\epsilon$. Conversely, a basic open set $\{ x \in \prod_{i=1}^\infty X_i \mid \delta_k(x_k, z_k) < \epsilon \}$ is open in $(A, d)$, as it is equal to the union of $B_d(y, \epsilon)$ over all $y \in \prod_{i=1}^\infty X_i$ with $y_k = z_k$. So it suffices to show that $(A, d) / G$ is compact.

As each $(X, \delta_k)$ is complete, $(A, d)$ is complete. (The limit of a Cauchy sequence $(x^n)$ is the point $x \in A$, where $x_k$ is equal to the pointwise limit of the sequence $(x^n_k)$ in $X_k$, which exists since $(x^n_k)$ is a Cauchy sequence in $X_k$.) By Proposition 8.8 $(A, d) / G$ is complete. So it suffices to show that $(A, d / G)$ is totally bounded. Let $\epsilon > 0$. Set $t := \lceil \epsilon^{-1} \rceil$. Let $B := \prod_{k=1}^t X_k$ and $C := \prod_{k=t+1}^\infty X_k$, with supremum metrics $d_B$ and $d_C$ respectively. As $B / G$ is compact (by assumption), it can be covered by finitely many $d_B / G$-balls of radius $\epsilon$. As $C$ has diameter at most $1/(t + 1) \leq \epsilon$, $A = B \times C$ can be covered by finitely many $d / G$-balls of radius $\epsilon$.

This proposition allows us to prove Theorem 8.4.
8.4. Proofs of Theorem 8.3 and 8.4

**Theorem 8.4.** The space \((\prod_{k=0}^{\infty}(B_k, d_{R_k}))/O(H)\) is compact.

**Proof.** As each \((B_k, d_{R_k})\) is complete by Proposition 8.7, it suffices by Proposition 8.11 to show that for each \(t\), \((\prod_{k=0}^{t}(B_k, d_{R_k}))/O(H)\) is compact. Consider the Hilbert space \(\prod_{k=0}^{t}H_k\), and let \(W := \prod_{k=0}^{t}B_k\) and \(R := \prod_{k=0}^{t}R_k\). Then the identity function is a homeomorphism from \((W, d_R)\) to \(\prod_{k=0}^{t}(B_k, d_{R_k})\). So it suffices to show that \((W, d_R)/O(H)\) is compact. Now for each \(n\), \(R^n/O(H)\) is compact, as it is the continuous image of \(B(H_1)^m/O(H)\), with \(m := n(1 + 2 + \ldots + t)\). The latter space is compact, as it is the continuous image of the compact space \(B(\mathbb{R}^m)^m\) in case \(\mathbb{F} = \mathbb{R}\). Since \(B(l^2(C, \mathbb{C}))^m\) can be seen as a closed subset of \(B(l^2(C, \mathbb{R}))^{2m}\), the previous argument implies that also for \(\mathbb{F} = \mathbb{C}\), \(B(H_1)^m/O(H)\) is compact. (Assuming \(|C| = \infty\) in both cases, otherwise \(B(H_1)\) is itself compact). So by Theorem 8.2, \((W, d_R)/O(H)\) is compact. 

Note that the proof also shows that for any fixed \(\lambda_0, \lambda_1, \ldots \in \mathbb{F}\) the space \((\prod_{k=0}^{\infty}(\lambda_k B_k, d_{R_k}))/O(H)\) is compact.