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CAUCHY PROBLEMS RELATED TO INTEGRABLE MATRIX HIERARCHIES

G. F. Helminck*

We discuss the solvability of two Cauchy problems in matrix pseudodifferential operators. The first is associated with a set of matrix pseudodifferential operators of negative order, a prominent example being the set of strict integral operator parts of products of a solution $(L, \{U_\alpha\})$ of the $\mathbf{h}[\partial]$-hierarchy, where $\mathbf{h}$ is a maximal commutative subalgebra of $gl_n(\mathbb{C})$. We show that it can be solved in the case of compatibility completeness of the adopted setting. The second Cauchy problem is slightly more general and relates to a set of matrix pseudodifferential operators of order zero or less. The key example here is the collection of integral operator parts of the different products of a solution $\{V_\alpha\}$ of the strict $\mathbf{h}[\partial]$-hierarchy. This system is solvable if two properties hold: the Cauchy solvability in dimension n and the compatibility completeness. Both conditions are shown to hold in the formal power series setting.

Keywords: Cauchy problem, formal power series, integrable deformations, matrix pseudodifferential operators, $\mathbf{h}[\partial]$-hierarchy, strict $\mathbf{h}[\partial]$-hierarchy, zero-curvature equations

Dedicated to Leonid Olegovich Chekhov and Nikita Andreevich Slavnov
on the occasion of their 60th birthday

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1. Introduction

In [1], for each commutative subalgebra $\mathbf{h}$ of $M_n(\mathbb{C})$ of maximal dimension, we considered deformations of two commutative Lie subalgebras of the algebra MPsd of matrix pseudodifferential operators in ∂. The first deforms the Lie subalgebra $\mathbf{h}[\partial]$ and the second, wider, deformation twists the Lie subalgebra $\mathbf{h}[\partial]\partial$ of $\mathbf{h}[\partial]$ of all elements without a constant term. The evolution equations we required of the deformed generators of each Lie algebra were a set of Lax equations depending on two different decompositions of MPsd. The first deformation and its set of Lax equations were originally called the $\mathbf{h}$-hierarchy, but a better name would be the $\mathbf{h}[\partial]$-hierarchy after the commutative algebra that is deformed, and for the same reason the second deformation can be called the strict $\mathbf{h}[\partial]$-hierarchy instead of strict $\mathbf{h}$-hierarchy. The Lax form of each system implies a set of zero-curvature equations for the projections of various products of the deformed generators on one of the components of the related decomposition, a good indication that there might be linear systems for which they are the compatibility relations.

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Here, we present two Cauchy problems in MPsd. One, for an appropriate choice of the operators involved, gives zero-curvature relations of the $\mathbf{h}[\partial]$-hierarchy and the other, those of the strict $\mathbf{h}[\partial]$-hierarchy. The freedom we have in solving these systems consists of right multiplication by operators that are constant with respect to all the parameters involved. If MPsd has a suitable specialization, like putting all parameters to zero in the formal power series setting, this allows gauging the solutions of the Cauchy problems. For the systems related to the two hierarchies, this freedom does not affect the solutions of the hierarchies that can be obtained in this way. As regards the solvability of the Cauchy problems, we need compatibility completeness in the first case, and in the second case we also need Cauchy solvability in dimension n, where $n \times n$ is the size of the matrices that occur as coefficients in the operators from MPsd. Both properties hold in the formal power series context.

The content of the various sections is as follows: in Sec. 2, we recall the necessary properties of MPsd, the Lax form, and the zero-curvature form of both hierarchies. In Sec. 3, we formulate the first associated Cauchy problem, analyze the freedom it allows and show under which condition it has a solution. In Sec. 4, we discuss the second Cauchy problem and present sufficient conditions under which it can be solved. Both problems turn out to be solvable in the formal power series setting.

2. The integrable matrix hierarchies

The central algebra for the Cauchy problems under investigation in this paper is that of the matrix pseudodifferential operators MPsd. We briefly recall its main features [1]. We start with a commutative complex algebra R from which the matrix coefficients of the elements of MPsd are chosen. We then assume that the algebra R has a privileged $\mathbb{C}$-linear derivation $\partial\colon R \mapsto R$. Next, we introduce matrix differential operators in ∂. For this, as usual, we put $\partial^0 := \mathrm{Id}$ and extend the action of all ∂^i, $i \geqslant 0$, to an endomorphism of R^n by letting ∂^i act on each component of a vector in R^n. Further, we let all these ∂^i act in this fashion on each of the columns of an $n \times n$-matrix with coefficients from R, yielding a $\mathbb{C}$-linear endomorphism of the algebra $M_n(R)$ that is denoted by the same symbol ∂^i.

Each $m \in M_n(R)$ also determines, by its natural action on vectors from R^n, a $\mathbb{C}$-linear endomorphism of R^n. Thus, given R and ∂, we can form the matrix differential operators in ∂ with coefficients from $M_n(R)$. They are the collection $M_n(R)[\partial]$ of all $\mathbb{C}$-linear endomorphisms of R^n of the form $\sum_{i=0}^{n} m_i\partial^i$, $m_i \in M_n(R)$, meaning that they are the maps

$$\vec{r} \to \sum_{i=0}^{n} m_i \partial^i(\vec{r})$$

from R^n to R^n. The composition of two endomorphisms from $M_n(R)[\partial]$ is determined by the Leibnitz rule. It might be that the powers of ∂ are not $M_n(R)$-linearly independent. In that case, we can go to an extension of $M_n(R)[\partial]$ where these relations decouple, see [2]. To avoid this technicality, we introduce the following assumption.

Assumption 1. *$M_n(R)[\partial]$ acts faithfully on R^n.*

Thanks to Assumption 1, the algebra of differential operators $M_n(R)[\partial]$ can be extended to the algebra MPsd of matrix pseudodifferential operators by adding the inverses of all powers of ∂ and by allowing infinite sums of products of elements in $M_n(R)$ and negative powers of ∂. This leads to the description of MPsd as all series

$$m = \sum_{j=-\infty}^{N} m_j \partial^j, \qquad m_j \in M_n(R), \tag{1}$$

where two series $\sum_j m_j\partial^j$ and $\sum_j n_j\partial^j$ in MPsd are the same if $m_j = n_j$ for all j. In particular, we can speak of the *degree* of an element m in MPsd. If $m = \sum_{j=-\infty}^{N} m_j\partial^j$, with $m_N \neq 0$, then its degree is N

and the degree of the zero element is $-\infty$. Addition and multiplication with scalars from $\mathbb{C}$ for such series is defined coefficient wise. The composition of ∂^{-1} and $m \in M_n(R)$ in this extended algebra is given by

$$\partial^{-1} m = \sum_{s=0}^{\infty} (-1)^s \partial^s(m) \partial^{-1-s} \tag{2}$$

and the product of two general series in MPsd is a matter of repeatedly applying the multiplication rules for $M_n(R)[\partial]$ and relation (2). Thus MPsd obtains the structure of an associative algebra and a Lie algebra with respect to the commutator.

In the sequel, we use the following observation. Any other $\mathbb{C}$-linear derivation $\Delta\colon R \to R$, by acting on the coefficients of a matrix in $M_n(R)$ determines a derivation of this algebra. We use the same symbol Δ for it. Under a suitable condition, it extends to the algebra MPsd.

Lemma 2.1. *Let a $\mathbb{C}$-linear derivation $\Delta\colon R \to R$ commute with ∂. Then the following formula defines a $\mathbb{C}$-linear derivation of* MPsd:

$$\Delta\biggl(\sum_{j=-\infty}^{N} m_j \partial^j\biggr) = \sum_{j=-\infty}^{N} \Delta(m_j) \partial^j.$$

Unlike $M_n(R)[\partial]$, the algebra MPsd possesses a rich collection of invertible elements. Let $M_n(R)^*$ be the set of invertible elements in $M_n(R)$. Similarly to the scalar case, we can show the following lemma.

Lemma 2.2. *Every $M = \sum_{j \leqslant k} m_j \partial^j$ whose leading coefficient m_k belongs to $M_n(R)^*$, the collection of invertible elements of $M_n(R)$, is invertible in* MPsd.

This provides good opportunities for the dressing procedure: an element $m \in$ MPsd is said to be obtained by *dressing* another element $n \in$ MPsd by K if K is invertible in MPsd and $m = KnK^{-1}$.

Inside MPsd, we use two decompositions. First, we split any matrix pseudodifferential operator $M = \sum_j m_j \partial^j \in$ MPsd as

$$\begin{aligned} &M = \pi_{\geqslant 0}(M) + \pi_{<0}(M), \\ &\pi_{\geqslant 0}(M) = \sum_{j \geqslant 0} m_j \partial^j, \qquad \pi_{<0}(M) = \sum_{j<0} m_j \partial^j. \end{aligned} \tag{3}$$

This decomposition splits the Lie algebra MPsd into a direct sum of two Lie subalgebras:

$$\begin{aligned} &\mathrm{MPsd} = \mathrm{MPsd}_{\geqslant 0} \oplus \mathrm{MPsd}_{<0}, \\ &\mathrm{MPsd}_{\geqslant 0} = \{M \in \mathrm{MPsd} \mid M = \pi_{\geqslant 0}(M)\}, \\ &\mathrm{MPsd}_{<0} = \{M \in \mathrm{MPsd} \mid M = \pi_{<0}(M)\}. \end{aligned}$$

According to Lemma 2.2, the collection

$$\mathcal{K}_{<0} = \mathcal{K}_{<0}(R) = \biggl\{P = \mathrm{Id} + \sum_{j<0} p_j \partial^j \biggm| p_j \in M_n(R)\biggr\}$$

forms a group with respect to multiplication and we see $\mathcal{K}_{<0}$ as the group corresponding to the Lie algebra $\mathrm{MPsd}_{<0}$ in decomposition (3).

The second decomposition we use is

$$\begin{aligned} &M = \pi_{>0}(M) + \pi_{\leqslant 0}(M), \\ &\pi_{>0}(M) = \sum_{j>0} m_j \partial^j, \qquad \pi_{\leqslant 0}(M) = \sum_{j \leqslant s} m_j \partial^j. \end{aligned} \tag{4}$$

This yields another splitting of MPsd into a direct sum of two Lie subalgebras,

$$\begin{aligned} &\mathrm{MPsd} = \mathrm{MPsd}_{>0} \oplus \mathrm{MPsd}_{\leqslant 0}, \\ &\mathrm{MPsd}_{<0} = \{M \in \mathrm{MPsd} \mid M = \pi_{>0}(M)\}, \\ &\mathrm{MPsd}_{\leqslant 0} = \{M \in \mathrm{MPsd} \mid M = \pi_{\leqslant 0}(M)\}. \end{aligned}$$

Again, we have an appropriate group

$$\mathcal{K}_{\leqslant 0} = \mathcal{K}_{\leqslant 0}(R) = \left\{ M = \sum_{j \leqslant 0} m_j \partial^j \,\middle|\, m_j \in M_n(R), m_0 \in M_n(R)^* \right\}$$

corresponding to the second Lie algebra $\mathrm{MPsd}_{\leqslant 0}$ in decomposition (4).

The next step is to choose, in the first Lie algebra of each decomposition, a commutative Lie subalgebras that is deformed.

We first describe our choice inside $\mathrm{MPsd}_{\geqslant 0}$. Let $\mathbf{h}$ be a commutative complex subalgebra of $M_n(\mathbb{C})$. Then each

$$\mathbf{h}[\partial] := \left\{ \sum_{i=0}^{n} h_i \partial^i \,\middle|\, \text{all } h_i \in \mathbf{h} \right\}$$

is a commutative subalgebra of $\mathrm{MPsd}_{\geqslant 0}$. If the $\{E_\alpha \mid 1 \leqslant \alpha \leqslant r\}$ are a basis of $\mathbf{h}$, then the $\{E_\beta \partial^i \mid 1 \leqslant \beta \leqslant r,\, i \geqslant 0\}$ are a basis of $\mathbf{h}[\partial]$ and they determine the various commuting flows of the hierarchy associated with the deformation of this commutative Lie algebra. We let the index set of the basis of $\mathbf{h}[\partial]$ be denoted as

$$I_1 = \{\sigma = (\beta, i) \mid 1 \leqslant \beta \leqslant r,\, i \geqslant 0\}.$$

The $\{\partial, \{E_\alpha\}\}$ are called the basic generators of $\mathbf{h}[\partial]$. Deforming these basic generators in accordance with decomposition (3) leads to the $\mathbf{h}[\partial]$-hierarchy. To include as many commuting flows as possible, we assume from now on that the algebra $\mathbf{h}$ has maximal dimension.

We digress for a moment to discuss examples of such algebras and give estimates of their dimension.

Example 2.1. We first note that the conjugate $N\mathbf{h}N^{-1}$, $N \in GL_n(\mathbb{C})$, of a maximal commutative $\mathbf{h}$ is again maximal. The first example of a maximal commutative $\mathbf{h}$ is the collection of diagonal matrices in $M_n(\mathbb{C})$. In that case, the $\mathbf{h}[\partial]$-hierarchy is the multicomponent KP hierarchy as discussed in [3] and [4].

The next two examples have a more nilpotent character.

Example 2.2. We consider the $n \times n$ matrix B given by

$$B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}.$$

Any $A = (a_{ij})$ in the centralizer of B in $M_n(\mathbb{C})$ satisfies

$$AB = \begin{pmatrix} 0 & a_{11} & \dots & a_{1,n-1} \\ 0 & a_{21} & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1,n-1} \\ 0 & a_{n1} & \dots & a_{n,n-1} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ a_{n1} & \dots & a_{n,n-1} & a_{nn} \\ 0 & \dots & 0 & 0 \end{pmatrix} = BA.$$

These relations imply that $a_{ij} = 0$ for all $i > j$ and $a_{ij} = a_{i+1,j+1}$ for all $i \leqslant j$, $j < n$. Hence the centralizer of B equals

$$\mathbf{h} = \left\{ c_0 \operatorname{Id} + \sum_{i=1}^{n-1} c_i B^i \right\},$$

and this is a maximal commutative subalgebra of $M_n(\mathbb{C})$ of dimension n.

Example 2.3. We consider an even $n = 2m$, $m \geqslant 1$, and decompose each matrix in $M_{2m}(\mathbb{C})$ in four $m \times m$-blocks. The nilpotent algebra

$$\mathbf{h}(0) = \left\{ \begin{pmatrix} 0 & H \\ 0 & 0 \end{pmatrix} \;\middle|\; H \in M_m(\mathbb{C}) \right\}$$

is clearly commutative and, for all $H \in M_m(\mathbb{C})$, any matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in the centralizer of $\mathbf{h}(0)$ has to satisfy the relation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & H \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & AH \\ 0 & CH \end{pmatrix} = \begin{pmatrix} 0 & H \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} HC & HD \\ 0 & 0 \end{pmatrix}.$$

Because H is arbitrary, this implies that $C = 0$, $A = D$, and A commutes with all H. Hence

$$\mathbf{h} = \left\{ \begin{pmatrix} \lambda \operatorname{Id} & H \\ 0 & \lambda \operatorname{Id} \end{pmatrix} \;\middle|\; H \in M_m(\mathbb{C}) \right\}$$

is a maximal commutative subalgebra of $M_{2m}(\mathbb{C})$ of dimension $r = m^2 + 1$.

For a long period of time (see [5]), it was conjectured that n was the lower bound for r, until in 1965 Courter [6] came with the following example that refuted that claim.

Example 2.4. In $M_{14}(\mathbb{C})$, let the algebra $\mathbf{h}$ be spanned by the basis $\{E_i \mid 1 \leqslant i \leqslant 13\}$ where $E_{13} = \operatorname{Id}$ and each $\sum_{i=1}^{12} c_i E_i$ equals

$$\begin{pmatrix}
0 & 0 & c_1 & 0 & c_2 & 0 & c_3 & c_5 & 0 & c_6 & 0 & c_7 & c_9 & c_{10} \\
0 & 0 & 0 & c_1 & 0 & c_2 & c_4 & 0 & c_5 & 0 & c_6 & c_8 & c_{11} & c_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & c_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_5 & c_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Then this is a maximal commutative subalgebra.

In 1985, Laffey showed in [7] that r satisfies the inequality $r > (2n)^{2/3} - 1$. This yields the following lower bound for r: $r \geqslant \lceil (2n)^{2/3} \rceil - 1$. An upper bound for r had been found much earlier: in 1905, Schur showed in [8] that the dimension of any maximal commutative complex subalgebra $\mathbf{h}$ of $M_n(\mathbb{C})$ satisfies the inequality $r \leqslant \lfloor n^2/4 \rfloor + 1$. For even n, this upper bound of r is met by the algebra in Example 2.3. In 1944, Jacobson showed in [9] that the same estimate holds for the dimension $r(k)$ of a maximal commutative k-subalgebra $\mathbf{h}(k)$ of $M_n(k)$ for any field k. More recently (1998), Mirzhakani [10] gave a simple one-page proof of the theorem of Schur.

In the case of decomposition (4), we choose the commutative Lie subalgebras to be the intersection $\mathbf{h}[\partial]\partial$ of $\mathrm{MPsd}_{>0}$ with the Lie subalgebras $\mathbf{h}[\partial]$ used in decomposition (3):

$$\mathbf{h}[\partial]\partial := \left\{ \sum_{i=1}^{n} h_i \partial^i \,\middle|\, \text{all } \; h_i \in \mathbf{h} \right\}.$$

The basic generators of $\mathbf{h}[\partial]\partial$ are the $\{E_\alpha \partial\}$ and we let the index set of the basis $\{E_\beta \partial^i\}$ of $\mathbf{h}[\partial]\partial$ be denoted by

$$I_2 = \{\sigma = (\beta, i) \mid 1 \leqslant \beta \leqslant r, \, i \geqslant 1\}.$$

The deformation of these generators leads to the strict version of the $\mathbf{h}[\partial]$-hierarchy.

In the $\mathbf{h}[\partial]$-hierarchy, we consider deformations of the basic generators by dressing them with an element from the group $\mathcal{K}_{<0}$ corresponding to $\mathrm{MPsd}_{<0}$:

$$\begin{aligned} L &= K\partial K^{-1} = \partial + \sum_{i<0} l_i \partial^i, \\ U_\alpha &= K E_\alpha K^{-1} = E_\alpha + \sum_{i<0} u_{\alpha i} \partial^i, \quad \text{with} \quad K = \mathrm{Id} + \sum_{j<0} k_j \partial^j \in \mathcal{K}_{<0}. \end{aligned} \tag{5}$$

We call the $(L, \{U_\alpha\})$ a $\mathcal{K}_{<0}$-*deformation* of the basic generators of $\mathbf{h}[\partial]$. They generate the commutative $\mathbb{C}$-algebra $K\mathbf{h}[\partial]K^{-1}$. For each such deformation, we use the short notation $\mathcal{B}_\sigma$ or $\mathcal{B}_{\beta i}$ for the matrix differential operator $\pi_{\geqslant 0}(U_\beta L^i)$, for all $\sigma = (\beta, i) \in I_1$. We note that if $\mathrm{Id} = \sum_{\beta=1}^{r} i_\beta E_\beta$, this relation implies that the U_β satisfy the relation

$$\sum_{\beta=1}^{r} i_\beta U_\beta = \mathrm{Id}\,.$$

Because each $\mathcal{K}_{<0}$-deformation $U_\beta L^i = L^i U_\beta$ of $E_\beta \partial^i$ commutes with L and the $\{U_\alpha\}$, it follows that for all $\sigma = (\beta, i) \in I_1$, we have

$$\begin{aligned} [\mathcal{B}_\sigma, L] &= -[\pi_{<0}(U_\beta L^i), L] = [\mathcal{A}_\sigma, L], \\ [\mathcal{B}_\sigma, U_\alpha] &= -[\pi_{<0}(U_\beta L^i), U_\alpha] = [\mathcal{A}_\sigma, U_\alpha]. \end{aligned} \tag{6}$$

Here, $\mathcal{A}_\sigma$ or $\mathcal{A}_{\beta i}$ denotes the cutoff $-\pi_{<0}(U_\beta L^i)$. In particular, we have $\mathcal{B}_{\beta 0} = E_\beta$ and $\mathcal{B}_{\beta 1} = E_\beta \partial$. We note that the right-hand side of the Eqs. (6) is of negative degree in ∂ just as the operators $\Delta(L)$ and $\Delta(U_\alpha)$ for any $\mathbb{C}$-linear derivation Δ of R commuting with ∂. Hence, it makes sense to seek a set of $\mathbb{C}$-linear commuting derivations $\{\partial_\sigma = \partial_{\beta i} \mid \sigma = (\beta, i) \in I_1\}$ of R, all commuting with ∂. The data $(R, \partial, \{\partial_\sigma\})$, where σ varies over I_1, is called a *setting* for the $\mathbf{h}[\partial]$-hierarchy.

Given a setting for the $\mathbf{h}[\partial]$-hierarchy, our aim is to find $\mathcal{K}_{<0}$-deformations $(L, \{U_\alpha\})$ such that for all $\sigma \in I_1$,

$$\begin{aligned} \partial_\sigma(L) &= [\mathcal{B}_\sigma, L] = -[\pi_{<0}(U_\beta L^i), L] = [\mathcal{A}_\sigma, L], \\ \partial_\sigma(U_\alpha) &= [\mathcal{B}_\sigma, U_\alpha] = -[\pi_{<0}(U_\beta L^i), U_\alpha] = [\mathcal{A}_\sigma, U_\alpha]. \end{aligned} \tag{7}$$

Such a deformation $(L, \{U_\alpha\})$ is called a *solution* of the $\mathbf{h}[\partial]$-hierarchy in the setting under consideration and Eqs. (7) are the Lax equations of this hierarchy. It can be directly verified that $(\partial, \{E_\alpha\})$ is a solution of the $\mathbf{h}[\partial]$-hierarchy, and we call it *trivial*. We note that for all solutions $(L, \{U_\alpha\})$ of the $\mathbf{h}[\partial]$-hierarchy, the action of the derivation $\sum_{\beta=1}^r i_\beta \partial_{\beta 1}$ satisfies

$$\begin{aligned} \sum_{\beta=1}^r i_\beta \partial_{\beta 1}(L) &= \left[\sum_{\beta=1}^r i_\beta \mathcal{B}_{\beta 1}, L \right] = [\partial, L] = \partial(L), \\ \sum_{\beta=1}^r i_\beta \partial_{\beta 1}(U_\alpha) &= \left[\sum_{\beta=1}^r i_\beta \mathcal{B}_{\beta 1}, U_\alpha \right] = \partial(U_\alpha). \end{aligned} \tag{8}$$

These relations tell us that the matrix coefficients of all the coefficients of L and the U_α belong to the $\mathbb{C}$-subalgebra R_1 of R, where ∂ and $\sum_{\beta=1}^r i_\beta \partial_{\beta 1}$ are equal. We call a setting $(R, \partial, \{\partial_\sigma\})$ *standard* if and only if $\partial = \sum_{\beta=1}^r i_\beta \partial_{\beta 1}$. By replacing R by R_1, we can always assume that we work in a standard setting.

Remark 2.1. Let $(L, \{U_\alpha\})$ be a solution of the $\mathbf{h}[\partial]$-hierarchy and N an element of $GL_n(\mathbb{C})$. Straightforward verification then shows that $(NLN^{-1}, \{NU_\alpha N^{-1}\})$ is a solution of the $N\mathbf{h}N^{-1}[\partial]$-hierarchy.

Next we pass to the strict $\mathbf{h}[\partial]$-hierarchies. In that situation, we consider deformations of the basic generators of $\mathbf{h}[\partial]\partial$ by dressing them with an element from the group $\mathcal{K}_{\leqslant 0}$, i.e.,

$$V_\alpha = K E_\alpha \partial K^{-1} = \sum_{j \leqslant 1} v_j(\alpha) \partial^j, \quad \text{with} \quad K = \sum_{j \leqslant 0} k_j \partial^j, \quad k_0 \in M_n(R)^*. \tag{9}$$

Similarly to the $\mathbf{h}[\partial]$-case, this is called a $\mathcal{K}_{\leqslant 0}$-*deformation* of the basic generators of $\mathbf{h}[\partial]\partial$.

Let $M = \sum_{\alpha=1}^r i_\alpha V_\alpha$, then $M = K\partial K^{-1}$, with K as in Eq. (9). For each $\mathcal{K}_{\leqslant 0}$-deformation of the basic generators of $\mathbf{h}[\partial]\partial$ and each $\sigma = (\beta, i) \in I_2$, we let $\mathcal{C}_\sigma$ or $\mathcal{C}_{\beta i}$ denote the matrix differential operator without a constant term $\pi_{>0}(V_\beta M^{i-1})$. Because all $V_\beta M^{i-1}$, $\sigma = (\beta, i) \in I_2$, commute with the V_α, for all those i and β we have

$$[\pi_{>0}(V_\beta M^{i-1}), V_\alpha] = [\mathcal{C}_{\beta i}, V_\alpha] = -[\pi_{\leqslant 0}(V_\beta M^{i-1}), V_\alpha] = [\mathcal{D}_{\beta i}, V_\alpha], \tag{10}$$

where $\mathcal{D}_\sigma$ or $\mathcal{D}_{\beta i}$ is a shorthand notation for the projection $-\pi_{\leqslant 0}(V_\beta M^{i-1})$. The right-hand side of Eqs. (10) has a degree in ∂ equal to or smaller than one, just as the operators $\Delta(V_\alpha)$, where Δ is any $\mathbb{C}$-linear derivation of R commuting with ∂. Therefore, we seek an algebra R that has a set of $\mathbb{C}$-linear commuting derivations $\{\partial_\sigma := \partial_{\beta i} \mid \sigma = (\beta, i) \in I_2\}$ of R that all commute with ∂. Following the terminology for the $\mathbf{h}[\partial]$-hierarchy, we call the data $(R, \partial, \{\partial_\sigma\})$, where σ runs through I_2, a *setting* for the strict $\mathbf{h}[\partial]$-hierarchy.

Next, we seek the deformations $\{V_\alpha\}$ in the setting $(R, \partial, \{\partial_\sigma\})$ such that for all $\sigma = (\beta, i) \in I_2$,

$$\partial_\sigma(V_\alpha) = [\mathcal{C}_\sigma, V_\alpha] = -[\pi_{\leqslant 0}(V_\beta M^{i-1}), V_\alpha] = [\mathcal{D}_\sigma, V_\alpha]. \tag{11}$$

We call such a $\mathcal{K}_{\leqslant 0}$-deformation $\{V_\alpha\}$ of the basic generators of $\mathbf{h}[\partial]\partial$ a *solution* of the strict $\mathbf{h}[\partial]$-hierarchy in the given setting, and Eqs. (11) are the Lax equations of this hierarchy. The solution $\{E_\alpha \partial\}$ of the strict $\mathbf{h}[\partial]$-hierarchy is called the *trivial* solution. For all solutions $\{V_\alpha\}$ of the strict $\mathbf{h}[\partial]$-hierarchy, the action of the derivation $\sum_{\beta=1}^r i_\beta \partial_{\beta 1}$ on the solution satisfies

$$\sum_{\beta=1}^r i_\beta \partial_{\beta 1}(V_\alpha) = \left[\sum_{\beta=1}^r i_\beta \mathcal{C}_{\beta 1}, V_\alpha \right] = [\partial, V_\alpha] = \partial(V_\alpha). \tag{12}$$

We call a setting $(R, \partial, \{\partial_{\beta i}\})$ of the strict $\mathbf{h}[\partial]$-hierarchy *standard* if $\partial = \sum_{\beta=1}^r i_\beta \partial_{\beta 1}$ and by passing to the $\mathbb{C}$-subalgebra of R, where these two derivations are equal, we can always assume that this is the case.

Example 2.5. The standard settings for both hierarchies that we meet in the next sections are as follows. For R, in both cases, we choose the complex formal power series $\mathbb{C}[[t_\sigma]]$ in the variables $\{t_\sigma\}$ with $\sigma \in I_1$ for the $\mathbf{h}[\partial]$-hierarchy and $\sigma \in I_2$ for its strict version. For each relevant derivation $\partial_\sigma = \partial_{\beta i}$ we take $\partial/\partial t_\sigma$ and finally choose $\partial := \sum_{\beta=1}^{r} i_\beta \partial_{\beta 1}$.

Remark 2.2. In the light of Remark 2.1, the following holds in the strict case. Let $\{V_\alpha\}$ be a solution of the strict $\mathbf{h}[\partial]$-hierarchy and N an element of $GL_n(\mathbb{C})$. It can then be verified directly that $(NLN^{-1}, \{NU_\alpha N^{-1}\})$ is also a solution of the strict $N\mathbf{h}N^{-1}[\partial]$-hierarchy.

In [1], it was shown that the Lax form of the $\mathbf{h}[\partial]$-hierarchy and that of its strict version are equivalent to the following sets of zero-curvature equations, one for the projections $\{\mathcal{B}_\sigma, \sigma \in I_1\}$ and the other for the projections $\{\mathcal{C}_\sigma, \sigma \in I_2\}$.

Theorem 2.1. *The following relations hold.*

1. *Let $(L, \{U_\alpha\})$ be a $\mathcal{K}_{<0}$-deformation of the basic generators of $\mathbf{h}[\partial]$ and let $\{\mathcal{B}_\sigma, \sigma \in I_1\}$ be the associated set of projections. Then $(L, \{U_\alpha\})$ is a solution of the $\mathbf{h}[\partial]$-hierarchy if and only if the $\{\mathcal{B}_\sigma\}$ satisfy*
$$\partial_{\sigma_1}(\mathcal{B}_{\sigma_2}) - \partial_{\sigma_2}(\mathcal{B}_{\sigma_1}) - [\mathcal{B}_{\sigma_1}, \mathcal{B}_{\sigma_2}] = 0 \quad \text{for all } \sigma_1, \sigma_2 \in I_1. \tag{13}$$

2. *Let $\{V_\alpha\}$ be a $\mathcal{K}_{\leqslant 0}$-deformation of the basic generators of $\mathbf{h}[\partial]\partial$ and let $\{\mathcal{C}_\sigma, \sigma \in I_2\}$ be the associated set of projections. Then $\{V_\alpha\}$ is a solution of the strict $\mathbf{h}[\partial]$-hierarchy if and only if the $\{\mathcal{C}_\sigma\}$ satisfy*
$$\partial_{\sigma_1}(\mathcal{C}_{\sigma_2}) - \partial_{\sigma_2}(\mathcal{C}_{\sigma_1}) - [\mathcal{C}_{\sigma_1}, \mathcal{C}_{\sigma_2}] = 0 \quad \text{for all } \sigma_1, \sigma_2 \in I_2. \tag{14}$$

Because of the equivalence stated in Theorem 2.1 we call the respective equations (13) and (14) the zero-curvature form of the $\mathbf{h}[\partial]$-hierarchy and of its strict version. These equations for $\{\mathcal{B}_\sigma\}$ and $\{\mathcal{C}_\sigma\}$ imply additional equations.

Corollary 2.1. *The following statements hold.*

1. *If the $\mathcal{K}_{<0}$-deformation $(L, \{U_\alpha\})$ is a solution of the $\mathbf{h}[\partial]$-hierarchy, then the $\{\mathcal{A}_\sigma = -\pi_{<0}(U_\beta L^i) \mid \sigma = (\beta, 1) \in I_1\}$ satisfy*
$$\partial_{\sigma_1}(\mathcal{A}_{\sigma_2}) - \partial_{\sigma_2}(\mathcal{A}_{\sigma_1}) - [\mathcal{A}_{\sigma_1}, \mathcal{A}_{\sigma_2}] = 0.$$

2. *If the $\mathcal{K}_{\leqslant 0}$-deformation $\{V_\alpha\}$ is a solution of the strict $\mathbf{h}[\partial]$-hierarchy, then the $\{\mathcal{D}_\sigma = \pi_{\leqslant 0}(V_\beta M^{i-1}) \mid \sigma = (\beta, i) \in I_2\}$ satisfy*
$$\partial_{\sigma_1}(\mathcal{D}_{\sigma_2}) - \partial_{\sigma_2}(\mathcal{D}_{\sigma_1}) - [\mathcal{D}_{\sigma_1}, \mathcal{D}_{\sigma_2}] = 0.$$

Proof. We show the result for the $\mathbf{h}[\partial]$-hierarchy, the proof in the strict case is similar. We recall that the $U_\alpha L^j$ satisfy Lax equations similar to that for L and U_α: for all $\sigma \in I_1$,

$$\partial_\sigma(U_\alpha L^j) = [\mathcal{A}_\sigma, U_\alpha L^j].$$

We now substitute $\mathcal{B}_\sigma = \mathcal{A}_\sigma + U_\beta L^i$ for $\{\mathcal{B}_\sigma\}$ in the zero-curvature relations and use the above Lax equations and the fact that all the $\{U_\alpha L^i\}$ commute. This yields zero-curvature equations for the $\{\mathcal{A}_\sigma\}$. ■

3. The Cauchy problem related to the h[∂]-hierarchy

Throughout this section, $(R, \partial, \{\partial_\sigma\})$ denotes a setting for the $\mathbf{h}[\partial]$-hierarchy. For the $\mathbf{h}[\partial]$-hierarchy in MPsd, an analogue of the Sato–Wilson equations for the KP hierarchy holds (see, e.g., formula (2.3) in [11]).

Proposition 3.1. *Let $(\mathcal{L}, \{U_\alpha\})$ be a $\mathcal{K}_{<0}$-deformation of $(\partial, \{E_\alpha\})$ in* MPsd *and for each $\sigma = (\beta, i) \in I_1$, let $\mathcal{A}_\sigma$ be as in Corollary* 2.1. *If the dressing operator $K_1 \in \mathcal{K}_{<0}$ associated with $(\mathcal{L}, \{U_\alpha\})$ satisfies the equations*

$$\partial_\sigma(K_1) = \mathcal{A}_\sigma K_1, \quad \text{for all } \sigma \in I_1, \tag{15}$$

then $(\mathcal{L}, \{U_\alpha\})$ is a solution of the $\mathbf{h}[\partial]$*-hierarchy.*

We note that it makes sense to consider Eqs. (15) because both sides have a negative degree in ∂. Equations (15) are called the Sato–Wilson equations of the $\mathbf{h}[\partial]$-hierarchy.

Proof. Because the $\{E_\alpha\}$ and $\operatorname{Id}\partial$ are constant with respect to the $\{\partial_\sigma\}$, in general we have

$$\begin{aligned}
\partial_\sigma(K_1 E_\alpha K_1^{-1}) &= \partial_\sigma(K_1)K_1^{-1}K_1 E_\alpha K_1^{-1} - K_1 E_\alpha K_1^{-1}\partial_\sigma(K_1)K_1^{-1} = \\
&= [\partial_\sigma(K_1)K_1^{-1}, U_\alpha], \\
\partial_\sigma(K_1 \partial K_1^{-1}) &= \partial_\sigma(K_1)K_1^{-1}K_1 \partial K_1^{-1} - K_1 \partial K_1^{-1}\partial_\sigma(K_1)K_1^{-1} = \\
&= [\partial_\sigma(K_1)K_1^{-1}, \mathcal{L}].
\end{aligned}$$

According to (15), each $\partial_\sigma(K_1)K_1^{-1} = \mathcal{A}_\sigma$, and the substitution in both equations results in

$$\begin{aligned}
\partial_\sigma(U_\alpha) &= [\mathcal{A}_\sigma, U_\alpha] = [\mathcal{B}_\sigma - U_\beta \mathcal{L}^i, U_\alpha] = [\mathcal{B}_\sigma, U_\alpha], \\
\partial_\sigma(\mathcal{L}) &= [\mathcal{A}_\sigma, \mathcal{L}] = [\mathcal{B}_\sigma - U_\beta \mathcal{L}^i, \mathcal{L}] = [\mathcal{B}_\sigma, \mathcal{L}],
\end{aligned}$$

which proves the claim. ■

In what follows, we take a closer look at the solvability in $\mathcal{K}_{<0}$ of more general systems of equations than (15). We assume that in MPsd we have a set of matrix pseudodifferential operators $\{A_\sigma \mid \sigma \in I_1 \text{ or } \sigma \in I_2\}$ of strictly negative degree in ∂, i.e., each A_σ is

$$A_\sigma = \sum_{j>0} A_\sigma(j)\partial^{-j}. \tag{16}$$

For example, $\{A_\sigma = \mathcal{A}_\sigma \mid \sigma \in I_1\}$ with the $\{\mathcal{A}_\sigma\}$ as in Proposition 3.1. We then seek operators K in $\mathcal{K}_{<0}$,

$$K = \sum_{s \geqslant 0} K_s \partial^{-s}, \qquad K_s \in M_n(R), \quad K_0 = \operatorname{Id}, \tag{17}$$

that satisfy the set of equations

$$\partial_\sigma(K) = A_\sigma K, \quad \text{for all } \sigma \in I_1 \text{ or } \sigma \in I_2. \tag{18}$$

System (18) is a Cauchy problem in MPsd.

If $K(1)$ is another solution of system (18) for the same set of $\{A_\sigma\}$, then $K(1) = KK(0)$, where $K(0)$ is a an operator in $\mathcal{K}_{<0}$ that is constant for all the $\{\partial_\sigma\}$, i.e., $\partial_\sigma(K(0)) = 0$, for all σ. Indeed, we have

$$\partial_\sigma(K(1)) = A_\sigma K(1) = \partial_\sigma(K)K^{-1}K(1) + K\partial_\sigma(K(0)) = A_\sigma K(1) + \partial_\sigma(K(0)).$$

This implies that $K\partial_\sigma(K(0)) = 0$ and the desired identity holds because K is invertible. Conversely, for any constant $K(0)$ in $\mathcal{K}_{<0}$ and any solution K of system (18), the operator $KK(0)$ is another solution of (18). In particular, the equations

$$\partial_\sigma(K(0)^{-1}KZK^{-1}K(0)) = [K(0)^{-1}A_\sigma K(0), K(0)^{-1}KZU^{-1}U(0)], \qquad \sigma \in I_1$$

hold for the basic generators $Z = \partial$ and $Z = E_\alpha$ of $\mathbf{h}[\partial]$. Hence, the freedom in the Cauchy problem for the $\mathbf{h}[\partial]$-hierarchy corresponds to the transition of a solution of the $\mathbf{h}[\partial]$-hierarchy to a solution of the hierarchy associated with the commutative algebra $K(0)^{-1}\mathbf{h}K(0)[\partial]$.

Next, we focus on the existence of solutions of Cauchy problem (18). To prove the existence of a solution of this system, we need the algebra of matrix coefficients R to satisfy the following crucial property.

Definition 3.1. The setting $(R, \{\partial_\sigma\})$, with σ varying in one of the index sets I_1 or I_2, is said to be *compatibility complete* if for each collection $\{g(\sigma)\}$ in R that satisfies the compatibility conditions

$$\partial_{\sigma_1}(g(\sigma_2)) = \partial_{\sigma_2}(g(\sigma_1)), \tag{19}$$

for all σ_1, σ_2 from the index set under consideration, there exists a $\kappa \in R$ such that

$$\partial_\sigma(\kappa) = g(\sigma) \quad \text{for all } \sigma.$$

The setting $\left(\mathbb{C}[[t_\sigma]], \sum_{\beta=1}^{r} i_\beta \partial_{\beta 1}, \{\partial_\sigma := \frac{\partial}{\partial t_\sigma}\}\right)$ satisfies this property (see, e.g., [12]). We need the matrix version of compatibility completeness.

Lemma 3.1. *Let $(R, \partial, \{\partial_\sigma\})$ be compatibility complete, where the index set σ is either I_1 or I_2. For each collection $\{G(\sigma) \in M_n(R)\}$ such that the compatibility conditions*

$$\partial_{\sigma_1}(G(\sigma_2)) = \partial_{\sigma_2}(G(\sigma_1)) \tag{20}$$

are satisfied for all σ_1 and σ_2 from the relevant index set, there then exists $G \in M_n(R)$ such that

$$\partial_\sigma(G) = G(\sigma) \quad \text{for all } \sigma.$$

In considering the existence of solutions of system (18) in MPsd, zero-curvature relations make their appearance.

Theorem 3.1. *Let the setting $(R, \partial, \{\partial_\sigma\})$ be compatibility complete, where the index set is either I_1 or I_2, and let $\{A_\sigma \mid \sigma \in I_1 \text{ or } \sigma \in I_2\}$ be a set of matrix pseudodifferential operators of form (16). Then there is a solution $K \in \mathcal{K}_{<0}(R)$ of system (18) if and only if for all σ_1 and σ_2 from the relevant index set, the $\{A_\sigma\}$ satisfy the zero-curvature relations*

$$\partial_{\sigma_1}(A_{\sigma_2}) - \partial_{\sigma_2}(A_{\sigma_1}) - [A_{\sigma_1}, A_{\sigma_2}] = 0. \tag{21}$$

Proof. The necessity of the zero-curvature relations follows by "cross differentiation" of (18) and by using the fact that the $\{\partial_\sigma\}$ commute. On one hand, this gives

$$\partial_{\sigma_1}\partial_{\sigma_2}(K) = \partial_{\sigma_1}(A_{\sigma_2})K + A_{\sigma_1}\partial_{\sigma_2}(K) = (\partial_{\sigma_1}(A_{\sigma_2}) + A_{\sigma_1}A_{\sigma_2})K$$

and on the other hand,

$$\partial_{\sigma_2}\partial_{\sigma_1}(K) = \partial_{\sigma_2}(A_{\sigma_1})K + A_{\sigma_1}\partial_{\sigma_2}(K) = (\partial_{\sigma_2}(A_{\sigma_1}) + A_{\sigma_1}A_{\sigma_2})K.$$

Their difference yields

$$(\partial_{\sigma_1}(A_{\sigma_2}) - \partial_{\sigma_2}(A_{\sigma_1}) - [A_{\sigma_1}, A_{\sigma_2}])K = 0,$$

and because K is invertible, we obtain zero-curvature relations (21).

We now assume that zero-curvature relations (21) hold for all A_σ and seek $K = \mathrm{Id} + \sum_{j\geq 1} K_j\partial^{-j}$ such that for all $\sigma \in I_1$ or $\sigma \in I_2$ we have

$$\begin{aligned}
\partial_\sigma(K) &= \sum_{s\geqslant 1}\partial_\sigma(K_s)\partial^{-s} = A_\sigma K = \sum_{j\geqslant 1} A_\sigma(j)\partial^{-j} + \left(\sum_{j\geqslant 1} A_\sigma(j)\partial^{-j}\right)\sum_{s\geqslant 1} K_s\partial^{-s} = \\
&= \sum_{j\geqslant 1} A_\sigma(j)\partial^{-j} + \sum_{k\geqslant 0}\sum_{j\geqslant 1}\sum_{s\geqslant 1}\binom{-j}{k} A_\sigma(j)\partial^k(K_s)\partial^{-j-s-k} = \\
&= \sum_{m\geqslant 1} B_\sigma(m)\partial^{-m},
\end{aligned} \tag{22}$$

where for all σ, the $B_\sigma(m) \in M_n(R)$ are given by

$$\begin{aligned}
&B_\sigma(1) = A_\sigma(1), \\
&B_\sigma(m) = A_\sigma(m) + \sum_{1\leqslant s\leqslant m-1}\ \sum_{0\leqslant k\leqslant m-2}\binom{-m+s+k}{k} A_\sigma(m-s-k)\partial^k(K_s)
\end{aligned} \tag{23}$$

for $m \geqslant 2$. We note that in each $B_\sigma(m)$ only occur the $A_\sigma(n)$ with $n \leqslant m$ and the $\partial^k(K_s)$ with $s \leqslant m-1$ and $k \leqslant m-2$. Because we have to solve the equations

$$\partial_\sigma(K_m) = B_\sigma(m) \tag{24}$$

for all $m \geqslant 1$ and all $\sigma \in I_1$ or $\sigma \in I_2$, we can apply the following induction procedure.

First solve Eqs. (24) for all σ from the index set and $m = 1$. Next, assume that we have found $K_1, \ldots, K_m$ such that for all $s \leqslant m$, Eqs. (24) hold, and prove that we can find an $K_{m+1} \in M_n(R)$ for which Eqs. (24) with $m+1$ hold. Then $K = \mathrm{Id} + \sum_{j>0} K_j\partial^{-j}$ is a solution of system (18).

In view of (23), Eqs. (24) with $m = 1$ become

$$\partial_\sigma(K_1) = A_\sigma(1), \quad \text{for all } \sigma.$$

Because all the A_σ have a strictly negative degree in ∂, the commutator $[A_{\sigma_1}, A_{\sigma_2}]$ does not contribute to the term with ∂^{-1} in the zero-curvature relations, and equating this term to zero yields $\partial_{\sigma_2}(A_{\sigma_1}(1)) = \partial_{\sigma_1}(A_{\sigma_2}(1))$ for all σ_1 and σ_2 from the index set. Hence, by the compatibility completeness of the setting and Lemma 3.1, we know that there is a $K_1 \in M_n(R)$ that satisfies $\partial_\sigma(K_1) = A_\sigma(1)$ for all relevant σ. Having constructed K_1, we take the next step in the induction procedure.

Assume that we have found matrices $K_1, \ldots, K_m$ in $M_n(R)$ such that for all s, $1 \leqslant s \leqslant m$, Eqs. (24) hold for $K_1, \ldots, K_s$. To find a $K_{m+1} \in M_n(R)$ that satisfies $\partial_\sigma(K_{m+1}) = B_\sigma(m+1)$, it suffices to prove for all σ_1 and σ_2 from the relevant index set that

$$\partial_{\sigma_2}(B_{\sigma_1}(m+1)) = \partial_{\sigma_1}(B_{\sigma_2}(m+1)). \tag{25}$$

The same argument as in the proof of the existence of K_1 then gives the existence of K_{m+1}. We prove identities (25) by using the induction hypothesis and the zero-curvature relations for $\{A_\sigma\}$. The induction hypothesis can be formulated as an identity in MPsd. We let $K_{\geqslant -m}$ denote the operator $\mathrm{Id} + \sum_{n=1}^{m} K_n \partial^{-n}$. Then the identity

$$\partial_\sigma(K_{\geqslant -m}) = \pi_{\geqslant -m}(\pi_{\geqslant -m}(A_\sigma) K_{\geqslant 1-m}) \tag{26}$$

is equivalent to the following statement: for all K_s, $1 \leqslant s \leqslant m$, Eqs. (24) hold.

The right-hand side of the $(m+1)$-counterpart of (26) is the operator $\pi_{\geqslant -m-1}(\pi_{\geqslant -m-1}(A_\sigma) K_{\geqslant -m})$, which depends on the known matrices $A_\sigma(n)$, $n \leqslant m+1$, and $\partial^k(K_s)$ with $s \leqslant m$ and $k \leqslant m-1$. To obtain Eqs. (25), it suffices to prove compatibility for these operators, i.e., that for all σ_1 and σ_2 from the index set, we have

$$\partial_{\sigma_1}(\pi_{\geqslant -m-1}(\pi_{\geqslant -m-1}(A_{\sigma_2}) K_{\geqslant -m})) = \partial_{\sigma_2}(\pi_{\geqslant -m-1}(\pi_{\geqslant -m-1}(A_{\sigma_1}) K_{\geqslant -m})). \tag{27}$$

Using Eq. (26) and replacing $K_{\geqslant 1-m}$ with $K_{\geqslant -m}$, which does not affect the cutoff process at ∂^{-m-1}, we reduce the left-hand side of Eq. (27) to

$$\begin{aligned} &\pi_{\geqslant -m-1}(\partial_{\sigma_1}(\pi_{\geqslant -m-1}(A_{\sigma_2})) K_{\geqslant -m} + \pi_{\geqslant -m-1}(A_{\sigma_2}) \pi_{\geqslant -m}(A_{\sigma_1}) K_{\geqslant 1-m}) = \\ &= \pi_{\geqslant -m-1}([\pi_{\geqslant -m-1}(\partial_{\sigma_1}((A_{\sigma_2}))) + \pi_{\geqslant -m-1}(A_{\sigma_2}) \pi_{\geqslant -m}(A_{\sigma_1})] K_{\geqslant -m}). \end{aligned} \tag{28}$$

We handle the right-hand side of (27) in a similar way; it then becomes

$$\begin{aligned} &\pi_{\geqslant -m-1}(\pi_{\geqslant -m-1}(\partial_{\sigma_2}(A_{\sigma_1})) K_{\geqslant -m} + \pi_{\geqslant -m-1}(A_{\sigma_1}) \pi_{\geqslant -m}(A_{\sigma_2}) K_{\geqslant 1-m}) = \\ &= \pi_{\geqslant -m-1}([\pi_{\geqslant -m-1}(\partial_{\sigma_2}(A_{\sigma_1})) + \pi_{\geqslant -m-1}(A_{\sigma_1}) \pi_{\geqslant -m}(A_{\sigma_2})] K_{\geqslant -m}). \end{aligned} \tag{29}$$

We verify directly that the operators

$$\partial_{\sigma_1}((A_{\sigma_2})_{\geqslant -m-1}) + (A_{\sigma_2})_{\geqslant -m-1}(A_{\sigma_1})_{\geqslant -m}, \quad \partial_{\sigma_2}((A_{\sigma_1})_{\geqslant -m-1}) + (A_{\sigma_1})_{\geqslant -s-1}(A_{\sigma_2})_{\geqslant -m}$$

respectively differ from

$$\partial_{\sigma_1}(A_{\sigma_2}) + A_{\sigma_2} A_{\sigma_1}, \quad \partial_{\sigma_2}(A_{\sigma_1}) + A_{\sigma_1} A_{\sigma_2}$$

by a pseudodifferential operator of an order smaller than or equal to $-m-2$. Hence, the zero-curvature relations for $\{A_\sigma\}$ imply that the two cutoffs in (28) and (29) are equal at the order $-m-1$. This implies the compatibility in (25), and hence there is a K_{m+1} such that

$$\partial_\sigma(K_{\geqslant -m-1}) = \pi_{\geqslant -m-1}(\pi_{\geqslant -m-1}(A_\sigma) K_{\geqslant -m}). \tag{30}$$

This proves the theorem. ■

In the basic setting $\left(\mathbb{C}[[t_\sigma]], \sum_{\beta=1}^{r} i_\beta \partial_{\beta 1}, \left\{\frac{\partial}{\partial t_\sigma}\right\}\right)$, with the index set I_1 or I_2, the constant matrices are elements of $\mathrm{MPsd}(\mathbb{C})$, and there is an algebra map

$$S_0 \colon \mathrm{MPsd}(\mathbb{C}[[t_\sigma]]) \to \mathrm{MPsd}(\mathbb{C})$$

given by substituting $t_\sigma = 0$ for all σ. This allows the solution K of system (18), which exists due to Lemma 3.1 and Theorem 3.1, to be gauged in this setting.

Corollary 3.1. *We consider the standard setting* $\left(\mathbb{C}[[t_\sigma]], \{\frac{\partial}{\partial t_\sigma}\}\right)$, *with the index set* I_1 *or* I_2, *and assume we have a set* $\{A_\sigma\}$ *of matrices in* $\mathrm{MPsd}(\mathbb{C}[[t_\sigma]])$ *of a strictly negative degree in* ∂ *that satisfy zero-curvature equations* (21)*. Let* $K(0)$ *be a constant element in* $\mathcal{K}_{<0} \cap \mathrm{MPsd}(\mathbb{C})$*. Then Eqs.* (18) *have a unique solution* $K \in \mathrm{MPsd}(\mathbb{C}[[t_\sigma]])$ *such that* $S_0(K) = K(0)$*.*

4. The Cauchy problem related to the strict h[∂]-hierarchy

Let $(R, \partial, \{\partial_\sigma\})$ be a setting for the strict $\mathbf{h}[\partial]$-hierarchy. An analogue of Sato–Wilson equations (15) holds in MPsd for this hierarchy.

Proposition 4.1. *Let the* $\{V_\alpha\}$ *be a* $\mathcal{K}_{\leqslant 0}$*-deformation of* $\{E_\alpha \partial\}$ *in* MPsd*. If the dressing operator* $P \in \mathcal{K}_{\leqslant 0}$ *related to* $\{V_\alpha\}$ *satisfies the equations*

$$\partial_\sigma(P) = \mathcal{D}_\sigma P, \quad \text{for all } \sigma \in I_2, \tag{31}$$

then the $\{V_\alpha\}$ *are a solution of the strict* $\mathbf{h}[\partial]$*-hierarchy.*

We note that both sides of Eqs. (31) are of degree zero or lower in ∂. Equations (31) are called the Sato–Wilson equations of the strict $\mathbf{h}[\partial]$-hierarchy.

We next discuss how to generalize Eqs. (31). This time, we start with a set $\{D_\sigma \mid \sigma = (\beta, i) \in I_2\}$ of matrix pseudodifferential operators of degree zero or less in ∂. For example, $\{D_\sigma = -\pi_{\leqslant 0}(V_\beta \mathcal{M}^{i-1})\}$ with $\{V_\beta\}$ being a potential solution of the strict $\mathbf{h}[\partial]$-hierarchy. In $\mathcal{K}_{\leqslant 0}$, we seek P that satisfies the set of equations

$$\partial_\sigma(P) = D_\sigma P, \quad \text{for all } \sigma \in I_2. \tag{32}$$

If $\widetilde{P}$ is another solution of this system for the same set of $\{D_\sigma\}$, then $\widetilde{P} = PP(0)$, where $P(0)$ is an element of $LT_{\mathbb{N}}(R)$ that is constant for all ∂_σ, and hence $\partial_\sigma(P(0)) = 0$ for all $\sigma \in I_2$. The proof is the same as for the Cauchy problem in (18).

Next, we focus on the existence of solutions of Cauchy problem (32). To prove the existence of a solution of this system, besides the compatibility completeness we met before, we also need another property of the algebra R.

Definition 4.1. The setting $(R, \partial, \{\partial_\sigma, \sigma \in I_2\})$ is called *Cauchy solvable in dimension* n if for each set of matrices $\{C_\sigma \in M_n(R), \sigma \in I_2\}$ such that

$$\partial_{\sigma_1}(C_{\sigma_2}) - \partial_{\sigma_2}(C_{\sigma_1}) - [C_{\sigma_1}, C_{\sigma_2}] = 0$$

holds for all $\sigma_1, \sigma_2 \in I_2$, there is an $F \in M_n(R)^*$ (in the group of invertible elements in $M_n(R)$) such that for all $\sigma \in I_2$, we have

$$\partial_\sigma(F) = C_\sigma F. \tag{33}$$

Before presenting a proof that $\left(\mathbb{C}[[t_\sigma | \sigma \in I_2]], \{\frac{\partial}{\partial t_\sigma}\}\right)$ is an example of a standard setting that is Cauchy solvable in dimension n, we introduce some notation. The variables are $\{t_\sigma \,|\, \sigma \in I_2\}$. We use the multiindex notation for monomials in these variables: for any $\alpha = (\alpha_\sigma) \in \mathbb{Z}_{\geqslant 0}^{I_2}$ where only a finite number of α_σ are nonzero, we write

$$t^\alpha := \prod_{\sigma \in I_2} t_\sigma^{\alpha_\sigma}.$$

We introduce an order relation between these multiindices: $\alpha \leqslant \beta$ if $\alpha_\sigma \leqslant \beta_\sigma$ for all $\sigma \in I_2$, and $\alpha < \beta$ if $\alpha_\sigma < \beta_\sigma$ for one index. For simplicity, the zero index is denoted by 0. Under differentiation with respect to the t_σ, the multiindex $1(\sigma)$, $\sigma \in I_2$, occurs, which is defined by

$$1(\sigma_1)_{\sigma_2} = \begin{cases} 0 & \text{for } \sigma_1 \neq \sigma_2, \\ 1 & \text{for } \sigma_1 = \sigma_2. \end{cases}$$

The degree $\deg(\alpha)$ of the multiindex α is

$$\deg(\alpha) := \sum_{\sigma \in I_2} \alpha_\sigma.$$

Any matrix $M \in M_n(\mathbb{C}[[t_\sigma | \sigma \in I_2]])$ decomposes as

$$M = \sum_{\alpha \geqslant 0} M(\alpha) t^\alpha,$$

where each $M(\alpha)$ is an $n \times n$-matrix over $\mathbb{C}$. For the invertible elements in $M_n(\mathbb{C}[[t_\sigma]])$, we have the following lemma.

Lemma 4.1. *An element $M \in M_n(\mathbb{C}[[t_\sigma]])$ has an inverse in $M_n(\mathbb{C}[[t_\sigma]])$ if and only if $M(0)$ is invertible in $M_n(\mathbb{C})$.*

Proof. If the matrix $M = \sum_{\alpha \geqslant 0} M(\alpha) t^\alpha$ has a multiplicative inverse

$$G = \sum_{\beta \geqslant 0} G(\beta) t^\beta = G(0) + \sum_{\beta > 0} G(\beta) t^\beta,$$

then the constant term of the product satisfies $M(0)G(0) = \mathrm{Id}$, and this shows that the condition is necessary and that we can assume $M(0) = \mathrm{Id}$. Multiplying the two formal series M and G yields, for each $\gamma > 0$,

$$\sum_{0 \leqslant \alpha \leqslant \gamma} M(\alpha) G(\gamma - \alpha) = 0 \quad \Longleftrightarrow \quad G(\gamma) = -\sum_{0 < \alpha \leqslant \gamma} M(\alpha) G(\gamma - \alpha).$$

For the multiindices α of degree 1, this yields $G(\alpha) = -F(\alpha)$ and by induction on the degree of the relevant multiindices, we show that each $G(\alpha)$ with $\deg(\alpha) = m$ is a polynomial of degree utmost m in the coefficients $\{M(\beta) \mid \deg(\beta) \leqslant m\}$. ■

We can now show the following proposition,

Proposition 4.2. *The standard setting $\left(\mathbb{C}[[t_\sigma]], \left\{\frac{\partial}{\partial t_\sigma}\right\}\right)$, where $\sigma \in I_1$ or $\sigma \in I_2$, is Cauchy solvable in dimension n.*

Proof. The necessity of the zero-curvature conditions in Definition 4.1 is proved in the same way as for system (18). They are also sufficient, as we show now. We suppose we have a collection

$$\{C_\sigma \in M_n(\mathbb{C}[[t_\sigma]]), \sigma \in I_2\}, \quad \text{i.e.} \quad C_\sigma = \sum_{\alpha \geqslant 0} C_\sigma(\alpha) t^\alpha, \tag{34}$$

that satisfies the zero-curvature relations in Definition 4.1. Because of Lemma 4.1, it suffices to find an $F \in M_n(\mathbb{C}[[t_\sigma]])$ with $F(0) = \mathrm{Id}$ that satisfies Eqs. (33).

We need to show that the power series coefficients $F(\alpha)$ of F are defined uniquely by the relations

$$(\gamma_\sigma + 1)F(\gamma + 1(\sigma)) = \sum_{0 \leqslant \alpha \leqslant \gamma} C_\sigma(\alpha)F(\gamma - \alpha) \tag{35}$$

for all $\gamma \geqslant 0$. First, we take a look at the multiindices α in which only one variable occurs. Then we have the recursion

$$(k_\sigma + 1)F((k_\sigma + 1)1(\sigma)) = \sum_{0 \leqslant m \leqslant k_\sigma} C_\sigma(m1(\sigma))F((k_\sigma - m)1(\sigma)), \tag{36}$$

and this inductively determines the $F(k_\sigma 1(\sigma))$ uniquely starting from $F(0) = \mathrm{Id}$. Moreover, the power series $F(\ldots, 0, t_\sigma, 0, \ldots)$ is built such that it satisfies

$$\frac{\partial}{\partial t_\sigma}(F(\ldots, 0, t_\sigma, 0, \ldots)) = C_\sigma(\ldots, 0, t_\sigma, 0, \ldots)F(\ldots, 0, t_\sigma, 0, \ldots).$$

The idea is now to use induction on m, the number of variables actually present in the multiindex α, i.e., the number of elements in $\{\sigma \,|\, \alpha_\sigma > 0\}$. We therefore assume that any coefficient $F(\beta)$ corresponding to m or less variables is well defined and consider a coefficient $F(\alpha)$ in which $m+1$ variables occur. We can assume that

$$\alpha = \sum_{k=1}^{m+1} \alpha_{\sigma_k} 1(\sigma_k),$$

otherwise we can reduce to this case by rearranging. It suffices to show that the compatibility conditions allow introducing a well-defined power series

$$F(t_{\sigma_1}, \ldots, t_{\sigma_{m+1}}, 0, \ldots)$$

that satisfies the equations

$$\partial_{\sigma_k}(F(t_{\sigma_1}, \ldots, t_{\sigma_{m+1}}, 0, \ldots)) = C_{\sigma_k}(t_{\sigma_1}, \ldots, t_{\sigma_{m+1}}, 0, \ldots)F(t_{\sigma_1}, \ldots, t_{\sigma_{m+1}}, 0, \ldots)$$

for all $k = 1, \ldots, m+1$. To define the coefficients $F(\alpha)$, $\alpha_{\sigma_k} > 0$ for all $k = 1, \ldots, m+1$, we have $n+1$ possibilities each corresponding to the variable t_{σ_k} for which we use the recursion

$$\alpha_{\sigma_k} F(\alpha) = \sum_{0 \leqslant \gamma \leqslant \alpha - 1(\sigma_k)} C_{\sigma_k}(\gamma)F(\alpha - 1(\sigma_k) - \gamma). \tag{37}$$

Each choice gives a power series $F_k(t_{\sigma_1}, \ldots, t_{\sigma_{m+1}}, 0, \ldots)$. We note that each F_k is constructed such that with respect to t_{σ_k}, it satisfies the equation

$$\partial_{\sigma_k}(F_k) = C_{\sigma_k}(t_{\sigma_1}, \ldots, t_{\sigma_{m+1}}, 0, \ldots)F_k. \tag{38}$$

Next, we show that $F_k = F_{m+1}$. We note that the two series agree on terms with m variables or less, and hence they agree in particular in the terms with only one variable, and therefore it suffices to prove the equality

$$\partial_{\sigma_{m+1}}\partial_{\sigma_k}(F_k) - \partial_{\sigma_k}\partial_{\sigma_{m+1}}(F_{m+1}) = 0.$$

In the left-hand side, we use Eqs. (38) and substitute the zero-curvature relations, which gives

$$\begin{aligned}
&\partial_{\sigma_{m+1}}(C_{\sigma_k})F_k + C_{\sigma_k}\partial_{\sigma_{m+1}}(F_k) - \partial_{\sigma_k}(C_{\sigma_{m+1}})F_k - C_{\sigma_{m+1}}\partial_{\sigma_k}(F_{m+1}) = \\
&= \partial_{\sigma_{m+1}}(C_{\sigma_k})F_{m+1} + C_{\sigma_k}\partial_{\sigma_{m+1}}(F_{m+1}) + \partial_{\sigma_{m+1}}(C_{\sigma_k})(F_k - F_{m+1}) + \\
&\quad + C_{\sigma_k}\partial_{\sigma_{m+1}}(F_k - F_{m+1}) - \partial_{\sigma_k}(C_{\sigma_{m+1}})F_{m+1} - C_{\sigma_{m+1}}\partial_{\sigma_k}(F_k) - \\
&\quad - \partial_{\sigma_k}(C_{\sigma_{m+1}})(F_k - F_{m+1}) - C_{\sigma_{m+1}}\partial_{\sigma_k}(F_{n+1} - F_k) = \\
&= \partial_{\sigma_{m+1}}(C_{\sigma_k})F_{m+1} + C_{\sigma_k}C_{\sigma_{m+1}}F_{m+1} + \partial_{\sigma_{m+1}}(C_{\sigma_k})(F_k - F_{m+1}) + \\
&\quad + C_{\sigma_k}\partial_{\sigma_{m+1}}(F_k - F_{m+1}) - \partial_{\sigma_k}(C_{\sigma_{m+1}})F_{m+1} - C_{\sigma_{m+1}}C_{\sigma_k}F_k - \\
&\quad - \partial_{\sigma_k}(C_{\sigma_{m+1}})(F_k - F_{m+1}) - C_{\sigma_{m+1}}\partial_{\sigma_k}(F_{m+1} - F_k) = \\
&= \partial_{\sigma_{m+1}}(C_{\sigma_k})(F_k - F_{m+1}) + C_{\sigma_k}\partial_{\sigma_{m+1}}(F_k - F_{m+1}) - \\
&\quad - \partial_{\sigma_k}(C_{\sigma_{m+1}})(F_k - F_{m+1}) - C_{\sigma_{m+1}}\partial_{\sigma_k}(F_{m+1} - F_k).
\end{aligned}$$

We assume that $F_k - F_{m+1}$ is nonzero and has a nontrivial coefficient for a multiindex γ; then $\partial_{\sigma_{m+1}}\partial_{\sigma_k}(F_k) - \partial_{\sigma_k}\partial_{\sigma_{m+1}}(F_{m+1})$ has a nontrivial coefficient for the multiindex $\gamma - 1(\sigma_k) - 1(\sigma_{m+1})$. If we choose γ to be one of the lowest multiindices of $F_k - F_{m+1}$ for which there is a nonzero coefficient, then this contradicts the fact that $F_k - F_{m+1}$, $\partial_{\sigma_k}(F_{m+1} - F_k)$ and $\partial_{\sigma_{m+1}}(F_{m+1} - F_k)$ do not have such coefficients. Hence, $F_k - F_{m+1}$ is zero. This proves the claim in the proposition. ■

Theorem 4.1. *Let the setting $(R, \partial, \{\partial_\sigma\}, \sigma \in I_2)$ be compatibility complete and Cauchy-solvable in dimension n. Then there is a solution $P \in \mathcal{K}_{\leqslant 0}(R)$ of system (32) if and only if the $\{D_\sigma\}$ satisfy the zero-curvature relations*

$$\partial_{\sigma_1}(D_{\sigma_2}) - \partial_{\sigma_2}(D_{\sigma_1}) - [D_{\sigma_1}, D_{\sigma_2}] = 0. \tag{39}$$

Proof. Because the operators P in system (32) are invertible, the proof of the necessity of the zero-curvature relations is the same as that for system (18). It remains to show the sufficiency.

We assume that zero-curvature relations (39) hold for the $\{D_\sigma\}$. The D_σ have the form

$$D_\sigma = \sum_{j\geqslant 0} d_j^{(\sigma)}\partial^{-j}, \qquad d_j^{(\sigma)} \in M_n(R) \text{ for all } j \geqslant 0.$$

We find an operator $P = \sum_{j\geqslant 0} P_j\partial^{-j}$ with all $P_j \in M_n(R)$ and $P_0 \in M_n(R)^*$ that satisfies system (32). We find such a P in two steps. We write $P = P_0\widetilde{P}$ with $\widetilde{P} \in \mathcal{K}_{<0}$ and determine the equations that P_0 and $\widetilde{P}$ have to satisfy.

We start by finding the equations for P_0. Comparing the constant terms in Eqs. (32) yields

$$\partial_\sigma(P_0) = d_0^{(\sigma)}P_0, \qquad \sigma \in I_2. \tag{40}$$

Zero-curvature relations (39) for $\{D_\sigma\}$ imply, for the coefficient of ∂^0, zero-curvature relations for the $\{d_0^{(\sigma)}\}$. Because the setting is Cauchy solvable in dimension n, all equations (40) can be solved and yield a $P_0 \in M_n(R)^*$ satisfying Eqs. (40). Assuming that P satisfies Eqs. (32), we obtain that $\widetilde{P}$ necessarily satisfies the following Cauchy problem: for all $\sigma \in I_2$,

$$\partial_\sigma(\widetilde{P}) = (P_0^{-1}D_\sigma P_0 - P_0^{-1}\partial_\sigma(P_0))\widetilde{P} =: \widehat{D}_\sigma\widetilde{P}. \tag{41}$$

We see from Eqs. (40) that all $\widehat{D}_\sigma \in$ MPsd are elements of a strictly negative order in ∂. Hence, if we can show that the zero-curvature relations hold for all $\widehat{D}_\sigma$, then we can apply Theorem 3.1 to Cauchy problem (41) deduce the existence of a solution $\widetilde{P}$ of this system in $\mathcal{K}_{<0}$.

First of all, we have

$$\begin{aligned}\partial_{\sigma_1}(\widehat{D}_{\sigma_2}) &= \partial_{\sigma_1}(P_0^{-1})D_{\sigma_2}P_0 + P_0^{-1}\partial_{\sigma_1}(D_{\sigma_2})P_0 + P_0^{-1}D_{\sigma_2}\partial_{\sigma_1}(P_0) - \partial_{\sigma_1}(P_0^{-1}\partial_{\sigma_2}(P_0)) = \\ &= -P_0^{-1}\partial_{\sigma_1}(P_0)P_0^{-1}D_{\sigma_2}P_0 + P_0^{-1}\partial_{\sigma_1}(D_{\sigma_2})P_0 + P_0^{-1}D_{\sigma_2}\partial_{\sigma_1}(P_0) + \\ &\quad + P_0^{-1}\partial_{\sigma_1}(P_0)P_0^{-1}\partial_{\sigma_2}(P_0) - P_0^{-1}\partial_{\sigma_1}\partial_{\sigma_2}(P_0)\end{aligned}$$

and similarly,

$$\begin{aligned}\partial_{\sigma_2}(\widehat{D}_{\sigma_1}) &= \partial_{\sigma_2}(P_0^{-1})D_{\sigma_1}P_0 + P_0^{-1}\partial_{\sigma_2}(D_{\sigma_1})P_0 + P_0^{-1}D_{\sigma_1}\partial_{\sigma_2}(P_0) - \partial_{\sigma_2}(P_0^{-1}\partial_{\sigma_1}(P_0)) = \\ &= -P_0^{-1}\partial_{\sigma_2}(P_0)P_0^{-1}D_{\sigma_1}P_0 + P_0^{-1}\partial_{\sigma_2}(D_{\sigma_1})P_0 + P_0^{-1}D_{\sigma_1}\partial_{\sigma_2}(P_0) + \\ &\quad + P_0^{-1}\partial_{\sigma_2}(P_0)P_0^{-1}\partial_{\sigma_1}(P_0) - P_0^{-1}\partial_{\sigma_2}\partial_{\sigma_1}(P_0).\end{aligned}$$

Further, the commutator $[\widehat{D}_{\sigma_2}, \widehat{D}_{\sigma_1}]$ is equal to

$$P_0^{-1}[D_{\sigma_2}, D_{\sigma_1}]P_0 - [P_0^{-1}\partial_{\sigma_2}(P_0), P_0^{-1}D_{\sigma_1}P_0] - [P_0^{-1}D_{\sigma_2}P_0, P_0^{-1}\partial_{\sigma}(P_0)].$$

We have

$$\begin{aligned}-[P_0^{-1}\partial_{\sigma_2}(P_0), P_0^{-1}D_{\sigma_1}P_0] &= -P_0^{-1}\partial_{\sigma_2}(P_0)P_0^{-1}D_{\sigma_1}P_0 + P_0^{-1}D_{\sigma_1}\partial_{\sigma_2}(P_0), \\ -[P_0^{-1}D_{\sigma_2}P_0, P_0^{-1}\partial_{\sigma_1}(P_0)] &= -P_0^{-1}D_{\sigma_2}\partial_{\sigma_1}(P_0) + P_0^{-1}\partial_{\sigma_1}(P_0)P_0^{-1}D_{\sigma_2}P_0,\end{aligned}$$

and therefore all these relations combine into the sought zero-curvature equations

$$\partial_{\sigma_1}(\widehat{D}_{\sigma_2}) - \partial_{\sigma_2}(\widehat{D}_{\sigma_1}) - [\widehat{D}_{\sigma_1}, \widehat{D}_{\sigma_2}] = 0,$$

Hence, there exists a solution $\widetilde{P}$ of system (41). It is now straightforward to verify that the operator $P = P_0\widetilde{P}$ is a solution of system (32). ■

We recall that in the basic standard setting $\left(\mathbb{C}[[t_\sigma]], \left\{\frac{\partial}{\partial t_\sigma}\right\}\right)$, we have the possibility to put all variables t_σ equal to zero, which defines the substitution map $S_0 \colon \mathrm{MPsd}(\mathbb{C}[[t_\sigma]]) \to \mathrm{MPsd}(\mathbb{C})$. In this setting, it is possible to gauge the solutions P of system (32) given by Theorem 4.1.

Corollary 4.1. *Let $\{D_\sigma, \sigma \in I_2\}$ in the basic standard setting be a set of elements in* MPsd *of degree zero or less that satisfy zero-curvature equations* (39), *and let*

$$P(0) = \sum_{j \geqslant 0} p_j \partial^{-j}, \qquad p_j \in M_n(\mathbb{C}), \quad p_0 \in Gl_n(\mathbb{C}),$$

be any element of $\mathcal{K}_{\leqslant 0}(\mathbb{C})$. Then Eqs. (32) *have a unique solution $P \in \mathcal{K}_{\leqslant 0}(\mathbb{C}[[t_\sigma]])$ that satisfies the condition $S_0(P) = P(0)$.*

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