Quantum entanglement in non-local games, graph parameters and zero-error information theory
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Chapter 3
Quantum Graph Parameters

This chapter is mainly based on three papers. (Section 3.5 is based on unpublished work.) The first paper is “Kochen-Specker Sets and the Rank-1 Quantum Chromatic Number”, by the author and S. Severini. The paper was presented as a poster at the Quantum Information Processing conference in December 2011, and published in the IEEE Transactions on Information Theory in April 2012.

The second paper is “New Separations in Zero-error Channel Capacity through Projective Kochen-Specker Sets and Quantum Coloring”, by L. Mančinska, the author and S. Severini. The paper was presented at the Asian Quantum Information Science conference and as an invited talk at the Workshop on Quantum Physics of Information in August 2012. It was then presented as a poster at the Quantum Information Processing conference in January 2013. The paper was published in the IEEE Transactions on Information Theory in June 2013.

The third paper is “Exclusivity structures and graph representatives of local complementation orbits”, by A. Cabello, M. G. Parker, the author and S. Severini. The paper was published in the Journal of Mathematical Physics in July 2013.

3.1 Introduction

The chromatic number and the independence number are important and well-studied graph parameters.

The notion of quantum chromatic number was described in its generality by Cameron et al. [CMN+07] in 2007, but it has been studied in the context of quantum non-locality since the late ’90s (see the seminal work by Brassard, Cleve, and Tapp [BCT99]; see also the recent survey by Galliard, Wolf and Tapp [GTW10] and the references therein). It also appears implicitly in works on communication complexity (see, for example, [BCW98] and [dW01, pages 148–150]).

The notion of quantum independence number was implicitly present in many works about zero-error information theory, because of its link with channel capacities (see, for example, [CLMW10, LMM+12, BCGSM]). We worked on the
quantum independence number in the same framework in [MSS13]. In the meanwhile, a new definition related to graph homomorphisms appeared in [RM12]. We will use such definition because it allows us to better express the quantum independence number as a graph parameter.

The value of these notions is at least twofold: first, they have a natural use as tools for isolating the difference between quantum and classical behavior, second, they are a new approach for studying many combinatorial parameters between well-known NP-hard quantities like the clique and the chromatic number (e.g., the Lovász $\vartheta$-function, the orthogonal rank, etc.).

The rest of the chapter is structured as follows. In Section 3.2 we give some definitions that will be useful later.

In Section 3.3 we focus on the quantum chromatic number. We give a formal definition in terms of a non-local game. We reprove a result from [CMN+07] concerning the form of the strategies for the above non-local game. Then, we prove some of the basic properties of the quantum chromatic number, including the relation with other common graph-theoretic parameters.

In Section 3.3.1, we focus on the rank-1 quantum chromatic number. This quantity is obtained by using only rank-1 measurement operators in the quantum strategies for the above-mentioned non-local game. We prove that the rank-1 quantum chromatic number is equal to orthogonal rank of a particular Cartesian product graph. Then, we exhibit graphs where the rank-1 quantum chromatic number is strictly greater than the orthogonal rank thereby solving an open problem stated in [CMN+07]. The proof technique is not based on a specific example, but on a general result which connects rank-1 quantum chromatic number and Kochen-Specker sets. These are collections of vectors originally used to prove the inadequacy of local hidden variable theories to model quantum mechanical behavior deterministically [KS67, PMMM05].

In Section 3.3.2, we use our newly-defined notion of projective Kochen-Specker set, to show that there is a separation between quantum and classical chromatic number. The characterization settles the graph-theoretic discussion started in [CMN+07]. A characterization for the rank-1 case was already given in [SS12] but it is subsumed by this result. Interestingly, [FIG11] observed a separation between rank-1 and general rank quantum chromatic number. From their result it follows that the use of projective KS sets is necessary. This full characterization is valuable because until now the only examples of the separation were some orthogonality graphs, specifically the Hadamard graphs considered by Avis et al. [AHKS06] and introduced in [FR87], and an isolated example with 18 vertices [CMN+07].

In Section 3.4 we focus on the quantum independence number. We give the definition as a graph parameter related to non-local games. We will see in Chapter 4 how this definition relates to zero-error channel capacity. Our main contribution about the quantum independence number is to exhibit three different constructions of graphs with a separation between quantum and classical inde-
3.2 Preliminaries

3.2.1 Notions of graph theory

A simple graph $G = (V, E)$ consists of a finite vertex set $V$ and its edge set $E \subseteq V \times V$ (the inclusion here is strict because there are no edges of the form $(v, v)$). Two vertices $(v, w) \in E$ are “adjacent” or equivalently “form an edge”.

All graphs considered in this chapter, unless otherwise specified, are simple graphs. For a graph $G = (V, E)$, we also denote its vertex set with $V(G)$ and its edge set with $E(G)$ whenever confusion has to be avoided.

A proper $c$-coloring of a graph is an assignment of $c$ colors to the vertices of the graph such that every two adjacent vertices have different colors. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimum number of colors $c$ such that there exists a proper $c$-coloring of $G$.

An independent set of a graph is a subset $I$ of $V(G)$ such that no two elements of $I$ are adjacent. The independence number of a graph $G$, denoted by $\alpha(G)$, is the maximum size of an independent set of $G$. A little thought shows that $\alpha(G) \cdot \chi(G) \geq |V(G)|$.

The complement of $G$ is $\overline{G}$, the graph with vertex set $V(G)$ where distinct vertices are adjacent if and only if they are not adjacent in $G$. A clique is a subset of vertices in which each pair is adjacent and the clique number $\omega(G)$ is the maximum cardinality of a clique in $G$. Clearly $\alpha(G) = \omega(\overline{G})$.

A homomorphism from a graph $G$ to a graph $H$ is a map $\phi : V(G) \to V(H)$ such that every edge $\{u, v\} \in E(G)$ in $G$ is mapped to an edge $\{\phi(u), \phi(v)\} \in E(H)$ in $H$. If such a map exists, we write $G \to H$.

Since simple graphs are non-directed, it is equivalent to address the edges as pairs $(u, v)$ or sets $\{u, v\}$. We will use both notations depending on the context.
A \(d\)-dimensional orthogonal representation of \(G = (V, E)\) is a map \(\phi : V \to \mathbb{C}^d\) such that for all \((v, w) \in E\), \(\langle \phi(v) | \phi(w) \rangle = 0\). (If all the vectors have unit norm, this is called orthonormal representation.) The \textit{orthogonal rank} of a graph \(G\), denoted by \(\xi(G)\), is defined as the minimum \(d\) such that there exists an orthogonal representation of \(G\) in \(\mathbb{C}^d\).

We relate every multiset of projectors \(T\) to a graph. The \textit{orthogonality graph} of \(T\) is the graph with vertex set \(T\) and edge set \(\{(P, P') : \text{Tr}(PP') = 0\}\). Let us denote by \(\uplus\) the multiset union. Some authors construct orthogonality graphs from a set of vectors. This is just a special case of the above definition: associate to each vector \(v\) the rank-1 projector \(|v\rangle\langle v|\).

For all pairs of graphs \(G\) and \(H\), define their \textit{Cartesian product} \(G \square H\) as follows. The vertex set \(V(G \square H) = V(G) \times V(H)\) is the Cartesian product of the vertex sets of \(G\) and \(H\). We can therefore identify each vertex in \(V(G \square H)\) with a pair of vertices, one from each of the two original graphs. There is an edge in \(E(G \square H)\) between vertices \((v, i)\) and \((w, j)\) if either \(v = w\) and \((i, j) \in E(H)\) or \((v, w) \in E(G)\) and \(i = j\).

\textbf{Lemma 3.2.1.} \cite{Viz63} For all graphs \(G, H\), the independence number of their Cartesian product satisfies

\[
\alpha(G \square H) \leq \min\{\alpha(G) \cdot |V(H)|, \alpha(H) \cdot |V(G)|\}.
\]

\textit{Proof.} Assume w.l.o.g. that \(\alpha(G) \cdot |V(H)| \leq \alpha(H) \cdot |V(G)|\). Suppose, towards a contradiction, that \(\alpha(G \square H) > \alpha(G) \cdot |V(H)|\). Then, there is a \(i \in V(H)\) such that there are more than \(\alpha(G)\) non-adjacent vertices of \(G \square H\) of the form \((*, i)\). But this implies the existence of an independent set of \(G\) of size larger than \(\alpha(G)\), because there is an edge between \((v, i)\) and \((w, i)\) whenever \((v, w) \in E(G)\).

For all pairs of graphs \(G\) and \(H\), the \textit{strong product} \(G \boxtimes H\) is the graph whose vertex set is the cartesian product \(V(G) \times V(H)\) and where two distinct vertices \((v, i)\) and \((w, j)\) are adjacent if and only if \((v, w) \in E(G)\) or \(v = w\) and \((i, j) \in E(H)\) or \(i = j\).

We finally introduce an important graph parameter: the \textit{theta number} (a.k.a. Lovász number or theta function). It was originally defined by Lovász \cite{Lov79} to solve a long-standing problem posed by Shannon \cite{Sha56}: computing the Shannon capacity of the five-cycle. There are many equivalent formulations of the theta number (see \cite{KD93} for a detailed survey). In this thesis we use two of them. The one that we use in this chapter is the following:

\[
\vartheta(G) = \max \sum_{v \in V(G)} |\langle \psi | \psi_v \rangle|^2,
\]

where the max is over unit vectors \(\psi\) and orthonormal representations \(\{\psi_v\}_{v \in V(G)}\).
3.2. Preliminaries

The second one is an SDP formulation that we use in Chapter 4.

\[ \vartheta(G) = \min \left\{ \lambda : \exists Z \in \mathbb{R}^{V(G) \times V(G)}, \quad Z \succeq 0, \right. \]
\[ \left. Z(u, u) = \lambda - 1 \text{ for } u \in V(G), \quad Z(u, v) = -1 \text{ for } \{u, v\} \notin E(G) \right\} \]  

(3.2)

Lovász [Lov79] proved that \( \alpha(G) \leq \vartheta(G) \leq \chi(G) \) holds (this inequality is often referred to as the sandwich theorem [KD93]). The theta number can be approximated to within arbitrary precision in polynomial time, hence it gives a tractable and in many cases useful bound for both \( \alpha \) and \( \chi \). Lovász proved that \( \vartheta \) is multiplicative under the strong graph product, that is,

\[ \vartheta(G\boxtimes H) = \vartheta(G)\vartheta(H). \]  

(3.3)

### 3.2.2 Kochen-Specker sets

Consider a set of \( n \)-dimensional (complex) vectors \( S \subseteq \mathbb{C}^n \).

**Definition 3.2.2.** A function \( f : S \rightarrow \{0, 1\} \) is a marking function for \( S \) if for all orthonormal bases \( b \subseteq S \) we have \( \sum_{u \in b} f(u) = 1 \).

Gleason’s theorem [Gle57] implies that for any \( n \geq 3 \) there does not exist a marking function for \( \mathbb{C}^n \). Bell [Bel66] and independently Kochen and Specker [KS67] interpreted this statement in the framework of contextuality of physical theories. For this reason, this statement is also known as the (Bell-)Kochen-Specker theorem. Since then, finite sets of vectors in some given dimension giving rise to a proof of this theorem are known as Kochen-Specker (KS) sets. Note that although in principle there are KS sets of infinite size, in this thesis we are only interested in finite sets, since we will use them as a tool to work on finite graphs. In general, much importance is given to finding the smallest possible KS set (see [AOW11]).

**Definition 3.2.3 (KS set).** A set of unit vectors \( S \subseteq \mathbb{C}^n \) is a Kochen-Specker set if there is no marking function for \( S \).

Renner and Wolf [RW04] considered a generalization of KS sets called weak KS sets. Intuitively, for a weak KS set there can be marking functions, but every such function evaluates to 1 for two orthogonal vectors in the set.

**Definition 3.2.4 (weak KS set).** A set of unit vectors \( S \subseteq \mathbb{C}^n \) is a weak Kochen-Specker set if for all marking functions \( f \) for \( S \) there exist orthogonal vectors \( u, v \in S \) such that \( f(u) = f(v) = 1 \).

As explained in [RW04], every KS set is clearly a weak KS set but the converse does not always hold. However, every weak KS set can be completed to a KS
set by adding $O(|S|^2)$ elements. Hence, a weak KS set also gives a proof of the Kochen-Specker theorem in some specific dimension. In fact it is more convenient to deal with weak KS sets, since they capture the essence of KS sets and can contain fewer vectors.

**Generalizations of KS sets** We introduce a natural generalization of weak KS sets by considering subsets of $Q_n$, the set of all $n \times n$ projectors, instead of subsets of $C^n$. Recall that an orthogonal projector is a Hermitian matrix $P$ such that $P^2 = P$. For brevity, from now on we omit the word “orthogonal” when we talk about projectors.

Let $M_n$ be the set of $n \times n$ matrices.

**Definition 3.2.5.** A marking function $f$ for $S \subseteq M_n$ is a function $f : S \to \{0, 1\}$ such that for all $M \subseteq S$ with $\sum_{P \in M} P = I$, we have $\sum_{P \in M} f(P) = 1$.

**Definition 3.2.6 (Projective KS set).** A set $S \subseteq Q_n$ is a projective Kochen-Specker set if for all marking functions $f$ for $S$, there exist $P, P' \in S$ for which $\text{Tr}(PP') = 0$ and $f(P) = f(P') = 1$.

Note that each set $M \subseteq Q_n$ with $\sum_{P \in M} P = I$ is a projective measurement. Also, note that weak KS sets are a special case of projective KS sets, if we identify a vector with the corresponding rank-1 projector. Conversely, starting from any projective KS set one can construct (usually infinitely many) underlying weak KS sets. For example, one can take for each projector an arbitrary orthonormal basis of its span. It can be verified that union of all the bases is a weak KS set (see [MSS13, Appendix A]).

Although in the rest of the thesis we will only deal with projective KS sets, we can further generalize weak KS sets by considering subsets of $S_n^+$, the set of all $n \times n$ positive semidefinite matrices.

**Definition 3.2.7 (Generalized KS set).** A set $S \subseteq S_n^+$ is a generalized Kochen-Specker set if for all marking functions $f$ for $S$ there exist $E, E' \in S$ for which $E + E' \leq I$ and $f(E) = f(E') = 1$.

Note that each set $M \subseteq S_n^+$ with $\sum_{E \in M} E = I$ is a POVM (where, as usual, “POVM” stands for positive operator-valued measure). Projective KS sets are a special case of generalized KS sets, because when $S$ is a set of projectors the condition $E + E' \leq I$ is equivalent to $\text{Tr}(EE') = 0$.

KS-like sets consisting of positive semidefinite matrices have already been considered by Cabello [Cab03]. Motivated by a recent analogue of Gleason’s theorem for positive semidefinite operators in two dimensions [Bus03, CFMR04], Cabello exhibits what we here call a generalized KS set in $S_2^+$. Hence, generalized KS sets exist even in two dimensions and have turned out to be useful for scenarios where regular KS sets do not apply (recall that there are no KS sets in $C^2$).
KS sets and pseudo-telepathy games Here we generalize the results of [RW04] concerning the relationship between weak KS sets and a class of pseudo-telepathy games. We show that there is a relationship between projective KS sets and a class of pseudo-telepathy games that is larger than the one above.

Informally, a non-local game is called a pseudo-telepathy game if players sharing the entangled state win with certainty, while classical players have non-zero probability to lose. More formally:

**Definition 3.2.8 (Pseudo-telepathy game).** A non-local game with input sets \(X, Y\), output sets \(A, B\), input distribution \(\pi\) and verification function \(V\) is called a pseudo-telepathy game if:

1. There exists a quantum strategy consisting of a shared bipartite entangled state \(|\psi\rangle\) and POVMs \(\{E^x_a\}_{a \in A}\) for every \(x \in X\) for Alice, and \(\{F^y_b\}_{b \in B}\) for every \(y \in Y\) for Bob, with the following property. For any \((a, b, x, y)\) with \(\pi(x, y) > 0\) it holds that \(\langle \psi | E^x_a \otimes F^y_b | \psi \rangle \neq 0\) implies \(V(a, b, x, y) = 1\).

2. For all deterministic classical strategies \(s_A, s_B\), there exists a tuple \((a, b, x, y)\) with \(\pi(x, y) > 0\) such that \(V(a, b, x, y) = 0\) but \(s_A(x) = a\) and \(s_B(y) = b\).

The following theorem relates projective KS sets and a special kind of pseudo-telepathy games. We will only consider KS sets for which each projector is part of some projective measurement from \(S\). Note that we can delete elements from any projective KS set until the resulting set is still a projective KS set and satisfies the above property. Therefore, projectors not contained in any measurement from \(S\) are inessential.

For all natural numbers \(n\), let \(|\psi_n\rangle = \frac{1}{\sqrt{n}} \sum_{i \in [n]} |i\rangle |i\rangle\), where \(\{|i\rangle\}_{i \in [n]}\) is the standard basis of \(\mathbb{C}^n\). Let \(\overline{P}\) be the entry-wise complex conjugate of \(P\). A useful property of \(|\psi_n\rangle\) is the following. For all \(A, B \in \mathbb{C}^{n \times n}\),

\[
\langle \psi_n | A \otimes B | \psi_n \rangle = \sum_{ij} (ii | A \otimes B | jj) = \sum_{ij} (i | A | j) (i | B | j) = \sum_{ij} A_{ij} B_{ij} = \text{Tr}(A B)
\]

(3.4)

**Theorem 3.2.9.** Let \(n\) be any natural number and let \(X, Y, A, B\) be sets. Consider projective measurements \(\{P^x_a\}_{a \in A}\) for every \(x \in X\), and \(\{Q^y_b\}_{b \in B}\) for every \(y \in Y\), acting on \(\mathbb{C}^n\). The following statements are equivalent:

1. \(S = \{P^x_a\}_{(x,a)} \cup \{\overline{Q^y_b}\}_{(y,b)}\) is a projective KS set.

2. There exists a pseudo-telepathy game with input sets \(X, Y\) and output set \(A, B\) for which there is a winning quantum strategy consisting of a shared state \(|\psi_n\rangle\) and projective measurements \(\{P^x_a\}_{a \in A}\) for every \(x \in X\) for Alice, \(\{Q^y_b\}_{b \in B}\) for every \(y \in Y\) for Bob.
Proof. (1 \Rightarrow 2) Assume \( S \) is a projective KS set. Let \( \{S_1, \ldots, S_k\} \) be the set of all projective measurements contained in \( S \) (i.e. each \( S_i \) is a set of projectors that sum to identity). Consider a non-local game \( G \) where \( X, Y = \{1, \ldots, n\} \), \( A, B = \{\max_i |S_i|\} \), \( \pi \) is the uniform distribution, and the verification function is defined by

\[
V(a, b, x, y) = 1 \Leftrightarrow \langle \psi_n | P^x_a \otimes P^y_b | \psi_n \rangle \neq 0 \\
\Leftrightarrow \text{Tr}(P^x_a P^y_b) \neq 0.
\]

By definition, this game has an optimal quantum strategy in which Alice and Bob share \( |\psi_n\rangle \) and they measure their part of the state using projective measurements \( S_x \) and \( S_y \) respectively upon receiving inputs \( (x, y) \).

Towards a contradiction, suppose there is an optimal classical strategy \( (s_A, s_B) \). Note that \( s_A = s_B \) by definition of \( V \), because if \( x = y \) then the only winning outputs satisfy \( a = b \). Let us now construct a marking function \( f \) for \( S \) from \( s_A \):

\[
\forall P \in S, \ f(P) = 1 \Leftrightarrow \exists a \in A, x \in X \text{ such that} \ P = P^x_a \text{ and } s_A(x) = a.
\]

It is clear that \( f \) marks one projector per measurement and since it is based on a winning strategy, it never marks two orthogonal projectors. However, to show that \( f \) is indeed a function defined on \( S \), we need to address a last issue. It can happen that the same projector appears in more than one measurement.

We need to show that if \( f \) marks a projector, it marks the same projector in all the measurements that contain it. Suppose, towards a contradiction, that there exists \( x, x' \) such that \( s_A(x) = a, P^x_a \in S'_x, s_A(x') = a' \) but \( P^x_a \neq P^{x'}_{a'} \). Then, since \( s_A = s_B \) and all elements of \( S_{x'} \) different from \( P^x_a \) are orthogonal to it, the players would lose the game on input \( (x, x') \). This contradicts the fact that \( s_A, s_B \) are winning strategies.

We have proved that \( f \) is a marking function for \( S \) constructed from \( s_A, s_B \). This contradicts that \( S \) is a projective KS set, therefore classical players cannot have an optimal strategy and the desired statement follows.

\( \Leftarrow \) Assume \( G \) is a pseudo-telepathy game and \( S = \{P^x_a\}_{(a,x)} \cup \{Q^y_b\}_{(b,y)} \) together with \( |\psi_n\rangle \) is a winning quantum strategy. Every marking function \( f \) for \( S \) can be mapped to a classical strategy in the following way:

\[
s_A(x) = a \Leftrightarrow f(P^x_a) = 1 \text{ and } s_B(y) = b \Leftrightarrow f(Q^y_b) = 1.
\]

Since \( G \) is a pseudo-telepathy game, for every \( f \) there exists a tuple \( (a, b, x, y) \) such that \( s_A(x) = a \) and \( s_B(y) = b \) (and therefore \( f(P^x_a) = f(Q^y_b) = 1 \)), but \( V(a, b, x, y) = 0 \). Since the quantum players always win the game, we have that \( \langle \Psi | P^x_a \otimes Q^y_b | \Psi \rangle = 0 \) and this implies \( \text{Tr}(P^x_a Q^y_b) = 0 \) by (3.4). Therefore, for any marking function \( f \) for \( S \) we can find orthogonal projectors \( P^x_a, Q^y_b \in S \) such that \( f(P^x_a) = f(Q^y_b) = 1 \). Hence, \( S \) is projective KS set. \( \square \)
3.3 Quantum Chromatic Number

In this section we define the quantum chromatic number of a graph. For the sake of completeness and to fix some useful facts, we present a comprehensive statement about its basic properties. This is done by extending and completing some observations contained in [CMN+07].

Informally, the \( c \)-coloring game for a graph \( G = (V, E) \) is as follows. Two players, Alice and Bob, claim that they have a proper \( c \)-coloring for \( G \). A referee wants to test this claim with a one-round game, so he forbids communication between the players and separately asks Alice the color \( \alpha \) for the vertex \( v \) and Bob the color \( \beta \) for the vertex \( w \). The players are required to give the same color as output if \( v = w \), and to give a different color if \( (v, w) \in E \). A formal definition follows.

**Definition 3.3.1.** The \( c \)-coloring game on the graph \( G = (V, E) \) is a non-local game with input sets \( X = Y = V \), output sets \( A = B = [c] \) and uniform distribution on the inputs.\(^2\) Alice gets input \( v \) and outputs \( \alpha \), Bob gets input \( w \) and outputs \( \beta \). The players lose the game in the following two cases:

1. \( v = w \) and \( \alpha \neq \beta \)
2. \( (v, w) \in E \) and \( \alpha = \beta \)

A classical strategy consists w.l.o.g. of two deterministic functions \( c_A : V \to [c] \) for Alice and \( c_B : V \to [c] \) for Bob. We can introduce shared randomness, but since this results in a probability distribution over deterministic strategies, it is not beneficial. A little thought will show that to win with probability 1, we must have \( c_A = c_B \) (to avoid the first losing condition) and that \( c_A \) must be a valid \( c \)-coloring of the graph (to avoid the second losing condition). It follows that the classical players cannot win the \( c \)-coloring game with probability 1 when \( c < \chi(G) \).

A quantum strategy for the \( c \)-coloring game uses an entangled state \( |\psi\rangle \) of local dimension \( d \) and two families of POVMs: for all \( v \in V \), Alice has \( \{E^v_\alpha\}_{\alpha=1,...,c} \) and Bob has \( \{F^v_\beta\}_{\beta=1,...,c} \). According to her input \( v \), Alice applies the corresponding POVM \( \{E^v_\alpha\}_{\alpha=1,...,c} \) to her part of the entangled state and outputs the outcome \( \alpha \). Bob acts similarly and outputs \( \beta \). The requirements for the game translate into the following consistency conditions. Alice and Bob win the coloring game with certainty, using a quantum strategy as described above, if and only if

\[
\forall v \in V, \forall \alpha \neq \beta, \quad \langle \psi | E^v_\alpha \otimes F^v_\beta | \psi \rangle = 0 \quad (3.5)
\]

\[
\forall (v, w) \in E, \forall \alpha, \quad \langle \psi | E^v_\alpha \otimes F^w_\alpha | \psi \rangle = 0. \quad (3.6)
\]

\(^2\)Another variant of the game uses an uniform distribution over pairs of inputs that equal or form an edge (see for example [AHKS06]).
In this case, we call the strategy a *winning strategy* or a *quantum c-coloring of G*. Note that we do not bound the dimension of the entangled state or the rank of the measurement operators, we only care about the number of measurement operators needed to win the game with certainty (i.e., the number of colors). We are now ready to give the central definition of this section.

**Definition 3.3.2.** For all graphs $G$, the quantum chromatic number $\chi_q(G)$ is the minimum number $c$ such that there exists a quantum $c$-coloring of $G$.

We will see that without loss of generality a quantum $c$-coloring has a *normal form*, a clean and simple structure defined as follows.

**Definition 3.3.3.** A quantum $c$-coloring of $G$ is in normal form if there exists an integer $r$ such that:

1. All POVMs are projective measurements with $c$ real-valued projectors of rank $r$.
2. The shared state is $|\psi\rangle = \frac{1}{\sqrt{rc}} \sum_{i \in [rc]} |i\rangle |i\rangle$.
3. Alice’s projectors are related to Bob’s as follows: for all $v, \alpha$, $E^v_\alpha = F^w_\alpha$.
4. The consistency conditions (3.5) and (3.6) can be expressed as the single condition:

$$\forall (v, w) \in E, \forall \alpha \in [c], \quad \text{Tr}(E^v_\alpha E^w_\alpha) = 0. \quad (3.7)$$

Notice that we do not know whether the rank $r$ of the projectors in the best normal form is equal to the minimum rank of a general strategy. It might be necessary to increase the rank to obtain the normal form.

The next proposition is a collection of statements from [CMN+07], expanded and rearranged, that are useful to direct our discussion.

**Proposition 3.3.4 ([CMN+07, SS12]).** If $G$ has a quantum $c$-coloring, then it has a quantum $c$-coloring in normal form.

**Proof.** We start with a generic winning strategy consisting of entangled state $|\psi'^m\rangle$, and POVMs $\{(E^v_\alpha)^m\}_{v \in V, \alpha \in [c]}$ for Alice and $\{(F^w_\beta)^m\}_{w \in V, \beta \in [c]}$ for Bob. Then, we will gradually construct an equivalent strategy with the desired properties. We will prove the statements in a few steps. Each bullet in the following list is a small statement that is proved right after. At the end of the steps, we will have the final strategy in normal form. The number of prime symbols of the notation for the entangled state and the POVM elements will reduce as soon as we get close to the final strategy, consisting of $|\psi\rangle$ and $\{E^v_\alpha\}_{v \in V, \alpha \in [c]}$, $\{F^w_\beta\}_{w \in V, \beta \in [c]}$. 

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- **The entangled state has full Schmidt rank.**

  Start with the entangled state $|\psi''\rangle$, with local dimension $d'$. Consider the Schmidt decomposition $|\psi''\rangle = \sum_{i=0}^{d'-1} \lambda_i |i\rangle |i\rangle$, where without loss of generality $\{i\} = \{0, \ldots, d'-1\}$ is the computational basis. Say there are $d$ non-zero $\lambda_i$. Then we define the new entangled state as $|\psi''\rangle = \sum_{i: \lambda_i \neq 0} \lambda_i |i\rangle |i\rangle$, and restrict the measurement operators to the respective supports of the reduced states as follows. Consider the projector $P = \sum_{i: \lambda_i \neq 0} |i\rangle \langle i|$. Then for all the POVM elements of Alice define $(E_\alpha^v)' = P(E_\alpha^v)^m P$, and do the same for Bob’s POVM elements. These restricted POVMs are valid measurements on $|\psi''\rangle$, and they still form a valid quantum coloring: $\sum_{\alpha} (E_\alpha^v)' = I$ (on the $d$-dimensional subspace on which $P$ projects) and

  \[
  \langle \psi''| (E_\alpha^v)^m \otimes (F_\beta^v)^m |\psi''\rangle = \langle \psi''| P(E_\alpha^v)^m P \otimes P(F_\beta^v)^m P |\psi''\rangle \\
  = \langle \psi''| (E_\alpha^v)^m \otimes (F_\beta^v)^m |\psi''\rangle.
  \]

  We have that $|\psi''\rangle$ has full Schmidt rank $d$ and together with the new POVMs is a winning strategy.

- **All POVM elements are projectors.**

  With some abuse of notation, we identify a projector with the support of the space on which it projects. We denote the support of an operator $A$ by $\text{supp}(A)$.

  It follows from consistency condition (3.5) that

  \[
  \forall v, \alpha, \sum_{\beta \neq \alpha} \langle (E_\alpha^v)^m, \text{Tr}_B(I \otimes (F_\beta^v)^m |\psi''\rangle \langle \psi''|) = 0, \quad (3.8)
  \]

  Since both $(E_\alpha^v)^m$ and $\text{Tr}_B(I \otimes (F_\beta^v)^m |\psi''\rangle \langle \psi''|$ are positive semidefinite operators, it follows from (3.8) that $\text{supp}((E_\alpha^v)^m)$ is a subspace of $\text{supp}(\text{Tr}_B(I \otimes (F_\beta^v)^m |\psi''\rangle \langle \psi''|))$. By a symmetric argument, it also follows that $\text{supp}(\text{Tr}_B(I \otimes (F_\beta^v)^m |\psi''\rangle \langle \psi''|)$ is a subspace of $\text{supp}((E_\alpha^v)^m)$.

  Therefore, without loss of generality for all $v$ and $\alpha$ we can replace $(E_\alpha^v)^m$ with $(E_\alpha^v)' = \text{supp}(\text{Tr}_B(I \otimes (F_\alpha^v)^m |\psi''\rangle \langle \psi''|))$. A similar replacement can be done for Bob’s POVM elements.

- **The state $|\psi''\rangle$ can be replaced by the maximally entangled state.**

  The winning strategies do not depend on the values of the Schmidt coefficients $\{\lambda_i\}$ of $|\psi''\rangle$, as long as they are non-zero. Thus we can set for all $i, \lambda_i = 1/\sqrt{d}$ and define $|\psi\rangle = |\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i \in [d]} |i\rangle |i\rangle$.

  - **All projectors can be real-valued, of the same rank $r$ and the maximally entangled state can have local dimension $r$.**
We first map every complex-valued $d \times d$ projector into a real-valued $2d \times 2d$ projector using the following map.

$$R(A) = \begin{pmatrix} \mathcal{R}(A) & \mathcal{I}(A) \\ -\mathcal{I}(A) & \mathcal{R}(A) \end{pmatrix},$$

where $\mathcal{R}(A)$ and $\mathcal{I}(A)$ are respectively the real and imaginary part of $A$. One can verify that this map preserves matrix sum and matrix product, and that for all $v$ we have $\sum_{\alpha \in [c]} R(E^v_{\alpha}) = I$.

Second, we extend the entangled state to $|\psi\rangle = |\psi_{2d}\rangle \otimes |\psi_c\rangle$ and then define new projectors for Alice $E^v_{\alpha} = \sum_{i,j=0}^{c-1} R((E^v_{\alpha+i[mod \, c]})' \otimes |i\rangle \langle i|)$ (and similarly for Bob). All have rank $r = \sum_{\alpha} \text{rank}(R((E^v_{\alpha})'))$ and act on the new state of local dimension $rc$. One can see that the new projectors still satisfy the consistency conditions.

* We have for all $v, \alpha$, $E^v_{\alpha} = F^v_{\alpha}$.

Call $\rho = \text{Tr}_B(|\psi\rangle\langle\psi|) = \sum_i \frac{1}{\sqrt{rc}} |i\rangle \langle i| = \text{Tr}_A(|\psi\rangle\langle\psi|)$. Then

$$E^v_{\alpha} = \text{supp}(\text{Tr}_B(I \otimes F^v_{\alpha}|\psi\rangle\langle\psi|)) = \text{supp} \left( \text{Tr}_B(I \otimes F^v_{\alpha} \sum_i \frac{1}{\sqrt{rc}} |i\rangle \langle i| \sum_j \frac{1}{\sqrt{rc}} |j\rangle \langle j|) \right) = \text{supp} \left( \sum_{i,j} \frac{1}{rc} \text{Tr}_B(|i\rangle \langle j| \otimes F^v_{\alpha}|i\rangle \langle j|) \right) = \text{supp} \left( \sum_{i,j} \frac{1}{rc} |i\rangle \langle j| (F^v_{\alpha})_{j,i} \right) = \text{supp} \left( \sum_{i,j} \frac{1}{rc} |i\rangle \langle i| (F^v_{\alpha})_{j,j} \right) = \text{supp} \left( \text{Tr}_B(F^v_{\alpha}) \right) = \text{supp} \left( \sqrt{\rho} (F^v_{\alpha}) \sqrt{\rho} \right).$$

Since all measurements are projective measurements, we have that $\sum_{\alpha} E^v_{\alpha} = I$ and that for $\alpha \neq \beta$

$$F^v_{\alpha} F^v_{\beta} = 0 \Rightarrow \sqrt{\rho} E^v_{\alpha} \rho E^v_{\beta} \sqrt{\rho} = 0 \Rightarrow \sum_{i,j} \lambda_i \lambda_j |i\rangle \langle i| (E^v_{\alpha} \rho E^v_{\beta}) |j\rangle \langle j| = 0 \Rightarrow E^v_{\alpha} \rho E^v_{\beta} = 0.$$
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Hence, we have:

\[ \rho = I \rho I = \sum_{\alpha} E_{\alpha}^{v} \sum_{\beta} E_{\beta}^{v} = \sum_{\alpha, \beta} E_{\alpha}^{v} \rho E_{\beta}^{v} = \sum_{\alpha} E_{\alpha}^{v} \rho E_{\alpha}^{v}. \]

This last fact implies that \( \rho \) commutes with all operators (to see this, use the fact that \( (E_{\alpha}^{v})^{2} = E_{\alpha}^{v} \)). Hence we have, using the fact that \( |\psi\rangle \) has full Schmidt rank,

\[ E_{\alpha}^{v} = \text{supp}(\sqrt{\rho} F_{\alpha}^{w} \sqrt{\rho}) = \text{supp}(F_{\alpha}^{w} \rho) = F_{\alpha}^{w}. \tag{3.9} \]

- We can express the consistency conditions (3.5) and (3.6) just in terms of Alice’s projectors as: \( \forall (v, w) \in E \) and \( \forall \alpha, \langle E_{\alpha}^{v}, E_{\alpha}^{w} \rangle = 0 \).

We have that \( |\psi\rangle \) is the maximally entangled state with local dimension \( r_{c} \).

It follows that for all \( v, \alpha \) and \( \beta \), \( \text{Tr}(E_{\alpha}^{v} \otimes F_{\beta}^{w} |\psi\rangle \langle \psi|) = \frac{1}{r_{c}} \text{Tr}(E_{\alpha}^{v} F_{\beta}^{w}) \). We also have that for all \( v \) and \( \alpha, E_{\alpha}^{v} = F_{\alpha}^{v} \). Then \( \frac{1}{r_{c}} \text{Tr}(E_{\alpha}^{v} F_{\alpha}^{w}) = 0 \) if and only if \( \langle E_{\alpha}^{v}, E_{\alpha}^{w} \rangle = 0 \), and we can write the consistency conditions as wanted.

Starting from any quantum \( c \)-coloring of \( G \), we have constructed a quantum \( c \)-coloring in normal form, consisting of \( |\psi\rangle \), \( \{E_{\alpha}^{v}\}_{v \in V, \alpha \in [c]} \).

It is natural to distinguish between different types of quantum chromatic number according to the rank of the POVM elements used in the strategies of Alice and Bob.

**Definition 3.3.5.** The rank-\( r \) quantum chromatic number \( \chi_{q}^{(r)}(G) \) of \( G \) is the minimum number of colors \( c \) such that \( G \) has a quantum \( c \)-coloring consisting of projectors of rank at most \( r \) and a maximally entangled state of local dimension \( r_{c} \).

It follows that for all \( r \geq s \) we have \( \chi_{q}^{(r)}(G) \leq \chi_{q}^{(s)}(G) \) and therefore

\[ \chi_{q}(G) = \min_{r} \{\chi_{q}^{(r)}(G)\}. \tag{3.10} \]

We remark that in this last definition the \( c \)-coloring need not be in normal form.

### 3.3.1 Rank-1 Quantum Chromatic Number

We now restrict our attention to the rank-1 quantum chromatic number \( \chi_{q}^{(1)}(G) \). It follows from (3.10) that the rank-1 quantum chromatic number is an upper bound on the quantum chromatic number. We also know an example where the rank-1 quantum chromatic number is strictly greater than the quantum chromatic number [FIG11]. In a rank-1 quantum \( c \)-coloring, we have that the maximally
entangled state has local dimension \( c \) and that the rank-1 projectors for each vertex \( v \) can be seen as outer products \( |a_{v\alpha}\rangle\langle a_{v\alpha}| \) of an orthonormal basis \( \{|a_{v\alpha}\rangle\}_{\alpha \in [c]} \). Then the consistency condition (3.7) becomes
\[
\forall (v, w) \in E, \forall \alpha \in [c], \quad \langle a_{v\alpha} | a_{w\alpha} \rangle = 0.
\] (3.11)

As explained in [CMN+07], a rank-1 quantum \( c \)-coloring of \( G \) induces a matrix representation of \( G \), which is a map \( \Phi : V \to \mathbb{C}^{c \times c} \) such that for all \( (v, w) \in E \), the diagonal of \( \Phi(v)^\dagger\Phi(w) \) is 0. This is obtained as follows. For all vertices \( v \in V \) consider the unitary matrix \( U_v \) mapping the computational basis \( \{|i\rangle\}_{i \in [c]} \) to \( \{|a_{v\alpha}\rangle\}_{\alpha \in [c]} \). This is a \( c \times c \) matrix and because of condition (3.11), if \( (v, w) \) is an edge then the diagonal entries of \( U_v^\dagger U_w \) are zero.

The results in [CMN+07, Proposition 7], give the following:

**Proposition 3.3.6.** For all graphs \( G \),
\[
\xi(G) \leq \chi_q^{(1)}(G) \leq \chi(G).
\]

Our results in this section answer some questions about the relation between these three quantities that were left open in [CMN+07]. For all graphs \( G \), we give a necessary and sufficient condition for \( \xi(G) = \chi_q^{(1)}(G) \), using a relation between the rank-1 quantum chromatic number and the orthogonal representation of a particular Cartesian product. Then, using the properties of Kochen-Specker sets in two different ways, we first give a class of graphs for which the rank-1 quantum chromatic number is strictly greater than the orthogonal rank. Later, for all graphs \( G \), we give a necessary and sufficient condition for \( \chi_q^{(1)}(G) < \chi(G) \).

**Equality between rank-1 quantum chromatic number and orthogonal rank of product graphs**

The following proposition will help us to characterize the graphs for which there is equality between orthogonal rank and the rank-1 quantum chromatic number. Let \( K_c \) be the complete graph on \( c \) vertices and let \( \Box \) denote the graph Cartesian product, as defined in Section 3.2.

**Proposition 3.3.7.** For all graphs \( G \),
\[
\chi_q^{(1)}(G) = \min\{c : \xi(G \Box K_c) = c\}.
\]

**Proof.** We first prove that we can map any orthogonal representation in \( \mathbb{C}^c \) of \( G' = G \Box K_c \) to a matrix representation in \( \mathbb{C}^{c \times c} \) of \( G \), and vice versa.

Let \( \{1, \ldots, c\} \) be the vertex set of \( K_c \). The vertex set of \( G' \) is \( V(G') = V(G) \times \{1, \ldots, c\} \). There is an edge in \( E(G') \) between vertices \( (v, i) \) and \( (w, j) \) if either \( v = w \) and \( i \neq j \) or \( (v, w) \in E(G) \) and \( i = j \). Thus an orthogonal representation \( \{a_{(v, i)}\}_{(v, i) \in V(G')} \) of \( G' \) can be mapped to a matrix representation
of $G$ as follows: for all $v \in V(G)$, let the $i$-th column of $U_v$ be $a_{(v,i)}/\|a_{(v,i)}\|$. One can check that this is a valid matrix representation in $\mathbb{C}^{n \times c}$ for $G$. Similarly we can map matrix representations of $G$ to orthogonal representations of $G'$.

We now prove the main statement. Let $G$ be a graph with $\chi_q^{(1)}(G) = d$, then by the discussion above we can find an orthogonal representation of $G \Box K_d$ in $d$ dimensions starting from the matrix representation.

We also know that $\xi(G \Box K_d) \geq d$, because there exist subgraphs of $G \Box K_d$ isomorphic to $K_d$ and $\xi(K_d) = d$. Hence we have $\xi(G \Box K_d) = \chi_q^{(1)}(G) = d$. Now suppose that $\min\{c : \xi(G \Box K_c) = c\} = d' < d$. Then there exists an orthogonal representation of $G \Box K_{d'}$ in $\mathbb{C}^{d'}$. We can map such orthogonal representation to a matrix representation for $G$ in $\mathbb{C}^{d' \times d'}$. But then $\chi_q^{(1)}(G) = d'$, contradicting the assumption. Therefore we have $\chi_q^{(1)}(G) = d = \min\{c : \xi(G \Box K_c) = c\}$.

We are now able to prove the following theorem.

**Theorem 3.3.8.** For all graphs $G$,

$$\chi_q^{(1)}(G) = \xi(G) \iff \xi(G \Box K_{\xi(G)}) = \xi(G).$$

**Proof.** One can see that for all pairs of graphs $G, H$ we have $\xi(G \Box H) \geq \max\{\xi(G), \xi(H)\}$ because there exist subgraphs of $G \Box H$ isomorphic to $G$ and subgraphs of $G \Box H$ isomorphic to $H$. We also have that for all $c$, $\xi(K_c) = c$. Hence, we have that $\xi(G \Box K_c) \geq \max\{\xi(G), c\}$. Using this and Proposition 3.3.7, we observe that

$$\chi_q^{(1)}(G) = \min\{c : \xi(G \Box K_c) = c\} \geq \xi(G),$$

with equality if and only if $\xi(G \Box K_{\xi(G)}) = \xi(G)$. \hfill \qed

On the basis of Proposition 3.3.7 and following [Hog07] we can upper bound the rank-1 quantum chromatic number, in terms of a positive-semidefinite rank. Let $S_n$ denote the set of $n \times n$ real symmetric matrices. Then for $A \in S_n$, the graph $G(A) = (V, E)$ is the graph with vertex set $V = \{1, \ldots, n\}$ and edge set $E = \{\{i, j\} : A_{ij} \neq 0\}$. The set of positive-semidefinite matrices of the graph $G$ is

$$S_+(G) = \{A \in S_n : A \succeq 0, \ G(A) = G\},$$

and the **positive-semidefinite minimum rank** of $G$ is

$$\text{mr}_+(G) = \min\{\text{rank}(A) : A \in S_+(G)\}.$$

From [Hog07, Observation 1.2] we have $\xi(G) \leq \text{mr}_+(G)$, and from Proposition 3.3.7 we have:

$$\chi_q^{(1)}(G) \leq \min\{c : \text{mr}_+(G \Box K_c) = c\}.$$

This observation may be useful for future work about the complexity of computing the quantum chromatic number.
Separation between rank-1 quantum chromatic number and orthogonal rank using KS sets

We know that for all graphs $G$, we have $\xi(G) \leq \chi_q(1)(G)$. One can see that, given a matrix representation of $G$ in $\mathbb{C}^{n \times n}$, one can obtain an orthogonal representation of $G$ in $\mathbb{C}^n$ (take the first row of each matrix). We now exhibit graphs with rank-1 quantum chromatic number strictly larger than the orthogonal rank. These graphs are the orthogonality graphs of Kochen-Specker sets (see Section 3.2.2).

**Theorem 3.3.9.** There are graphs $G$ such that $\xi(G) < \chi_q(1)(G)$.

**Proof.** Let $S \subseteq \mathbb{C}^3$ be a KS set. We exhibit a graph $G_S$ such that $\xi(G_S) = 3$ but $\chi_q(1)(G_S) > 3$. The vertices of $G_S$ are all the vectors in $S$, and there is an edge between orthogonal vectors. The graph $G_S$ has obviously an orthogonal representation of dimension 3. We show now that it is not 3-colorable. Suppose we are able to 3-color the graph, and let $f$ be a function that maps a vector in $S$ to 1 if it has color 1, and to zero otherwise. Every orthonormal basis $b \subseteq S$ is a clique in $G_S$, so we have $\sum_{u \in b} f(u) = 1$, contradicting the assumption that $S$ is a KS set. In [CMN+07, Proposition 11] it is proven that for all graphs $G$, $\chi_q(1)(G) = 3$ if and only if $\chi(G) = 3$, so we conclude that $\chi_q(1)(G_S) > 3$.

Orthogonality graphs of KS sets are not the only ones that exhibit a separation between rank-1 quantum chromatic number and chromatic number. We know about the existence of a small graph with rank-1 quantum chromatic number and chromatic number equal to 4, but orthogonal rank 3, based on [CMN+07, Proposition 11]. It is the orthogonality graph of a set of 13 vectors of dimension 3 [MO]. This set is not a Kochen-Specker set, as there are no KS sets in $\mathbb{C}^3$ smaller than 18 vectors [AOW11].

### 3.3.2 Quantum chromatic number and KS sets

We now prove that projective KS sets characterize all the graphs with a separation between the chromatic number and the quantum chromatic number.

Theorem 3.3.4 allows us to identify a quantum $c$-coloring with Alice’s multiset of projectors, denoted as $\{P_{\alpha}^v\}_{v \in V, \alpha \in [c]}$.

**Theorem 3.3.10.** For all graphs $G$, we have that $\chi(G) > \chi_q(G) := c$ if and only if for all quantum $c$-colorings in normal form, $S = \bigcup_{v \in V, \alpha \in [c]} \{P_{\alpha}^v\}$ is a projective KS set.

**Proof.** \( \Rightarrow \) Let $\chi(G) > \chi_q(G) := c$ and let $S$ be the union of Alice’s projectors in a quantum $c$-coloring in normal form. We now show that if $S$ is not a projective KS set, then we can properly $c$-color the graph, contradicting the assumption that $\chi(G) > c$. If $S$ is not a projective KS set then there exists a marking function
3.4 Quantum Independence Number

$f : S \to \{0, 1\}$ such that for all orthogonal $P, P' \in S$ we have $f(P) = 0$ or $f(P') = 0$. We can use the function $f$ to $c$-color the graph as follows:

$$\text{color}(v) = \alpha \text{ if } f(P^v_\alpha) = 1.$$ 

This is a proper $c$-coloring for the following two reasons. First, the quantum coloring associates each vertex to a projective measurement, and since $f$ is a marking function, exactly one projector per measurement is mapped to 1. Second, this property of $f$ and the consistency condition (3.7) ensure that we never color adjacent vertices with the same color.

$\Leftarrow$ Let $\chi_q(G) = c$ and assume that for all quantum $c$-colorings in normal form, the union of Alice’s projectors is a projective KS set. Now suppose, towards a contradiction, that it is possible to classically $c$-color the graph. Then for each $v \in V$ with classical color $\alpha$, define the projective measurement $\{P^v_i = | i + \alpha \rangle \langle i + \alpha | \}_{i \in \{0, \ldots, c-1\}}$ (where the addition is modulo $c$). One can see that this is a valid quantum $c$-coloring, and the union of its vectors consists of one projective measurement only. Thus it is not a projective KS set, because you can define a function that maps $| 1 \rangle \langle 1 |$ to 1 and all other projectors to 0. This is a contradiction with the assumption that the union of Alice’s projectors is a projective KS set.

We remark that Theorem 3.3.10 can also be proven starting from Theorem 3.2.9. However, we prefer the direct approach to underline the structural relationship between graphs with $\chi(G) > \chi_q(G)$ and orthogonality graphs of projective KS sets. Also notice that, because of the bijection between vectors and rank-1 projectors, weak KS sets characterize all the graphs with a separation between the chromatic number and the rank-1 quantum chromatic number.

3.4 Quantum Independence Number

In this section we define the quantum independence number of a graph and study its properties.

In [MSS13] we presented the results of this chapter in terms of zero-error information theory. However, a new definition came in [RM12], after our paper. We decided to adopt this new definition, since it better captures the nature of the problem. The work by Roberson and Mančinska discusses a framework that defines in the quantum regime all the graph parameters that can be expressed as graph homomorphisms. For simplicity, here we leave homomorphisms apart and give a direct definition of the quantum independence number.

As with the chromatic number, the quantum independence number can be defined in terms of a non-local game. Informally, the independent set game with parameter $t$ for a graph $G = (V, E)$ is as follows. Two players, Alice and Bob, claim that they know an independent set $I$ of $G$ consisting of $t$ vertices. A
referee wants to test this claim with a one-round game. He forbids communication between the players, generates two random numbers $x, y \in [t]$ and separately asks Alice to provide the $x$-th vertex of $I$ and Bob to provide the $y$-th vertex of $I$. The players are required to output the same vertex if $x = y$, and to output non-adjacent vertices if $x \neq y$. A formal definition follows.

**Definition 3.4.1.** The independent set game with parameter $t$ on the graph $G = (V, E)$ is a non-local game with input sets $X = Y = [t]$, output sets $A = B = V$. Alice gets input $x$ and outputs $v$, Bob gets input $y$ and outputs $w$. The players lose the game in the following two cases:

1. $x = y$ and $v \neq w$
2. $x \neq y$ and $(v, w) \in E$ or $v = w$

A classical strategy consists w.l.o.g. of two deterministic functions $f_A : [t] \to V$ for Alice and $f_B : [t] \to V$ for Bob. Shared randomness, as seen for the coloring game, is not beneficial. A little thought will show that to win with probability 1, we must have $f_A = f_B$ (to avoid the first losing condition) and that \{f_A(1), \ldots, f_A(t)\} must be a valid independent set of the graph of size $t$ (to avoid the second losing condition). It follows that the classical players cannot win the game with probability 1 when $t > \alpha(G)$.

It is proven in [RM12] (with a proof almost identical to the proof of Proposition 3.3.4) that w.l.o.g. quantum strategies for the independent set game consist of real-valued projective measurements on a maximally entangled state and that the projective measurements of Alice and Bob are the same. Therefore we can define a quantum independent set of size $t$ as a collection of $t$ real-valued projective measurements $\{P_v^x\}_{v \in V}$ for all $x \in [t]$ that have the whole vertex set as outputs, with the following consistency condition:

$$
\text{for all } (u, v) \in E \text{ or } u = v \text{ and for all } x \neq x', \quad P_u^x P_v^{x'} = 0. \quad (3.12)
$$

**Definition 3.4.2.** For all graphs $G$, the quantum independence number $\alpha_q(G)$ is the maximum number $t$ such that there exists a quantum independent set of $G$ of size $t$.

In the following sections we show three different ways to construct graphs with a separation between quantum and classical independence number. In Chapter 4 we will see that the quantum independence number is a lower bound on the one-shot entanglement-assisted channel capacity in zero-error information theory. Since the classical independence number is related to the classical one-shot channel capacity, all the separations in this chapter imply separations in the information theory framework.
3.4. Quantum Independence Number

3.4.1 Separation using projective KS sets

It follows from a result in [CLMW10] in the zero-error information theory context that every weak Kochen-Specker set can be used to construct a graph for which \( \alpha(G) < \alpha_q(G) \). By the same line of argument, we now prove that also projective KS sets can be used for this purpose (Theorem 3.4.4) as well as a weak converse of this statement (Theorem 3.4.5).

Let \( |\Psi⟩ = \frac{1}{\sqrt{n}} \sum_{i \in [n]} |i⟩|i⟩ \), where \( \{ |i⟩ \}_{i \in [n]} \) is the standard basis of \( \mathbb{C}^n \). We start by proving the following technical lemma.

**Lemma 3.4.3.** Let \( G = (V,E) \) be a graph whose vertex set \( V \) can be partitioned into \( k \) cliques \( S_1, \ldots, S_k \), not necessarily of the same size. Assume that there is an assignment of a projector \( P_v \) to each vertex \( v \) such that:

1. for all \( i \in [k] \), we have \( \sum_{v \in S_i} P_v = I \).
2. for all edges \( (v,w) \), we have \( \text{Tr}(P_v P_w) = 0 \).

Then, \( \alpha_q(G) \geq k \).

**Proof.** We define projective measurements \( \{ P^x_v \}_{v \in V} \) for all \( x \in [k] \) as follows: \( P^x_v = P_v \) if \( v \in S_x \) and \( P^x_v = 0 \) otherwise.

We now check that this is a winning strategy for the independent set game on \( G \). By the first property, each projector is a valid projective measurement. By the second property and by the disjointness of the cliques, such projectors satisfy the consistency condition (3.12). Hence, the above strategy is a winning strategy and we conclude that \( \alpha_q(G) \geq k \). \(\square\)

We are now ready to prove the following.

**Theorem 3.4.4** (Projective KS set \( \Rightarrow \) separation). Let \( S \) be a projective KS set. Let \( S_1, \ldots, S_k \subseteq Q_n \) be all the subsets of \( S \) such that \( \sum_{P \in S_i} P = I \). Then the orthogonality graph \( G \) of the multiset \( S_1 \uplus \cdots \uplus S_k \) satisfies \( \alpha(G) < \alpha_q(G) \).

**Proof.** Observe that every \( S_i \) is a projective measurement, so the vertices of \( G \) can be partitioned in \( k \) cliques \( S_1, \ldots, S_k \). Let \( T \) be a maximal independent set in \( G \). Suppose towards a contradiction that \( |T| = k \); i.e., \( T \) is a multiset of projectors containing exactly one element per clique. Since \( G \) contains one clique per measurement and orthogonal elements are joined by edges, we have that if \( P \in T \) is part of \( \ell \) measurements, then \( T \) contains \( \ell \) copies of \( P \). Define a marking function for \( S \) as:

\[ f(P) = 1 \iff P \in T. \]

It is a marking function for \( S \) because by assumption \( S_1, \ldots, S_k \subseteq Q_n \) are all the projective measurements in \( S \) and \( f \) selects exactly one element from each \( S_i \). Moreover, \( f \) does not mark any pair of orthogonal elements because \( T \) is an independent set and \( G \) is an orthogonality graph. The existence of \( f \) contradicts
the assumption that $S$ is a projective KS set. Therefore, $\alpha(G) < k$. To see that $\alpha_q(G) \geq k$, partition the vertices of $G$ into $k$ cliques $S_1, \ldots, S_k$ and use Lemma 3.4.3.

The following theorem provides a converse of Theorem 3.4.4.

**Theorem 3.4.5** (Separation $\Rightarrow$ projective KS set). Let $G = (V, E)$ be a graph with $\alpha(G) < k$. If there exists a quantum independent set $\{P^x_v\}_{v \in V}$ for all $x \in [k]$, then

$$S = \{P^x_v : v \in V, x \in [k]\}$$

is a projective KS set.

**Proof.** We prove that if $S$ is not a projective KS set, then we can construct an independent set of size $k$ and thus $\alpha(G) \geq k$. If $S$ is not a projective KS set, then there exists a marking function $f : S \to \{0, 1\}$ such that for any $P, P' \in S$ for which $\text{Tr}(PP') = 0$, $f(P) = 0$ or $f(P') = 0$. Consider the set

$$J = \{v \in V : f(P^x_v) = 1, \text{ for some } x \in [k]\}.$$

We now show that $J$ is an independent set of size $k$. From the fact that $f$ selects one projector from each of the $k$ measurements, we obtain that $|J| = k$. From the fact that $f$ cannot mark two orthogonal projectors and from Condition 3.12 we get that $J$ is an independent set.

We remark that Theorems 3.4.4 and 3.4.5 also follow from Theorem 3.2.9, by using a reduction from one-shot zero-error channel capacity to pseudo-telepathy games given in [CLMW10]. However, the direct approach taken in the proofs above more clearly shows the relationship between orthogonality graphs of projective KS sets and graphs having separations between quantum and classical independence number.

### 3.4.2 Separation using properties of chromatic numbers

Here we use the relationships described in Section 3.3.2 to show that every graph with a separation between quantum and classical chromatic number can be used to construct a graph with separation between quantum and classical independence number. Using this fact we find a new class of graphs with large separation.

The main result of this section needs the following lemmas. Here, the symbol $\square$ denotes the Cartesian graph product (see 3.2.1) and $K_k$ is the complete graph on $k$ vertices.

**Lemma 3.4.6.** Let $G$ be a graph on $n$ vertices with $\chi(G) > k$. Then we have $\alpha(G \square K_k) < n$. 


Proof. The vertex set of $G \square K_k$ can be partitioned into $n$ disjoint cliques of size $k$. Towards a contradiction, suppose $\alpha(G \square K_k) \geq n$. Then an independent set of size $n$ must contain exactly one vertex from each clique in the partition. We can get a $k$-coloring for $G$, as follows: if $(v, i)$ belongs to the independent set, color $v \in E(G)$ with the $i$-th color. This is a proper coloring because, by definition of the Cartesian product of graphs, for all $(u, v) \in E(G)$ we have $((u, i), (v, i)) \in E(G \square K_k)$, and hence $u$ and $v$ will not both get color $i$. This contradicts the assumption that $\chi(G) > k$.

Lemma 3.4.7. Let $G$ be a graph on $n$ vertices and $\chi_q(G) \leq k$. Then we have $\alpha_q(G \square K_k) = n$.

Proof. Let $G' = G \square K_k$. We first show that $\alpha_q(G') \geq n$. Note that the vertex set of $G'$ can be partitioned into $n$ disjoint cliques of size $k$. Now consider a quantum $k$-coloring of $G$ in normal form, $\{P^v_{i, j}\}_{v \in V, i, j \in [k]}$. Assign each projector $P^v_{i, j}$ to the vertex $(v, i)$ of $G'$. By the properties of a quantum coloring in normal form, this assignment satisfies the requirements of Lemma 3.4.3. Therefore, $\alpha_q(G') \geq n$.

Now note that $\overline{K_n} \boxtimes K_k$ is a subgraph of $G \square K_k$, where $\boxtimes$ denotes the strong product of graphs and $\overline{K_n}$ is the complement of $K_n$. Since for all graphs $H$ and subgraphs $H'$ we have $\vartheta(H) \leq \vartheta(H')$ (see [Lov79]) and since the theta number is multiplicative under the strong product (see (3.3)), we obtain

$$\vartheta(G \square K_k) \leq \vartheta(\overline{K_n} \boxtimes K_k) = \vartheta(\overline{K_n})\vartheta(K_k) = n.$$ 

Lovász $\vartheta$ is an upper bound for the quantum independence number [DSW13, Bei10], therefore we conclude

$$n \leq \alpha_q(G \square K_k) \leq \vartheta(G \square K_k) \leq n.$$ 

Combining the results from the above lemmas we obtain the following.

Theorem 3.4.8. Let $G$ be a graph on $n$ vertices with $\chi(G) > \chi_q(G) =: k$. Then for $G' = G \square K_k$:

1. $\alpha(G') < \alpha_q(G') = n$
2. $\alpha(G') \leq \alpha(G) \cdot k$.

Proof. Since $\chi(G) > k$, $\chi_q(G) = k$ and $G$ has $n$ vertices, we have from Lemma 3.4.6 and Lemma 3.4.7 that $\alpha(G') < \alpha_q(G') = n$, as desired. The second bound follows directly from Lemma 3.2.1.

Note that the second upper bound on Theorem 3.4.8 is very interesting in the case $\alpha(G) \cdot \chi_q(G) \ll n$. This happens only when there is a separation between quantum and classical chromatic number, because for the chromatic
number we have $\alpha(G) \cdot \chi(G) \geq n$. Therefore, as we show in the next section, some graphs with a large separation between quantum and classical chromatic number induce graphs with large separation between quantum and classical independence number.

**Orthogonality graphs**

In this section we apply the observations made above. Specifically, we isolate a new family of graphs for which the quantum independence number is exponentially larger than its classical counterpart. Each member of this family is a Cartesian product of an orthogonality graph with a complete graph. Chromatic number and quantum chromatic number are known to be different for orthogonality graphs with a sufficiently large number of vertices [dKP07, AHKS06, GN08].

For each integer $n$ multiple of 2, the orthogonality graph $\Omega_n$ is a graph with vertex set $\{\pm 1\}^n$ and edge set $\{(u, v) : \langle u, v \rangle = 0\}$. Some of these graphs (for certain values of $n$) are also known in the literature as Hadamard graphs and Deutsch-Jozsa graphs.

**Theorem 3.4.9.** For all $n > 8$ that are divisible by 4, there exists $\epsilon > 0$ such that

$$\frac{\alpha_q(\Omega_n \square K_n)}{\alpha(\Omega_n \square K_n)} \geq \frac{1}{n} \left( \frac{2}{2 - \epsilon} \right)^n.$$

**Proof.** It is shown in [AHKS06] that $\chi(\Omega_n) \leq n$ for all $n \in \mathbb{Z}_+$. Since $|V(\Omega_n)| = 2^n$, using Lemma 3.4.7 we conclude that $\alpha_q(\Omega_n \square K_n) \geq 2^n$.

On the other hand, from Theorem 1.11 in [FR87] it follows that for all $n$ divisible by 4 and greater than 8, there exists $\epsilon > 0$ such that $\alpha(\Omega_n) \leq (2 - \epsilon)^n$. Hence, by Lemma 3.2.1, we have that $\alpha(\Omega_n \square K_n) \leq (2 - \epsilon)^n \cdot n$. By putting the two observations together we obtain the desired statement.

To conclude, we give an example that also for small $n$ we can find a large ratio $\frac{\alpha_q(\Omega_n \square K_n)}{\alpha(\Omega_n \square K_n)}$. The following properties are proven in [dKP07, AHKS06]:

1. $\alpha(\Omega_{16}) = 2304$
2. $\chi(\Omega_{16}) \geq 29$
3. $\chi_q(\Omega_{16}) = 16$.

Take a graph $\Omega_{16} \square K_{16}$. It follows from Theorem 3.4.8 that $\alpha_q(G) = 2^{16}$ while $\alpha(\Omega_{16} \square K_{16}) \leq \alpha(\Omega_{16}) \cdot 16 = 36864$.

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3We have defined orthogonality graphs in general in Section 3.2.1. The ones discussed here are a special case, but we keep the naming consistent with literature.
3.4. Quantum Independence Number

3.4.3 Separation using graph states

In this section we show how starting from any graph $G$ we can construct a graph $H(G)$ with separation between quantum and classical independence number. The drawback of this method is that we do not know how to quantify such separation. The construction is used in [CPSS12] to study orbits of graphs under local complementation, but this is out of the scope of this thesis.

The useful tools for this construction are graph states, which we define below. Given a graph $G = (V, E)$, a generator for $i \in V(G)$ is defined as

$$g_i = X^{(i)} \bigotimes_{j \in N(i)} Z^{(j)},$$

(3.13)

where $X^{(i)}$, $Y^{(i)}$, and $Z^{(i)}$ denote the Pauli matrices acting on the $i$-th qubit and $N(i)$ is the neighborhood of $i$. Identity matrices act on the qubits corresponding to non-adjacent vertices, but we omit them to simplify the notation. Therefore, $g_i$ can be obtained directly from $G$. The graph state $|G\rangle$ (see, for example, [HEB04, SW01]) associated to $G$ is the unique (up to a phase) $n$-qubit state such that

$$g_i|G\rangle = |G\rangle \text{ for } i = 1, \ldots, n. \quad (3.14)$$

The stabilizer group of the state $|G\rangle$ is the set $S$ of the stabilizing operators $s_j$ of $|G\rangle$ defined by the product of any number of generators $g_i$. For convenience, we remove the identity element from $S$. Therefore, the set $S$ contains $2^n - 1$ elements.

We will now explain how to construct a graph $H(G)$ from any graph $G$. Let $G$ be a graph on $n$ vertices and consider the $n$-qubit graph state $|G\rangle$. Let $S$ be the stabilizer group of $G$. For each $s_j \in S$ of the form $s_j = \bigotimes_{k=1}^{n} O^{(k)}$, where each $O^{(k)}$ is a Pauli matrix, let $w_j = |\{O^{(k)} : O^{(k)} \neq I\}|$ be the weight of $s_j$. Let $S_j = \{S_{(i,j)} : i = 1, 2, \ldots, 2^{w_j}\}$ be the set of the events of $s_j$, i.e. the measurement outcomes that can occur with non-zero probability when the system is in state $|G\rangle$ and the stabilizing operators $s_j$ are measured with single-qubit measurements. The set of all events is $S = \bigcup_{j=1,2,\ldots,2^{w_j}} S_j$. Two events are exclusive if there exists a $k \in \{1, \ldots, n\}$ for which the same single-qubit measurement gives a different outcome.

We give an example for events and exclusiveness. Let $n = 3$ and $s_2 = ZZX$ (we omit the superscripts for simplicity). This means that $ZXZ(G) = |G\rangle$, i.e., if the system is prepared in $|G\rangle$ and $s_2$ is measured by measuring $Z$ on the first qubit (with possible results $-1$ or $1$), $X$ on the second qubit, and $Z$ on the third qubit, then the product of the three results must be $1$. Therefore, $S_2 = \{zzz, zzz, zxz, zxz\}$, where hereafter $zzz$ denotes the event “the result 1 is obtained when $Z$ is measured on qubit 1, the result $-1$ is obtained when $X$ is measured on qubit 2, and the result $-1$ is obtained when $Z$ is measured on qubit 3”. As another example: if $n = 2$ and $s_1 = XZ$, then $S_1 = \{zx, zy\}$.

We now define the graph $H(G)$ starting from the exclusivity structure of the events explained above.
Definition 3.4.10. Let $G$ be a graph. Let $S$ be the stabilizer group of the graph state of $G$. We denote by $H(G)$ the graph whose vertices are the events in $S$ and the edges are all the pairs of exclusive events.

We give an example of the construction. Let us consider $G = P_3$, the path on three vertices, with $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}\}$. We construct $H(P_3)$. The stabilizer group $S$ (minus the identity) has the following elements: $s_1 = g_1 = XZI$, $s_2 = g_2 = ZXZ$, $s_3 = g_3 = IZX$, $s_4 = g_1g_2 = YYZ$, $s_5 = g_1g_3 = XIX$, $s_6 = g_2g_3 = ZYY$, and $s_7 = g_1g_2g_3 = -YXY$. For all $j = 1, \ldots, 2^3 - 1$, obtain all possible events (i.e., those which can happen with non-zero probability) when three qubits are prepared in the state $|\psi\rangle$ and three parties measure the observables corresponding to $s_j$. For instance, when $j = 1$, Alice measures $X^{(1)}$, Bob measures $Z^{(2)}$, and Charlie does not perform any measurement. Since the three qubits are in state $|\psi\rangle$, there are only two possible outcomes: Alice obtains $X^{(1)} = +1$ and Bob obtains $Z^{(2)} = +1$, denoted as $xI$; or Alice obtains $X^{(1)} = -1$ and Bob obtains $Z^{(2)} = -1$, denoted as $zI$. For $j = 2$, the only events that can occur are $zxx$, $zxz$, $zxz$, and $xzx$. The other events for the remaining $j$'s are obtained in a similar way. Now, let us construct the graph $H(P_3)$: the vertices represent possible events; two vertices are adjacent if and only events are exclusive (e.g., $xzI$ and $gIz$). Notice that each $s_j$ of weight $w_j$ generates $2^{w_j-1}$ vertices. A drawing of $H(P_3)$ is in Fig. 3.1.

We now prove that this graph has separation between quantum and classical independence number. We now prove the classical upper bound.

Theorem 3.4.11. Let $G$ be a graph on $n > 2$ vertices and let $H(G)$ be as in Definition 3.4.10. Then,

$$\alpha(H) < 2^n - 1. \quad (3.15)$$

Proof. For simplicity let $H = H(G)$. We use an argument very similar to [GTHB05, Lemma 1 and Theorem 1]. Each connected graph with more than two vertices has a subgraph with three vertices. For each of those we can see that $\alpha(H) < 7$ (by direct calculation, see [CPSS12, Table 1]). Therefore, we just need to show that if $G'$ is a subgraph of $G$ with $n'$ vertices and $\alpha(H') < 2^{n'} - 1$, where $H' = H(G')$, then $\alpha(H) < 2^n - 1$ for $n > 2$. Notice that $S'$, the stabilizer group of $G'$, is a subset of $S$. Therefore, in the graph $H$ we find cliques associated with $S'$, but containing slightly different events. For each $s_i \in S'$, the corresponding $s_i \in S$ has the same structure, with eventually some additional $Z$ operators. Let $\tilde{H}$ be the subgraph of $H$ induced by the vertices in cliques associated with the elements of $S'$. We need to show that if in $H'$ there is no vertex per clique to form a maximal independent set then neither are there in $\tilde{H}$. Therefore, $\alpha(H) < 2^n - 1$. Towards a contradiction, suppose there is an independent set $L$ of $\tilde{H}$ such that $|L| = 2^{n'} - 1$. We distinguish two cases:

- If the events at the vertices in $L$ do not have any $\pm$ element then we can map
them to an independent set in $H'$ of size $2^n' - 1$, just by ignoring the additional $Z$ operators. This contradicts the hypothesis that $\alpha(H') < 2^n' - 1$.

- If the events at the vertices in $L$ do have $\overline{z}$ elements then we can find another independent set $J$ with the same cardinality such that the events at its vertices do not have any $\overline{z}$ element. We can find $J$ as follows. One can check that an operator $s_i$ has the form $O^{(1)} \cdots Z^{(\ell)} \cdots O^{(n)}$ if and only if it has an odd number of $X^{(k)}$ and $Y^{(k)}$, with $\{\ell, k\} \in E(H)$. Therefore, complementing $\overline{z}^{(\ell)}$ and all occurrences of $X^{(k)}$ and $Y^{(k)}$ in the events at the vertices of the independent set $L$, we obtain the events in $J$ with the desired properties, and so we are back to the previous case.

For each $H(G)$, we define an orthogonal representation and use it to calculate the Lovász number.

**Definition 3.4.12** (Canonical orthogonal representation). Let $H = (V, E)$ be a graph as in Definition 3.4.10. Consider the vertex of $H$ associated to an event $S_{(i,j)} = \left(s_{(i,j)}^{(1)}, s_{(i,j)}^{(2)}, \ldots, s_{(i,j)}^{(n)}\right)$, where $i = 1 \ldots 2^n - 1$, $j = 1 \ldots 2^{w_i} - 1$, and $s_{(i,j)}^{(k)} \in$
\{I, x, \bar{x}, y, \bar{y}, z, \bar{z}\}, \text{ for each } k = 1, 2, \ldots, n. \text{ Let } |s^{(k)}_{(i,j)}\rangle \text{ be defined as follows:}

\begin{align*}
|s^{(k)}_{(i,j)}\rangle &\quad |x\rangle \quad |\bar{x}\rangle \quad |y\rangle \quad |\bar{y}\rangle \quad |z\rangle \quad |\bar{z}\rangle \quad I. \\
\sum_{i,j} |s^{(k)}_{(i,j)}\rangle &= |s^{(1)}_{(i,j)}\rangle \otimes |s^{(2)}_{(i,j)}\rangle \otimes \cdots \otimes |s^{(n)}_{(i,j)}\rangle : S_{(i,j)} \in V(H). \tag{3.16}
\end{align*}

Here, |\psi\rangle \text{ is an arbitrary unit vector in } \mathbb{C}^2 \text{ and } |y_+\rangle, |y_-\rangle \text{ are the eigenvectors of the Pauli matrix } Y \text{ with eigenvalue } +1 \text{ and } -1, \text{ respectively. The canonical orthogonal representation of } H \text{ is the set of vectors } \{|s_{(i,j)}\rangle = |s^{(1)}_{(i,j)}\rangle \otimes |s^{(2)}_{(i,j)}\rangle \otimes \cdots \otimes |s^{(n)}_{(i,j)}\rangle : S_{(i,j)} \in V(H)\}.

For example, in \(H(P_3)\) (see Fig. 3.1), the element of the canonical orthogonal representation of the vertex labeled by \(xI\bar{x}\) is \(|-\rangle \otimes |\psi\rangle \otimes |\bar{y}\rangle\).

**Lemma 3.4.13.** Let \(G\) be a graph on \(n\) vertices and \(H(G)\) as in Definition 3.4.10. Then,

\[\vartheta(H(G)) = 2^n - 1.\]

**Proof.** Let \(H = H(G)\). We first prove that \(\vartheta(H) \geq 2^n - 1\). It follows directly from Eq. (3.14) that \(\sum_{i=1}^{2^n-1} \langle G|s_i\rangle G = 2^n - 1\). We know that the eigenvectors with eigenvalue +1 of each operator \(s_i\) are in one-to-one correspondence with the vertices of a clique in \(H: |s_{(i,1)}, s_{(i,2)}, \ldots, s_{(i,2^n-1)}\rangle\). These are elements of the canonical orthogonal representation of \(H\). From the definition of the stabilizer group, for all \(s_i \in S\) and for all eigenvectors \(|s^{(i,j)}_{(i,j)}\rangle\) \((j = 1, 2, \ldots, 2^n-1)\) with eigenvalue \(-1\), we have \(\langle s^{(i,j)}_{(i,j)}|G = 0\), because \(|G\rangle\) is in the +1 eigenspace. Now, let \(s_i = \sum_j \lambda_{ij} |s_{(i,j)}\rangle \langle s_{(i,j)}|\) be an Hermitian eigendecomposition of \(s_i\). Thus,

\[2^n - 1 = \sum_{i=1}^{2^n-1} \langle G|s_i\rangle G\]

\[= \sum_{i=1}^{2^n-1} \sum_j \lambda_{ij} \langle G|s_{(i,j)}\rangle \langle s_{(i,j)}|G\]

\[= \sum_{i=1}^{2^n-1} \sum_{j: \lambda_{ij} = 1} \langle G|s_{(i,j)}\rangle \langle s_{(i,j)}|G\]

\[= \sum_{i=1}^{2^n-1} \sum_{j: \lambda_{ij} = 1} |\langle G|s_{(i,j)}\rangle|^2\]

\[\leq \vartheta(H),\]

where the inequality in the last line follows because a canonical orthogonal representation of \(H\) together with the state \(|G\rangle\) represents a feasible solution for the formulation (3.1) of the Lovász number.

We now prove the upper bound \(\vartheta(H) \leq 2^n - 1\). We can partition \(H\) into \(2^n - 1\) disjoint cliques by considering the events associated with each \(s_i\). Since
each clique of $H$ is an independent set in the complement $\bar{H}$, we can associate each independent set to a color and obtain $\chi(\bar{H}) \leq 2^n - 1$. It follows from the sandwich theorem [KD93] that

$$\vartheta(H) \leq \chi(\bar{H}) \leq 2^n - 1.$$ 

Combining the two directions, we have the desired result.

We are now ready to show that for every graph $G$ on $n$ vertices, the graph $H(G)$ has quantum independence number equal to $2^n - 1$, therefore strictly larger than the classical independence number by Theorem 3.4.11. This result opens directions for future studies, for example identifying subclasses or hierarchies where the separation is large or is easy to quantify.

**Theorem 3.4.14.** Let $G$ be a graph on $n$ vertices and let $H(G)$ be as in Definition 3.4.10. Then,

$$\alpha_q(H(G)) = 2^n - 1.$$ 

**Proof.** Let $H = H(G)$. We have the upper bound

$$\alpha_q(H) \leq \vartheta(H) = 2^n - 1,$$

where the inequality is [DSW13, Corollary 14] and the equality is Lemma 3.4.13. We need to show a matching lower bound on $\alpha_q(H)$. We do this by exhibiting a strategy for quantum players to win the independent set game on $H$ with $t = 2^n - 1$. Observe that $H$ can be partitioned into $2^n - 1$ cliques, one for each element of the stabilizer group. We denote by $S_i$ the set of events related to the element of the stabilizer group $s_i$. The clique corresponding to $s_i \in S$ consists of the vertices associated with the mutually exclusive events in the set $S_i$. The strategy is as follows. Alice and Bob share a maximally entangled state of local dimension $2^n(n-1)$. Since w.l.o.g. Alice and Bob use the same strategy, we can describe only Alice’s side. For each $i \in \{1, 2, \ldots, 2^n - 1\}$, Alice performs a projective measurement on her part of the shared state. The outcomes of the measurement are the elements of $S_i$. The strategy has to satisfy two properties to be correct:

1. For each $i \in \{1, 2, \ldots, 2^n - 1\}$, the projectors associated to elements of $S_i$ form a projective measurement.

2. For each edge $\{u, v\} \in E(H)$, projectors associated with $u$ and $v$ must be orthogonal (to satisfy the consistency condition (3.12)).

The next step is to exhibit the projectors and show that both properties are satisfied. In what follows we use the notation in Definition 3.4.12.

We begin by examining the case where $s_i$ does not contain any identity operator. In this case, each projective measurement will consist of projectors of rank
1 acting on $\mathbb{C}^{2^n(n-1)}$. Order the elements of $S_i$ arbitrarily. Let $s_i$ be of the form $O^{(1)} \cdots O^{(n)}$, where $O^{(k)} \in \{X, Y, Z\}$. Define for each $s^{(k)}_{(i,j)}$ the occurrence number $\nu(i,j,k)$ based on a chosen ordering: if the same eigenvector of $O^{(k)}$ occurs in $s^{(k)}_{(i,j)}$ for the $\ell$-th time in the chosen ordering then $\nu(i,j,k) = \ell$. Construct projectors starting from the canonical orthogonal representation and an ancillary space of dimension $n - 1$. For $s^{(k)}_{(i,j)}$, let

$$P_{(i,j)} = \bigotimes_{k=1}^{n} \langle s^{(k)}_{(i,j)} | \otimes | \nu(i,j,k) \rangle \langle \nu(i,j,k) | .$$

(3.17)

We show that Property 1 is satisfied. These projectors are mutually orthogonal for all vertices $(i,j)$. We need to prove that their sum is the identity. From the structure of the events in $S_i$ we observe that, for each $O^{(k)}$, the eigenvectors with eigenvalue $+1$ (and $-1$) occur in half of the elements of $S_i$. Therefore, in the construction of the projectors, a pair of $\pm 1$ eigenvectors for each $O^{(k)}$ is summed for each ancillary subspace. The sum of each subspace is the identity. Hence, the total sum is the identity for the whole space. We show now that also Property 2 is satisfied. If two projectors are in the same clique, orthogonality follows from the discussion above. Consider now two projectors of adjacent vertices from two different cliques that project to the same ancillary subspace. Since we started from an orthogonal representation, those projectors are orthogonal.

Now, consider the more general case where $s_i$ can contain identity operators. Let $s_i$ be of the form $O^{(1)} \cdots O^{(n)}$, where $O^{(k)} \in \{I, X, Y, Z\}$. We assume that $s_i$ has weight $w$. First consider the case where the first $w$ operators are different from Identity, $O^{(1)}, O^{(2)}, \ldots, O^{(w)} \neq I$. To construct the projective measurement for $S_i$, we initially construct the projectors for the first $w$ operators as in the previous case. We obtain rank-1 projectors acting on $\mathbb{C}^{2^w(w-1)}$. Choose a basis for $\mathbb{C}^{2^n(n-1)-2^w(w-1)}$ and let the projectors be

$$Q_{(i,j)} = \sum_{\ell=1}^{2^n(n-1)-2^w(w-1)} P_{(i,j)} \otimes | \ell \rangle \langle \ell | .$$

(3.18)

This ensures that the dimensions match and that Properties 1 and 2 hold. To finish the proof, we need to prove the general case where identity operators are in arbitrary positions and not all at the end. In this case, split the construction into subspaces so that each subspace has all the identities at the end. Obtain the projectors for the subspaces as described above and then obtain the final projectors by making tensor products of the projectors for the subspaces.
3.5 Bounds on the value of non-local games through game graphs

This section contains previously unpublished results. It is based on a collaboration with A. Chailloux, L. Mančinska and S. Severini. We consider a construction of graphs associated with non-local games from [CSW10]. It is known that the independence number of such graphs corresponds to the classical value of the game. Here we show that quantum independence number and the Lovász theta number are bounds on the quantum value of the game.

Consider a two-prover game $G$ with input sets $X, Y$, output sets $A, B$, predicate $\lambda : X \times Y \times A \times B \rightarrow \{0, 1\}$ and uniform distribution on the inputs.\(^4\)

**Definition 3.5.1.** A graph $G = (V, E)$ associated to the game $G$ has:

1. $V = \{xyab \mid x \in X, y \in Y, a \in A, b \in B \text{ and } \lambda(x, y, a, b) = 1\}$,
2. $E = \{\{xyab, x'y'a'b'\} \mid (x = x' \land a \neq a') \lor (y = y' \land b \neq b')\}$.

This definition is inspired by a construction in [CSW10] in the framework of contextuality of physical theories. The authors used something similar to Definition 3.5.1 for the special case of the CHSH game. Here we generalize to all games.

For simplicity, we prove the results in this section for the case where the game has the uniform distribution on the inputs and $\lambda$ is a boolean function. It is straightforward to generalize to games with real-valued predicate and any probability distribution $\pi$ of the inputs, as follows. Consider the (vertex) weighted graph with all the quadruples $xyab$ in the vertex set, labelled with weight($xyab$) = $\lambda(x, y, a, b) \cdot \pi(x, y)$, and the same edge set as before. The classical bound and the Lovász theta bound that we will prove later can be adapted by considering the weighted versions of these parameters. However, we do not know how to generalize our last result because we do not define the quantum independence number for a weighted graph.

Now we prove that that the classical value of a game can be expressed in terms of the independence number of its game graph.

**Theorem 3.5.2.** Let $\mathcal{G}$ be a non-local game with input sets $X$ and $Y$, uniform input distribution and associated graph $G$. Then

$$\omega(\mathcal{G}) = \frac{\alpha(G)}{|X \times Y|}.$$ 

\(^4\)Note the change of notation: in this section we denote the game as $\mathcal{G}$ and its predicate as $\lambda$ to avoid confusion with the standard notation for a graph $G$ and for its vertex set $V$. 
Proof. Let \( k = |X \times Y| \). We begin by proving that \( \omega(G) \geq \alpha(G)/k \). Namely, we show that given a maximal independent set \( I \subseteq V \) of size \( \ell \), we can exhibit a strategy for \( G \) that answers correctly to at least \( \ell \) of the \( k \) questions. By the structure of \( G \), the independent set \( I \) cannot contain vertices \( xyab \) and \( x'y'a'b' \) such that \( a \neq a' \). Similarly, \( I \) cannot contain vertices \( xyab \) and \( x'ya'b' \) such that \( b \neq b' \). Hence, we have the following strategy: on input \( x \), Alice outputs the unique \( a \) determined by the vertices in the independent set \( I \). Bob behaves similarly. Since \( V \) contains only winning quadruples \( xyab \), the size \( \ell \) of the independent set means Alice and Bob answer correctly to at least \( \ell \) input pairs. Hence, \( \omega(G) \geq \ell/k \).

Now we show that \( \omega(G) \leq \alpha(G)/k \), i.e., if there exists a strategy that wins on \( \ell \) of the \( k \) input pairs, then there exists an independent set with weight \( \ell \). We have that w.l.o.g. classical strategies consist of a pair of deterministic functions. Fix Alice and Bob’s deterministic functions \( f_A \) and \( f_B \) that win on \( \ell \) input pairs. Now take the set of quadruples \( S = \{(x, y, f_A(x), f_B(y))\}_{x \in X, y \in Y} \). We have that \( I = S \cap V \) is a set of \( \ell \) vertices of \( G \). Since \( f_A \) and \( f_B \) are deterministic, I cannot contain vertices \( xyab \) and \( x'ya'b' \) such that \( a \neq a' \) nor vertices \( xyab \) and \( x'ya'b' \) such that \( b \neq b' \). Therefore, there cannot be an edge between any pair of the elements of \( I \) and we have that \( I \) is an independent set of \( G \) of size \( \ell \). Hence, \( \alpha(G) \geq \ell \). Combining the two directions proves the theorem. \( \square \)

Bounds on the quantum value of a game

Cabello et al. [CSW10] observe that the quantum value of the CHSH game is equal to the theta number of its associated graph divided by the number of questions. We have found by direct calculation that this is not always true for general games, for example in the case of the 2-fold parallel repetition of CHSH. Instead, we have the following upper bound.

**Theorem 3.5.3.** Let \( G \) be a non-local game with input sets \( X \) and \( Y \), uniform input distribution and associated graph \( G = (V, E) \). Then

\[
\omega^*(G) \leq \frac{\vartheta(G)}{|X \times Y|}.
\]

**Proof.** Let \( k = |X \times Y| \). Consider a quantum strategy for \( G \) that achieves the value \( \omega^*(G) \). It consists of a shared entangled state \( |\psi\rangle \) and a collection of projective measurements \( \{P^x_a\}, \{Q^y_b\} \), such that

\[
\sum_{xyab} \frac{1}{k} \lambda(x, y, a, b) \langle \psi | P^x_a \otimes Q^y_b | \psi \rangle = \frac{1}{k} \sum_{xyab \in V} \langle \psi | P^x_a \otimes Q^y_b | \psi \rangle = \omega^*(G).
\]

For each quadruple \( xyab \) let \( |\psi_{xyab}\rangle = P^x_a \otimes Q^y_b | \psi \rangle \). This is an orthogonal representation of \( G \), since for every edge \( (xyab, x'y'a'b') \) either \( P^x_a P^x_a' = 0 \) or...
$Q_b^y Q_{b'}^y = 0$. Now for each $xyab$ consider the normalized vector

$$|\psi'_{xyab}⟩ = \frac{|\psi_{xyab}⟩}{||\psi_{xyab}||} = \frac{|\psi_{xyab}⟩}{\sqrt{⟨\psi|P_a^x \otimes Q_b^y|ψ⟩}}.$$  

We have that $\{\psi'_{xyab}\}_{xyab \in V}$ and $ψ$ are a feasible solution for the formulation (3.1) of $ϑ(G)$.

We conclude

$$ϑ(G) ≥ \sum_{xyab \in V} |⟨ψ|ψ_{xyab}⟩|^2$$

$$= \sum_{xyab \in V} \left|\frac{⟨ψ|ψ_{xyab}⟩}{||ψ_{xyab}||}\right|^2$$

$$= \sum_{xyab \in V} \frac{⟨ψ|P_a^x \otimes Q_b^y|ψ⟩^2}{⟨ψ|P_a^x \otimes Q_b^y|ψ⟩}$$

$$= \sum_{xyab \in V} ⟨ψ|P_a^x \otimes Q_b^y|ψ⟩$$

$$= k \cdot ω^*(G).$$

We now have the following lower bound in terms of the quantum independence number.

**Theorem 3.5.4.** Let $G$ be a non-local game with input sets $X$ and $Y$, uniform input distribution and associated graph $G = (V, E)$. Then

$$ω^*(G) ≥ \frac{α_q(G)}{|X \times Y|}$$

To prove the theorem, we will use the following lemma.

**Lemma 3.5.5.** Let $M, N$ be positive semidefinite matrices. Then for any vector $|v⟩$, we have that

$$⟨v|\text{supp}(M + N)|v⟩ ≥ ⟨v|\text{supp}(M)|v⟩,$$

where $\text{supp}(M)$ denotes the projector onto the support (i.e., the column space) of $M$.

**Proof.** If $P$ is a projector onto a subspace $Π$ then $⟨v|P|v⟩$ is the squared length of the projection of $|v⟩$ into $Π$. Hence, to prove the lemma it suffices to show that $\text{supp}(M) ⊆ \text{supp}(M + N)$, where by abusing the notation we use $\text{supp}$ to denote the support itself (rather than the projection onto it).
For contradiction, suppose that \( \text{supp}(M) \nsubseteq \text{supp}(M + N) \). Then the orthogonal complement of \( \text{supp}(M) \) (i.e. the nullspace \( \text{Null}(M) \)) does not contain \( \text{Null}(M + N) \). Hence we can pick a vector \( |w\rangle \) such that \( (M + N)|w\rangle = 0 \) but \( M|w\rangle \neq 0 \). This further implies that

\[
\langle w|N|w\rangle = \langle w|(M + N)|w\rangle - \langle w|M|w\rangle = -\langle w|M|w\rangle < 0,
\]

since \( M \) is positive semidefinite and \( M|w\rangle \neq 0 \). This completes the proof as we have reached a contradiction with the initial assumption that \( N \) is positive semidefinite.

\[\square\]

**Proof of Theorem 3.5.4.** Given a quantum strategy \( \{P_{xyab}\} \) for the independent set game on \( G \) with parameter \( t \), we construct a strategy to win the game \( G \) with probability at least \( t/|X \times Y| \), as follows.

Players share a maximally entangled state with local dimension \( d \) (which is the dimension of the projectors above). On input \( x \), Alice measures her half of the state using the projective measurement \( \{P_x^a\}_{a \in A \cup \{I - \sum_a P_x^a\}} \), where the individual elements are defined as follows:

\[
P_x^a = \text{supp} \left( \sum_{xayb \in V} \sum_i P_i^{xayb} \right),
\]

where we use \( \text{supp}(M) \) to denote the projector onto the image of \( M \). We show that this is a valid projective measurement. For all \( y, b, y', b' \) there is an edge \( (xyab, xy'a'b') \in E \). Therefore in the strategy for the independent set game we have that for all \( i, j \) each projector \( P_{xyab}^i \) is orthogonal to \( P_{xy'a'b'}^j \). Hence, for all \( a \neq a' \) we have \( P_{xyab}^a \cdot P_{xyab}^{a'} = 0 \). Bob constructs projectors \( P_{yb}^y \) similarly.

Now we lower bound the quantum value of \( G \) as follows:

\[
|X \times Y| \cdot \omega^*(G) \geq \sum_{xyab \in V} \langle \psi|P_x^x \otimes P_y^y|\psi\rangle = \sum_{xyab \in V} \langle \psi|\text{supp} \left( \sum_{i,j} \sum_{y'b'} \sum_{x'y'ab} P_{x'y'ab}^{i} \otimes P_{x'y'ab}^{j} \right)|\psi\rangle,
\]

where we have used the fact that \( \text{supp}(M \otimes N) = \text{supp}(M) \otimes \text{supp}(N) \) for all matrices \( M, N \) to obtain the last equality. Now by applying Lemma 3.5.5, we drop all the terms except the ones with \( i = j, a = a', b = b', x = x', \) and \( y = y' \).
and we have that
\[ |X \times Y| \cdot \omega^*(G) \geq \sum_{xyab \in V} \langle \psi | \text{supp} \left( \sum_i P^i_{xayb} \otimes P^i_{xayb} \right) |\psi \rangle \] (3.19)
\[ = \sum_{xyab \in V} \langle \psi | \left( \sum_i P^i_{xayb} \otimes P^i_{xayb} \right) |\psi \rangle \] (3.20)
\[ = \sum_{xyab \in V} \sum_i \frac{1}{d} \text{Tr}(P^i_{xayb}) \] (3.21)
\[ = \sum_i \frac{1}{d} \text{Tr}(I_d) \] (3.22)
\[ = \alpha_q(G). \] (3.23)

In the above we have observed that supp\((P + Q) = P + Q\) for mutually orthogonal projectors \(P\) and \(Q\) to get Expression (3.20). We have used properties of \(|\psi\rangle = \frac{1}{\sqrt{d}} \sum_i |i, i\rangle\) to obtain Expression (3.21). We have used the fact that, for all \(i\), \(\{P^i_{xayb} : \lambda(x,a,y,b) = 1\}\) forms a measurement to obtain Expression (3.22).

**Pseudo-telepathy games**

Here we show that from a class of pseudo-telepathy games (i.e., games with quantum value 1 and classical value strictly less than 1), one can obtain graphs with separation between independence number and quantum independence number.

**Theorem 3.5.6.** Let \(G\) be a pseudo-telepathy game with a 0-1 valued verification function \(\lambda\), such that the best quantum strategy uses a maximally entangled state \(|\psi\rangle\). Let \(G\) be the corresponding game graph. Then,
\[ \omega^*(G) = \frac{\alpha_q(G)}{|X \times Y|}. \]

**Proof.** From Theorem 3.5.4 we have \(\alpha_q(G) \leq |X \times Y| \cdot \omega^*(G)\). We need to prove the other direction.

Let \(\{P^x_a\}, \{Q^y_b\}\) be the strategies that win the game \(G\) on \(|\psi\rangle\). We have:
\[ \sum_{xy} \pi(x, y) \sum_{ab : \lambda(xyab) = 1} \langle \psi | P^x_a \otimes Q^y_b |\psi \rangle = 1, \]
so for all \((x, y)\) we must have
\[ \sum_{ab : \lambda(xyab) = 1} \langle \psi | P^x_a \otimes Q^y_b |\psi \rangle = 1 \]
and for all quadruples \((x, y, a, b)\) such that \(\lambda(xyab) = 0\) we have \(P^x_a Q^y_b = 0\).

Let \(\Pi_{xyab} = P^x_a Q^y_b\). We observe:
1. For all \((x, y)\) we have
\[
\sum_{a,b: \lambda(xyab) = 1} P_x^a Q_y^b = \sum_{a,b} P_x^a Q_y^b = \sum_a P_x^a \sum_b Q_y^b = I,
\]
where the second equality follows from \(Q_y^b Q_y^{b'} = \delta_{bb'}\).

2. For each edge \((x, y, a, b), (x', y', a', b')\) we have
\[
\Pi_{xyab} \Pi_{x'y'a'b'} = 0,
\]
because if \(x = x'\) and \(a \neq a'\) then \(P_x^a P_x^{a'} = 0\), and if \(y = y'\) and \(b \neq b'\) then \(Q_y^b Q_y^{b'} = 0\).

Therefore, we can construct \(|X \times Y|\) projective measurements that are a winning strategy for the independent set game with \(t = |X \times Y|\) as follows. For each pair \((x, y)\) consider the projective measurement \(\{\Pi_{xyab}\}_{a,b: \lambda(xyab) = 1}\) (and zero matrices for the other vertices of the graph). The first observation above proves that those are valid projective measurements; the second observation shows that they respect the consistency condition (3.12).

\[\square\]

3.6 Concluding remarks and open problems

The main contribution of Sections 3.3 and 3.4 is the introduction and use of a formal generalization of Kochen-Specker sets. In particular, we showed that projective KS sets lead to graphs for which the quantum independence number is strictly larger than the independence number. We have also shown that projective KS sets completely characterize the graphs for which the quantum chromatic number is strictly smaller than the chromatic number. Furthermore, we used projective KS sets to relate quantum chromatic number to quantum independence number. For all graphs obtained with our construction the Lovász theta function is equal to the quantum independence number. Hence, although our construction contributes to shed light on the link between the Lovász theta function and quantum graph parameters, it cannot directly be used to resolve whether or not the quantum independence number equals the Lovász theta function [DSW13]. An open question is: can we use a graph \(G\) with \(\alpha(G) < \alpha_q(G)\) to construct a \(G'\) with \(\chi_q(G') < \chi(G')\)? Such a construction would be complementary to Theorem 3.4.8.

We showed in Section 3.5, with the use of a specific graph construction, that the quantum independence number is a bound to the value of non-local games. Moreover, we have shown that for a class of pseudo-telepathy games that quantum players can win using projective measurements on maximally entangled state, this bound is tight. The same class of games is shown in Section 3.2.2 to be in one-to-one correspondence with projective KS sets. It is not clear to us if those
two results together could be used to prove something stronger. Perhaps the whole class could be interpreted as pseudo-telepathy games based on some graph parameter (maybe the homomorphism games in [RM12]) and the relationship to the quantum independence number would be a consequence of this. Another open question is whether generalized KS sets of Definition 3.2.7 that are not projective KS sets can be used to construct pseudo-telepathy games. Furthermore, can a wider class of pseudo-telepathy games be characterized using generalized KS sets than projective ones?

Finally, a fundamental open question. Determining the computational complexity of $\chi_q(G)$ and $\alpha_q(G)$ as a function of the number of vertices in $G$ is now a long standing open question. Can we use the relationship with KS sets to answer this question?

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