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The impact of noise and instantaneous mixing on measures of Granger causal flux direction

This work is in preparation as:

Abstract

We consider the problem that measurement noise affects measures of Granger (G-) causal flux direction. A popular measure of causal directionality is the integrated slope of the coherency (PSI), and it has been argued that it is robust against noise. We show that for realistic (normal and Minnesota) priors that encompass bidirectionally coupled systems, PSI is not a valid measure of G-causality for a substantial % of systems. Further, we find that PSI, like G-causality measures, can be strongly affected by the addition of linearly mixed noise sources. Our simulations show that testing for G-causality on time-reversed signals provides a more conservative measure of causal directionality producing fewer false positives, both in the absence and presence of noise, at the cost of yielding fewer true positives than classic measures of Granger-causality. We propose an additional rejection criterion on the cross-covariance function that strongly reduces the % of false positives.
4.0 Introduction

The Wiener-Granger definition of causality allows inference of causal relationships from time series (Granger, 1969; Wiener, 1956). A discrete-time variable \(x_1(t)\) is said to \(G\)-cause \(x_2(t)\), if \((x_1(t - 1), \ldots, x_1(t - M))\) can forecast \(x_2(t)\) after conditioning on the universal information set (excluding \(x_2\)) available at time \(t - 1\) (Granger, 1969). This idea is commonly implemented by VAR (vector autoregressive) modelling, with \(G\)-causality from \(x_i\) to \(x_j\) measured by examining \(x_i\)'s effect on the residual errors in forecasting \(x_j(t)\) (Granger, 1969; Geweke, 1982). \(G\)-causality has attracted considerable attention in the neurosciences (e.g. (Bosman et al., 2012; Gregoriou et al., 2009; Bressler & Seth, 2011; Ding et al., 2006)). A characteristic of neuroscience experiments is that measurements can be strongly corrupted by noise, which affects \(G\)-causality measures (Newbold, 1978; Nalatore et al., 2007; Nolte et al., 2008; Haufe et al., 2012a). This poses a problem especially for techniques such as the scalp EEG (electroencephalogram), with currents from single sources spreading instantaneously to multiple sensors (‘volume conduction’) (Nolte et al., 2004; Stam et al., 2007; Vinck et al., 2011), distorting the measurement of \(G\)-causal flux direction (Nolte et al., 2008).

To address this problem, Nolte et al. (2008) proposed to measure \(G\)-causal directionality by the sign of the integrated slope of the coherency (phase slope index; PSI). A non-zero PSI implies temporal precedence of one signal to the other, which was taken as indicative of a \(G\)-causal relationship (Nolte et al., 2008). It was stated that PSI is insensitive to volume conduction, because instantaneous mixing adds only a symmetric component to the cross-covariance \(s_{12}(\tau) \equiv E((x_1 - \mu_1)(x_2 - \mu_2))\) (Nolte et al., 2008; Haufe et al., 2012a). Another approach to protect causality measures against common noise is to compare \(G\)-causality values with those for time-reversed signals \(x_1(-t)\) and \(x_2(-t)\) (Haufe et al., 2012a) (henceforth referred to as reversed \(G\)-causality testing).

A first aim of this letter is to address several open issues concerning these alternative causality measures, namely: (i) It is not \textit{a priori} obvious that statistics like PSI are in fact valid measures of \(G\)-causality. To resolve this issue, they should be evaluated not only for ‘easy’ priors with unidirectional \(G\)-causal flow (Nolte et al., 2008; Haufe et al., 2012a), but also for ‘difficult’ prior distributions that include bidirectional \(G\)-causal flow, considering the brain’s interconnected nature. (ii) Differential effects of independent and dependent noise on \(G\)-causality measures have not been studied. This question is relevant because for some applications (e.g. spike trains (Mitzdorf, 1985)), noise is largely independent, while for others (e.g. EEG (Nolte et al., 2004)) it is not. (iii) Nolte et al. (2008) provided simulations only for finite samples, with the sum of fractions of true (TP) and false positives (FP) for PSI (maximum \(\approx 5\%\)) smaller than 1. It remains unclear what the asymptotic behavior of alternative \(G\)-causality measures like PSI is. A second main aim of this letter is to explore whether other properties of the auto-covariance function besides symmetry (as used by (Nolte et al., 2008)) can be used to further reduce sensitivity to mixed noise. We will show that the presence of a peak at \(\tau = 0\), i.e. no delay between signals, is a good indicator of a low FP fraction.

4.1 Theoretical considerations

Let \(\mathbf{x}(t) = [x_1(t), x_2(t)]\) be discrete-time wide-sense stationary time series for which the spectral density matrix \(\mathbf{S}(\omega)\) exists, with VAR model \(\mathbf{x}(t) = \sum_{\tau=1}^{M} \mathbf{A}(\tau)\mathbf{x}(t-\tau) + \mathbf{U}(t)\), where \(\mathbf{U}(t)\)
is innovation noise with covariance matrix $\Sigma \equiv \text{Cov}\{\mathbf{U}(t), \mathbf{U}(t)\}$, and $\mathbf{A}(\tau)$ the model coefficients. If $A_{ij}(\tau) = 0$ for $i \neq j$, we denote $\Sigma_{(\text{restr})}$ (restricted model). Granger’s time-domain measure of (prima facie) G-causal flow is defined $f_{j \rightarrow i} \equiv \ln(\Sigma_{ii}^{(\text{restr})}) - \ln(\Sigma_{ij}) (i \neq j)$ (Granger, 1969). We define $x_1$ to be $G$-causal dominant over $x_2$ (and vice versa), if $g \equiv f_{1 \rightarrow 2} - f_{2 \rightarrow 1} > 0$. The problem is stated as providing a measure of $\text{sgn}(g)$ that maximizes the number of TP and minimizes the number of FP in the presence of noise (where FP outweigh TP in importance by some factor).

PSI is an alternative measure of $\text{sgn}(g)$. Nolte et al. (2008), defined $\psi \equiv \int \mathcal{S}[C'(\omega)]d\omega$, with $C(\omega) \equiv S_{12}(\omega)/\sqrt{S_{11}(\omega)S_{22}(\omega)}$ the coherency, and slope $C'(\omega) \equiv \lim_{d\omega \to 0} C(\omega)C^*(\omega + d\omega)/d\omega$. It was motivated by the argument that causal dominance implies temporal precedence (Nolte et al., 2008), with a shift of $s_{12}(\tau)$ by $\Delta \tau$ corresponding to multiplication of $S_{12}(\omega)$ by $e^{-i\omega\Delta \tau}$. Note that coherency can be related to linear prediction in the context of constructing an optimal noncausal (Wiener) filter. Suppose we seek a a filter kernel $h(\tau)$ that minimizes the error $E[(x_2(t) - \hat{x}_2(t))^2]$, with $\hat{x}_2(t) = \sum_{\tau=-\infty}^{\infty} h(\tau)x_1(t-\tau)$, corresponding to a linear regression model comparable to the VAR model, except that only $A_{12}(\tau) \neq 0$ here. The optimal filter kernel is given by $h(\tau)$ with Fourier Transform $H(\omega) = S_{21}(\omega)/S_{11}(\omega)$, such that the minimum error equals $\epsilon_{\text{min}} = \int S_{22}(\omega)\left(1 - |C(\omega)|^2\right)d\omega$, where $|C(\omega)|^2$ plays a similar role as in linear regression analysis. If prediction errors tend to be more reduced by the $\tau > 0$ than the $\tau < 0$ (or vice versa) components of $h(\tau)$, then $C(\omega)$ will typically, but not necessarily, have a negative (or positive) slope (as follows from the properties of the Fourier Transform).

Nolte et al. (2008) argued that the main advantage of PSI over $g$ is that it should have reduced noise sensitivity, since $\psi = 0$ if $s_{12}(\tau) = s_{12}(-\tau)$ for all $\tau$ ($\Rightarrow \mathcal{S}[S_{12}] = 0 \Rightarrow \psi = 0$), and because instantaneous mixing adds only a symmetric component to $s_{12}(\tau)$.

We also consider a measure of $\text{sgn}(g)$ based on computing $f_{j \rightarrow i}$ for $x^{\text{rev}}(t) = x(-t)$ (Haufe et al., 2012a), with $S^{\text{rev}}(\omega) = S(-\omega)$ and $S^{\text{rev}}(\tau) = s(-\tau)$. Reversed G-causality testing holds that $x_1$ is G-causal dominant over $x_2$ if $g \equiv f_{2 \rightarrow 1} - f_{1 \rightarrow 2} > 0$, i.e. if the G-causal dominance flips for the time-reversed signals. No decision about G-causal dominance is taken when $\text{sgn}(g) \neq \text{sgn}(g^{\text{rev}})$. Haufe et al. (2012a,b) concluded, based on their simulations (for unidirectional systems), that PSI and reversed Granger testing yielded quantitatively similar results, although it was observed that PSI yielded better performance in terms of the TP/FP mix (Haufe et al., 2012b).

### 4.1.1 Case of no noise

We decompose $f_{j \rightarrow i}$ in terms of auto- and cross-correlations, revealing the limiting case where reversed Granger testing performs well. Treating the VAR model as a multiple regression model (Pierce, 1979) yields

$$ f_{j \rightarrow i} = -\ln\left(\frac{1 - R_{i,j \rightarrow i}^2}{1 - R_{i,j \rightarrow i}^2}\right), $$

with coefficient of determination

$$ R_{i,j \rightarrow i}^2 = \mathbf{u}_{i,j \rightarrow i} [\mathbf{D}_{i,j \rightarrow i}]^{-1} \mathbf{u}_{i,j \rightarrow i}^T, $$

where

$$ \mathbf{u}_{i,j \rightarrow i} \equiv (\rho_{ii}(1), \ldots, \rho_{ii}(M), \rho_{ij}(1), \ldots, \rho_{ij}(M)), $$

$$ \mathbf{D}_{i,j \rightarrow i} = \text{Cov}\{\mathbf{U}(t), \mathbf{U}(t+\tau)\}, $$

and

$$ M \equiv \min\{M_1, M_2\}, $$

where $M_1$ and $M_2$ denote the lags for $\mathbf{U}(t)$ and $\mathbf{U}(t+\tau)$, respectively.
with cross-correlation function
\[ \rho_{ij}(\tau) \equiv \frac{s_{ij}(\tau)}{\sqrt{s_{ii}(0)s_{jj}(0)}}. \] (4.4)

(Subscript $i, j \rightarrow i$ indicates that $x_i$ and $x_j$ are used to predict $x_i$). The symmetric block matrix
\[ D_{i,j-\rightarrow i} \equiv \begin{pmatrix} D_{ii} & D_{ij} \\ D_{ij}^T & D_{jj} \end{pmatrix} \] (4.5)
holds the correlations among the predictors, where $D_{ii,km} \equiv \rho_{ii}(k-m), D_{12,km} \equiv \rho_{12}(m-k)$. Define
\[ [D_{i,j-\rightarrow i}]^{-1} \equiv E \equiv \begin{pmatrix} E_{ii} & E_{12} \\ E_{12}^T & E_{jj} \end{pmatrix} \] (4.6)

Note that $E$ is symmetric. For the restricted model $v_{i-\rightarrow i} \equiv (\rho_{ii}(1), \cdots \rho_{ii}(M))$, and
\[ R^2_{i-\rightarrow i} = v_{i-\rightarrow i} [D_{ii}]^{-1} v_{i-\rightarrow i}^T. \] (4.7)

We now show the behavior of reversed Granger testing in two limiting cases.

1) Suppose that the cross- and autocorrelations are small, that is $\forall \tau, |\rho_{ij}(\tau)| \leq \mu << 1$ (note that $\rho_{ii}(0) = 1$). Then, $E_{ij} \approx 1 + O(\mu^2)$ for $i = j$ and $E_{ij} \approx -[D_{i,j-\rightarrow i}]_{ij} + O(\mu^2)$ for $i \neq j$. Taylor expansion of $\ln(z)$ around $z = 1$ (which is justified because of the small Granger values) yields
\[ f_{j-\rightarrow i} \approx \sum_{\tau=1}^{M} \rho_{12}^2(\tau) + O(\mu^3) \approx f_{i-\rightarrow j}^{\text{rev}}. \] (4.8)

Thus, for small auto and cross-correlations, Granger values approximately equal the energy of the left or right-sided part of the cross-correlation function. Indeed, for small $A(\tau)$'s, reversed G-causality testing invariably identifies $\text{sgn}(g)$, and tends to fail for a larger $\%$ of systems when the squares of $A(\tau)$'s are larger (corresponding to larger squared correlations) (Fig 1A-B, main text).

2) If the signals have similar autocorrelation functions, i.e. if $D_{11} \approx D_{22}$, then it follows directly from eq. 4.1 that $f_{j-\rightarrow i} \approx f_{i-\rightarrow j}^{\text{rev}}$.

4.1.2 Case of independent noise

Now consider the problem that noise is added to the measurements, such that $x^{(\epsilon)}(t) = (1 - \gamma)x(t) + \gamma \epsilon(t)$ where $\gamma$ sets the signal-to-noise ratio (SNR), and $\text{Cov}[x_i(t), \epsilon_j(t+\tau)] = 0$ for all $(\tau, i, j)$. The equalities $S^{(\epsilon)}(\omega) = (1 - \gamma)^2 S(\omega) + \gamma^2 S_\epsilon(\omega)$ and $s^{(\epsilon)}(\tau) = (1 - \gamma)^2 s(\tau) + \gamma^2 s_\epsilon(\tau)$ hold. Independent noise only affects $S_{ij}(\omega)$ and $s_{ij}(\tau)$. Hence, it affects PSI only via $C(\omega)$’s denominator.

The effect of measurement noise is fundamentally different from innovation noise: Whereas the latter affects future values of $x$ through $A$, the former merely distorts the measurement of
the present value of $x$, such that it cannot be subsumed by the innovation noise $U$ term in the VAR model. Now suppose that uncorrelated noise $\eta(t)$ is added only to $x_1(t)$. Then,

$$
\rho_{11}^{(e)}(\tau) = \frac{s_{11}(\tau) + \rho_{\eta\eta}(\tau)\sigma_\eta^2}{\sigma_1^2 + \sigma_\eta^2}.
$$

(4.9)

Suppose that, in addition, $\rho_{\eta\eta}(\tau)$ is small (i.e., nearly white noise). Then, $\rho_{11}^{(e)}(\tau) \approx \rho_{11}/\vartheta$ for $\tau \neq 0$ where $\vartheta = (\sigma_1^2 + \sigma_\eta^2)/\sigma_1^2$. At the same time,

$$
\rho_{12}^{(e)}(\tau) = \frac{\rho_{12}(\tau)}{(\sigma_1^2 + \sigma_2^2)^{1/2}\sigma_2}
$$

(4.10)

is decreased relative to $\rho_{12}$ by only $1/\sqrt{\vartheta}$, uncorrelated of the noise’s color. As for $f_{1 \rightarrow 2}$, note that $\rho_{22}^{(e)}(\tau) = \rho_{22}(\tau)$, while $\rho_{21}^{(e)}(\tau)$ decreases at rate $1/\sqrt{\vartheta}$.

The impact of uncorrelated noise on Granger values can be analyzed in the limiting case that the magnitudes of the cross- and autocorrelations are small, that is $\forall \tau, |\rho_{ij}(\tau)| \leq \mu << 1$. It follows from eq. 4.8 that $\text{sgn}(g)$ (and $f_{j \rightarrow i}/f_{i \rightarrow j}$) tends to be unaffected by noise, even though $f_{j \rightarrow i}$ and $f_{i \rightarrow j}$ decrease at rate $1/\vartheta$.

### 4.1.3 Case of mixed noise

Now take the case of mixed noise, by letting $\epsilon = BZ$ where $B$ is a $2 \times S$ mixing matrix and $Z$ is an $S \times 1$ noise vector. Now, $s_{\epsilon}(\tau) = s_{\epsilon}(-\tau)$, implying $\Im\{s_{\epsilon}(\omega)\} = 0$. Nolte et al. (2008) has shown that standard G-causality measure $g$ fails for large $\gamma$, and concluded that PSI is insensitive to mixed noise. The PSI (like metrics in (Nolte et al., 2004; Vinck et al., 2011)) is based on the notion that mixed noise adds a symmetric component to $s_{12}(\tau)$, and is designed to suppress this component. We note however that symmetry is not the only usable property of the scaled noise autocorrelation function $B_{1k}B_{2k}s_{12}(\tau)$; another property is a global peak or valley at $\tau = 0$ (symmetry does not imply this property). Hence, if the noise is dominant ($\gamma$ large), $s_{12}^{(e)}$ will tend to have a global peak or valley at $\tau = 0$ (provided that the sampling frequency is not too high). We propose that examination of the occurrence of a global peak/valley at $\tau = 0$ may provide a very useful indication of the expected % of FP (see below).

PSI can, by construction, be affected by adding mixed noise, because: (i) $\Re\{C(\omega)\}$ does play a role in the PSI measure, as $\Im\{C'(\omega)\}$ cannot be rewritten as $\lim_{d\omega \rightarrow 0} \Im\{C(\omega)\}/d\omega$; (ii) adding a symmetric function $s_{\epsilon}(\tau)$ to an asymmetric function $s(\tau)$ can create asymmetries in the opposite direction, because $s(\tau)$ can be either negative or positive. That is, $|s_{12}(\tau)| > |s_{12}(-\tau)|$ does not imply $|s_{12}^{(e)}(\tau)| > |s_{12}^{(e)}(-\tau)|$ (Fig 4.1). For example, suppose that $s_{12}(\tau) = u(\tau)e^{-\tau} \cos(\tau)$ with $u(\tau)$ the Heaviside step function. If now $s_{\epsilon_{1,\epsilon_{2}}}(\tau) = -e^{-|\tau|} \cos(\tau)$, then $s_{12}^{(e)}(\tau) = 0$ for $\tau > 0$ and $s_{12}^{(e)}(\tau) = -e^{\tau} \cos(\tau)$ for $\tau < 0$, flipping $\text{sgn}(\vartheta)$. Thus, the effect of measurement noise is fundamentally different for the case of dependent noise as it affects $s_{12}^{(e)}(\tau)$ and can flip asymmetries in the opposite direction, unless the noise is white.


4.2 Simulations

4.2.1 Validity of G-causality measures

We first address the question under which conditions \( \text{sgn}(\psi) \) and \( \text{sgn}(g^{\text{rev}}) \) `typically' equate \( \text{sgn}(g) \). Simulations performed by (Nolte et al., 2008) comprised `easy' problems with unidirectional G-causal flow. The notion of `typicality' requires the specification of a prior distribution \( P(A, \Sigma) \) on the space of VAR models, where \( A = [A(1), \ldots, A(M)] \). We assume that variables interact at a delay, which is justified for nervous systems if the sampling frequency is high enough, hence taking \( \Sigma \) diagonal. As the values on the diagonal of \( \Sigma \) represent only a scaling of the signals, we take \( P(\Sigma = I) = 1 \). Our prior beliefs are now specified by \( P(A) \). The expected fraction of TP equals

\[
\zeta = \frac{1}{2} + \frac{1}{2} \int_A P(A) \frac{g(A, \Sigma) \psi(A, \Sigma)}{|g(A, \Sigma) \psi(A, \Sigma)|} dA , \tag{4.11}
\]

where we substitute \( \psi \) by \( g^{\text{rev}} \) for reversed G-causality testing. The expected fraction of FP is defined similarly, however, by construction, reversed G-causality testing does not yield FP in the absence of noise.

We first consider an informative normal prior \( A_{ij}(\tau) \sim N(0, \sigma^2_{ij}) \). For many \( A \)'s, the system is unstable, `biasing' \( P(A, \Sigma) \) towards small \( ||A||_F \) (Frobenius norm). Because the integral in eq. 4.11 is analytically intractable, we performed numerical approximations of \( \zeta \) by generating random VAR models according to \( P(A) \). Since most systems became unstable for large \( \sigma_A \), we iteratively updated draws by iteratively replacing the maximum squared element of \( A \) with a new random variate from \( N(0, \sigma^2_A) \), until the system was stable. We also examined the unidirectional case \( P(A_{12}(\tau) = 0) = 1 \), as in (Nolte et al., 2008). To generalize our results across priors, we evaluated different \( \sigma_A \) and \( M \). PSI and \( g^{\text{rev}} \) often failed to identify the G-causal dominant variable (i.e. \( \zeta < 1 \)) (Fig 4.1), especially for large \( M \) and \( \sigma^2_A \). PSI did not control the FP fraction for many \( (M, \sigma^2_A) \). Both PSI and \( g^{\text{rev}} \) performed best for small \( M \) and \( \sigma_A \) (as predicted above), which yields coherences that are typical for electrophysiological data, i.e. few peaks in the coherence spectrum, and medium coherence values (Gregoriou et al., 2009; Bosman et al., 2012) (Fig 4.1). Reversed G-causality testing yielded more or less TP than PSI when \( \sigma^2_A \) was small or large, respectively, and failed to detect unidirectional G-causality in some cases, as opposed to PSI (Fig 4.1). 

We did not evaluate the constant prior \( P(A) = 1 \), being an improper prior having infinite support, with almost all systems unstable for large \( ||A||_F \). Instead, we consider a popular prior in macro-economic forecasting, the shrinkage Minnesota prior (Litterman, 1986), which has \( A_{ij}(m) \sim N(\mu, \sigma^2_{ij}(m)) \), with \( \sigma^2_{ij}(m) = \lambda^2/m^2 \) for \( i = j \) and \( \sigma^2_{ij}(m) = \theta \lambda^2/m^2 \) for \( i \neq j \) (Litterman, 1986). Economical variables are often characterized by high persistence, implying \( \mu = 1 \). EEG and MEG signals have a strong tendency for mean reversals and lack sustained drifts, entailing \( \mu = 0 \), the prior belief of colored noise. The Minnesota prior is based on the beliefs that recent lags provide more information than distant lags \( (1/m^2) \), matching the decay of synaptic potentials, and that a time series is a better forecaster of itself than of other time series \( (\theta \in [0, 1]) \), matching the preponderance of local over global neural connectivity. Yet, it remains to be seen which settings of \( (\lambda, \theta) \) capture the typical characteristics of multivariate brain signals. An advantage of the Minnesota prior is that a reasonable \% of systems is stable, even for large \( M \) and \( \lambda \). We evaluated the performance of the G-causal measures
for \( M = 20 \) and to generalize results, we took different values of \((\theta, \lambda)\) (systems that were not stable were directly rejected). Both PSI and reversed G-causality testing failed to detect G-causal dominance for a substantial \% of systems (Fig 4.1). Even for small \( \lambda \), PSI had a FP rate that largely exceeded the standard 5\% criterion. The \% of TP for reversed G-causality testing was highest for small \( \lambda \), but declined more rapidly with an increase in \( \lambda \). The Minnesota prior reveals the most difficult regime (that still yields realistic peak coherences) for reversed Granger testing, namely \( \lambda \) large and \( \theta \) small (Fig 4.1).

### 4.2.2 Effect of independent measurement noise

We evaluated the G-causality measures for both the normal and the Minnesota prior. We drew \( \epsilon \) from the same prior as \( x \). Our analysis does not make any assumptions about the
Figure 4.2: (Color online) (A) Independent noise, unidirectional prior. Fraction of TP and FP for PSI (black, squares), reversed G-causality testing (green, diamonds), and $g$ (Granger; red, triangles), as a function of noise level $\gamma$. Narrow-dashed, wide-dashed and solid correspond to $\sigma_\epsilon = (0.01, 0.1, 0.3)$. (B) As (A), but now for bidirectional normal prior. (C) As (A), but now with dependent noise. (D) As (C), but now for bidirectional normal prior. (E) As (D), but rejecting systems for which cross-covariance function has global peak at $\tau = 0$. (F) Minnesota prior with $\theta = 0.3$, $M = 10$ and $\lambda = 0.01, 0.1, 0.3$. (Left) independent noise. (Middle) dependent noise. (Right) same as (Middle), but now rejecting systems for which cross-covariance function has a global peak at $\tau = 0$. (A-F) Values of $f_{j \rightarrow i}$ and $f_{j \rightarrow i}^{rev}$ were obtained by analytically solving the Yule-Walker equations for $s(\tau)$ and $s_\epsilon(\tau)$, and then solving for $\Lambda^{(\epsilon)}$. Because of added noise, $\Lambda^{(\epsilon)}(\tau)$ can be non-zero for $\tau > M$, but decays at steep exponential rate. We computed the order $M^{(\epsilon)}$ by increasing $M^{(\epsilon)}$ until all $\Lambda^{(\epsilon)}(M^{(\epsilon)} + 1), \ldots, \Lambda^{(\epsilon)}(M^{(\epsilon)}/2)$ coefficients had absolute values $< 10^{-4}$ smaller than the respective coefficients in $\Lambda^{(\epsilon)}(1), \ldots, \Lambda^{(\epsilon)}(M)$ (the original order of $\mathbf{x}$ and $\mathbf{\epsilon}$). Using $M^{(\epsilon)} = M$ (as in (Nolte et al., 2008)) or spectral matrix factorization yielded highly similar results. PSI was directly computed from $S^{(\epsilon)}(\omega)$, with the same settings as above.
distribution of the noise. All measures were minimally affected by noise for unidirectional systems (Fig 4.2A). For bidirectional systems, PSI was minimally affected by noise, whereas $g$ was moderately affected, but only for large, not for small $\sigma_A$ (Fig 4.2B and F), as predicted above. While the number of TP was strongly decreased for high variance normal priors for reversed G-causality testing, it yielded relatively few (<0.1) FP (Fig 4.2A, B and F).

### 4.2.3 Effect of dependent measurement noise

Previous simulations by (Nolte et al., 2008) were performed for unidirectional systems with finite sample data traces. The question remains whether the observed lack of FP (maximum $\approx 5\%$) was due to limited sampling, and what the asymptotic behavior of the PSI is. We took a normal prior on $B \sim N(\mu, \sigma_B)$ with $\sigma_B = 1$, $\mu = 0$, as volume conduction coefficients can be either negative or positive, and $S = 2$, like (Nolte et al., 2008). The scaling of the noise through $\gamma$ makes our analysis invariant to $\sigma_B$. Our simulations conform Nolte et al. (2008)’s observation that $g = f_{1 \to 2} - f_{2 \to 1}$ fails with mixed noise. However, they show that PSI also has a large number of FP with mixed noise (Fig 4.2C-D), even for unidirectional systems (Fig 4.2C). In fact, $g$ performed better than PSI for a normal (bidirectional) prior (Fig 4.2D). Similar results were obtained for the Minnesota prior (Fig 4.2F). On the other hand, we find that reversed G-causality testing yielded a much smaller fraction of FP (maximum <0.1-0.15) even for high $\gamma$ (Fig 4.2C-F)). The discrepancy with the results for PSI of Nolte et al. (2008) are explained by Nolte et al. (2008) considering only finite samples ($M = 5$, $n = 6 \cdot 10^4$) and uni-directional systems, which decreased the % of FP (we verified that the fraction of TP and FP converged to those shown in Fig 2 with increasing sample size). Thus, the paradoxical conclusion is that PSI performs better (in terms of FP) for limited samples; it is not a priori obvious however how limited the sample size must be, as this depends on our prior on $A$.

Thus, reversed Granger testing is a much more conservative procedure than the PSI. A further ‘safety handle’ is suggested by our finding that the % of FP for $g$, PSI and $g^{\text{rev}}$ is much higher in the presence than in the absence of a global peak or valley in $s_{12}^{(\epsilon)}$ at $\tau = 0$, i.e. no observable time-delay in the interactions. Thus, the absence of such a peak/valley is a good indicator of a low expected FP ratio.

### 4.3 Discussion

We have investigated the question to what extent a causally dominant variable can be identified without noise, and with independent or dependent noise, in the limit of infinite sampling. We summarize: 1) For bidirectional VAR priors, alternative causality measures like PSI and reversed Granger testing fail to identify the causal dominant variable in up to 65-80% of cases. These failures are more problematic for PSI than reversed Granger testing as PSI (in the infinite sampling regime) always identifies a causally dominant variable, while reversed Granger testing does not have FP without noise. The shrinkage Minnesota prior yields a particularly difficult regime, with compromised TP fraction and realistic coherence values. 2) Independent noise has moderate effects on standard Granger causality (in terms of identifying the dominant variable), and has no effect in the limiting case that signals are close to white and coherence values are small. PSI and reversed Granger testing have compromised fractions of TP for a broad range of priors, making them less attractive measures for the inde-
pendent noise case. 3) Dependent noise (i.e. volume conduction) has much stronger effects on Granger causality than independent noise. Theoretical analysis shows that PSI can be affected by volume conduction, and our simulations show that it is strongly affected, contrary to previous results in the finite sampling regime (Nolte et al., 2008). Reversed Granger testing has compromised TP fraction relative to standard Granger measures (Granger, 1969), but yields a much smaller fraction of FP and thus provides a viable solution for identifying the causally dominant variable in the presence of volume conduction. Shrinkage Minnesota prior yields the most difficult regime with largest FP fractions for standard Granger and PSI, and smallest TP fraction for reversed Granger testing. 4) If volume conduction dominates, then the cross-covariance function tends to have a peak or valley at t = 0 (besides becoming more symmetric, a property utilized by PSI). We show that the presence of a peak/valley at delay τ = 0 indicates a much smaller (larger) fraction of FP (TP) than the absence of a peak/valley.

4.4 Appendix A: Computational details for figures

Fig 1 and 2: PSI values were obtained by computing $S(\omega)$ analytically and using the broadband frequency spectrum, sampled at $10^3$ frequencies (without loss of generalization). Starting from the VAR model

$$x(t) = \sum_{\tau=1}^{M} A(\tau)x(t-\tau) + U(t), \quad (4.12)$$

the spectral density matrix is obtained by using the stochastic Cramer representation for the wide-sense stationary time series $x_j = \int_{-\pi}^{\pi} e^{i\omega \tau} dZ_j(\omega)$ (Granger, 1969; Percival & Walden, 1993) (in analogy to the Fourier Transform of a deterministic signal) such that

$$\begin{pmatrix} a_{11}(\omega) & a_{12}(\omega) \\ a_{21}(\omega) & a_{22}(\omega) \end{pmatrix} \begin{pmatrix} dZ_1(\omega) \\ dZ_2(\omega) \end{pmatrix} = \begin{pmatrix} dZ_u_1(\omega) \\ dZ_u_2(\omega) \end{pmatrix}, \quad (4.13)$$

where $a_{ij}$ is defined in terms of the Discrete Fourier Transform of the coefficients

$$a_{ij}(\omega) = \delta_{ij} - \sum_{\tau=1}^{M} A_{ij}(\tau)e^{-i\omega\tau}. \quad (4.14)$$

The Hermitian spectral density matrix is now given as

$$S(\omega) \equiv \begin{pmatrix} dZ_1(\omega) \\ dZ_2(\omega) \end{pmatrix} \begin{pmatrix} \bar{dZ}_1(\omega) & \bar{dZ}_2(\omega) \end{pmatrix} = H(\omega)\Sigma H^*(\omega) \quad (4.15)$$

where $H(\omega) \equiv a(\omega)^{-1}$. Thus, the spectral density matrix can be easily computed starting from the coefficients of a given VAR model. Note that using matrix spectral factorization, the VAR model can also be obtained from the $S$ in return (Dhamala et al., 2008).