Testing distributional assumptions in psychometric measurement models with substantive applications in psychology

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Publication date
2012

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Non-normality in the Second-order Factor Model: Modeling Ability Differentiation

In this Chapter we present factor models to test for ability differentiation. Ability differentiation predicts that the size of IQ subtest correlations decreases as a function of the general intelligence factor. In the Schmid-Leiman decomposition of the second-order factor model, we model differentiation by introducing heteroscedastic residuals, non-linear factor loadings, and a skew-normal second-order factor distribution. Using marginal maximum likelihood, we fit this model to Spanish standardization data of the WAIS III to test the differentiation hypothesis.

4.1 Introduction

This Chapter concerns modeling of ability differentiation in the second-order factor model. Ability differentiation refers to the possible dependence of the magnitude of the positive correlations among IQ subtest scores on the level of the common factor "general intelligence" or *g* (Jensen, 1998). Spearman (1927) hypothesized that these correlation are higher in low *g*-samples than in high *g*-samples. Attempts to test this hypothesis have usually involved standard statistical modeling, the results obtained have been mixed. For instance, Jensen (2003) found support for the effect, Hartmann & Teasdale (2004) found the opposite effect, and Facon (2004) found no effect. One dominant approach involves the creation and subsequent comparison of two or more subgroups that are supposed to differ with respect to the mean level of *g*. These groups are formed on the basis of observed scores, such as scores on one or more of the IQ subtests. These subgroups are then compared with respect to the first principal component of the covariance matrix of the subtests not used in creating the groups (e.g., Deary et al., 1996), or with respect to the average correlation among the subtests (e.g., Detterman & Daniel, 1989). The creation of subgroups on the basis of observed criteria has the drawback that the number of subgroups is arbitrary, and that such selection results in a distortion of the covariance structure in the subgroups (Muthén, 1989; Nesselroade & Thompson, 1995). In subsequent analyses, any non-normality in the data

* This chapter is published as: Molenaar, D., Dolan, C.V., & van der Maas, H.L.J. (2011). Modeling ability differentiation in the second-order factor model. *Structural Equation Modeling*, 18, 578-594.

may suggest differentiation, while the non-normality is unrelated to differentiation (e.g., due to floor or ceiling effects). Carlstedt (2001) used structure preserving factor scores to create the subgroups and fitted multi-group factor models. Whether such factor scores are structure preserving in selected subgroups is doubtful. However, this procedure is certainly less sensitive to factor structure distortion, and therefore preferable.

Recently, other approaches have been suggested to model ability differentiation. These have the advantage that they do not require the creation of an arbitrary number of subgroups, and that they allow one to investigate various hypotheses concerning the exact source of differentiation in the factor model. The first approach involves a class of models based on *Marginal Maximum Likelihood estimation* (MML; Bock & Aitkin, 1981). Using this approach, Hessen & Dolan (2009) presented a one factor model, in which the residual variances depended on the common factor. From the perspective of ability differentiation, this model identifies heteroscedasticity of the residuals as the source of the differentiation. That is, if a higher level of the common factor is associated with more subtest specific residual variance, correlations among the subtests will be lower for an increasing level of the common factor. Tucker-Drob (2009) and Molenaar, Dolan, and Verhelst (2010) showed that differentiation may arise when the magnitude of the factor loadings depends on the common factor. If a higher level of the common factor is associated with smaller factor loadings, subtest correlations will be lower for an increasing level of the common factor. In addition, Molenaar, et al. (2010) established that a non-normal common factor distribution can account for a differentiation effect. Specifically, if the distribution of the common factor is negatively skewed, less probability mass is present in the upper range of the common factor-scores, resulting in lower correlations among the indicators of the common factor in this range. In the model proposed in that paper, all effects are included simultaneously, i.e., ability differentiation may be manifested in the factor loadings, the factor distribution, and/or the residual variances.[17]

A second, related, approach involves *moderated non-linear factor models* recently discussed by Bauer & Hussong (2009; see also Purcell, 2002). In this approach, the parameters in the one factor model are modeled as a function of an observed moderator variable (e.g., age). These models have important applications as models for measurement invariance with respect to a continuous background variable, and as models to analyze data from multiple studies (see Bauer & Hussong, 2009). Models for ability differentiation as developed by Hessen & Dolan (2009), Tucker-Drob (2009) and Molenaar et al. (2010) can be interpreted as an instance of these

[17] As Molenaar et al. (2010) show, the effect on the factor distribution and the factor loadings cannot be combined as this results in an empirically underidentified model. However, the effect on the residuals could be combined with each of these two effects. We elaborate on this below.

moderated non-linear factor models in which the parameters of the common factor model are moderated by the common factor itself.

The approaches to modeling ability differentiation discussed above have the drawback that they are limited to the one factor model. The aim of the present Chapter is to present an extension to the second-order factor model, in which the covariances among first order common factors are modeled by posting a second order *g* factor (Jensen, 1998). Specifically, we focus on the Schmid-Leiman hierarchical factor model (Schmid & Leiman, 1957), and the bi-factor factor model (Holzinger, & Swineford, 1937). Following Hessen and Dolan (2009), Tucker-Drob (2009), and Molenaar et al (2010), we fit the model using MML. In so doing, we cast ability differentiation in terms of heteroscedastic residuals in the measurement model, non-linear second-order factor loadings, and/or the distribution of the second-order common factor.

The outline of the present Chapter is as follows: first we introduce the Schmid-Leiman hierarchical factor model. In this model, we introduce heteroscedastic residuals, non-linear second-order factor loadings, and a non-normal factor distribution. Subsequently, MML estimation of the parameters is discussed. We then apply the model to Spanish standardization data of the WAIS-III, and evaluate the presence of ability differentiation in these data. We end with a general discussion.

4.2 The Schmid-Leiman Hierarchical Factor Model

Let $\mathbf{y}_i$ denote the *p*-dimensional column vector of observed variables for subject *i*. In standard factor analysis, $\mathbf{y}_i$ is linearly related to *q* first-order common factors. We refer to the associated factor model as the measurement model (Mellenbergh, 1994), i.e.,

$$\mathbf{y}_i = \boldsymbol{\upsilon}_i + \boldsymbol{\Lambda}\, \boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i, \qquad (1)$$

where $\boldsymbol{\upsilon}$ is a *p*-dimensional column vector of intercepts, $\boldsymbol{\Lambda}$ is a full column rank *p* x *q* matrix of factor loadings, $\boldsymbol{\eta}_i$ is a *q*-dimensional column vector of latent common factor scores, and $\boldsymbol{\varepsilon}_i$ is a *q*-dimensional column vector of residuals. We note that $\boldsymbol{\Lambda}$ need not necessarily to display simple structure (each indicator loading only on one common factor), but should contain sufficient elements fixed to 0 to avoid rotational indeterminacy. In the structural model, the first-order common factors are regressed on *r* higher-order factors, where $r < q$, i.e.,

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \boldsymbol{\Gamma}\, \boldsymbol{\xi}_i + \boldsymbol{\delta}_i, \qquad (2)$$

where $\boldsymbol{\alpha}$ denotes the *q*-dimensional column vector of intercepts, $\boldsymbol{\Gamma}$ denotes the full column rank *q* x *r* matrix of second-order factor loadings, $\boldsymbol{\xi}_i$ denotes the *r*-dimensional column vector of second-order factor scores, and $\boldsymbol{\delta}_i$ denotes the q-dimensional vector of second-order residuals. Here we limit our presentation to the situation in which r = 1, i.e., the second-order common factor model, with general intelligence as a single second-order common factor. By assuming cov($\boldsymbol{\xi}$, $\boldsymbol{\varepsilon}$) = cov($\boldsymbol{\xi}$, $\boldsymbol{\delta}$) = cov($\boldsymbol{\delta}$, $\boldsymbol{\varepsilon}$) = 0, the model implied covariance matrix equals

$$\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\,(\boldsymbol{\Gamma}\,\boldsymbol{\varphi}\,\boldsymbol{\Gamma}^{\mathrm{T}} + \boldsymbol{\Psi})\;\boldsymbol{\Lambda}^{\mathrm{T}} + \boldsymbol{\Theta}, \tag{3}$$

where $\boldsymbol{\varphi}$ is a 1 x 1 covariance matrix of ξ, $\boldsymbol{\Psi}$ is the q x q diagonal covariance matrix of $\boldsymbol{\delta}$, and $\boldsymbol{\Theta}$ is the p x p diagonal covariance matrix of $\boldsymbol{\varepsilon}$. By further assuming that E($\boldsymbol{\varepsilon}$) = E($\boldsymbol{\delta}$) = 0, the mean vector, $\boldsymbol{\mu}$, equals

$$\boldsymbol{\mu} = \boldsymbol{\upsilon} + \boldsymbol{\Lambda}\,\boldsymbol{\Gamma}\,\boldsymbol{\tau} + \boldsymbol{\Lambda}\,\boldsymbol{\alpha}, \tag{4}$$

where $\boldsymbol{\tau}$ denotes the mean of $\boldsymbol{\xi}$. See Figure 4.1 for a graphical representation of the second-order common factor model including the parameters.

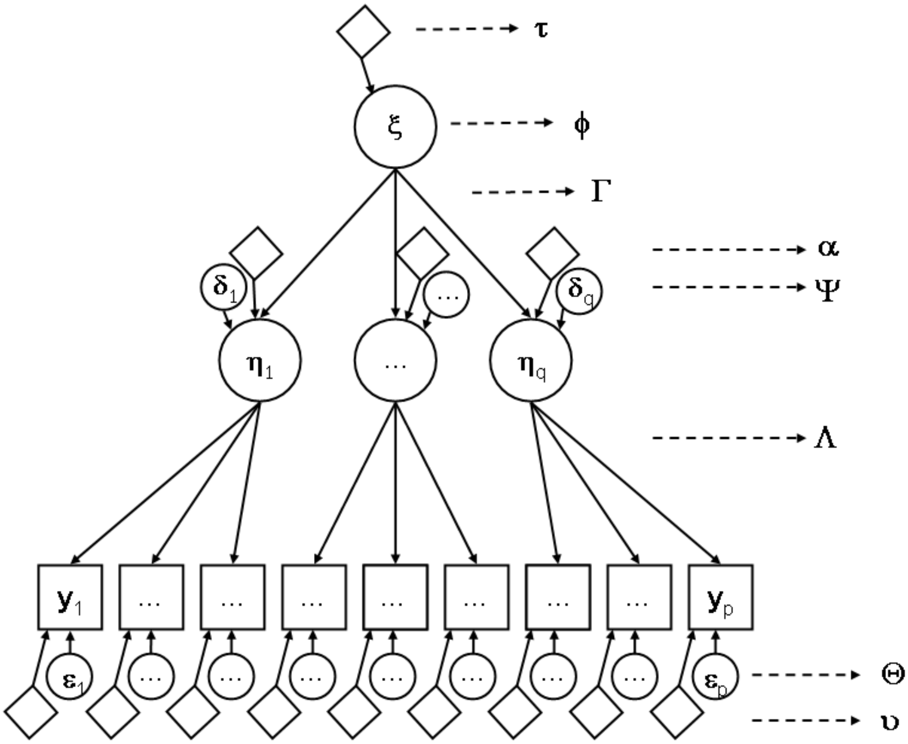

Figure 4.1. A graphical representation of the second-order common factor model. Circles denote latent variables, squares denote observed variables, and diamonds denote constants. Note that in this representation the model does not include identification constraints.

In the Schmid-Leiman decomposition of the second-order factor model, $\boldsymbol{\Sigma}$ is modeled as

$$\mathbf{\Sigma} = \mathbf{\Lambda}^{*}\,\mathbf{\Psi}^{*}\,\mathbf{\Lambda}^{*} + \mathbf{\Theta}, \tag{5}$$

where

$$\mathbf{\Lambda}^{*} = [\mathbf{\Lambda\Gamma} \mid \mathbf{\Lambda}], \quad \text{and} \tag{6}$$

$$\mathbf{\Psi}^{*} = \begin{bmatrix} \boldsymbol{\varphi} & \mathbf{0}^{T} \\ \mathbf{0} & \mathbf{\Psi} \end{bmatrix}, \tag{7}$$

where ‘|’ denotes horizontal adhesion, and **0** denotes a column vector of zeros. Equation 5 may be written as

$$\mathbf{\Sigma} = \mathbf{\Lambda\Gamma}\,\boldsymbol{\varphi}\,(\mathbf{\Lambda\Gamma})^{\mathbf{T}} + \mathbf{\Lambda\Psi\Lambda}^{\mathbf{T}} + \mathbf{\Theta}. \tag{8}$$

We note that $\mathbf{\Lambda\Gamma}\,\boldsymbol{\varphi}\,(\mathbf{\Lambda\Gamma})^{\mathbf{T}}$ is a rank one matrix. If $\mathbf{\Lambda}$ is characterized by simple structure, $\mathbf{\Lambda\Psi\Lambda}^{T} + \mathbf{\Theta}$ is block diagonal. In Equation 8, if one substitutes the unconstrained p x 1 matrix **K** for $\mathbf{\Lambda\Gamma}$, i.e.,

$$\mathbf{\Sigma} = \mathbf{K}\,\boldsymbol{\varphi}\,\mathbf{K}^{\mathbf{T}} + \mathbf{\Lambda\Psi\Lambda}^{\mathbf{T}} + \mathbf{\Theta}, \tag{9}$$

the model is known as the bi-factor model (Holzinger, & Swineford, 1937; Yung, Thissen, & McLeod, 1999). See Figure 4.2 for a graphical representation of the Schmidt-Leimand factor model and the bi-factor model. Assuming that $\mathbf{\Theta}$ is diagonal, this may be viewed as a first-order orthogonal common factor model, with one general factor and q group factors (or actually q residual group factors, as these factors account for the residual common variance, not explained by the general first-order g factor).

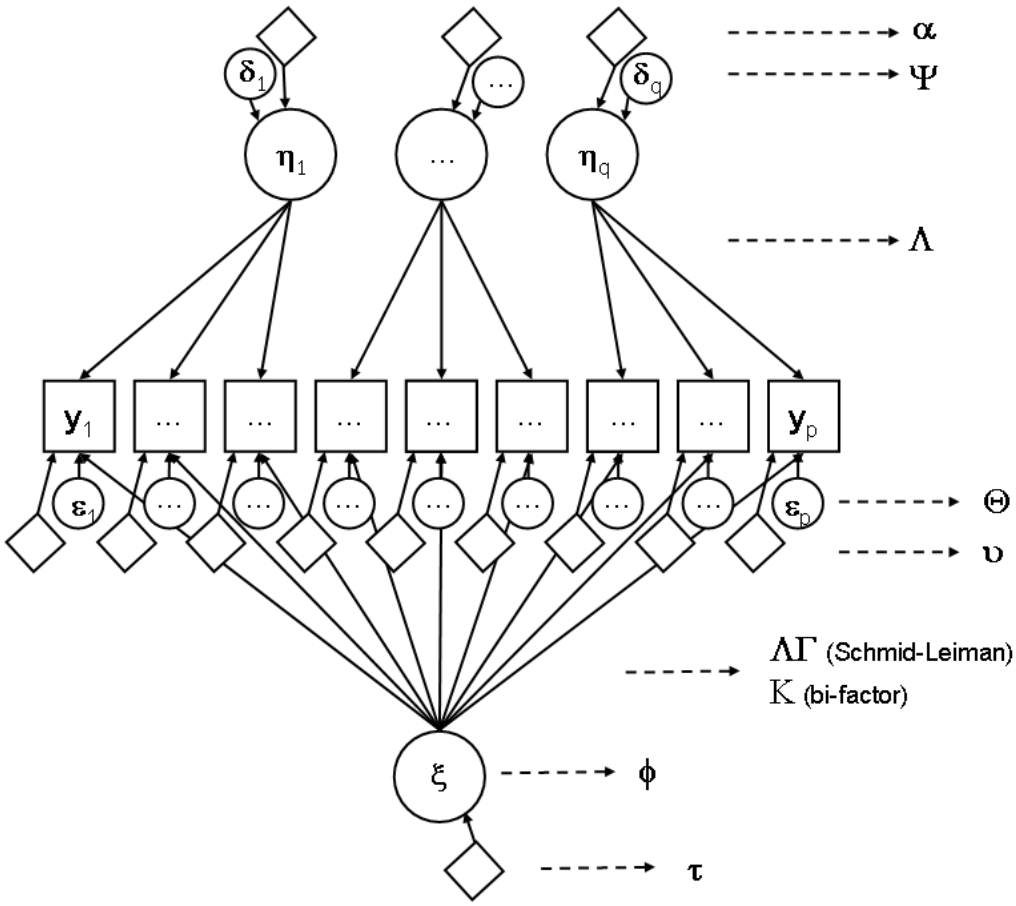

Figure 4.2. A graphical representation of the Schmid-Leiman factor model and the bi-factor model. The model as depicted again does not include identification constraints.

Assuming **y** is multivariate normally distributed; the parameters can be estimated using normal theory maximum likelihood (Lawley, 1943; Azzelini, 1996). The standard assumptions are that the common factors (first and second-order) are normally distributed, that the residuals are normally distributed and homoscedastic, and that the regressions are all linear. As discussed, ability differentiation implies specific violations of (all or some of) these assumptions. Using MML as outlined in Hessen and Dolan (2009), Molenaar et al (2010), these assumptions may be relaxed by introducing a non-normal common factor distribution, non-linear factor loadings, and heteroscedasticity of the residuals. This enables statistical tests of differentiation, as operationalized at different locations within the factor model. Here, we consider non-normality of the second-order common factor *g*, non linearity of the loadings in the *q* x

1 matrix $\mathbf{\Gamma}$ (in the second order model) or in the *p* x 1 matrix $\mathbf{K}$ (in the bi-factor model), and, again, heteroscedastic residuals.

4.3 Model extensions

4.3.1 Non-normal factor distribution

In the one-factor model, Molenaar et al. (2010) used the so-called skew-normal density (Azzalini, 1985; 1986) to study the normality of the common factor distribution. The skew-normal density function is given by

$$h(\xi \mid \kappa, \omega, \zeta) = \frac{2}{\omega} \times \Phi\left(\zeta \frac{\eta - \kappa}{\omega}\right) \times \phi\left(\frac{\eta - \kappa}{\omega}\right), \tag{10}$$

where Φ(.) and φ(.) refer to the standard normal distribution and density function respectively. In the skew-normal density function, κ and ω are a location and scale parameter, and ζ is a shape parameter. From these parameters, the mean and variance of the common factor is calculated according to (Azzalini & Capatanio, 1990):

$$E(\xi) = \kappa + \omega \sqrt{\frac{2}{\pi}} \delta \tag{11}$$

and

$$Var(\xi) = \omega^2 \left(1 - \frac{2\delta^2}{\pi}\right), \tag{12}$$

where

$$\delta = \frac{\zeta}{\sqrt{1 + \zeta^2}}. \tag{13}$$

If ζ equals 0, the skew-normal density function simplifies to a normal density function, therefore a test of normality may be formulated in terms of $\zeta = 0$ vs $\zeta \neq 0$. As differentiation predicts decreasing correlations across the common factor, a negatively skewed common factor distribution is taken as evidence for the differentiation hypothesis, i.e., $\zeta < 0$.

4.3.2 Non-linear factor loadings

Tucker-Drob (2009) and Molenaar et al. (2009) proposed the notion of level dependent factor loadings to test for differentiation. Within the second-order factor model this generalizes to

$$\mathbf{\Gamma}_{\xi} = f\,(\,\mathbf{\Gamma}_0 + \mathbf{\Gamma}_1\,\xi\,), \tag{14}$$

where the subscript in $\mathbf{\Gamma}_{\xi}$ indicates that the second-order factor loadings are conditional on ξ. In addition, $\mathbf{\Gamma}_0$ is a q x 1 matrix of second-order baseline factor loadings, $\mathbf{\Gamma}_1$ is a q x 1 matrix of level-dependency parameters with elements γ_{k1} for $k = 1,\ldots, q$, and $f(.)$ is a suitable link-function. In the one factor model, Tucker-Drob (2009) used the identity link, in which case, Equation 14 simplifies to quadratic factor loadings (e.g., McDonald, 1967). Molenaar et al. (2010) also investigated the logistic and exponential link functions. The hypothesis that $\gamma_{k1} < 0$ for all k, is consistent with differentiation, as then correlations among the subtest scores decrease across the level of the second-order common factor.

4.3.3 Heteroscedastic residuals

In the one factor model, Hessen & Dolan (2009) proposed to study heteroscedasticity of the residual variances by modeling the logarithm of the residual variances as a linear function of the common factor. We adopt this approach in the Schmid-Leiman model, i.e.,

$$\log \mathit{diag}(\,\mathbf{\Theta}_{\xi}\,) = \boldsymbol{\beta}_0 + \boldsymbol{\beta}_1\,\xi, \tag{15}$$

where the subscript in $\mathbf{\Theta}_{\xi}$ indicates that the covariance matrix of the residual variances is conditional on ξ. In addition, $\boldsymbol{\beta}_0$ is a p-dimensional vector with baseline parameters, and $\boldsymbol{\beta}_1$ is a p-dimensional vector of heteroscedasticity parameters with elements β_{j1} for $j = 1, \ldots, \boldsymbol{p}$**.** A test of homoscedasticity involves testing whether the vector $\boldsymbol{\beta}_1$ departs significantly from a vector of zeros using a likelihood ratio test (Buse, 1982; Hessen & Dolan, 2009). More specifically, we can investigated which (if any) β_{i1} departs from zero, i.e., establish heteroscedasticity in a given indicator in the model. However, as differentiation predicts decreasing subtest correlations across the common factor, residual variances are hypothesized to increase across the common factor, i.e., $\beta_{j1} > 0$ for all j.

2.2.4 Marginal maximum likelihood

Using MML estimation, we can estimate the parameters in the Schmid-Leiman hierarchical factor model with heteroscedastic residuals, non-linear second-order factor loadings, and a skew-normal second-order factor distribution. We will focus on a single sample. Consequently in Equation 4, elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\tau}$ are fixed to zero for reasons of identification. Using a multivariate normal distribution conditional on ξ in the measurement model, we obtain the following conditional distribution for the data

$$f(\mathbf{y}_i \mid \mathbf{v}, \xi) = (2\pi)^{-p/2} \left|\boldsymbol{\Sigma}_\xi\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\left(\mathbf{y}_i - \boldsymbol{\mu}_\xi\right)^T \left[\boldsymbol{\Sigma}_\xi\right]^{-1}\left(\mathbf{y}_i - \boldsymbol{\mu}_\xi\right)\right) \tag{16}$$

where $\mathbf{v}$ is the vector of unknown parameters from the Schmid-Leiman factor model or bi-factor model, and $\boldsymbol{\Sigma}_\xi$ and $\boldsymbol{\mu}_\xi$ are the expected conditional covariance matrix and conditional mean vector under the measurement model. The conditional covariance matrix is obtained by conditioning on ξ in Equation 3,

$$\boldsymbol{\Sigma}_\xi = \boldsymbol{\Lambda\Psi\Lambda}^T + \boldsymbol{\Theta}_\xi \tag{17}$$

which is block diagonal if $\boldsymbol{\Lambda}$ is characterized by simple structure. In Equation 17, the conditional residual variance matrix, $\boldsymbol{\Theta}_\xi$, is given by Equation 15. We now obtain the conditional mean vector by conditioning on ξ in Equation 4, resulting in

$$\boldsymbol{\mu}_\xi = \boldsymbol{\upsilon} + \boldsymbol{\Lambda\Gamma}_\xi\xi \tag{18}$$

where $\boldsymbol{\Gamma}_\xi$ is given by Equation 14. When a bi-factor model is considered as opposed to the Schmid-Leiman model, we have

$$\boldsymbol{\mu}_\xi = \boldsymbol{\upsilon} + \mathbf{K}_\xi\xi \tag{19}$$

where $\mathbf{K}_\xi$ could then be level dependent in a similar way as Equation 14, i.e.,

$$\mathbf{K}_\xi = f\,(\,\mathbf{K}_0 + \mathbf{K}_1\xi\,) \tag{20}$$

where $\mathbf{K}_1$ contains the level dependency parameters, k_{j1}, for $j = 1,...,p$.

As ξ is a nuisance parameter, it can be integrated out, resulting in the marginal density of the data, i.e.,

$$g(\mathbf{y}_i \mid \mathbf{v}) = \int_{-\infty}^{\infty} f(\mathbf{y}_i \mid \mathbf{v}, \xi) h(\xi \mid \kappa, \omega, \zeta) d\xi \tag{21}$$

where *h*(.) is the distribution of the common factor in the structural model, i.e., the skew-normal density function given in Equation 10. From Equation 21, it follows that the log-marginal likelihood of the data of a random draw of *N* subjects is given by

$$\ell(\mathbf{v} \mid \mathbf{y}_i) = \sum_{i=1}^{N} \log \int_{-\infty}^{\infty} f(\mathbf{y}_i \mid \mathbf{v}, \xi) h(\xi \mid \kappa, \omega, \zeta) d\xi \tag{22}$$

The integral can be approximated using Gauss-Hermite Quadrature (GHQ) approximation, yielding

$$\ell(\mathbf{v} \mid \mathbf{y}_i) \approx \sum_{i=1}^{N} \log \left(\sum_{t=1}^{T} W_t^{'} \times f(\mathbf{y}_i \mid \mathbf{v}, N_t^{'}) \right) \quad , \tag{23}$$

where

$$N_t^{'} = \sqrt{2}\omega N_t + \kappa \quad , \tag{24}$$

$$W_t^{'} = \frac{1}{\sqrt{\pi}} W_t \times \Phi\left(\zeta \sqrt{2}\omega N_t\right), \tag{25}$$

i.e., $N_t^{'}$ and $W_t^{'}$ are transformations of the weights (W_t) and nodes (N_t) of the GHQ-approximation as found in standard tables (e.g., Stroud & Secrest, 1966). See the appendix of Molenaar et al. (2010) for a detailed derivation within the one-factor model.

4.3.5 Identification

To identify the model in equation 23, standard scaling constraints should be imposed (see Bollen, 1989, p. 238). That is, for each factor in the model a factor loading or the factor variance should be constrained to equal 1. Molenaar et al (2010) established that, given these standard scaling constraints, 1) heteroscedastic residuals and non-linear factor loadings can be combined in a single model; and 2) heteroscedastic residuals can be combined with a skew-normal second-order factor distribution. However, the effects on the residual variances and the factor distribution are not estimated simultaneously as this results in severe empirical underidentification.

4.3.6 Fitting the model

The model can be fitted using MML as outlined above in the freely available software package Mx (Neale, Boker, Xie, & Maes, 2006). The syntax files are available from www.dylanmolenaar.nl.

4.4 Application

In this section we apply the present approach to modeling differentiation in the Schmid-Leiman factor model. The data comprise Spanish standardization data of the WAIS-III (García, Ruiz, & Abad, 2003). The dataset consists of the scores of 1369 subjects on 14 cognitive ability subtests: vocabulary (VO), similarities (SI), arithmetic (AR), digit span (DS), information (IN), comprehension (CO), letter–number series (LN), picture completion (PC), digit–symbol coding (DC), block design (BD), matrix reasoning (MA), picture arrangement (PA), symbol search (SS), and object assembly (OA). The means, standard deviations and QQ-plots of the subtest scores are depicted in Figure 4.3. A Shapiro-Wilks test on univariate normality revealed that all subtests departed significantly from normality (all p-values < .001). The factor structure of the data is depicted in Figure 4.4 (see García, et al., 2003). In fitting the model, we used 15 quadrature points.[18]

4.4.1 Results

In the present application, analyses are limited to the data of young adults (aged 16 – 34 years; N = 588). The scores on the subtests are scaled to have approximately equal variances (between 10-20) to facilitate the model estimation procedure. We fitted four models to these data. For the goodness of fit statistics of these models, see Table 4.1. We considered the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), the sample size adjusted BIC (saBIC), and the Deviance Information

[18] We considered this as sufficient as it is generally recommended to use 10 (Vermunt & Magidson, 2005) or 15 (Muthén & Muthén, 2007) quadrature points per dimension. We also tried 25 quadrature points, but parameters estimates changed by less than 0.01 (relative to the 15 point solution).

Criterion (DIC). Additionally we conducted likelihood ratio tests to compare nested models.

We started by fitting the Schmid-Leiman decomposition of the common factor model shown in Figure 4.4. Following Garcia et al. (2003), we used a second-order common factor to account for the correlations between the first-order common factors. This model served as a baseline; see Table 4.1 for the fit indices obtained with MML.

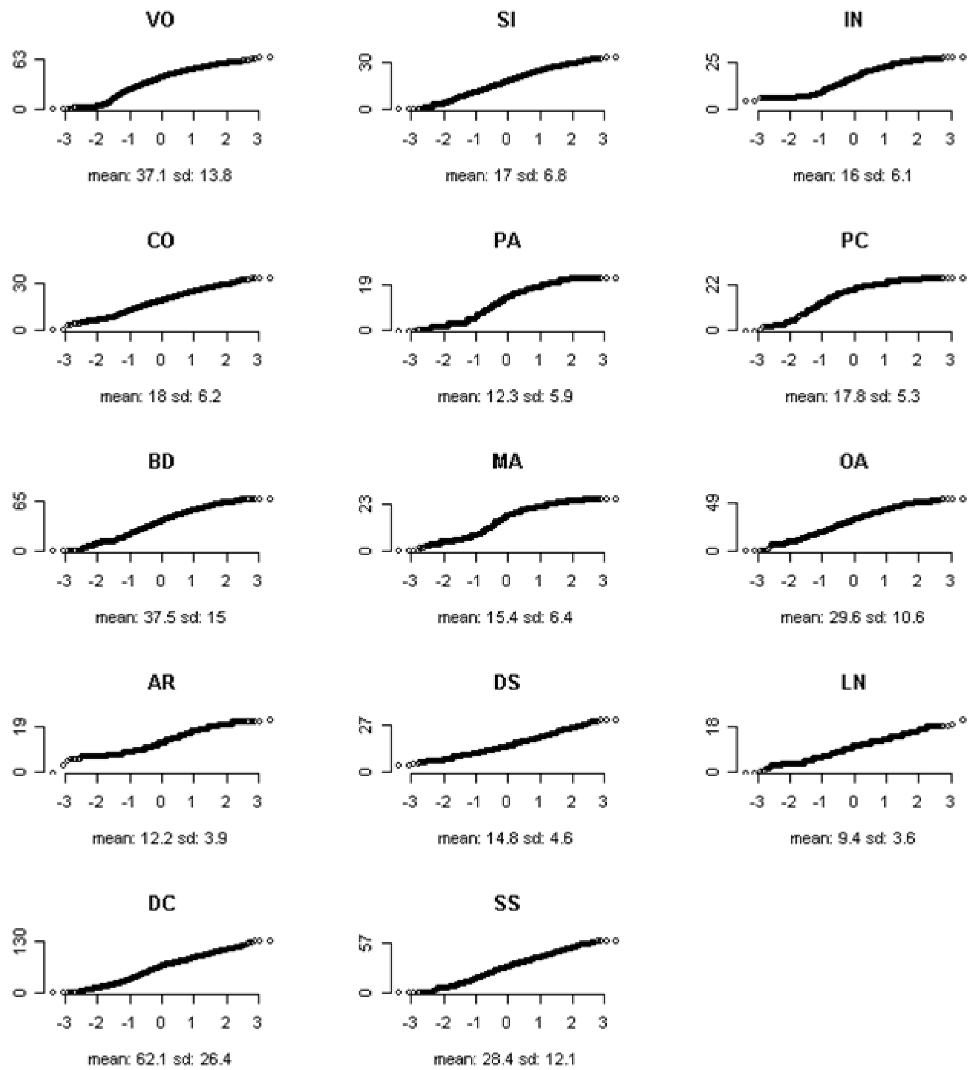

Figure 4.3. Means, standard deviations, and QQ-plots of the observed subtest scores.

Table 4.1.
Fit indices for the four models fitted

Model	-2ℓ	df	AIC	BIC	saBIC	DIC
1: baseline	41299.034	8179	24941.034	-5428.108	7554.657	2087.890
2: HSR	41059.783	8165	24729.783	-5503.096	7457.446	2000.037
3: HSR & NLF	41037.704	8161	24715.704	-5501.382	7452.811	1998.075
4: HSR & skew	41052.059	8164	24724.059	-5503.770	7455.185	1998.445
5:BF HSR & NLF	40983.092	8155	24673.092	-5509.558	7435.111	1984.385

Note. HSR: Heteroscedastic residuals. NLF: Non-linear factor loadings. BF: Bi-factor model. saBIC: sample size adjusted BIC.

As all the fit indices in Table 4.1 are comparative, we established the absolute fit of the baseline model by fitting it using standard full information maximum likelihood. We judged the baseline model to fit acceptable, χ^2 (71) = 175.2, RMSEA=.050.

Next, we fitted Model 2 using MML. In this model, the heteroscedasticity parameters β_{j1} were freed for all $j = 1,\ldots,14$. Parameter estimates and 99% confidence intervals are in Table 4.2.[19] The fit of Model 2 was better than that of Model 1, as judged by the AIC, BIC, saBIC, and DIC. In addition, the likelihood ratio test was significant: χ^2 (14) = 239.2. Judged by the confidence intervals, the subtests VO, IN, CO, PA, PC, BD, MA, and DS were associated with heteroscedastic residuals. Figure 4.5 shows how these eight residual variances vary as a function of *g*. As indicated by the negative estimates of β_{i1} for all these subtests bar DS, the residual variances get smaller for higher levels of the second-order common factor. This effect is thus not in line with the ability differentiation hypothesis, which predicts the opposite effect.

Next, in Model 3, we introduced quadratic second-order factor loadings (i.e., equation 14 with an identity link). This improved all model fit indices from Table 4.1 except the BIC. The likelihood ratio test was significant χ^2 (4) = 22.1. Non-linearity parameters, γ_{k1} were significantly different from 0, and in the predicted direction according to the differentiation hypothesis (i.e., smaller than zero) for the factors VC and PO; see Table 4.2. Figure 4.6 shows how the second-order factor loadings vary as a function of *g*.

[19] Confidence intervals are based on the likelihood profile (Neale & Miller, 1997).

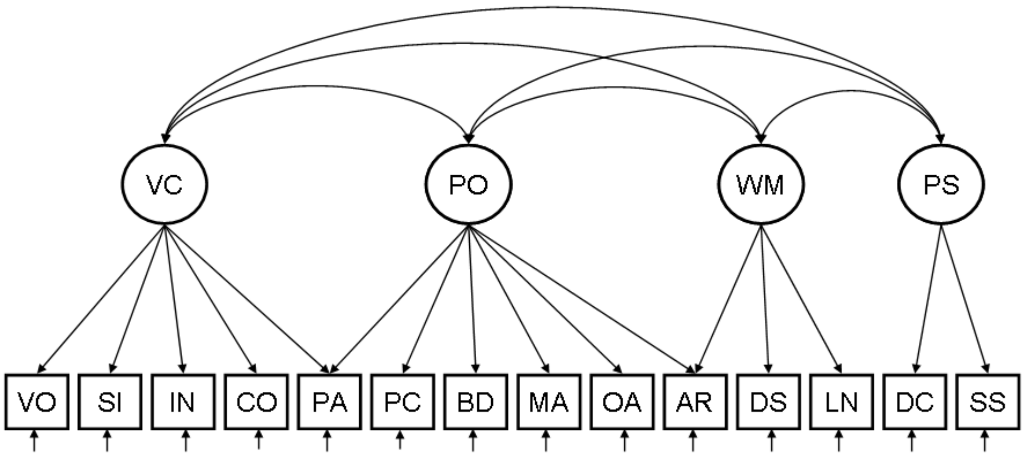

Figure 4.4. Factor structure of the Spanish standardization data of the WAIS-III.

As non-linear second-order factor loadings appear to be supported by the data, it may be feasible to account for these non-linear effects by introducing a skew-normal distribution in the model instead of the non-linear factor loadings. This model is parsimonious, as we estimate one parameter rather than four. Thus in Model 4, γ_{k1} are fixed to 0 for all *k*, and the shape-parameter, ζ, was freed to test for normality in the common factor distribution. The estimate of ζ was significant according and indicated that the second-order common factor was negatively skewed, see Table 4.2. The pattern of conclusions concerning the subtest residual variances remained largely unchanged, see Table 4.2.[20] In evaluating model fit, a significant improvement is obtained compared to Model 2, $\chi^2(1) = 7.7$.[21] However, when comparing the model to Model 3, it can be seen from Table 4.1 that the AIC, saBIC, and DIC favor Model 3 with the non-linear factor loadings, while only the BIC favor Model 4 with the skewed common factor distribution.

[20] The only difference is that in Model 2 subtest DS is associated with a significant estimate of β_{j1}, while in Model 3 this parameter is not significant anymore.
[21] Model 4 is not nested under Model 3, a comparison in terms of a likelihood ratio is therefore not possible.

Table 4.2.

Estimates (99% confidence intervals) of the heteroscedasticity parameters and shape parameter.

	Variable	Model 2	Model 3	Model 4	Model 5
Heteroscedasticity parameters, β_{j1}	VO	-0.520 (-0.745; -0.316)	-0.516 (-0.750; -0.303)	-0.508 (-0.729; -0.307)	-0.524 (-0.780; -0.314)
	SI	-0.214 (-0.445; 0.016)	-0.202 (-0.430; 0.024)	-0.193 (-0.418; 0.031)	-0.206 (-0.426; 0.010)
	IN	-0.346 (-0.592; -0.103)	-0.363 (-0.614; -0.117)	-0.314 (-0.554; -0.080)	-0.327 (-0.550; -0.105)
	CO	-0.194 (-0.381; -0.010)	-0.185 (-0.373; -0.003)	-0.189 (-0.373; -0.004)	-0.170 (-0.175; 0.034)
	PA	-0.429 (-0.629; -0.239)	-0.433 (-0.636; -0.239)	-0.400 (-0.596; -0.215)	-0.434 (-0.645; -0.240)
	PC	-0.498 (-0.721; -0.295)	-0.508 (-0.731; -0.297)	-0.455 (-0.692; -0.256)	-0.571 (-0.790; -0.346)
	BD	-0.240 (-0.426; -0.059)	-0.229 (-0.419; -0.046)	-0.245 (-0.426; -0.071)	-0.273 (-90.342; -0.033)
	MA	-0.673 (-0.695; -0.454)	-0.677 (-0.927; -0.445)	-0.620 (-0.852; -0.408)	-0.570 (-0.780; -0.375)
	OA	-0.164 (-0.367; 0.034)	-0.165 (-0.371; 0.036)	-0.166 (-0.373; 0.031)	-0.162 (-0.391; 0.047)
	AR	0.019 (-0.194; 0.231)	0.007 (-0.207; 0.221)	0.045 (-0.161; 0.251)	0.054 (-0.157; 0.267)
	DS	0.148 (0.060; 0.436)	0.141 (-0.105; 0.413)	0.161 (0.129; 0.451)	0.152 (-0.071; 0.619)
	LN	-0.096 (-0.470; 0.228)	-0.092 (-0.470; 0.231)	-0.112 (-0.485; 0.210)	-0.255 (-2.388; 134.897)
	DC	0.037 (-0.190; 0.275)	0.046 (-0.190; 0.289)	0.047 (-0.181; 0.285)	0.085 (-0.150; 0.325)
	SS	-0.005 (-0.267; 0.267)	0.025 (-0.242; 0.301)	0.004 (-0.243; 0.264)	0.000 (-0.220; 0.221)
Non-linearity parameters, γ_{k1}	VC	-	-0.279 (-0.479; -0.093)	-	-
	PO	-	-0.186 (-0.373; -0.030)	-	-
	WM	-	0.023 (-0.279; 0.328)	-	-
	PS	-	-0.164 (-0.413; 0.062)	-	-
ζ	g	-	-	-1.710 (-2.478; -0.608)	-1.944 (-2.557; -1.420)

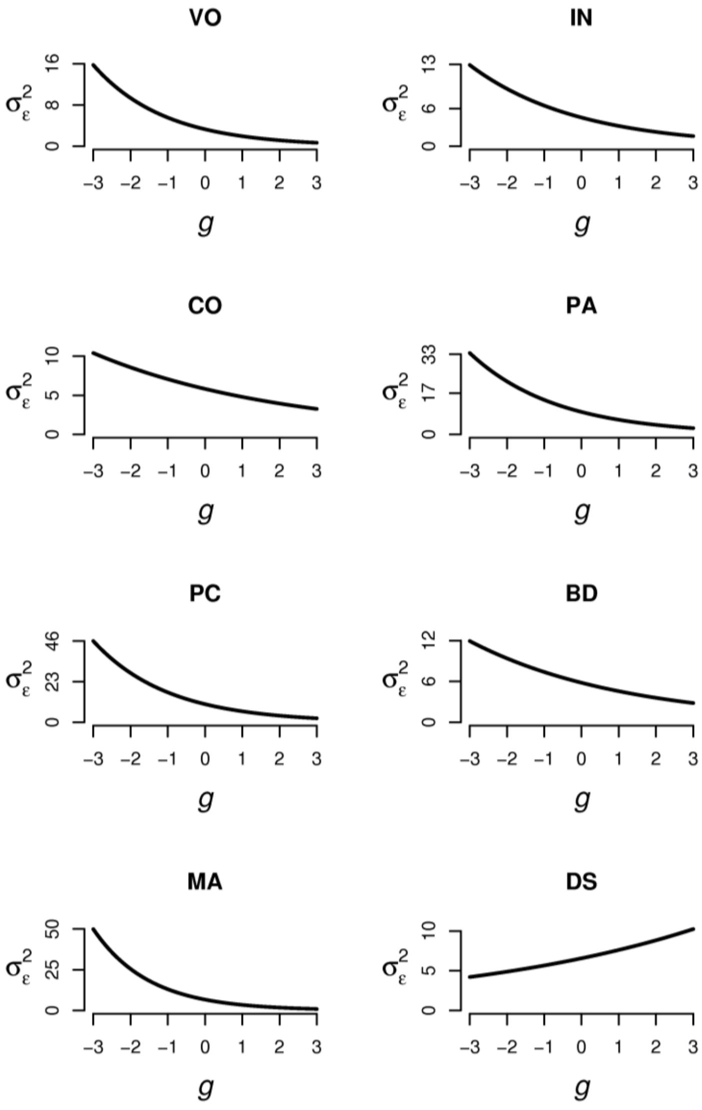

Figure 4.5. The residual variances as a function of *g* in Model 2. Only the residual variances that showed a significant effect are plotted.

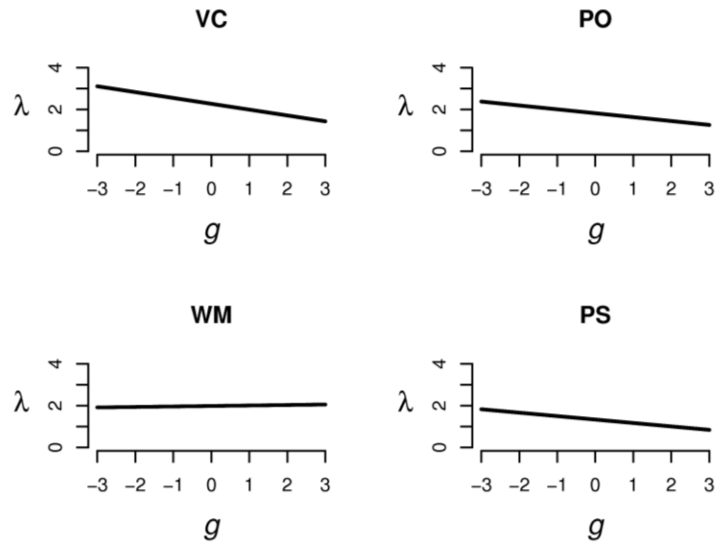

Figure 4.6. The second-order factor loadings as a function of *g* in Model 3.

Finally, we fitted Model 5, i.e., the bi-factor model, in which **K** is estimated instead of **Γ** (see Equation 19). Taking the previous modeling results into account, we fitted a bi-factor model with heteroscedastic residuals and a skew-normal factor distribution. We chose the latter as this model is more parsimonious. The goodness of fit of Model 5 is given in Table 4.1, the parameter estimates are in Table 4.2.[22] Pattern of results are largely the same as in the second-order factor model with skew-normal factor distribution and heteroscedastic residuals (Model 3). Only 2 residual variances (CO and DS) display significant heteroscedasticity in Model 3, but are non-significant in Model 5. As the second-order factor model is nested in the bi-factor model (Yung, et al 1999), we can compare the fit of Model 5 and Model 3 using a likelihood ratio. On the basis of the result of the test ($\chi^2(9) = 54.6$), Model 5 is preferable.

4.4.2 Conclusion

In the Spanish standardization data of the WAIS-III (Figure 4.4) we found some differentiation effects. We found a negatively skewed second-order common factor distribution and non-linear second order factor loadings, which are both consistent with the differentiation hypothesis. That is, *g* appears to be a weaker source of individual differences at the higher end of its range. It is difficult to choose among the model with non-linear second order factor loadings and the model with a skewed second-order factor distribution, as the fit indices of these models differed little. Our results

[22] We equated the factor loadings of both indicators of the PS factor, as without this restriction the model is unidentified.

concerning the residual variances are ambiguous. That is, 8 of the 14 subtests have heteroscedastic residual variances, but 7 of these are decreasing with increasing g, which is incompatible with the differentiation hypothesis.

We also applied the bi-factor model with a skew-normal factor distribution and heteroscedastic residuals. The results were similar to those obtained in the second-order factor model with these same effects. However, the bi-factor model fitted better compared to the second-order factor model. This is not unexpected as the bifactor model is less parsimonious (see also Gignac, 2008). However, the second-order factor model enjoys wider acceptance for substantive reasons (e.g., Johnson & Bouchard, 2004; Jensen, 1998). Taken the evidence together, we conclude that the evidence for differentiation in the Spanish standardization data is mixed. We return to the results in the discussion.

4.5 Discussion

Present undertaking was inspired by recent studies into modeling of ability differentiation in the one-factor model (Hessen & Dolan, 2009; Tucker-Drob, 2009; and Molenaar et al., 2010). We extended the one-factor approach to the second-order factor model by using the Schmid-Leiman decomposition of the second-order factor model. This resulted in a second-order factor model with heteroscedastic residuals, non-linear second-order factor loadings, and a non-normal second-order factor distribution. Using this model, it is possible to posit and test several possible loci of differentiation in the higher-order common factor model.

We consider the present approach to investigate ability differentiation an improvement over previous approaches, as these lacked an explicit statistical model of differentiation, and/or were limited to the one factor model. We have developed our approach to study differentiation in subtests of multidimensional intelligence tests, as it is at that level that ability differentiation has been defined and had been studied. We note that this may be problematic, because differentiation effects at item level may hard to detect at subtest level, as the subtest score is a linear combination of the item scores. A potential drawback of subtest scores is also that they may suffer floor and ceiling effects, which may produce heteroscedastic residual variances in the factor model. Of course, ceiling and floor effects can be evaluated by visual inspection. Still, a discrete or item level factor model approach to ability differentiation will be useful, as differentiation, if it exists, should be detectable at the item level. The discrete factor model itself is well developed (Wirth & Edwards, 2007), and in view of Bock and Aitkin (1981) is amenable to MML estimation

In applying the model to the Spanish standardization data of the WAIS-III, we found effects on the second-order factor loadings/the second-order factor distribution and the residual variances. As it turned out, the former effects were consistent with the differentiation hypothesis, while the effects on the residual variances were not. If ability differentiation is attributed strictly to the diminishing role of *g* as a source of individual differences in cognitive test batteries, the non-linear second-order factor loadings/skewed second-order factor distribution can be taken as evidence for differentiation. But the inconsistent heteroscedasticity of residual variances then require some other explanation. These results illustrate why it is important to consider various possible loci of differentiation in the common factor model. In the present approach we can pinpoint the locus of each effect, and evaluate these individually. In forming high and low *g* groups and conducting a PCA, for instance, it is not possible to disentangle distinct effects (e.g., a non-normal *g* distribution vs. subtest heteroscedasticity). The results of such analyses will reflect whichever effect happens to be the strongest or whichever effect happens to exert the greatest effect on the PCA. This may explain the very mixed results of differentiation research to date.

The present approach is related to Reynolds, Keith & Beretvas (2010), who propose the application of factor mixtures to test for ability differentiation. Bauer (2005b) showed how factor mixture models are related to non-linear factor model (i.e., the present approach). Like the present approach, the mixture approach is based on the assumption of conditional multivariate normality. However, the mixture approach assumes normality within class while the present approach assumes normality conditional on *g*, which is less stringent.

Finally, we note that the present model was developed for a single group. However, extensions to multi-group models are straightforward. Implementations in the software package Mx (Neale, et al., 2006) are available from www.dylanmolenaar.nl. The multi-group extension enables more elaborate tests on differentiation, especially when two groups differ with respect to the mean *g* score. It could then be investigated whether the source of differentiation is the same within the groups and between the groups. For instance, within groups the subtest scores could be associated with heteroscedastic residuals while between groups the *g* variance may differ. This may be an interesting undertaking in the near future.